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Journal of Symbolic Computation 41 (2006) 909–942
Journal of Symbolic Computation
www.elsevier.com/locate/jsc

Integration in finite terms with elementary functions and dilogarithms[☆]

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Received 11 October 2005; accepted 13 March 2006
Available online 30 May 2006

Abstract

In this paper, we report on a new theorem that generalizes Liouville's theorem on integration in finite terms. The new theorem allows dilogarithms to occur in the integral in addition to transcendental elementary functions. The proof is based on two identities for the dilogarithm, that characterize all the possible algebraic relations among dilogarithms of functions that are built up from the rational functions by taking transcendental exponentials, dilogarithms, and logarithms. This means that we assume the integral lies in a transcendental tower.

Keywords: Dilogarithms; Elementary functions; Integration in finite terms; Differential algebra

1. Introduction

In 1967, Rosenlicht (1968) published an algebraic proof of Liouville's theorem on the problem of integration in finite terms with elementary functions, based on the notions of differential algebra. In 1972, Moses (1972) started discussing the problem of extending Liouville's result to include non-elementary functions in the integral. He asked whether a given expression has an integral within a class of expressions of the form $F(V_i)$, where F is a given special function and (V_i) is a finite set of functions lying in the ground field. Singer et al. (1985) proved an extension of Liouville's theorem allowing logarithmic integrals and error functions to occur in addition

☆ This work was supported in part by the US Air Force Office of Scientific Research Grant No. AFOSR-85-0264.
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0747-7171/$ - see front matter
doi:10.1016/j.jsc.2006.03.003

to elementary functions. However the techniques used in their proofs don't apply directly to special functions such as the dilogarithm since the later has "non-elementary" identities. Also the dilogarithm is more complex than logarithmic integrals and error functions, in the sense that if an integrand has an integral which can be expressed using dilogarithms, these can have derivatives which contain logarithms transcendental over the field of integrand. Coleman (1982) produced an analytic characterization of the identities of the dilogarithm for rational functions. We show that two identities of the dilogarithm, in addition to the identities among primitives and the identities among exponentials, are required to generate all algebraic relations among dilogarithms and logarithms of functions built up from the rational functions by taking transcendental exponential logarithms and dilogarithms. Our proof uses Ostrowski's theorem (Ostrowski, 1946) in several places. Given these two identities we generalize Liouville's theorem to include dilogarithms in the integral, in addition to transcendental elementary functions. The basic conclusion is that an associated function to the dilogarithm, if dilogarithms appear in the integral, appears linearly, with logarithms appearing in a possible non-linear way.

2. Dilogarithmic elementary extensions

Definition. A differential ring is a commutative ring R together with a derivation of R into itself, that is, a map $R \to R$ which if $x \to x'$ satisfies the two rules

$$(x+y)' = x' + y'$$
$$(xy)' = x'y + xy'.$$

In a differential ring we have $(x^n)' = nx^{n-1}x'$ for $n = 1, 2, 3, \ldots$. In particular setting $x = 1, n = 2$ we have $1' = 0$.

Definition. A differential field is a differential ring that is a field. If u, v are elements of a differential field and $v \neq 0$ we have the relation $\left(\frac{u}{v}\right)' = \frac{(u'v-uv')}{v^2}$.

Elements of derivative zero are called constants and in a differential field the totality of constants is itself a field, the subfield of constants.

If u, v are elements of a differential field such that $v \neq 0$ and $u' = \frac{v'}{v}$, in analogy with the classical situation we say that u is a logarithm of v or that v is an exponential of u.

Definition. If k is a differential field of characteristic zero, we call K a differential extension of k if K is a field extension of k and is itself a differential field such that the derivation on K, when restricted to k, is identical to the derivation on k.

Let k be a differential field of characteristic zero. The subfield of constants of k will be denoted by C. Let K be a differential extension such that $K = k(t)$ for some $t \in K$. An element $t \in K$ is called elementary if the field of constants of k is the same field of constants of K and t satisfies one of the following:

(1) $t' = \frac{a'}{a}$ for some $a \in k^*$. In this case, we write $t = \log a$ and call t logarithmic over k.
(2) $t' = a't$ for some $a \in k$. In this case, we write $t = \exp a$ and call t exponential over k.
(3) t is algebraic over k.

Definition. A differential extension field of a differential field is said to be elementary if this extension has the same subfield of constants as the base field and if there exists a finite tower of intermediate fields starting with the given base field and ending with the given extension field, such that each field in the tower after the first is obtained from its predecessor by the adjunction of a single element that is elementary over the preceding field.

That is a differential field extension F of k is said to be elementary over k if F and k have the same field of constants and if F can be resolved into a tower:

$$F = F_n \supseteq F_{n-1} \supseteq \cdots \supseteq F_1 \supseteq F_0 = k$$

such that $F_i = F_{i-1}(\theta_i)$, where, for each i, $1 \leq i \leq n$ one of the following holds:

(i) $\theta_i' = \frac{\phi'}{\phi}$ for some nonzero ϕ in F_{i-1}, which we write as $\theta_i = \log \phi$. We say that θ_i is logarithmic over F_{i-1}.
(ii) $\theta_i' = \phi' \theta_i$ for some ϕ in F_{i-1}, which we write as $\theta_i = \exp \phi$. We call θ_i exponential over F_{i-1}.
(iii) θ_i is algebraic over F_{i-1}.

Proposition (*See Rosenlicht (1968)*). *Let F be a differential field of characteristic zero and K an extension field of F. Then there exists a differential field structure on K that is compatible with that of F and with the field structure of K. This differential field structure on K is unique if K is algebraic over F and in any case induces a differential field structure on any subfield of K that contains and is algebraic over F.*

Proof. Let D be a derivation on F. We want to show that D extends to a derivation on K. Assume first that $K = F(X)$, with X transcendental over F and consider the map:

$$D_0 : F[X] \to F[X]$$

defined by:

$$D_0 \left(\sum_{i=0}^{n} a_i X^i \right) = \sum_{i=0}^{n} D(a_i) X^i$$

if $a_0, a_1, \ldots, a_n \in F$. This is a derivation of $F[X]$ extending D.

We extend D_0 to the field $K = F(X)$ by setting, for $u, v \in F[X]$, $v \neq 0$,

$$D_0 \left(\frac{u}{v} \right) = \frac{((D_0 u)\, v - (D_0 v)\, u)}{v^2}.$$

Suppose next that $K = F(x)$ with x algebraic over F. Let X be an indeterminate over F and let $f(X) \in F[X]$ be the minimal polynomial of x over F. The map

$$\frac{\partial}{\partial X} : F[X] \to F[X]$$

defined by:

$$\frac{\partial}{\partial X} \sum_{i=0}^{n} a_i X^i = \sum_{i=0}^{n} i a_i X^{i-1}$$

if $a_0, a_1, \ldots, a_n \in F$, is a derivation of $F[X]$ that annuls each element of F. So for any $g(X) \in F[X]$ the additive map $D_0 + g(X) \frac{\partial}{\partial X}$ is a derivation of $F[X]$ that extends D.

Setting $f'(X) = \left(\frac{\partial}{\partial X} \right) f$, we have $f'(x) \neq 0$ and since $F(x) = F[x]$ we can find a particular $g(X) \in F[X]$ such that

$$(D_0 f)(x) + g(x) f'(x) = 0.$$

So $D_0 + g(X)\frac{\partial}{\partial X}$ maps $f(X)$ into a multiple of itself, hence maps the ideal $F[X]f(X)$ into itself, hence induces a derivation on the factor ring $F[X]/F[X]f(X)$ which is isomorphic to $F(x)$. This gives us the desired extension of D to $K = F(x)$.

Thus D can be extended to a derivation of any simple extension field of F. If K is an arbitrary extension field of F then using the above and Zorn's lemma D can be extended to K. To complete the proof it suffices to show that if D_1 and D_2 are two derivations of the field K that agree on the subfield F and $x \in K$ is algebraic over F then $D_1 x = D_2 x$. Considering the derivation $D_1 - D_2$ of K, we have to show that any derivation of K which annuls all of F also annuls each $x \in K$ that is algebraic over F. For this we note that if $f(X) \in F[X]$ is the minimal polynomial of x over F then we have $0 = (f(x))' = f'(x) \cdot x'$, so that $x' = 0$.

Let k be a differential field of characteristic zero. A differential field extension F of k is said to be dilogarithmic-elementary over k if F and k have the same subfield of constants and if F can be resolved into a tower:

$$F = F_n \supseteq F_{n-1} \supseteq \cdots \supseteq F_1 \supseteq F_0 = k$$

such that $F_i = F_{i-1}(\theta_i', \theta_i)$, where for each i, $1 \le i \le n$ one of the following holds:

(i) $\theta_i' = \frac{\phi'}{\phi}$ for some nonzero ϕ in F_{i-1}, which we write as $\theta_i = \log\phi$. We say that θ_i is logarithmic over F_{i-1}.
(ii) $\theta_i' = \phi'\theta_i$ for some ϕ in F_{i-1}, which we write as $\theta_i = \exp\phi$. We call θ_i exponential over F_{i-1}.
(iii) $\theta_i' = -\left(\frac{\phi'}{\phi}\right)u$, where $\phi \in F_{i-1} - \{0, 1\}$, and u is such that $u' = \frac{(1-\phi)'}{(1-\phi)}$. In this case, we write $\theta_i = \ell_2(\phi)$ and call θ_i dilogarithmic over F_{i-1}. We note, in this case, that θ_i is defined up to the addition of a constant multiple of a logarithm over F_{i-1} since u is defined up to a constant. We don't assume, however, that u lies in F_{i-1}.
(iv) θ_i is algebraic over F_{i-1}.

Roughly speaking condition (iii) means that θ_i is the composition of the function ϕ with the dilogarithmic function $\ell_2(x)$ defined as:

$$\ell_2(x) = -\int_0^x \frac{\log(1-t)}{t}\mathrm{d}t.$$

If K is a differential extension of k such that $K = k(t)$ for some $t \in K$ and $t' = a \in k$, we call t primitive over k and write $t = \int a$.

Definition. If k is a differential field of characteristic zero, K a differential field extension of k such that $K = k(t, u, v)$, we say that $t = D(\phi)$, D is the Bloch–Wigner–Spence function of ϕ, if ϕ is an element of $k - \{0, 1\}$ and:

$$t' = -\frac{1}{2}\frac{\phi'}{\phi}u + \frac{1}{2}\frac{(1-\phi)'}{(1-\phi)}v$$

where $u' = \frac{(1-\phi)'}{(1-\phi)}$ and $v' = \frac{\phi'}{\phi}$. From this definition, since u and v are defined up to additive constants, it follows that t is defined up to the addition of a linear combination of $\log\phi$ and $\log(1-\phi)$ with constant coefficients. Informally, t is equal to:

$$\ell_2(\phi) + \frac{1}{2}\log\phi\log(1-\phi).$$

Definition. For two differential fields k and K we say that K is a Liouvillian extension of k if there exist $t_1, \ldots, t_n \in K$ such that $K = (t_1, \ldots, t_n)$ and each t_i is either elementary or primitive over $k(t_1, \ldots, t_{i-1})$.

Here are a few results that are used repeatedly in what follows. First we recall a theorem of Kolchin (1968).

Theorem 1. *Let k be a differential field of characteristic zero and let $K = k(\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_r)$ where each η_i is primitive over k and each ξ_j is an exponential over k. We also assume that k and K have the same subfield of constants. If $\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_r$ are algebraically dependent over k then there exists either a nontrivial relation of the form $\sum_{i=1}^{n} c_i \eta_i \in k$ where each c_i is a constant or else one of the form $\prod_{j=1}^{r} \xi_j^{e_j} \in k$ where each e_j is an integer and there exists j_0 such that $e_{j_0} \neq 0$.*

As a result of the previous theorem we deduce Ostrowski's theorem (Ostrowski, 1946).

Corollary 1. *Let k be a differential field of characteristic zero and let $K = (\eta_1, \ldots, \eta_n)$ where each η_i is primitive over k. We also assume that k and K have the same subfield of constants. If $\eta_1, \ldots, \eta_n$ are algebraically dependent over k then there exist constants c_i $(1 \leq i \leq n)$, not all of which are zero, such that $\sum_{i=1}^{n} c_i \eta_i \in k$.*

We deduce from Corollary 1, using Linear Algebra, the following corollary.

Corollary 2. *Let k be a differential field of characteristic zero and let $K = k\ (\log v_1, \ldots, \log v_n)$ where $v_i \in k, 1 \leq i \leq n$, we also assume that k and K have the same field of constants. Suppose that $\log v_1, \ldots, \log v_r$ $(0 \leq r \leq n)$ are algebraically independent over k and that $k(\log v_1, \ldots, \log v_r)$ and K have the same transcendence degree r over k. Then, there exist constants c_{ij} $(1 \leq i \leq r, r < j \leq n)$, and $s_j \in k$ $(r < j \leq n)$ such that:*

$$\log v_j = \sum_{i=1}^{r} c_{ij} \log v_i + s_j, \quad \textit{for } j \in \{r+1, \ldots, n\}$$

and if $r = 0, \log v_j \in k$ for all $j \in \{1, \ldots, n\}$.

We use also the following lemma due to Rosenlicht and Singer (1977).

Lemma. *Let $k \subset K$ be differential fields of characteristic zero with the same field of constants C assumed to be algebraically closed. Assume that k is a Liouvillian extension of C and that K is algebraic over k. Suppose that $c_1, \ldots, c_n \in C$ are linearly independent over Q, that $u_1, \ldots, u_n \in K^*, v \in K$, and that we have:*

$$\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' \in k.$$

Then, $v \in k$ and there is a nonzero integer N such that $u_i^N \in k, i = 1, \ldots, n$.

