 Conditional limiting distribution of beta-independent random vectors

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Abstract

The paper deals with random vectors $X$ in $\mathbb{R}^d, d \geq 2$, possessing the stochastic representation $X \overset{d}{=} ARV$, where $R$ is a positive random radius independent of the random vector $V$ and $A \in \mathbb{R}^{d \times d}$ is a non-singular matrix. If $V$ is uniformly distributed on the unit sphere of $\mathbb{R}^d$, then for any integer $m < d$ we have the stochastic representations $(V_1, \ldots, V_m) \overset{d}{=} W(U_1, \ldots, U_m)$ and $(V_{m+1}, \ldots, V_d) \overset{d}{=} (1 - W^2)^{1/2}(U_{m+1}, \ldots, U_d)$, with $W \geq 0$, such that $W^2$ is a beta distributed random variable with parameters $m/2, (d - m)/2$ and $(U_1, \ldots, U_m), (U_{m+1}, \ldots, U_d)$ are independent uniformly distributed on the unit spheres of $\mathbb{R}^m$ and $\mathbb{R}^{d-m}$, respectively. Assuming a more general stochastic representation for $V$ in this paper we introduce the class of beta-independent random vectors. For this new class we derive several conditional limiting results assuming that $R$ has a distribution function in the max-domain of attraction of a univariate extreme value distribution function. We provide two applications concerning the Kotz approximation of the conditional distributions and the tail asymptotic behaviour of beta-independent bivariate random vectors.

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1. Introduction

Let $(S_1, S_2)$ be a bivariate random vector with distribution function invariant with respect to orthogonal transformations in $\mathbb{R}^2$. Denote by $R := \sqrt{S_1^2 + S_2^2}$ the associated random radius
and let $F$ be its distribution function with $F(0) = 0$ and upper endpoint $\infty$. Approximation of the distribution function of $S_1$ conditioning on the event $\{S_2 = u_n\}$ with $u_n, n \geq 1$, constants tending to $\infty$ is of some theoretical interest. Hashorva [16] shows that the distribution function of the random variable $Z_n := S_1|S_2 = u_n, n \geq 1$, can be approximated ($n \to \infty$) by a Gaussian distribution function on $\mathbb{R}$, provided that $F$ has a distribution function in the max-domain of attraction of the Gumbel distribution. Making use of this fact it follows that the maxima of elliptical triangular arrays can be approximated by the Hüsler–Reiss distribution function (see Hashorva [17]). Under the same assumption on $F$ it is further shown in the aforementioned paper that also the distribution function of the random variable $Z_n^* := S_1|S_2 > u_n$ can be approximated by the same Gaussian distribution function on $\mathbb{R}$. The latter Gaussian approximation is proved in the context of Berman processes in [5] (see also [3,4]), and in [13,8] in the context of the extremes of convex hulls.

Estimation of the conditional distribution (density) and the conditional expectation is an important topic in statistics. In various statistical applications estimation of the distribution or the density function of $Z_n$ is of interest (for fixed $u_n$), see [11,2,29] and the references therein.

In the latter monograph statistical methods dealing with the estimation of rare events related to the conditional random variable $Z_n$ (again for fixed $u_n$) are presented. Statistical applications arise also when the approximation of the distribution function of $Z_n$, or $Z_n^*$ for high thresholds $u_n$, is of interest. The first attempt in this direction is made in [1]. Utilising the Gaussian approximation of the distribution function of $Z_n^*$, in the aforementioned paper novel statistical techniques for the estimation of the distribution of $Z_n^*$ ($u_n$ being large) are presented.

In a multivariate setup if $S$ is a random vector in $\mathbb{R}^d, d \geq 2$, with the distribution function invariant with respect to orthogonal transformations in $\mathbb{R}^d$, then we have the stochastic representation (see e.g., [6] or [10])

$$S \overset{d}{=} R\tilde{U},$$

where $R > 0$ (almost surely) is the associated random radius of $S$ with distribution function $F$ independent of $\tilde{U}$, which is uniformly distributed on the unit sphere of $\mathbb{R}^d$ ($\overset{d}{=}$ stands for equality of the distribution functions). Commonly, $S$ is referred to as a spherical random vector, whereas linear transformations of $S$ as elliptical random vectors. Assuming that $F$ is in the max-domain of attraction of a univariate extreme value distribution, [16,18,19] obtains several conditional limiting results for the elliptical random vector $X$ with stochastic representation

$$X \overset{d}{=} AR\tilde{U},$$  \hspace{1cm} (1.1)

with $A \in \mathbb{R}^{d \times d}$ a non-singular matrix. Similar results for $\tilde{U}$ being an $L_p$-norm symmetrised Dirichlet random vector in $\mathbb{R}^d$ are derived in [23].