The rest of this section is devoted to the statement and proof of one of the main results of this paper.

Definition. Let k be a differential field of characteristic zero. We call an expression S a simple elementary-dilogarithmic expression over k if:

$$S = g + \sum_{i \in I} c_i \log w_i + \sum_{j \in J} (s_j \log(1 - h_j) + t_j \log h_j + d_j D(h_j))$$

where I and J are finite sets, $g, w_i, s_j, t_j, h_j \in k$ and c_i, d_j are constants.

Lemma 1. *Let k be a differential field of characteristic zero, which is a Liouvillian extension of its subfield of constants C assumed algebraically closed. Suppose that we have an expression of the form:*

$$\int f = g + \sum_{i \in I} c_i \log w_i + \sum_{j \in J} (s_j \log(1 - h_j) + t_j \log h_j + d_j D(h_j)) \tag{2.1}$$

where I and J are finite sets, $f \in k, s_j, t_j, g,$ and w_i are algebraic over k, $h_j \in k$ and c_i, d_j are constants. Then we can write $\int f = S$, where S is a simple elementary-dilogarithmic expression over k. (So, we get g, w_i, s_j, t_j in k instead of being algebraics.)

Proof. Let K be a finite normal algebraic extension field of k that contains $g, w_i (i \in I)$, $s_j, t_j (j \in J)$ (the smallest normal extension containing $k(g, w_1, \ldots, w_i, \ldots, s_1, \ldots, s_j, \ldots, t_1, \ldots, t_j, \ldots)$). Consider the vector space E over k spanned by the vectors

$$1, \log h_1, \ldots, \log h_j, \ldots, \log(1 - h_1), \ldots, \log(1 - h_j), \ldots.$$

Then, we choose among these vectors a k-basis $(1, e_1, \ldots, e_N)$ for E. By Corollary 2, we can write:

$$\log h_j = \sum_{m=1}^{N} a_{jm} e_m + p_j, \quad a_{jm} \in C, p_j \in k \tag{$*$}$$

$$\log(1 - h_j) = \sum_{m=1}^{N} b_{jm} e_m + q_j, \quad b_{jm} \in C, q_j \in k. \tag{$**$}$$

We claim that $1, e_1, \ldots, e_N$ are still linearly independent over K.

Otherwise, and by Corollary 2, there exist constants α_m $(2 \le m \le N)$ and $Q_0 \in K$ such that:

$$e_1 = \sum_{m=2}^{N} \alpha_m e_m + Q_0 \Rightarrow e_1' = \sum_{m=2}^{N} \alpha_m e_m' + Q_0'. \tag{2.2}$$

By assumption, $e_m = \log H_m$ $(1 \le m \le N)$, where $H_m \in \{(1 - h_1), \ldots, h_1, \ldots\}$.

Let $\gamma_{0=1}, \gamma_1, \ldots, \gamma_r$ be a vector space basis for the Q-span of $1, \alpha_2, \ldots, \alpha_N$, and write:

$$\alpha_m = \sum_{i=0}^{r} n_{mi} \gamma_i$$

with each $n_{mi} \in Q$. Replacing each γ_i by $\frac{\gamma_i}{LCD(n_{mi})}$ if necessary, we can assume $n_{mi} \in Z$ (where *LCD* means Least Common Denominator).

So we can write (2.2) as:

$$\frac{(H_1)'}{H_1}\gamma_0 = \sum_{i=0}^{r} \gamma_i \frac{(H_2^{n_{2i}} H_3^{n_{3i}} \dots H_N^{n_{Ni}})'}{H_2^{n_{2i}} H_3^{n_{3i}} \dots H_N^{n_{Ni}}} + Q_0'$$

which can be written as:

$$\gamma_0 \frac{(H_1^{-1} H_2^{n_{20}} \dots H_N^{n_{N0}})'}{H_1^{-1} H_2^{n_{20}} \dots H_N^{n_{N0}}} + \sum_{i=1}^{r} \gamma_i \frac{(H_2^{n_{2i}} H_3^{n_{3i}} \dots H_N^{n_{Ni}})'}{H_2^{n_{2i}} H_3^{n_{3i}} \dots H_N^{n_{Ni}}} + Q_0' = 0. \tag{2.3}$$

Using the Rosenlicht–Singer lemma, we deduce that $Q_0 \in k$. Investigating (2.2) again, we get:

$$e_1 = \sum_{m=2}^{N} \alpha_m e_m + Q_0$$

with $Q_0 \in k, \alpha_m \in k \cap C = C$. This is a contradiction, since the e_m $(1 \le m \le N)$ and 1 were assumed to be linearly independent over k. So, $1, e_1, \dots, e_N$ are linearly independent over K.

Now, we write (2.1) in terms of the relations $(*)$ and $(**)$:

$$\int f = g_0 + \sum_{m=1}^{N} r_m e_m + \sum_{j\in J} d_j D(h_j) + \sum_{i\in I} c_i \log w_i \tag{2.4}$$

(where $g_0 \in K, r_m \in K$).

Taking the derivative of the previous relation, we obtain:

$$\begin{aligned} f &= g_0' + \sum_{i\in I} c_i \frac{w_i'}{w_i} + \sum_{m=1}^{N} r_m e_m' + \sum_{m=1}^{N} r_m' e_m - \frac{1}{2}\sum_{j\in j} d_j \frac{h_j'}{h_j}\log(1-h_j) \\ &\quad + \frac{1}{2}\sum_{j\in J} d_j \frac{(1-h_j)'}{(1-h_j)}\log h_j. \end{aligned} \tag{2.5}$$

Using again the relations $(*)$ and $(**)$ for $\log h_j$ and $\log(1-h_j)$, and assembling coefficients of (2.5) according to the K-basis $(1, e_1, \dots, e_N)$, we obtain:

$$f = g_0' + \sum_{i\in I} c_i \frac{w_i'}{w_i} + \sum_{m=1}^{N} r_m e_m' - \frac{1}{2}\sum_{j\in J} d_j \frac{h_j'}{h_j} q_j + \frac{1}{2}\sum_{j\in J} d_j \frac{(1-h_j)'}{(1-h_j)} p_j \tag{2.6}$$

(the above is the coefficient of the vector 1), and:

$$r_m' - \sum_{j\in J}\frac{1}{2} d_j b_{jm}\frac{h_j'}{h_j} + \sum_{j\in J}\frac{1}{2} d_j a_{jm}\frac{(1-h_j)'}{(1-h_j)} = 0 \quad 1 \le m \le N \tag{2.7}$$

(the above is the coefficient of the vector e_m).

From (2.7) we deduce that $r_m \in k$ (using the Rosenlicht–Singer lemma and exactly the same argument used in the above proof).

Assume that $M = [K : k]$. For any $\sigma \in \mathrm{Aut}(K/k)$, we have using (2.6):

$$\begin{aligned} f &= \sigma(f) = \sigma(g_0') + \sum_{i\in I} c_i \frac{\sigma(w_i)'}{\sigma(w_i)} + \sum_{m=1}^{N} r_m e_m' - \frac{1}{2}\sum_{j\in J} d_j \frac{h_j'}{h_j} q_j \\ &\quad + \frac{1}{2}\sum_{j\in J} d_j \frac{(1-h_j)'}{(1-h_j)} p_j. \end{aligned}$$

Taking the sum over all the σ's in $\mathrm{Aut}(K/k)$, we obtain:

$$\begin{aligned} Mf &= \sum_{\sigma} \sigma(g_0') + \sum_{i\in I} c_i \sum_{\sigma} \frac{\sigma(w_i)'}{\sigma(w_i)} \\ &\quad + M\left[\sum_{m=1}^{N} r_m e_m' - \frac{1}{2}\sum_{j\in J} d_j q_j \frac{h_j'}{h_j} + \frac{1}{2}\sum_{j\in J} d_j p_j \frac{(1-h_j)'}{(1-h_j)}\right] \end{aligned}$$

which implies that:

$$\begin{aligned} &-f + \left(\frac{\mathrm{Tr}(g_0)'}{M}\right) + \sum_{i\in I} \frac{c_i}{M}\frac{N(w_i)'}{N(w_i)} \\ &\quad + \left[\sum_{m=1}^{N} r_m e_m' - \frac{1}{2}\sum_{j\in J} d_j q_j \frac{h_j'}{h_j} + \frac{1}{2}\sum_{j\in J} d_j p_j \frac{(1-h_j)'}{(1-h_j)}\right] = 0 \end{aligned} \tag{2.8}$$

where $\mathrm{Tr}(\,)$ and $N(\,)$ are the trace and norm maps, respectively, from K to k.

Now, multiplying (2.8) by 1 and each (2.7) by e_m, adding them using again the relations $(*)$ and $(**)$, and integrating, we get:

$$\int f = \frac{\mathrm{Tr}(g_0)}{M} + \sum_{i\in I} \frac{c_i}{M} \log N(w_i) + \sum_{m=1}^{N} r_m e_m + \sum_{j\in J} d_j D(h_j). \tag{2.9}$$

Note that $\frac{\mathrm{Tr}(g_0)}{M} \in k$ and $N(w_i) \in k$, and also $e_m = \log H_m$, where $H_m \in \{h_1, h_2, \ldots, 1-h_1, 1-h_2, \ldots\}$. So, the right-hand side of (2.9) is a simple elementary-dilogarithmic expression over k, which is what we wanted to prove.

Definition. Let k be a differential field of characteristic zero. K is a finite algebraic extension of k, and $\log h_1, \ldots, \log h_m$ are logarithmic over k (that is, $h_1, \ldots, h_m \in k$). Assume that the fields k and $K(\log h_1, \ldots, \log h_m)$ have the same field of constants C. We call L a linear logarithmic expression over $K(\log h_1, \ldots, \log h_m)$ if:

$$L = \sum_{i=1}^{m} c_i \log h_i + r$$

where the c_i are constants and $r \in K$. L is said to be dependent on $\log h_j$ $(1 \le j \le m)$ if $c_j \ne 0$.

Proposition 1 (*See Baddoura (1987)*). *Let k be a differential field of characteristic zero which is a Liouvillian extension of its field of constants C assumed to be algebraically closed. Suppose that $f \in k$; $h_1, \ldots, h_n \in k$; K a finite algebraic extension of k; $a_1, \ldots, a_m \in C$; $d_1, \ldots, d_n \in C$; and $L_1, \ldots, L_m$ are linear logarithmic expressions over*

$$K(\log(1-h_1), \ldots, \log(1-h_n)).$$

Then, if:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i=1}^{m} a_i \log L_i \in K(\log(1-h_1), \ldots, \log(1-h_n)) \tag{2.10}$$

$\int f$ is a simple elementary-dilogarithmic expression over k.

Proof. Let r = trans-degree $K(\log(1-h_1), \ldots, \log(1-h_n))$ over k. If $r = 0$, then, by Corollary 2, $\log(1-h_j) \in k (1 \leq j \leq n) \Rightarrow K(\log(1-h_1), \ldots, \log(1-h_n)) = K$, and $L_i \in K$ $(1 \leq i \leq m)$. So, (2.10) implies that:

$$\int f = \sum_{j=1}^{n} d_j \ell_2(h_j) + \sum_{i=1}^{m} a_i \log L_i + g, \quad g \in K, L_i \in K$$

$$\Rightarrow \int f = \sum_{j=1}^{n} d_j D(h_j) + g + \sum_{i=1}^{m} a_i \log L_i - \frac{1}{2} \sum_{j=1}^{n} d_j \log(1-h_j) \log h_j.$$

So, if:

$$s_j = -\frac{1}{2} d_j \log(1-h_j) \in k, \quad \text{we get :}$$

$$\int f = \sum_{j=1}^{n} d_j D(h_j) + g + \sum_{i=1}^{m} a_i \log L_i + \sum_{j=1}^{n} s_j \log h_j, \quad s_j \in k, g \in K, L_i \in K.$$

So, by Lemma 1, $\int f$ is a simple elementary-dilogarithmic expression over k and the proposition is proved for $r = 0$. Let r be greater than 0 and assume without loss of generality that $\log(1-h_1), \ldots, \log(1-h_r)$ are algebraically independent over K so that by Corollary 2 again we find constants c_{jp} such that:

$$\log(1-h_j) = \sum_{p=1}^{r} c_{jp} \log(1-h_p) + R_j \tag{***}$$

where $R_j \in k, r < j \leq n$.

So, $K(\log(1-h_1), \ldots, \log(1-h_n)) = K(\log(1-h_1), \ldots, \log(1-h_r))$.

Let $K_{i_0} = K(\log(1-h_1), \ldots, \log(1-h_{i_0-1}), \log(1-h_{i_0+1}), \ldots, \log(1-h_r))(1 \leq i_0 \leq r)$.

Clearly, $t_{i_0} = \log(1-h_{i_0})$ is transcendental over K_{i_0} since we have assumed that $\log(1-h_j)(1 \leq j \leq r)$ are algebraically independent over K. For each $i_0 \in \{1, 2, \ldots, r\}$, let I_0 be the subset of $\{1, 2, \ldots, m\}$ such that, for all $i \in I_0$, L_i is dependent on $t_{i_0} = \log(1-h_{i_0})$. Then, (2.10) implies that:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i=1}^{m} a_i \log L_i \in K_{i_0}(t_{i_0}). \tag{2.11}$$

We want to prove that:

$$\left[\sum_{i \in I_{i_0}} a_i \log L_i \right]' = 0$$

and that

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \in I_{i_0}} a_i \log L_i \in K_{i_0}[t_{i_0}]. \tag{2.12}$$

Once (2.12) is proved for each index $i_0 \in \{1, 2, \ldots, r\}$, we deduce that:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \in I_{00}} a_i \log L_i \in \bigcap_{i_0 \in \{1,2,\ldots,r\}} K_{i_0}[t_{i_0}] = K[t_1, t_2, \ldots, t_r]$$

where I_{00} is such that, for all $i \in I_{00}$, L_i is not dependent on any $t_j = \log(1 - h_j)$, for all $j \in \{1, 2, \ldots, r\}$. So, $L_i \in K$ for all $i \in I_{00}$, and:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \in I_{00}} a_i \log L_i = P(t_1, \ldots, t_r)$$

where P is a polynomial.

So, let $K_0 = K_{i_0} = K(t_1, \ldots, t_{i_0-1}, t_{i_0+1}, \ldots, t_r)$ and $t = t_{i_0}$. Then if L_i depends on t, $L_i = b_i t + r_i$, where $r_i \in K_0$, and b_i is a constant, $b_i \neq 0$. By assumption, we had:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i=1}^{m} a_i \log L_i = g(t) \in K_0(t). \tag{2.13}$$

If K^0 is a finite algebraic extension of K_0 where $g(t)$ splits into linear factors, we write:

$$g(t) = g_0(t) + \sum_{\alpha,\beta} \frac{r_{\alpha,\beta}}{(t - T_\alpha)^\beta}, \quad r_{\alpha,\beta} \in K^0, T_\alpha \in K^0, \beta \in N^*$$

α and β range over a finite set of positive integers, and $g_0(t) \in K^0[t]$. (2.13) yields:

$$f + \sum_{j=1}^{n} d_j \frac{h'_j}{h_j} \log(1 - h_j) - \sum_{i=1}^{m} a_i \frac{L'_i}{L_i} - g'_0(t) - \sum_{\alpha,\beta} \frac{r'_{\alpha,\beta}}{(t - T_\alpha)^\beta} + \sum_{\alpha,\beta} \frac{\beta r_{\alpha,\beta}(t' - T'_\alpha)}{(t - T_\alpha)^{\beta+1}} = 0. \tag{2.14}$$

The key idea in the on-going proof is that, when we use the relations ($***$) the expression:

$$f + \sum_{j=1}^{n} d_j \frac{h'_j}{h_j} \log(1 - h_j)$$

is a linear polynomial in t over K_0. Also, $g'_0(t)$ is a polynomial in t since $t' = \frac{(1-h_{i_0})'}{1-h_{i_0}} \in k$. So:

$$\sum_{L_i \text{ depends on } t} -a_i \frac{L'_i}{L_i} - \sum_{\alpha,\beta} \frac{r'_{\alpha,\beta}}{(t - T_\alpha)^\beta} + \sum_{\alpha,\beta} \frac{\beta r_{\alpha,\beta}(t' - T'_\alpha)}{(t - T_\alpha)^{\beta+1}}$$

must cancel.

Let $I_t = \{i \text{ such that } L_i = b_i t + r_i, b_i \neq 0\}$ and $I_t^0 = \{1, \ldots, m\} - I_t$. (2.14) then becomes:

$$f + \sum_{j=1}^{n} d_j \frac{h'_j}{h_j} \log(1 - h_j) - \sum_{i \in I_t^0} \frac{r'_i}{r_i} a_i - \sum_{i \in I_t} a_i \frac{(b_i t' + r'_i)}{(b_i t + r_i)} - g'_0(t) - \sum_{\alpha,\beta} \frac{r'_{\alpha,\beta}}{(t - T_\alpha)^\beta} + \sum_{\alpha,\beta} \frac{\beta r_{\alpha,\beta}(t' - T'_\alpha)}{(t - T_\alpha)^{\beta+1}} = 0 \tag{2.15}$$

where $r_i \in K_0$.

First $t' - T'_\alpha \neq 0$, otherwise we would have $t' = T'_\alpha$; and for each $\sigma \in \mathrm{Aut}(K^0/K_0)$ we have $t' = \sigma(T_\alpha)' \Rightarrow \left[K^0 : K_0\right] t' = \mathrm{Tr}(T_\alpha)' \Rightarrow t = \frac{1}{[K^0:K_0]}\mathrm{Tr}(T_\alpha) + c$, where c is a constant and Tr is the trace map from K^0 to K_0. But this gives a contradiction since t was supposed to be transcendental over K_0.

So, if we look at the partial fraction decomposition we have in (2.15), we deduce that $r_{\alpha,\beta} = 0$ for all α, β, and we get:

$$f + \sum_{j=1}^{n} d_j \frac{h'_j}{h_j} \log(1 - h_j) - \sum_{i \in I_t^0} a_i \frac{r'_i}{r_i} - \sum_{i \in I_t} a_i \frac{(b_i t'_+ r'_i)}{(b_i t + r_i)} - g'_0(t) = 0$$

which also implies that:

$$\sum_{i \in I_t} a_i \frac{(b_i t' + r'_i)}{(b_i t + r_i)} = 0$$

(by looking at partial fraction decomposition). Also, $g_0(t) \in K^0[t] \cap K_0(t) \Rightarrow g_0 \in K_0[t]$, and:

$$\sum_{i \in I_t} a_i \log L_i$$

is a constant. So if we let $i_0 = r$, which means that $t = t_r$, we deduce from the above that:

$$\sum_{i=1}^{m} a_i \log L_i = \sum_{i \in I_{t_r}^0} a_i \log L_i + c_r$$

where c_r is a constant. Applying the same process above to:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \in I_{t_r}^0} a_i \log L_i + c_r = g_0(t_r)$$

when $t = t_{r-1}$, where L_i is not dependent on t_r and $g_0(t_r) \in K_r[t_r]$, we get;

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \in I_{r-1}} a_i \log L_i + c_{r-1} + c_r = g_{r-1}(t_{r-1}, t_r)$$

where I_{r-1} is a set over which L_i is not dependent on t_r or t_{r-1} and $g_{r-1}(t_{r-1}, t_r)$ is a polynomial in t_{r-1}, t_r over the field:

$$K(t_1, t_2, \ldots, t_{r-2})$$

and c_{r-1} is a constant.

Repeating the same process when $t = t_{r-2}, \ldots, t_1$ we get:

$$g(t_1, \ldots, t_r) = \int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \in I_{00}} a_i \log L_i \in K[t_1, \ldots, t_r] \quad \text{and} \quad L_i \in K. \tag{2.16}$$

We claim that g is a polynomial of degree 2, with constant coefficients for all terms in $t_1, \ldots, t_r$ of degree 2. In fact, let $A_{\alpha_1\alpha_2\ldots\alpha_r} t_1^{\alpha_1} \ldots t_r^{\alpha_r}$ be one monomial in the leading homogeneous term of g, with $A_{\alpha_1\alpha_2\ldots\alpha_r} \neq 0$. Then:

$$\left(A_{\alpha_1\alpha_2\ldots\alpha_r} t_1^{\alpha_1} \ldots t_r^{\alpha_r}\right)' = A'_{\alpha_1\alpha_2\ldots\alpha_r} t_1^{\alpha_1} \ldots t_r^{\alpha_r} + \sum_{j=1}^{r} A_{\alpha_1\alpha_2\ldots\alpha_r} \alpha_j t'_j t_1^{\alpha_1} \ldots t_j^{\alpha_j - 1} \ldots t_r^{\alpha_r}.$$

Assuming:

$$\sum_{j=1}^{r} \alpha_j \geq 2$$

and noticing that the derivative of the right-hand side of (2.16) is of degree 1 in $t_1, \ldots, t_r$, we deduce that:

$$A'_{\alpha_1\alpha_2\ldots\alpha_r} = 0 \Rightarrow A_{\alpha_1\alpha_2\ldots\alpha_r} \quad \text{is a constant.}$$

Assume that the leading homogeneous term of g is of degree strictly larger than 2 then the coefficient of $t_1^{\beta_1} \ldots t_r^{\beta_r}$, where $\sum_{i=1}^{r} \beta_i = \sum_{j=1}^{r} \alpha_j - 1$, in the derivative of g is:

$\sum_{i=1}^{r} (\beta_i + 1) t'_i A_{\beta_1\ldots\beta_i+1\ldots\beta_r} + A'_{\beta_1\ldots\beta_r}$ which is equal to zero, since we assumed that $\sum_{i=1}^{r} \alpha_i > 2$. Taking the trace of the last equality from K to k, we obtain:

$$\sum_{i=1}^{r} [K:k](\beta_i + 1) A_{\beta_1\ldots\beta_i+1\ldots\beta_r} t'_i + \left(\mathrm{Tr}(A_{\beta_1\ldots\beta_i\ldots\beta_r})\right)' = 0.$$

Integrating we get:

$$\sum_{i=1}^{r} [K:k](\beta_i + 1) A_{\beta_1\ldots\beta_i+1\ldots\beta_r} t_i + \mathrm{Tr}(A_{\beta_1\ldots\beta_i\ldots\beta_r}) = c$$

where c is a constant. But this is a contradiction since $t_1, \ldots, t_r$ were assumed to be algebraically independent over k.

So, we deduce that g is a polynomial of degree 2, with constant coefficients for all the terms in $t_1, \ldots, t_r$ of degree 2. That is:

$$g(t_1, \ldots, t_r) = A_0 + \sum_{p=1}^{r} A_p t_p + \sum_{\alpha,\beta\ \beta\geq\alpha} A_{\alpha,\beta} t_\alpha t_\beta$$

where $t_\alpha, t_\beta \in \{t_1, \ldots, t_r\}$, and $A_{\alpha,\beta}$ are constants.

$$g'(t_1, \ldots, t_r) = A'_0 + \sum_{p=1}^{r} A'_p t_p + \sum_{p=1}^{r} A_p t'_p + \sum_{\alpha,\beta\ \beta\geq\alpha} A_{\alpha,\beta} t'_\alpha t_\beta + \sum_{\alpha,\beta\ \beta\geq\alpha} A_{\alpha,\beta} t_\alpha t'_\beta \quad (2.17)$$

and:

$$g'(t_1, \ldots, t_r) = f + \sum_{j=1}^{n} d_j \frac{h'_j}{h_j} \log(1 - h_j) - \sum_{i=1}^{m} a_i \frac{L'_i}{L_i}. \quad (2.18)$$

Using the dependency relations ($***$), we obtain from (2.17) and (2.18):

$$f - A_0' - \sum_{p=1}^{r} A_p t_p' + \sum_{j=r+1}^{n} d_j \frac{h_j'}{h_j} R_j - \sum_{i=1}^{m} a_i \frac{L_i'}{L_i}$$
$$= \sum_{p=1}^{r} \left[-d_p \frac{h_p'}{h_p} - \sum_{j=r+1}^{n} c_{jp} d_j \frac{h_j'}{h_j} + 2A_{pp} t_p' + \sum_{\alpha \neq p} A_{\alpha p} t_\alpha' + A_p' \right] t_p$$

(where $A_{\alpha p} = A_{p\alpha}$ if $\alpha > p$).

From the above, we deduce that:

$$2A_{pp} t_p' + \sum_{\alpha \neq p} A_{\alpha p} t_\alpha' = d_p \frac{h_p'}{h_p} + \sum_{j=r+1}^{n} c_{jp} d_j \frac{h_j'}{h_j} - A_p'$$

and, by integration, we get:

$$A_{pp} t_p + \sum_{\alpha \neq p} \frac{1}{2} A_{\alpha p} t_\alpha = \frac{1}{2} \left[d_p \log h_p + \sum_{j=r+1}^{n} c_{jp} d_j \log h_j - A_p \right] + c_p \tag{2.19}$$

where c_p is a constant.

Notice that we can write:

$$g(t_1, \ldots, t_r) = A_0 + \sum_{p=1}^{r} A_p t_p + \sum_{p=1}^{r} \left[A_{pp} t_p + \sum_{\alpha \neq p} \frac{1}{2} A_{\alpha p} t_\alpha \right] t_p$$

and, using (2.19) and (2.16),we get:

$$\int f = \sum_{j=1}^{n} d_j \ell_2(h_j) + A_0 + \sum_{p=1}^{r} A_p t_p - \frac{1}{2} \sum_{p=1}^{r} A_p t_p + \sum_{p=1}^{r} c_p t_p$$
$$+ \frac{1}{2} \sum_{p=1}^{r} \left[d_p \log h_p + \sum_{j=r+1}^{n} c_{jp} d_j \log h_j \right] t_p + \sum_{i=1}^{m} a_i \log L_i$$

which gives:

$$\int f = \sum_{p=1}^{r} d_p \left(\ell_2(h_p) + \frac{1}{2} (\log h_p) t_p \right) + \sum_{j=r+1}^{n} d_j \left[\ell_2(h_j) + \frac{1}{2} \left[\sum_{p=1}^{r} c_{jp} t_p \right] \log h_j \right]$$
$$+ A_0 + \frac{1}{2} \sum_{p=1}^{r} A_p t_p + \sum_{p=1}^{r} c_p t_p + \sum_{i=1}^{m} a_i \log L_i.$$

But we had:

$$\sum_{p=1}^{r} c_{jp} t_p = \log(1 - h_j) - R_j$$

for $j \in \{r + 1, \ldots, n\}$ and $t_p = \log(1 - h_p)$. So:

$$\int f = \sum_{j=1}^{n} d_j D(h_j) - \frac{1}{2}\sum_{j=r+1}^{n} d_j R_j \log h_j + A_0 + \frac{1}{2}\sum_{p=1}^{r} A_p \log(1-h_p)$$
$$+ \sum_{p=1}^{r} c_p \log(1-h_p) + \sum_{i=1}^{m} a_i \log L_i,$$

$R_j \in k$, $A_0, A_p \in K$, $L_i \in K$ and by Lemma 1, $\int f = S$, where S is a simple elementary-dilogarithmic expression over k. This completes the proof of Proposition 1.