A natural question that arises is whether conditional limiting results for $X$ with stochastic representation (1.1) can be stated for a general random vector $\tilde{U}$ without specifying its distribution function.

In this paper we show that the answer is positive, provided that $\tilde{U}$ inherits a crucial distributional property shared by the uniformly distributed random vectors. Explicitly, if $\tilde{U}$ is uniformly distributed on the unit sphere (with respect to $L_2$-norm) of $\mathbb{R}^d$, then for any integer $m < d$ (see Lemma 2 of [6]) the stochastic representation

$$\tilde{U} \overset{d}{=} (WU_1, (1 - W^2)^{1/2}U_2)$$  \hspace{1cm} (1.2)
holds with $U_1, U_2$ uniformly distributed on the unit spheres of $\mathbb{R}^m$ and $\mathbb{R}^{d-m}$, $1 \leq m < d$, respectively. Furthermore, $W \in [0, 1]$ with $W^2 \sim \text{Beta}(m/2, (d-m)/2)$, and $W, U_1, U_2$ are mutually independent. We refer to (1.2) as the beta-independent splitting. If instead a power function of $W$ is beta distributed we refer to (1.2) as the weak beta-independent splitting.

In this paper we introduce a large class of random vectors with stochastic representation (1.1), where the random vector $\tilde{U}$ satisfies the weak beta-independent splitting property, being independent of the associated random radius $R$. We refer to such random vectors as beta-independent random vectors.

We show that the weak beta-independent splitting property implies interesting conditional limiting results, provided that the associated random radius of the beta-independent random vectors has a distribution function in the max-domain of attraction of a univariate extreme value distribution $H$. In particular, if $H$ is the Gumbel distribution, then we show that instead of the Gaussian approximation (elliptical setup) for the conditional distributions of interest, the Kotz approximation is the adequate one in the more general setup of beta-independent random vectors.

Our paper is organised as follows. In Section 2 we introduce our notation, define a new class of beta-independent random vectors, and derive a stochastic representation related to conditional random vectors. In Section 3 we obtain asymptotic results for the conditional distribution of beta-independent random vectors assuming that the associated random radius $R$ has a distribution function in the max-domain of attraction of a univariate extreme value distribution. Two applications of the Kotz approximation are presented in Section 4. The proofs of all the results are relegated to Section 5.

2. Notation and preliminaries

First we present some standard notation. Then we introduce a new class of random vectors with stochastic representation (1.1). Under certain working assumptions we derive a useful stochastic representation for the conditional distribution of such random vectors.

Throughout the paper $I$ is a non-empty index subset of $\{1, \ldots, d\}$, $d \geq 2$, with $m := |I| < d$ elements and $J$ is defined by $J := \{1, \ldots, d\} \setminus I$. For any vector $x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ set $x_I := (x_i, i \in I)^\top, x_J := (x_i, i \in J)^\top$ (here $\top$ denotes the transpose sign). Similarly, if $A \in \mathbb{R}^{d \times d}$ is a given matrix, we write $A_{II}, A_{IJ}, A_{JI}, A_{JJ}$ for the submatrices of $A$ obtained by keeping the rows and columns with the indices in the corresponding index sets. If $y = (y_1, \ldots, y_d)^\top$ is another vector in $\mathbb{R}^d$ we use the following notation

$$
x + y := (x_1 + y_1, \ldots, x_d + y_d)^\top,
$$

$$
x > y, \text{ if } x_i > y_i, \quad \forall i = 1, \ldots, d,
$$

$$
x \geq y, \text{ if } x_i \geq y_i, \quad \forall i = 1, \ldots, d,
$$

$$
x \neq y, \text{ if for some } i \leq d \text{ } x_i \neq y_i,
$$

$$
ax := (a_1 x_1, \ldots, a_d x_d)^\top, \quad cx := (c x_1, \ldots, c x_d)^\top, \quad a \in \mathbb{R}^d, c \in \mathbb{R},
$$

$$
\|x_J\|_p := \sum_{i \in J} |x_i|^p, \quad \|x_J\|_{A,p} := \|A_{JJ}^{-1} x_J\|_p, \quad A \in \mathbb{R}^{d \times d},
$$

$$
S_{p}^{k-1} := \{x \in \mathbb{R}^k : \|x\|_p = 1\}, \quad \text{the unit sphere in } \mathbb{R}^k, \quad k \geq 1.
$$