3. The functional identities of the dilogarithm

In this section, we exhibit and prove two identities of the dilogarithm that will be shown in Section 4, in addition to the identities among primitives and the identities among exponentials, to be capable of generating all the algebraic relations among dilogarithms and logarithms built up from the rational functions by taking transcendental exponentials, logarithms and dilogarithms.

For a differential field k and t dilogarithmic over k we observe the following fact: t is defined up to the addition of a constant multiple of a logarithm or more precisely: if $t' = -\frac{a'}{a}\psi$, where $\psi' = \frac{(1-a)'}{(1-a)}$, ψ is defined up to the addition of a constant. So, if $\psi_1' = \frac{(1-a)'}{(1-a)}$ we deduce that $\psi_1 = \psi + c$, where c is a constant and $t' = -\left(\frac{a'}{a}\right)(\psi_1 - c) = -\frac{a'}{a}\psi_1 + c\frac{a'}{a}$ so t is defined up to the addition of $c \log a$.

Also, if ϕ is an element of $k - \{0, 1\}$ and $t = D(\phi)$ it follows that t is defined up to the addition of a linear combination of $\log \phi$ and $\log(1-\phi)$ with constant coefficients. Informally t is equal to:

$$\ell_2(\phi) + \frac{1}{2}\log\phi\log(1-\phi).$$

This motivates us considering the dilogarithm and the associated function D as defined Modulo the vector space generated by constant multiples of logarithms over k. We denote from now on this vector space by M_k for any differential field k. So, if $W \in M_k$, then there exist constants $c_1, \ldots, c_n$ and $u_1, \ldots, u_n$ such that u_i, $1 \le i \le n$, is logarithmic over k for all i, and:

$$W = \sum_{i=1}^{n} c_i u_i.$$

The first identity satisfied by the dilogarithm is given by the following lemma which is relatively easy to prove.

Lemma 2 (*See Baddoura (1987)*). *If k is a differential field of characteristic zero, then for all $f \in k - \{0, 1\}$:*

$$D\left(\frac{1}{f}\right) \equiv -D(f) \pmod{M_k}.$$

Proof.

$$D'\left(\frac{1}{f}\right) = \frac{1}{2}\frac{f'}{f}\phi + \frac{1}{2}\frac{\left(1-\frac{1}{f}\right)'}{\left(1-\frac{1}{f}\right)}\theta$$

where:

$$\phi' = \frac{\left(1-\frac{1}{f}\right)'}{\left(1-\frac{1}{f}\right)} \quad \text{and} \quad \theta' = \frac{\left(\frac{1}{f}\right)'}{\left(\frac{1}{f}\right)} = -\frac{f'}{f}.$$

So:

$$\phi' = \frac{(1-f)'}{(1-f)} - \frac{f'}{f}.$$

$$\Rightarrow D'\left(\frac{1}{f}\right) \equiv \frac{1}{2}\frac{f'}{f}(\log(1-f) - \log f) - \frac{1}{2}\left(\frac{(1-f)'}{(1-f)} - \frac{f'}{f}\right)\log f \pmod{M_k'}$$

$$\Rightarrow D'\left(\frac{1}{f}\right) \equiv \frac{1}{2}\frac{f'}{f}\log(1-f) - \frac{1}{2}\frac{(1-f)'}{(1-f)}\log f \pmod{M_k'}$$

$$\Rightarrow D\left(\frac{1}{f}\right) \equiv -D(f) \pmod{M_k}$$

(M_k' is the space of derivatives of M_k).

The second identity satisfied by the dilogarithm is one of the main discoveries of this paper. It is given in the following proposition whose proof, although lengthy and involved, uses only standard techniques from differential algebra.

Proposition 2 (*See Baddoura (1987)*). *Let k be a differential field of characteristic zero, and let θ be transcendental over k with $k(\theta)$ being a differential field having the same subfield of constants as k. Let $f(\theta) \in k(\theta)$ and K be the splitting of $f(\theta)$ and $1 - f(\theta)$. We define, if a is a zero or a pole of $f(\theta)$, $\mathrm{ord}_a f(\theta)$ to be the multiplicity of $(\theta - a)$; this is positive if a is a zero of $f(\theta)$ and negative if a is a pole of $f(\theta)$. Then, there exists $f_1 \in k$ such that:*

$$D(f(\theta)) \equiv D(f_1) + \sum_{a,b\ a\neq b} \mathrm{ord}_b(1-f)\mathrm{ord}_a(f) D\left(\frac{\theta-b}{\theta-a}\right) \pmod{M_{K(\theta)}} \tag{A}$$

where a runs over the zeros and poles of f, and b runs over the zeros and poles of $(1 - f)$.

Remark. The splitting field of a rational function $S(\theta) = \frac{T(\theta)}{U(\theta)}$ where T and U are relatively prime is the splitting field of the polynomial $T(\theta)U(\theta)$.

Proof. Let $f(\theta) = f_0\frac{P(\theta)}{Q(\theta)}$, where $f_0 \in k$, and $P(\theta)$, $Q(\theta)$ are relatively prime polynomials over k which are monic. We can also assume that $\deg P(\theta) \geq \deg Q(\theta)$, otherwise, using Lemma 2, we replace f by $\frac{1}{f}$

$$1 - f(\theta) = \frac{Q(\theta) - f_0 P(\theta)}{Q(\theta)} = g_0\frac{R(\theta)}{Q(\theta)}$$

where $g_0 \in k$, and $R(\theta)$ is a monic polynomial relatively prime with both P and Q.

First step:

$$D'(f) = -\frac{1}{2}\frac{f'}{f}\log(1-f) + \frac{1}{2}\frac{(1-f)'}{(1-f)}\log f$$

is well-defined mod $M'_{K(\theta)}$. We can check easily that, if $a \neq b$ and $a, b \in K$, then:

$$D'\left(\frac{\theta-b}{\theta-a}\right) \equiv \frac{1}{2}\left(\frac{\theta'-b'}{\theta-b}-\frac{b'-a'}{b-a}\right)\log(\theta-a)+\frac{1}{2}\left(\frac{b'-a'}{b-a}-\frac{\theta'-a'}{\theta-a}\right)\log(\theta-b)$$
$$+\frac{1}{2}\left(\frac{\theta'-a'}{\theta-a}-\frac{\theta'-b'}{\theta-b}\right)\log(b-a) \quad (\mathrm{mod}\ M'_{K(\theta)})$$

(this is because $\log(gh) = \log g + \log h + \text{constant}$ and $\log(\frac{1}{g}) = \log g + \text{constant}$.)
Second step: consider the set $I_1 = \{(a, b)$ such that a is a zero of P or of Q, b is zero of R or of Q, but whenever one of a and b is a zero of Q the other is not$\}$. (So the set (a, b), a zero of Q and b zero of Q is excluded.)

We have:

$$f_0\frac{P(\theta)}{Q(\theta)}+g_0\frac{R(\theta)}{Q(\theta)}=1 \tag{B}$$
$$\Leftrightarrow f_0P(\theta)+g_0R(\theta)=Q(\theta). \tag{C}$$

Let us compute:

$$-\frac{1}{2}\left[\sum_{(a,b)\in I_1}\mathrm{ord}_a(f)\mathrm{ord}_b(1-f)\frac{b'-a'}{b-a}\log(\theta-a)\right]$$
$$+\frac{1}{2}\left[\sum_{(a,b)\in I_1}\mathrm{ord}_a(f)\mathrm{ord}_b(1-f)\frac{b'-a'}{b-a}\log(\theta-b)\right] \quad \mathrm{mod}\ M'_{K(\theta)}.$$

We call the above sum S_1:

$$S_1 = -\frac{1}{2}\sum_{a\ \text{zero of}\ P}\mathrm{ord}_a(f)\left[\sum_{b\ \text{zero or pole of}\ (1-f)}\mathrm{ord}_b(1-f)\frac{b'-a'}{b-a}\right]\log(\theta-a)$$
$$-\frac{1}{2}\sum_{a\ \text{zero of}\ Q}\mathrm{ord}_a(f)\left[\sum_{b\ \text{zero of}\ R}\mathrm{ord}_b(1-f)\frac{b'-a'}{b-a}\right]\log(\theta-a)$$
$$+\frac{1}{2}\sum_{b\ \text{zero of}\ R}\mathrm{ord}_b(1-f)\left[\sum_{a\ \text{zero or pole of}\ f}\mathrm{ord}_a(f)\frac{b'-a'}{b-a}\right]\log(\theta-b)$$
$$+\frac{1}{2}\sum_{b\ \text{zero of}\ Q}\mathrm{ord}_b(1-f)\left[\sum_{a\ \text{zero of}\ P}\mathrm{ord}_a(f)\frac{b'-a'}{b-a}\right]\log(\theta-b)$$

since $(a, b) \in I_1$.

Now, (B) above implies, if a is a zero of P, that:

$$g_0\frac{R(a)}{Q(a)}=1 \Rightarrow \frac{g_0'}{g_0}+\frac{(R(a))'}{R(a)}-\frac{(Q(a))'}{Q(a)}=0$$

but, as we can easily check:

$$\sum_{b\ \text{zero or pole of}\ (1-f)}\mathrm{ord}_b(1-f)\frac{b'-a'}{b-a}=\frac{(R(a))'}{R(a)}-\frac{(Q(a))'}{Q(a)}=-\frac{g_0'}{g_0} \tag{3.1}$$

(where a is a zero of P).

Also, if b is a zero of R, we have, using (B) above:

$$f_0 \frac{P(b)}{Q(b)} = 1 \Rightarrow \frac{f_0'}{f_0} = -\frac{(P(b))'}{P(b)} + \frac{(Q(b))'}{Q(b)}.$$

So we get:

$$\sum_{a \text{ zero or pole of } f} \operatorname{ord}_a \frac{b' - a'}{b - a} = \frac{(P(b))'}{P(b)} - \frac{(Q(b))'}{Q(b)} = -\frac{f_0'}{f_0} \tag{3.2}$$

(where, in the above, $b$is a zero of R).

Now, we look at the sum:

$$\begin{aligned} S_2 &= -\frac{1}{2} \sum_{a \text{ zero of } Q} \operatorname{ord}_a(f) \left[\sum_{b \text{ zero of } R} \operatorname{ord}_b(1 - f) \frac{b' - a'}{b - a} \right] \log(\theta - a) \\ &\quad + \frac{1}{2} \sum_{b \text{ zero of } Q} \operatorname{ord}_b(1 - f) \left[\sum_{a \text{ zero of } P} \operatorname{ord}_a(f) \frac{b' - a'}{b - a} \right] \log(\theta - b) \\ \Rightarrow S_2 &= \frac{1}{2} \sum_{a \text{ zero of } Q} \operatorname{ord}_a(f) \left[\sum_{b \text{ zero of } R} -\operatorname{ord}_b(1 - f) \frac{b' - a'}{b - a} \right. \\ &\quad \left. + \sum_{b \text{ zero of } P} \operatorname{ord}_b(f) \frac{b' - a'}{b - a} \right] \log(\theta - a). \end{aligned}$$

But the relation $f_0 P(\theta) + g_0 R(\theta) = Q(\theta)$ implies, if a is a zero of Q, that:

$$\begin{aligned} & f_0 P(a) + g_0 R(a) = 0 \Rightarrow \frac{f_0'}{f_0} + \frac{(P(a))'}{P(a)} = \frac{g_0'}{g_0} + \frac{(R(a))'}{R(a)} \\ & \Rightarrow \frac{(P(a))'}{P(a)} - \frac{(R(a))'}{R(a)} = \frac{g_0'}{g_0} - \frac{f_0'}{f_0} \end{aligned} \tag{3.3}$$

and:

$$\begin{aligned} & -\sum_{b \text{ zero of } R} \operatorname{ord}_b(1 - f) \frac{b' - a'}{b - a} = -\frac{(R(a))'}{R(a)} \\ & \sum_{b \text{ zero of } P} \operatorname{ord}_b(f) \frac{b' - a'}{b - a} = \frac{(P(a))'}{P(a)} \end{aligned}$$

(if a is a zero of Q).

(3.3) and the above imply that:

$$S_2 = \frac{1}{2} \sum_{a \text{ zero of } Q} \operatorname{ord}_a(f) \left[\frac{g_0'}{g_0} - \frac{f_0'}{f_0} \right] \log(\theta - a)$$

which is exactly:

$$S_2 = -\frac{1}{2} \sum_{b \text{ zero of } Q} \operatorname{ord}_b(1 - f) \frac{f_0'}{f_0} \log(\theta - b) + \frac{1}{2} \sum_{a \text{ zero of } Q} \operatorname{ord}_a(f) \frac{g_0'}{g_0} \log(\theta - a)$$

(3.1) and (3.2) imply, respectively, that:

$$-\frac{1}{2}\sum_{a \text{ zero of } P}\operatorname{ord}_a(f)\left[\sum_{b \text{ zero or pole of } (1-f)}\operatorname{ord}_b(1-f)\frac{b'-a'}{b-a}\right]\log(\theta-a)$$
$$=\frac{1}{2}\sum_{a \text{ zero of } P}\operatorname{ord}_a(f)\frac{g_0'}{g_0}\log(\theta-a).$$

This sum will be denoted by S_3.

$$\frac{1}{2}\sum_{b \text{ zero of } R}\operatorname{ord}_b(1-f)\left[\sum_{a \text{ zero or pole of } f}\operatorname{ord}_a(f)\frac{b'-a'}{b-a}\right]\log(\theta-b)$$
$$=-\frac{1}{2}\sum_{b \text{ zero of } P}\operatorname{ord}_b(1-f)\frac{f_0'}{f_0}\log(\theta-b).$$

This sum will be denoted by S_4.

Now, $S_1 = S_2 + S_3 + S_4$, and by regrouping the terms in S_2, S_3 and S_4 we deduce that:

$$\begin{aligned} S_1 &= \frac{1}{2}\sum_{a \text{ zero or pole of } f}\operatorname{ord}_a(f)\frac{g_0'}{g_0}\log(\theta-a) \\ &\quad -\frac{1}{2}\sum_{b \text{ zero or pole of } (1-f)}\operatorname{ord}_b(1-f)\frac{f_0'}{f_0}\log(\theta-b). \end{aligned} \tag{3.4}$$

Now, consider the four following sums:

$$H_3=\frac{1}{2}\sum_{a \text{ zero of } P}\operatorname{ord}_a(f)\left[\sum_{b \text{ zero or pole of } (1-f)}\operatorname{ord}_b(1-f)\log(b-a)\right]\frac{\theta'-a'}{\theta-a}$$
$$H_4=-\frac{1}{2}\sum_{b \text{ zero of } R}\operatorname{ord}_b(1-f)\left[\sum_{a \text{ zero or pole of } f}\operatorname{ord}_a(f)\log(b-a)\right]\frac{\theta'-b'}{\theta-b}$$
$$\begin{aligned} H_2 &= \frac{1}{2}\sum_{a \text{ zero of } Q}\operatorname{ord}_a(f)\left[\sum_{b \text{ zero of } R}\operatorname{ord}_b(1-f)\log(b-a)\right]\frac{\theta'-a'}{\theta-a} \\ &\quad -\frac{1}{2}\sum_{b \text{ zero of } Q}\operatorname{ord}_b(1-f)\left[\sum_{a \text{ zero of } P}\operatorname{ord}_a(f)\log(b-a)\right]\frac{\theta'-b'}{\theta-b} \end{aligned}$$

and: $H_1 = H_2 + H_3 + H_4$. It follows immediately that:

$$H_1=\sum_{(a,b)\in I_1}\frac{1}{2}\operatorname{ord}_a(f)\operatorname{ord}_b(1-f)\left[\frac{\theta'-a'}{\theta-a}-\frac{\theta'-b'}{\theta-b}\right]\log(b-a).$$

Now, and as before, integrating (3.1)–(3.3), we deduce:

$$\begin{aligned} &\sum_{b \text{ zero or pole of } (1-f)}\operatorname{ord}_b(1-f)\log(b-a)=\log R(a)-\log Q(a)+\text{constant} \\ &\quad=-\log g_0+\text{constant},\quad \text{where } a \text{ is a zero of } P \end{aligned} \tag{3.1$'$}$$

$$\begin{aligned} &\sum_{a \text{ zero or pole of } f}\operatorname{ord}_a(f)\log(b-a)=\log P(b)-\log Q(b)+\text{constant} \\ &\quad=-\log f_0+\text{constant},\quad \text{where } b \text{ is a zero of } R \end{aligned} \tag{3.2$'$}$$

$$\sum_{b \text{ zero of } R} -\mathrm{ord}_b(1-f)\log(b-a) + \sum_{b \text{ zero of } P} \mathrm{ord}_b(f)\log(b-a)$$
$$= \log g_0 - \log f_0 + \text{constant}, \quad \text{where } a \text{ is a zero of } Q. \tag{3.3$'$}$$

Plugging (3.1′), (3.2′) and (3.3′) in H_3, H_4 and H_2, respectively, and regrouping, as we have done for computing S_1, we obtain:

$$H_1 \equiv -\frac{1}{2}\sum_{a \text{ zero or pole of } f} \mathrm{ord}_a(f)\frac{\theta'-a'}{\theta-a}\log g_0$$
$$+\frac{1}{2}\sum_{b \text{ zero or pole of } (1-f)} \mathrm{ord}_b(1-f)\frac{\theta'-b'}{\theta-b}\log f_0 \pmod{M'_{K(\theta)}}.$$

(This is because we had constants in relations (3.1′), (3.2′) and (3.3′).)

Third step: we compute $(D(f(\theta)))' \bmod M'_{K(\theta)}$, which can be immediately verified to be:

$$\begin{aligned}(D(f(\theta)))' \equiv \frac{1}{2}\Bigg[&-\sum_{a,b}\mathrm{ord}_a(f)\mathrm{ord}_b(1-f)\frac{(\theta-a)'}{\theta-a}\log(\theta-b)\\ &+\sum_{a,b}\mathrm{ord}_a(f)\mathrm{ord}_b(1-f)\frac{(\theta-b)'}{\theta-b}\log(\theta-a)\Bigg]\\ &-\frac{1}{2}\sum_a \mathrm{ord}_a(f)\frac{(\theta-a)'}{\theta-a}\log g_0+\frac{1}{2}\sum_b \mathrm{ord}_b(1-f)\frac{(\theta-b)'}{\theta-b}\log f_0\\ &-\frac{1}{2}\sum_b \mathrm{ord}_b(1-f)\frac{f_0'}{f_0}\log(\theta-b)+\frac{1}{2}\sum_a \mathrm{ord}_a(f)\frac{g_0'}{g_0}\log(\theta-a)\\ &-\frac{1}{2}\frac{f_0'}{f_0}\log g_0+\frac{1}{2}\frac{g_0'}{g_0}\log f_0 \pmod{M'_{K(\theta)}}\end{aligned} \tag{3.5}$$

(where $\sum_{a,b}$ runs over all zeros and poles of f and $(1-f)$, respectively, $\sum_a$ runs over the zeros and poles of f, and $\sum_b$ runs over the zeros and poles of $(1-f)$).

The term:

$$\sum_{(a,b)\notin I_1, a\neq b}\Bigg[-\mathrm{ord}_a(f)\mathrm{ord}_b(1-f)\frac{(\theta-a)'}{\theta-a}\log(\theta-b)$$
$$+\mathrm{ord}_a(f)\mathrm{ord}_b(1-f)\frac{(\theta-b)'}{\theta-b}\log(\theta-a)\Bigg]$$

is zero since a and b run over the roots of Q.

So:

$$(D(f(\theta)))' \equiv -\frac{1}{2}\frac{f_0'}{f_0}\log g_0+\frac{1}{2}\frac{g_0'}{g_0}\log f_0$$
$$-\frac{1}{2}\sum_{(a,b)\in I_1}\mathrm{ord}_a(f)\mathrm{ord}_b(1-f)\left[\frac{(\theta-b)'}{\theta-b}\log(\theta-a)-\frac{(\theta-a)'}{\theta-a}\log(\theta-b)\right]$$
$$+H_1+S_1 \pmod{M'_{K(\theta)}}$$

$$(3.5) \Rightarrow (D(f(\theta)))' \equiv \left[\sum_{(a,b)\in I_1} \mathrm{ord}_a(f)\mathrm{ord}_b(1-f)D\left(\frac{\theta-b}{\theta-a}\right)\right]' - \frac{1}{2}\frac{f_0'}{f_0}\log g_0 + \frac{1}{2}\frac{g_0'}{g_0}\log f_0 \pmod{M'_{K(\theta)}}.$$

Now we distinguish three cases:

Case 1:

$$\deg(P) > \deg(Q) \quad \text{(strict inequality)}$$
$$\Rightarrow \deg(Q(\theta) - f_0 P(\theta)) = \deg(P(\theta)) \Rightarrow g_0 = -f_0$$

and:

$$\log(-f_0) = \log g_0 = \log f_0 + \text{constant}.$$

So:

$$-\frac{1}{2}\frac{f_0'}{f_0}\log g_0 + \frac{1}{2}\frac{g_0'}{g_0}\log f_0 \equiv 0 \pmod{M'_{K(\theta)}}$$

and we take f_1 in Proposition 2 to be a constant. So $(D(f_1))' = 0$.

Case 2: if $\deg(P) = \deg(Q)$ (and $f_0 \neq 1$) then the leading coefficient of $Q(\theta) - f_0 P(\theta)$ is $1 - f_0 \Rightarrow g_0 = 1 - f_0$

$$\Rightarrow -\frac{1}{2}\frac{f_0'}{f_0}\log g_0 + \frac{1}{2}\frac{g_0'}{g_0}\log f_0 = -\frac{1}{2}\frac{f_0'}{f_0}\log(1-f_0) + \frac{1}{2}\frac{(1-f_0)'}{(1-f_0)}\log f_0$$

and we take f_1 in Proposition 2 to be f_0.

Case 3: $\deg P = \deg Q$ and $f_0 = 1$.

Let $I = \{(a, b)$ such that a pole or zero of f, b pole or zero of $(1 - f)\}$.

Then $I - I_1 = \{(a, b)$ such that a zero of Q, b zero of $Q\}$. But:

$$D\left(\frac{\theta-b}{\theta-a}\right) \equiv -D\left(\frac{\theta-a}{\theta-b}\right) \pmod{M_{K(\theta)}}$$
$$\Rightarrow \sum_{(a,b)\in I-I_1} \mathrm{ord}_a(f)\mathrm{ord}_b(1-f)D\left(\frac{\theta-a}{\theta-b}\right) \equiv 0 \pmod{M_{K(\theta)}}.$$

So:

$$\sum_{(a,b)\in I_1} \mathrm{ord}_a(f)\mathrm{ord}_b(1-f)D\left(\frac{\theta-b}{\theta-a}\right) \equiv \sum_{(a,b)\in I} \mathrm{ord}_a(f)\mathrm{ord}_b(1-f)D\left(\frac{\theta-b}{\theta-a}\right) \pmod{M_{K(\theta)}}. \tag{3.6}$$

Now, $\deg P = \deg Q$ and $f_0 = 1 \Rightarrow$

$$1 - f = \frac{Q(\theta) - P(\theta)}{Q(\theta)} \Rightarrow \deg(Q(\theta) - P(\theta)) < \deg Q(\theta).$$

But, since:

$$D(f) \equiv -D(1-f) \equiv D\left(\frac{1}{1-f}\right) \pmod{M_{K(\theta)}}$$

and:

$$\sum_{(a,b)\in I_1} \mathrm{ord}_a(f)\mathrm{ord}_b(1-f)D\left(\frac{\theta-b}{\theta-a}\right)$$

is unchanged if we replace f by $\frac{1}{1-f}$, we are again in case 1.

But, by the results of case 1 and case 2, and relations (3.5) and (3.6), Proposition 2 is proved.

We would like to end this section by giving one example that illustrate the power of these two identities in generating well known identities of the dilogarithm.

Example. Let $k = C(z)$, where z is transcendental over C and $z' = 1$, and C is the field of complex numbers. Applying Lemma 2 and Proposition 2 to $f(z) = z^2$, $f(z) = z$, and $f(z) = -z$, respectively, yields

$$D(z^2) \equiv 2D\left(\frac{z-1}{z}\right) + 2D\left(\frac{z+1}{z}\right) \pmod{M_{C(z)}}$$

$$D(z) \equiv D\left(\frac{z-1}{z}\right) \pmod{M_{C(z)}}$$

$$D(-z) \equiv D\left(\frac{z+1}{z}\right) \pmod{M_{C(z)}}.$$

So,

$$D(z^2) \equiv 2D(z) + 2D(-z) \pmod{M_{C(z)}}$$

which implies that

$$\begin{aligned}&\ell_2(z^2) + \frac{1}{2}\log z^2 \log(1-z^2)\\ &\quad\equiv 2\left[\ell_2(z) + \ell_2(-z) + \frac{1}{2}\log z \log(1-z) + \frac{1}{2}\log(-z)\log(1+z)\right] \pmod{M_{C(z)}}\end{aligned}$$

and we obtain

$$\ell_2(z^2) \equiv 2\ell_2(z) + 2\ell_2(-z) \pmod{M_{C(z)}}$$

which is a well known identity of the dilogarithm.

4. An extension of Liouville's theorem

In this section, we state and prove the major result of this paper. Our result is a new theorem that generalizes Liouville's theorem on integration in finite terms. It allows dilogarithms to occur in the integrals in addition to elementary functions. The proof is based on the two identities of the Bloch–Winger–Spence function given in Lemma 2 and Proposition 2 of the previous section. It also uses Proposition 1 of Section 2 in several places.

The statement of the theorem uses the following definition of a transcendental–dilogarithmic–elementary extension of a differential field:

Definition. A transcendental–dilogarithmic–elementary extension of a differential field k is a differential field extension K such that there is a tower of differential fields $k = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_N = K$ all having the same constant field and for each $i = 1, \ldots, N$ we have one of the following three cases:

(1″) $K_i = K_{i-1}(\theta_i)$, where θ_i is logarithmic over K_{i-1}.
(2″) $K_i = K_{i-1}(\theta_i)$, where θ_i is exponential over K_{i-1}. We also assume θ_i transcendental over K_{i-1} in this case.
(3″) $K_i = K_{i-1}(\theta_i, \theta_i')$, where $\theta_i = \ell_2(a)$ for some $a \in K_{i-1} - \{0, 1\}$.