In order to simplify the notation, we write next $x_I^\top$ instead of $(x_I)^\top$ and $\|x\|, \|x_J\|_{A,p}$ instead of $\|x\|_p, \|x_J\|_{A,p}$, respectively. We write $(A_{JJ})^{-1}$ instead of $A_{JJ}^{-1}$ provided it exists. Further set $\theta := (0, \ldots, 0)^\top \in \mathbb{R}^d, I := (1, \ldots, 1)^\top \in \mathbb{R}^d$, and write $Z \sim \text{Beta}(a, b)$ or $Z \sim \text{Beta}(a, b)$ or $Z \sim \text{Beta}(a, b)$.
Gamma($a, b$) if the random variable $Z$ is beta or gamma distributed with positive parameters $a, b$, respectively. In our notation the beta and the gamma distributions with positive parameters $a, b$ possess the density functions $x^{a-1}(1 - x)^{b-1} \Gamma(a + b)/(\Gamma(a)\Gamma(b))$, $x \in (0, 1)$, and $x^{a-1}\exp(-bx)b^a/\Gamma(a), x \in (0, \infty)$, respectively ($\Gamma(\cdot)$ denotes the gamma function).

Basic distributional results for spherical random vectors are derived in [6]. In Lemma 2 therein it was shown that for any two non-empty disjoint index sets $I, J$ such that $I \cup J = \{1, \ldots, d\}$, and a uniformly distributed random vector $\tilde{U}$ on the unit sphere $S_p^{d-1}$ (where $p := 2$) we have the stochastic representation

$$
\tilde{U}_I \overset{d}{=} WU_I, \quad \tilde{U}_J \overset{d}{=} (1 - W^2)^{1/2}U_J,
$$

with $W \in (0, 1), W^p \sim \text{Beta}(|I|/p, |J|/p)$, and $U_I, U_J$ are two uniformly distributed random vectors on the unit spheres $S_p^{|I|-1}$ and $S_p^{|J|-1}$, respectively. Furthermore, $W, U_I, U_J$ are independent.

The stochastic representation (2.1) shows that the random vector $\tilde{U}$ can be split into two subvectors utilising the positive random variable $W$ which is independent of both $U_I, U_J$. We refer to this property as the beta-independent splitting. The basic distributional properties of the spherical random vectors in general, can be derived by making use of the beta-independent splitting in (2.1) (see [6], or [26]).

The first natural generalisation aimed to enlarge the class of the spherical random vectors is achieved by dropping the distributional assumptions on $U_I, U_J$, and requiring only that $U_I$ and $U_J$ be independent. Further, the assumptions on $W$ can be relaxed, still involving the beta family of distribution functions. We give next a precise definition of the new class or random vectors, which will be the focus for our asymptotic results presented in Section 3.

Let $\alpha, \beta, \delta, p \in (0, \infty)$ be given constants, and let $U = (U_1, \ldots, U_d)\top$ be a random vector in $\mathbb{R}^d$ such that almost surely

$$
\|U_I\| = \|U_J\| = 1
$$

is valid. If $R, W_{\alpha, \beta}$ are two independent positive random variables being independent of $U$, then we define a new random vector $S = (S_1, \ldots, S_d)\top$ via the stochastic representation

$$
S_I \overset{d}{=} RW_{\alpha, \beta}U_I, \quad S_J \overset{d}{=} R(1 - W_{\alpha, \beta}^p)^{1/p}U_J.
$$

We impose throughout the paper the following distributional constraint on $W_{\alpha, \beta}$

$$
1 - W_{\alpha, \beta}^p \overset{d}{=} Z_{\alpha, \beta}^\delta, \quad \text{with} \ Z_{\alpha, \beta} \sim \text{Beta}(\beta, \alpha).
$$

For the random vector $S$ we have introduced thus a weak beta-independent splitting property. The class of random vectors $S$ defined in (2.3) is quite large. For instance any $L_p$-norm ($p > 0$) spherical random vector in $\mathbb{R}^d, d \geq 2$, belongs to this class, satisfying (2.4) with

$$
\alpha = m/p, \quad \beta = (d - m)/p, \quad \delta = 1
$$

for any non-empty index set $I \subset \{1, \ldots, d\}$ with $m < d$ elements. See [31] for further details on $L_p$-norm spherical random vectors.