The theorem reads as follows:

Theorem (*See Baddoura (1987)*). *Let k be a differential field of characteristic zero, which is a Liouvillian extension of its subfield of constants assumed algebraically closed. Let $f \in k$ and suppose that there is a transcendental–dilogarithmic–elementary extension K of k such that:*

$$\int f \in K.$$

Then, the integral $\int f$ is a simple elementary-dilogarithmic expression over k. That is:

$$\int f = g + \sum_{i=1}^{m} s_i \log v_i + \sum_{j=1}^{n} c_j D(h_j) \quad (n, m \textit{ are positive integers})$$

where $g, s_i, v_i, h_j \in k$, and the c_j's are constants.

The rest of this section is devoted to the proof of this theorem.
We start by stating a corollary to Theorem 1.

Corollary 3. *Let k be a differential field of characteristic zero. Assume that $u_1, \ldots, u_n$ are logarithmic and algebraically independent over k, that v is exponential over k and that $k(v, u_1, \ldots, u_n)$ and k have the same field of constants. Then, if v is algebraic over $k(u_1, \ldots, u_n)$ there exists an integer $n \neq 0$ such that $v^n \in k$.*

Corollary 3.1. *Let k be a differential field of characteristic zero. Assume that $u_1, \ldots, u_m$ are logarithmic over k, that v is exponential and transcendental over kand that $k(v, u_1, \ldots, u_m)$ and k have the same field of constants. Then, v is transcendental over $k(u_1, \ldots, u_m)$.*

Proof. We can assume using Corollary 2 and without loss of generality that there exists $n \leq m$ such that $k(u_1, \ldots, u_m)$ is algebraic over $k(u_1, \ldots, u_n)$ where $u_1, \ldots, u_n$ are assumed to be algebraically independent over k. If v were algebraic over $k(u_1, \ldots, u_m)$ it would be algebraic over $k(u_1, \ldots, u_n)$, but by the previous corollary there exists an integer $r \neq 0$ such that $v^r \in k$ and this contradicts the fact that v is transcendental over k.

Corollary 3.2. *Let k be a differential field of characteristic zero. Assume that $u_1, \ldots, u_n$ are logarithmic over k, that t is primitive over k and that $k(t, u_1, \ldots, u_n)$ and k have the same field of constants. If t is algebraic over $k(u_1, \ldots, u_n)$ then there exist constants $c_1, \ldots, c_n$ and an element $s \in k$ such that:*

$$t = \sum_{i=1}^{n} c_i u_i + s.$$

Proof. This follows easily from Corollary 1.

Proposition 3. *Let k be a differential field of characteristic zero, and let θ be primitive and transcendental over k. Let $\alpha_1, \dots, \alpha_n \in k$ ($\alpha_i \neq \alpha_j$, if $i \neq j$), $u_1, \dots, u_m \in k$ and assume the existence of constants $c_1, \dots, c_n, d_1, \dots, d_m$ such that:*

$$\sum_{i=1}^{n} c_i \log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \log u_j \in k(\theta)$$

(where k and $k(\theta)(\log(\theta - \alpha_1), \dots, \log(\theta - \alpha_n), \log u_1, \dots, \log u_m)$ have the same field of constants). Then $c_1 = c_2 = \cdots = c_n = 0$.

Proof. There exists $s(\theta) \in k(\theta)$ such that:

$$\sum_{i=1}^{n} c_i \log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \log u_j + s(\theta) = 0.$$

This implies that:

$$\sum_{i=1}^{n} c_i \frac{\theta' - \alpha_i'}{\theta - \alpha_i} + (s(\theta))' = -\sum_{j=1}^{m} d_j \frac{u_j'}{u_j}.$$

In a suitable finite normal extension field K of k $s(\theta)$ will split into linear factors so that we can write:

$$s(\theta) = \sum_{i,\nu} h_{\nu i}(\theta - \alpha_i)^{\nu} + \sum_{\alpha,j} l_{\alpha j}(\theta - \beta_j)^{\alpha} + (\text{element of } K[\theta])$$

where i ranges over the set $\{1, 2, \dots, n\}$, ν ranges over a finite set of negative integers, j ranges over a finite set of positive integers, α ranges over a finite set of negative integers and $h_{\nu i}, l_{\alpha j}$, $\beta_j \in K$ $(\beta_j \neq \alpha_i, \forall i, j)$.

We work in the differential field $K(\theta)$ which is an extension of $k(\theta)$. By assumption we have:

$$\sum_{i=1}^{n} c_i \frac{\theta' - \alpha_i'}{\theta - \alpha_i} + \sum_{i,\nu} (h_{\nu i}(\theta - \alpha_i)^{\nu})' + \sum_{j,\alpha} (l_{\alpha j}(\theta - \beta_j)^{\alpha})' \in K[\theta]. \qquad (*)$$

The basic idea of the proof is the following: when the various functions appearing in $(*)$ are expressed as quotients of polynomials in θ we get no pole cancellation, and therefore all the c_i's and $h_{\nu i}$'s will vanish.

Since θ is primitive over k we have $\theta' = a$, where a belongs to k.

$$\frac{\theta' - \alpha_i'}{\theta - \alpha_i} = \frac{a - \alpha_i'}{\theta - \alpha_i}.$$

We claim that $a - \alpha_i \neq 0$, that is the numerator and denominator in the previous fraction, are relatively prime (as polynomials in θ).

If $a - \alpha_i' = 0$ then $(\theta - \alpha_i)' = \theta' - \alpha_i' = 0$ which implies that $\theta - \alpha_i$ is a constant in k and that contradicts the fact that θ is transcendental over k.

Now:

$$(h_{\nu i}(\theta - \alpha_i)^{\nu})' = h_{\nu i}'(\theta - \alpha_i)^{\nu} + \nu h_{\nu i}(\theta - \alpha_i)^{\nu - 1}(\theta' - \alpha_i').$$

We notice that since $\theta' - \alpha_i' \in K$ and is different from zero and since $-\nu + 1 > 1$ the various terms of the left-hand side of (∗) would not cancel unless $h_{\nu i} = 0$ for all the ν's and the i's and this will imply that $c_i = 0$ for all $i \in \{1, 2, \dots, n\}$ which is what we want to prove.

Proposition 4. *Let k be a differential field of characteristic zero, and let θ be exponential and transcendental over k. Let $\alpha_1, \dots, \alpha_n \in k^*(\alpha_i \neq \alpha_j,\ \text{if } i \neq j)$, $u_1, \dots, u_m \in k$ and assume the existence of constants $c_1, \dots, c_n, d_1, \dots, d_m$ such that:*

$$\sum_{i=1}^{n} c_i \log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \log u_j \in k(\theta)$$

(where k and $k(\theta)(\log(\theta - \alpha_1), \dots, \log(\theta - \alpha_n), \log u_1, \dots, \log u_m)$ have the same field of constants). Then $c_1 = c_2 = \cdots = c_n = 0$.

Proof. There exists $s(\theta) \in k(\theta)$ such that:

$$\sum_{i=1}^{n} c_i \log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \log u_j + s(\theta) = 0.$$

This implies that:

$$\sum_{i=1}^{n} c_i \frac{\theta' - \alpha_i'}{\theta - \alpha_i} + (s(\theta))' = -\sum_{j=1}^{m} d_j \frac{u_j'}{u_j}.$$

In a suitable finite normal extension field K of k $s(\theta)$ will split into linear factors so that we can write:

$$s(\theta) = \sum_{i,\nu} h_{\nu i}(\theta - \alpha_i)^{\nu} + \sum_{\alpha, j} l_{\alpha j}(\theta - \beta_j)^{\alpha} + (\text{element of } K[\theta])$$

where i ranges over the set $\{1, 2, \dots, n\}$, ν ranges over a finite set of negative integers, j ranges over a finite set of positive integers, α ranges over a finite set of negative integers and $h_{\nu i}, l_{\alpha j}, \beta_j \in K$ ($\beta_j \neq \alpha_i, \forall i, j$).

We work in the differential field $K(\theta)$ which is an extension of $k(\theta)$. By assumption we have:

$$\sum_{i=1}^{n} c_i \frac{\theta' - \alpha_i'}{\theta - \alpha_i} + \sum_{i,\nu} (h_{\nu i}(\theta - \alpha_i)^{\nu})' + \sum_{j,\alpha} (l_{\alpha j}(\theta - \beta_j)^{\alpha})' \in K[\theta]. \tag{∗∗}$$

The basic idea of the proof is the following: when the various functions appearing in (∗∗) are expressed as quotients of polynomials in θ we get no pole cancellation, and therefore all the c_i's and $h_{\nu i}$'s will vanish.

Since θ is exponential over k we have $\theta' = a'\theta$ where a belongs to k.

$$\frac{\theta' - \alpha_i'}{\theta - \alpha_i} = \frac{a'\theta - \alpha_i'}{\theta - \alpha_i}.$$

We claim that the numerator and denominator in the previous fraction are relatively prime (as polynomials in θ). If not we would have $\alpha_i' = a'\alpha_i$ but since $\alpha_i \neq 0$ we get $\frac{\alpha_i'}{\alpha_i} = a' = \frac{\theta'}{\theta}$ which implies that $(\frac{\theta}{\alpha_i})' = 0$ and this gives $\frac{\theta}{\alpha_i}$ is a constant in k and that contradicts the fact that θ is transcendental over k.

Now:

$$(h_{\nu i}(\theta - \alpha_i)^{\nu})' = h'_{\nu i}(\theta - \alpha_i)^{\nu} + \nu h_{\nu i}(\theta - \alpha_i)^{\nu - 1}(\theta' - \alpha'_i).$$

By what has been done and since $-\nu + 1 > 1$ the various terms of the left-hand side of (∗∗) would not cancel unless $h_{\nu i} = 0$ for all the ν's and the i's and this will imply that $c_i = 0$ for all $i \in \{1, 2, \ldots, n\}$ which is what we wanted to prove.

Corollary 3.3. *In the conditions of Propositions 3 and 4,* $\log(\theta - \alpha_1), \ldots, \log(\theta - \alpha_n)$ *(where $\alpha_i \neq 0$ for all i if θ is exponential) are algebraically independent over* $k(\theta)(\log u_1, \ldots, \log u_m)$.

Proof. If $\log(\theta - \alpha_1), \ldots, \log(\theta - \alpha_n)$ were not algebraically independent and since $\log(\theta - \alpha_1), \ldots, \log(\theta - \alpha_n), \log u_1, \ldots, \log u_m$ are logarithmic over $k(\theta)$, we deduce by Corollary 1 that there exist constants $c_1, \ldots, c_n$ not all zero and constants $d_1, \ldots, d_m$ such that:

$$\sum_{i=1}^{n} c_i \log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \log u_j \in k(\theta)$$

and the above implies by Propositions 3 and 4 that $c_1 = c_2 = \cdots = c_n = 0$ which gives a contradiction.

Proposition 5. *Let k be a differential field of characteristic zero. Let θ be transcendental over k where we assume that k and $k(\theta)$ have the same field of constants. Let $s(\theta) \in k(\theta)$ be such that $(s(\theta))' \in k$. Then:*

(1″) *If θ is primitive over k, $s(\theta) = c\theta + v$, where c is a constant and $v \in k$.*
(2″) *If θ is exponential over k, $s(\theta) \in k$.*

Proof. (1″) Let $s(\theta) = \frac{p(\theta)}{q(\theta)}$ where $p(\theta), q(\theta) \in k[\theta]$. Then $q(\theta)s(\theta) = p(\theta)$ and both $s(\theta)$ and θ are primitive over k so by Corollary 1 $s(\theta) = c\theta + v$, where c is a constant and $v \in k$.

(2″) Let $s(\theta) = \frac{p(\theta)}{q(\theta)}$ where $p(\theta), q(\theta) \in k[\theta]$. Then $q(\theta)s(\theta) = p(\theta)$ and θ is exponential over k while $s(\theta)$ is primitive over k so by Theorem 1 $s(\theta) \in k$ otherwise there would exist a nonzero integer n such that $\theta^n \in k$ which is impossible since θ was assumed to be transcendental over k.

Now we are ready to prove the main theorem in this paper.

Theorem. *Let k be a differential field of characteristic zero, which is a Liouvillian extension of its subfield of constants assumed algebraically closed. Let $f \in k$ and suppose that there exists a transcendental–dilogarithmic–elementary extension K of k such that:*

$$\int f \in K.$$

Then, the integral $\int f$ is a simple elementary-dilogarithmic expression over k. That is:

$$\int f = g + \sum_{i=1}^{m} s_i \log v_i + \sum_{j=1}^{n} c_j D(h_j) \quad (n, m \text{ are positive integers})$$

where $g, s_i, v_i, h_j \in k$ and the c_j's are constants.

Proof. It is by induction on N, the length of K over k.

If $N = 0$ then $\int f = g \in k$ and the theorem is proved.

If $N > 0$, we apply the induction hypothesis to $f \in K_1$ and the tower $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_N = K$, to obtain:

$$\int f = g + \sum_{i=1}^{m} s_i \log v_i + \sum_{j=1}^{n} c_j D(h_j) \tag{4.1}$$

where $g, s_i, v_i, h_j \in K_1$ and the c_j's are constants.

We want to modify (4.1) in such a way that g, s_i, v_i, and h_j are in $k = K_0$.

For this we consider three major cases.

Case 1: $K_1 = k(\theta)$ and θ logarithmic over k: $\theta = \log a, a \in k$. If θ is algebraic over k, then by Corollary 3.2, $\theta \in k$ and there is nothing to prove.

So, we assume θ transcendental over k, and factor $v_i, h_j, 1-h_j$ over k. So we will be working over k^0 the splitting field of these quantities which we assume normal.

By Proposition 2:

$$D(h_j(\theta)) \equiv D(H_j) + \sum_{a,b} \mathrm{ord}_a(h_j)\mathrm{ord}_b(1-h_j) D\left(\frac{\theta - b}{\theta - a}\right) \pmod{M_{k^0(\theta)}}$$

where $H_j \in k, a, b \in k^0, a \neq b$ where a and b are the zeros and poles of h_j and $1 - h_j$ respectively. So (4.1) can be written as:

$$\begin{aligned}\int f \ &\equiv g(\theta) + \sum_{i=1}^{n} S_i(\theta) \log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p \\ &\quad + \sum_{j=1}^{m} d_j D(H_j) + \sum_{i,j} c_{ij} D\left(\frac{\theta - \alpha_i}{\theta - \alpha_j}\right) \pmod{M_{k^0(\theta)}}\end{aligned} \tag{4.1$'$}$$

where in the last sum $i \in \{1, \ldots, n\}, j \in \{1, \ldots, n\}, i \neq j$ and $\alpha_i \neq \alpha_j$ if $i \neq j$. Also d_j and c_{ij} are constants, $f_p, H_j \in k$ for $p \in \{1, \ldots, r\}$and $j \in \{1, \ldots, m\}$, $g(\theta), S_i(\theta), s_p(\theta) \in k(\theta)$ for $i \in \{1, \ldots, n\}$ and $p \in \{1, \ldots, r\}$.

We notice that the last sum can be written as:

$$\begin{aligned}&\sum_{i,j} c_{ij} D\left(\frac{\theta - \alpha_i}{\theta - \alpha_j}\right) \\ &\quad \equiv d_{12} D\left(\frac{\theta - \alpha_1}{\theta - \alpha_2}\right) + d_{13} D\left(\frac{\theta - \alpha_1}{\theta - \alpha_3}\right) + \cdots + d_{1n} D\left(\frac{\theta - \alpha_1}{\theta - \alpha_n}\right) \\ &\qquad + d_{23} D\left(\frac{\theta - \alpha_2}{\theta - \alpha_3}\right) + d_{24} D\left(\frac{\theta - \alpha_2}{\theta - \alpha_4}\right) + \cdots + d_{2n} D\left(\frac{\theta - \alpha_2}{\theta - \alpha_n}\right) \\ &\qquad + \cdots + \sum_{j>i} d_{ij} D\left(\frac{\theta - \alpha_i}{\theta - \alpha_j}\right) + \cdots + d_{(n-1)n} D\left(\frac{\theta - \alpha_{n-1}}{\theta - \alpha_n}\right) \\ &\qquad + \text{constant} \pmod{M_{k^0(\theta)}}.\end{aligned} \tag{4.2}$$

(This is possible because $D(\frac{\theta-\alpha_i}{\theta-\alpha_j}) \equiv -D(\frac{\theta-\alpha_j}{\theta-\alpha_i}) \pmod{M_{k^0(\theta)}}$.)

We call the above expression reduced, that is (4.2). For example:

$$d_1 D\left(\frac{\theta-\alpha_1}{\theta-\alpha_2}\right)+d_2 D\left(\frac{\theta-\alpha_1}{\theta-\alpha_2}\right)+d_3 D\left(\frac{\theta-\alpha_2}{\theta-\alpha_3}\right)$$

is reduced, while the expression:

$$d_1 D\left(\frac{\theta-\alpha_1}{\theta-\alpha_2}\right)+d_2 D\left(\frac{\theta-\alpha_1}{\theta-\alpha_2}\right)+d_3 D\left(\frac{\theta-\alpha_2}{\theta-\alpha_1}\right)$$

is not reduced.

Without changing the notation $S_i(\theta)$, (4.1′) becomes:

$$\begin{aligned}\int f &= g(\theta)+\sum_{i=1}^{n} S_i(\theta)\log(\theta-\alpha_i)+\sum_{p=1}^{r} s_p(\theta)\log f_p\\ &\quad+\sum_{j_0=1}^{m} d_{j_0} D(H_{j_0})+\sum_{i=1}^{n-1}\sum_{j>i} d_{ij} D\left(\frac{\theta-\alpha_i}{\theta-\alpha_j}\right)\end{aligned} \tag{4.3}$$

(with $\alpha_i \neq \alpha_j$ for all $i \neq j$) and $(1 < j \leq n)$.

Now, we take the derivative of (4.3), to get:

$$\begin{aligned} f &= g'(\theta)+\sum_{i=1}^{n} S_i(\theta)\frac{(\theta-\alpha_i)'}{(\theta-\alpha_i)}+\sum_{i=1}^{n} S_i'(\theta)\log(\theta-\alpha_i)\\ &\quad+\sum_{p=1}^{r} s_p(\theta)\frac{f_p'}{f_p}+\sum_{p=1}^{r} s_p'(\theta)\log f_p\\ &\quad+\sum_{j_0=1}^{m} d_{j_0}\left[-\frac{1}{2}\frac{H_{j_0}'}{H_{j_0}}\log(1-H_{j_0})+\frac{1}{2}\frac{(1-H_{j_0})'}{(1-H_{j_0})}\log H_{j_0}\right]\\ &\quad+\sum_{i=1}^{n-1}\sum_{j>i} d_{ij}\left[\frac{1}{2}\left(\frac{\theta'-\alpha_i'}{\theta-\alpha_i}-\frac{\alpha_i'-\alpha_j'}{\alpha_i-\alpha_j}\right)\log(\theta-\alpha_j)\right.\\ &\quad+\frac{1}{2}\left(\frac{\alpha_i'-\alpha_j'}{\alpha_i-\alpha_j}-\frac{\theta'-\alpha_j'}{\theta-\alpha_j}\right)\log(\theta-\alpha_i)\\ &\quad\left.+\frac{1}{2}\left(\frac{\theta'-\alpha_j'}{\theta-\alpha_j}-\frac{\theta'-\alpha_i'}{\theta-\alpha_i}\right)\log(\alpha_i-\alpha_j)\right]. \end{aligned} \tag{4.4}$$

Identifying the term which multiplies $\log(\theta-\alpha_1)$, we get:

$$S_1'(\theta)+\sum_{j>1}\frac{1}{2}\left(\frac{\alpha_1'-\alpha_j'}{\alpha_1-\alpha_j}-\frac{\theta'-\alpha_j'}{\theta-\alpha_j}\right)d_{1j}=0. \tag{4.5}$$

This is because the $\log(\theta-\alpha_i)(1 \leq i \leq n)$ are algebraically independent over the field: $k^0(\theta)(\log H_{j_0}(1 \leq j_0 \leq m), \log(1-H_{j_0})(1 \leq j_0 \leq m), \log f_p(1 \leq p \leq r), \log(\alpha_i-\alpha_j)\ (i < j))$, by Corollary 3.3. Now (4.5) implies:

$$S_1(\theta)+\sum_{j>1}\frac{1}{2}(\log(\alpha_1-\alpha_j)-\log(\theta-\alpha_j))d_{1j}+\text{constant}=0$$

which, by Proposition 3, gives $d_{1j} = 0$ for $j > 1$, and $S_1(\theta) = s_1$ is a constant. By induction we prove easily that $d_{ij} = 0$ for all i, j and that $S_i(\theta) = s_i$ is a constant. So we get:

$$\int f = g(\theta) + \sum_{i=1}^{n} s_i \log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p + \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) \tag{4.6}$$

where d_{j_0}, s_i are constants, $\alpha_i \in k^0$ and $f_p, H_{j_0} \in k$.

At this point, we distinguish two cases:

Case 1-a: θ is algebraic over the field:

$$F = k^0(\log H_{j_0}(1 \leq j_0 \leq m), \log(1 - H_{j_0})(1 \leq j_0 \leq m), \log f_p(1 \leq p \leq r))$$

So, by Corollary 3.2, we get:

$$\theta = \sum_{p=1}^{r} c_p \log f_p + \sum_{j_0=1}^{m} b_{j_0} \log H_{j_0} + \sum_{j_0=1}^{m} a_{j_0} \log(1 - H_{j_0}) + h$$

where c_p, b_{j_0}, a_{j_0} are constants, and $h \in k^0$.

So, $L_i = \theta - \alpha_i$ is a linear logarithmic expression over F, and (4.6) can be written as:

$$\begin{aligned} \int f &- \sum_{j_0=1}^{m} d_{j_0} \ell_2(H_{j_0}) - \sum_{j_0=1}^{m} 0.\ell_2(1 - H_{j_0}) \\ &- \sum_{p=1}^{r} 0.\ell_2(1 - f_p) - \sum_{i=1}^{n} s_i \log L_i \in F \end{aligned} \tag{4.7}$$

which implies, by Proposition 1, that $\int f$ is a simple elementary-dilogarithmic expression over k and our theorem is proved in this case.

Case 1-b: θ is transcendental over the field:

$$F = k^0(\log H_{j_0}(1 \leq j_0 \leq m), \log(1 - H_{j_0})(1 \leq j_0 \leq m), \log f_p(1 \leq p \leq r)).$$

In this (4.6) can be written as:

$$\int \left(f - \left[\sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) \right]' \right) = g(\theta) + \sum_{i=1}^{n} s_i \log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p. \tag{4.8}$$

From this, and as in the proof of Liouville's theorem, we deduce that $s_i = 0$ for all $1 \leq i \leq n$. Also, by Proposition 5, we deduce that there exists c, a constant, and $v \in F$ such that:

$$g(\theta) + \sum_{p=1}^{r} s_p(\theta) \log f_p = c\theta + v \quad (\theta = \log a)$$

so:

$$\begin{aligned} \int f &- \sum_{j_0=1}^{m} d_{j_0} \ell_2(H_{j_0}) - \sum_{j_0=1}^{m} 0.\ell_2(1 - H_{j_0}) \\ &- \sum_{p=1}^{r} 0.\ell_2(1 - f_p) - c \log a \in F \end{aligned}$$

$\Rightarrow$ by Proposition 1 that $\int f$ is a simple elementary-dilogarithmic over k, and the theorem is proved in case 1.

Case 2: $K_1 = k(\theta, \theta')$ and $\theta = \ell_2(a)$, where $a \in k - \{0, 1\}$. Let $k_1 = k(\log(1-a))$. So, $\theta' \in k_1$. If θ is algebraic over k_1, then by Corollary 3.2, $\theta \in k_1$. So, writing (4.1) again, we have:

$$\int f = g + \sum_{i=1}^{m} s_i \log v_i + \sum_{j=1}^{n} c_j D(h_j) \tag{4.9}$$

where $g, s_i, v_i, h_j \in k_1$ and the c_j's are constants.

Then, using case 1 (the logarithmic case), we deduce that $\int f$ is a simple elementary-dilogarithmic expression over k.

So, we consider the case θ transcendental over k_1. As in the previous case, (4.9) can be written:

$$\begin{aligned}\int f \ &= g(\theta) + \sum_{i=1}^{n} S_i(\theta)\log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p \\ &\quad + \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) + \sum_{i=1}^{n-1}\sum_{j>i} d_{ij} D\left(\frac{\theta-\alpha_i}{\theta-\alpha_j}\right)\end{aligned}$$

where $f_p, H_{j_0} \in k_1, \alpha_i \neq \alpha_j$ for $i \neq j, 1 < j \leq n$, and $\alpha_i \in k_1^0$ a normal finite extension of k_1 containing the roots and poles of v_i, h_j and $(1-h_j)$ for all i, j.

Now, we use the same argument as in case 1 ($\theta = \log a$) and Proposition 3 to deduce:

$$\int f = g(\theta) + \sum_{i=1}^{n} s_i \log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p + \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) \tag{4.10}$$

where $f_p, H_{j_0} \in k_1, \alpha_i \in k_1^0$ and s_i, d_{j_0} are constants.

We also distinguish two cases:

Case 2-a: θ is algebraic over the field:

$$F_1 = k_1^0(\log H_{j_0}(1 \leq j_0 \leq m), \log(1 - H_{j_0})(1 \leq j_0 \leq m), \log f_p(1 \leq p \leq r)).$$

We apply again the same argument as in case 1-a (using Corollary 3.2), and obtain: $\int f$ is a simple elementary-dilogarithmic expression over $k_1 \Rightarrow$ by case 1 and since $f \in k$ that $\int f$ is a simple elementary-dilogarithmic expression over k.

Case 2-b: θ is transcendental over the field:

$$F_1 = k_1^0(\log H_{j_0}(1 \leq j_0 \leq m), \log(1 - H_{j_0})(1 \leq j_0 \leq m), \log f_p(1 \leq p \leq r)).$$

Then, from (4.10), and as in case 1-b ($\theta = \log a$), we deduce that $s_i = 0$, for all $1 \leq i \leq n$ and that there exists c, a constant, and $v \in F_1$, such that:

$$g(\theta) + \sum_{p=1}^{r} s_p(\theta) \log f_p = c\theta + v \quad (\theta = \ell_2(a))$$

Now (4.10) implies that:

$$\int f - \sum_{j_0=1}^{m} d_{j_0}\ell_2(H_{j_0}) - \sum_{j_0=1}^{m} 0.\ell_2(1-H_{j_0})$$
$$- \sum_{p=1}^{r} 0.\ell_2(1-f_p) - c\ell_2(a) \in F_1 = F_1(\log(1-a))$$

(since $\log(1-a) \in k_1$) $\Rightarrow$ by Proposition 1 that $\int f$ is a simple elementary-dilogarithmic expression over $k_1 \Rightarrow$ by case 1 that $\int f$ is a simple elementary-dilogarithmic expression over k.

Case 3: $K_1 = k(\theta)$, $\theta = \exp a$, $a \in k$, and θ transcendental over k. As seen before, we can write (4.1) as:

$$\int f = g(\theta) + \sum_{i=1}^{n-1} S_i(\theta)\log(\theta-\alpha_i) + \sum_{p=1}^{r} s_p(\theta)\log f_p$$
$$+ \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) + \sum_{i=1}^{n-1}\sum_{j>i} d_{ij} D\left(\frac{\theta-\alpha_i}{\theta-\alpha_j}\right) \tag{4.11}$$

($\alpha_i \in k^0$), $(1 < j \le n)$ and k^0 is a finite normal extension of k.

In this case we have assumed that $\alpha_n = 0$ so $\log(\theta-\alpha_n) = \log\theta = \log(\exp a) = a \in k$, and that $\alpha_i \neq 0$ for $i \neq n$. This is why $S_n(\theta)\log(\theta-\alpha_n)$ does not appear in (4.11).

The derivative of (4.11) is exactly (4.4), from which we extract the coefficient of $\log(\theta-\alpha_1)$ and use Corollary 3.3 to obtain:

$$S_1'(\theta) + \sum_{j>1, j\neq n} \frac{1}{2}\left(\frac{\alpha_1'-\alpha_j'}{\alpha_1-\alpha_j} - \frac{\theta'-\alpha_j'}{\theta-\alpha_j}\right) d_{1j} + \frac{1}{2}d_{1n}\left(\frac{\alpha_1'}{\alpha_1} - \frac{\theta'}{\theta}\right) = 0$$
$$\Rightarrow S_1(\theta) + \sum_{j>1, j\neq n} \frac{1}{2}(\log(\alpha_1-\alpha_j) - \log(\theta-\alpha_j))d_{1j}$$
$$+ \frac{1}{2}d_{1n}\log\alpha_1 - \frac{1}{2}d_{1n}a = \text{constant}$$

since $\log\theta = a \in k) \Rightarrow$ by Proposition 4:

$$d_{1j} = 0 \quad \text{for all } j > 1,\ j \neq n$$

and:

$$S_1'(\theta) = \frac{1}{2}d_{1n}\left(\frac{\theta'}{\theta} - \frac{\alpha_1'}{\alpha_1}\right).$$

By induction on i, we can deduce that:

$$d_{ij} = 0, \quad \text{for all } i \text{ and for all } j > 1,\ j \neq n$$

and:

$$S_i'(\theta) = \frac{1}{2}d_{in}\left(\frac{\theta'}{\theta} - \frac{\alpha_i'}{\alpha_i}\right) \quad (1 \le i \le n-1). \tag{4.12}$$

So, (4.11) becomes:

$$\begin{aligned}
\int f &= g(\theta) + \sum_{i=1}^{n-1} S_i(\theta)\log(\theta-\alpha_i) + \sum_{p=1}^{r} s_p(\theta)\log f_p \\
&\quad + \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) + \sum_{i=1}^{n-1} d_{in} D\left(\frac{\theta-\alpha_i}{\theta}\right) \\
\Rightarrow f &= g'(\theta) + \sum_{i=1}^{n-1} S_i(\theta)\frac{(\theta-\alpha_i)'}{(\theta-\alpha_i)} + \sum_{i=1}^{n-1} S_i'(\theta)\log(\theta-\alpha_i) \\
&\quad + \sum_{p=1}^{r} s_p(\theta)\frac{f_p'}{f_p} + \sum_{p=1}^{r} s_p'(\theta)\log f_p \\
&\quad + \sum_{j_0=1}^{m} d_{j_0}\left[-\frac{1}{2}\frac{H_{j_0}'}{H_{j_0}}\log(1-H_{j_0}) + \frac{1}{2}\frac{(1-H_{j_0})'}{(1-H_{j_0})}\log H_{j_0}\right] \\
&\quad + \sum_{i=1}^{n-1} d_{in}\left[\frac{1}{2}\left(\frac{\theta'-\alpha_i'}{\theta-\alpha_i} - \frac{\alpha_i'}{\alpha_i}\right)(a+c) + \frac{1}{2}\left(\frac{\alpha_i'}{\alpha_i} - \frac{\theta'}{\theta}\right)\log(\theta-\alpha_i)\right. \\
&\quad \left. + \frac{1}{2}\left(\frac{\theta'}{\theta} - \frac{\theta'-\alpha_i'}{\theta-\alpha_i}\right)\log\alpha_i\right] \qquad (4.13)
\end{aligned}$$

(c is a constant such that $\log\theta = a + c$). In the above expression, the coefficient of $\log(\theta-\alpha_i)$ is zero, as we have seen before.

Now, by Corollary 3.1, θ is transcendental over the field:

$$\begin{aligned}
F_0 &= k^0(\log\alpha_i (1 \le i \le n-1), \log H_{j_0}(1 \le j_0 \le m), \log(1-H_{j_0})(1 \le j_0 \le m), \\
&\quad \log f_p (1 \le p \le r))
\end{aligned}$$

On the other hand, we choose the $\log f_p$ $(1 \le p \le r)$ in such a way that they are linearly independent and transcendental over k^0. Then, by Corollaries 3 and 3.1, they are algebraically independent over $k^0(\theta)$.

From (4.13), we deduce that there exist subsets J_p, I_p, T_p such that:

$$\begin{aligned}
&s_p'(\theta) + \sum_{j_0\in J_p}\left(-\frac{1}{2}\frac{H_{j_0}'}{H_{j_0}} d_{j_0}\right) + \sum_{j_0\in I_p}\left(\frac{1}{2}\frac{(1-H_{j_0})'}{(1-H_{j_0})} d_{j_0}\right) \\
&\quad + \sum_{i\in T_p}\frac{1}{2} d_{in}\left(\frac{\theta'}{\theta} - \frac{\theta'-\alpha_i'}{\theta-\alpha_i}\right) = 0 \qquad (4.14)
\end{aligned}$$

(this is the coefficient of $\log f_p$: J_p, I_p, T_p exist because $\log\alpha_i, \log H_{j_0}, \log(1-H_{j_0})$ could depend on $\log f_p$).

By Proposition 4, we deduce that d_{in}=0 for all $i \in T_p$. So:
$s_p'(\theta) \in k \Rightarrow s_p(\theta) = s_p \in k$ by Proposition 5 (for all p).
So, (4.13) becomes:

$$f = g'(\theta) + \sum_{i=1}^{n-1} S_i(\theta)\frac{(\theta-\alpha_i)'}{(\theta-\alpha_i)} + \sum_{p=1}^{r} s_p\frac{f_p'}{f_p} + \sum_{p=1}^{r} s_p'\log f_p$$

$$+\sum_{j_0=1}^{m} d_{j_0}\left[-\frac{1}{2}\frac{H'_{j_0}}{H_{j_0}}\log(1-H_{j_0})+\frac{1}{2}\frac{(1-H_{j_0})'}{(1-H_{j_0})}\log H_{j_0}\right]$$
$$+\sum_{i=1}^{n-1} d_{in}\left[\frac{1}{2}\left(\frac{\theta'-\alpha'_i}{\theta-\alpha_i}-\frac{\alpha'_i}{\alpha_i}\right)(a+c)+\frac{1}{2}\left(\frac{\theta'}{\theta}-\frac{\theta'-\alpha'_i}{\theta-\alpha_i}\right)\log\alpha_i\right]. \quad (4.15)$$

But, from (4.12), we had:

$$S'_i(\theta)=\frac{1}{2}d_{in}\left(\frac{\theta'}{\theta}-\frac{\alpha'_i}{\alpha_i}\right)\quad(1\le i\le n-1)$$
$$\Rightarrow S_i(\theta)=\frac{1}{2}d_{in}(a-\log\alpha_i)+c_i,\quad c_i \text{ is a constant} \quad (4.16)$$

So, $S_i(\theta)$ belongs to the field:

$$F_0=k^0(\log\alpha_i(1\le i\le n-1),\log H_{j_0}(1\le j_0\le m),\log(1-H_{j_0})\ (1\le j_0\le m),$$
$$\log f_p(1\le p\le r)).$$

Computing the coefficient of $\frac{(\theta-\alpha_i)'}{(\theta-\alpha_i)}$ in (4.15), we get:

$$g'(\theta)+\sum_{i=1}^{n-1}\left(S_i(\theta)+\frac{1}{2}d_{in}[(a+c)-\log\alpha_i]\right)\frac{\theta'-\alpha'_i}{\theta-\alpha_i}\in F_0.$$

Considering the partial fraction decomposition of $g(\theta)$, we can prove, as in the proof of Liouville's theorem, that (since $\alpha_i\neq 0$):

$$S_i(\theta)+\frac{1}{2}d_{in}[(a+c)-\log\alpha_i]=0\quad\text{for all } i\le n-1. \quad (4.17)$$

Comparing with (4.16), we deduce that:

$$d_{in}[a-\log\alpha_i]=\text{constant for all } i\le n-1. \quad (4.18)$$

We claim that $d_{in}=0$, otherwise we would have:

$$a'-\frac{\alpha'_i}{\alpha_i}=0\Rightarrow\frac{\theta'}{\theta}-\frac{\alpha'_i}{\alpha_i}=0\Rightarrow N_0\frac{\theta'}{\theta}-\frac{(\mathrm{Norm}(\alpha_i))'}{\mathrm{Norm}(\alpha_i)}=0 \quad (4.19)$$

where $N_0=[k^0:k]$, and Norm is the usual norm from k^0 to k.
So, (4.19) implies:

$$(\theta^{-N_0}\mathrm{Norm}(\alpha_i))'=0\Rightarrow\theta^{N_0}\in k\Rightarrow\text{contradiction}$$

and:

$$d_{in}=0\quad\text{for all } i,\ 1\le i\le n-1$$

which implies that $S_i(\theta)=0$ by (4.17). Now (4.15) becomes:

$$f=g'(\theta)+\sum_{p=1}^{r}s_p\frac{f'_p}{f_p}+\sum_{p=1}^{r}s'_p\log f_p$$
$$+\sum_{j_0=1}^{m} d_{j_0}\left[-\frac{1}{2}\frac{H'_{j_0}}{H_{j_0}}\log(1-H_{j_0})+\frac{1}{2}\frac{(1-H_{j_0})'}{(1-H_{j_0})}\log H_{j_0}\right].$$

Let $F_{00} = k^0(\log H_{j_0}(1 \le j_0 \le m), \log(1 - H_{j_0})\ (1 \le j_0 \le m), \log f_p(1 \le p \le r))$. θ is transcendental over F_{00} which implies, by Proposition 5, that $g(\theta) = g \in F_{00}$. So we get:

$$\int \left(f - \left[\sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) \right]' \right) = g + \sum_{p=1}^{r} s_p \log f_p$$

$$\Rightarrow \int f - \sum_{j_0=1}^{m} d_{j_0} \ell_2(H_{j_0}) - \sum_{j_0=1}^{m} 0.\ell_2(1 - H_{j_0})$$

$$- \sum_{p=1}^{r} 0.\ell_2(1 - f_p) \in F_{00}$$

$\Rightarrow \int f$ is a simple elementary-dilogarithmic expression over k by Proposition 1, so the theorem is proved.

We end this section by giving a nontrivial example that illustrates the fundamental concept behind our generalization of Liouville's theorem, which is that integration in finite terms is actually a simplification process.

In fact, what we have proved is:

Let k be a differential field of characteristic zero, which is a Liouvillian extension of its subfield of constants assumed algebraically closed. Let f be an element in k and suppose that f has a transcendental–dilogarithmic–elementary integral. Then

$$\int f = g + \sum_{i=1}^{m} s_i w_i + \sum_{j=1}^{n} d_j v_j$$

where n and m are positive integers, $g \in k$, $s_i \in k$, for all i, $1 \le i \le m$, w_i is logarithmic for all i, $1 \le i \le m$, d_j is a constant for all j, $1 \le j \le n$, and $v_j = D(\phi_j)$, where $\phi_j \in k - \{0, 1\}$ for all j, $1 \le j \le n$. In our proof of the theorem, we observed that, although v'_j does not in general belong to k, it can even be transcendental over k, as is illustrated in the following example.

Example. Let k be any differential field of characteristic zero. Assume that θ is primitive and transcendental over k. Let $p(\theta)$ and $q(\theta)$ be two irreducible polynomials over k such that $\deg p > \deg q \neq 0$.

We consider the differential field $K = k(\theta)(\phi_1, \phi_2)$, where ϕ_1 and ϕ_2 are such that

$$\phi'_1 = \frac{p'(\theta)}{p(\theta)} \quad \text{and} \quad \phi'_2 = \frac{q'(\theta)}{q(\theta)}.$$

It is immediate that ϕ_1 and ϕ_2 are algebraically independent over $k(\theta)$. It is also clear that, if ϕ_3 is such that

$$\phi'_3 = \frac{(p(\theta) + q(\theta))'}{p(\theta) + q(\theta)}$$

then ϕ_3 is transcendental over K. Consider the function:

$$f = \frac{1}{2}\left(\frac{q'}{q} - \frac{(p+q)'}{p+q}\right)\phi_1 - \frac{1}{2}\left(\frac{(p+q)'}{p+q} - \frac{p'}{p}\right)\phi_2 + \frac{1}{2}(\phi_1 + \phi_2)\frac{(p+q)'}{p+q}$$

$f \in K$, and we can check that

$$\left[D\left(\frac{-p}{q}\right) + \frac{1}{2}(\phi_1 + \phi_2)\phi_3 \right] \equiv \int f \quad (\text{mod } M_K)$$

but $\left(D(\frac{-p}{q})\right)'$ is transcendental over K since ϕ_3 is.

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