By the definition of the random vector $U$, utilising (2.3) we arrive at

$$
S \overset{d}{=} R\tilde{U}, \quad \|\tilde{U}\| = 1,
$$
with \( \tilde{U} \) independent of \( R \). On the other hand (2.3) can be written as
\[
S_I \overset{d}{=} R_1 U_I, \quad S_J \overset{d}{=} R_2 U_J, \quad \text{with } R_1 := R W_{\alpha, \beta}, \quad R_2 := R (1 - W_{\alpha, \beta})^{1/p},
\]
with \( W_{\alpha, \beta} \) satisfying (2.4). Thus the random vector \( S \) and both components \( S_I, S_J \) can be written as products of a positive random radius with an independent random vector taking values on the corresponding \( L_p \)-norm unit spheres.

Next, let \( X = (X_1, \ldots, X_d)^\top \) be a random vector with the stochastic representation
\[
X \overset{d}{=} AS \overset{d}{=} AR \tilde{U},
\]
where \( A \) is a non-singular \( d \times d \) real matrix.

If \( S \) is a spherical random vector (i.e., \( L_2 \)-norm spherically distributed), then \( X \) is elliptically distributed, and (see e.g., [26])
\[
BS \overset{d}{=} AS
\]
holds for any matrix \( B \in \mathbb{R}^{d \times d} \) such that
\[
BB^\top = AA^\top = \Sigma.
\]
Consequently, for the elliptical model the distribution function of \( X \) is completely known when the matrix \( \Sigma \) and the distribution function of \( S \) are known. The particular choice of \( A \) is therefore irrelevant for the distribution function of \( X \), which is not in general the case for a beta-independent random vector \( X \) with stochastic representation (2.8), i.e., (2.10) does not always imply (2.9). If \( X \) is not an elliptical random vector we shall impose a restriction on the matrix \( A \), namely \( A_{JI} \) has all entries equal to 0. This restriction is dropped if \( S \) is a (\( L_2 \)-norm) spherical random vector.

Our conditional limiting results make use of a convenient stochastic representation of \( X_I | X_J = x_J, x \in \mathbb{R}^d \), which we derive in Theorem 2.5 below. Depending on the distribution function of \( U_J \) and \( R \) the conditional random vector might not be defined for some \( x \in \mathbb{R}^d \). In order to avoid this problem, we impose throughout the paper the following assumption:

A1. If \( X \) is a random vector with stochastic representation (2.8), then for any \( x \in \mathbb{R}^d \) the conditional random vector \( X_I | X_J = x_J \) is well defined, and if \( |J| = 1 \), then \( P \{ X_J = 1 \} \in (0, 1] \).

**Definition 2.1 (Beta-independent Random Vector).** A random vector \( X \) in \( \mathbb{R}^d, d \geq 2 \), satisfying assumption A1 is called a beta-independent random vector with parameters \( A, \alpha, \beta, \delta, p \) and index sets \( I, J \), if \( X \) possesses the stochastic representation (2.8), where the random vector \( S \) satisfies (2.7) with \( R \sim F, W_{\alpha, \beta}, U_I, U_J \) mutually independent, with \( \|U_I\| = \|U_J\| = 1 \), and \( W_{\alpha, \beta}, \alpha, \beta > 0 \), such that (2.4) holds with \( \delta > 0 \). Further \( F(0) = 0 \), and \( A_{JI} \) has all entries equal to 0 if \( S \) is not an \( L_2 \)-norm spherical random vector.

In the following we shall write
\[
X \sim Bi(p, A, \alpha, \beta, \delta, F, U_I, U_J)
\]
whenever the random vector \( X \) is a beta-independent random vector as defined above.

We give next two illustrative examples.

**Example 2.2 (Kummer-Beta \( L_p \)GSD).** Let \( p > 0 \), and let \( \alpha_i, 1 \leq i \leq d, d \geq 2 \), be positive constants. Hashorva et al. [23] consider a Kummer-Beta \( L_p \)-norm generalised symmetrised
Dirichlet (LpGSD) random vector $S$ in $\mathbb{R}^d$ with density function

$$h(x) := c_1 \|x\|^{p(\delta - 1)} (1 - \|x\|^p)^{\gamma - 1} \exp(-\lambda \|x\|^p) \prod_{i=1}^d |x_i|^{\alpha_{1,i} - 1}, \quad x \in \mathbb{R}^d : \|x\| \in (0, 1),$$

with $\delta > 1 - \sum_{i=1}^d \alpha_i$, $\gamma \in (0, \infty)$, $\lambda \geq 0$, and

$$c_1 := \left( \frac{p}{2} \right)^k \frac{\Gamma(\alpha)}{\prod_{i=1}^k \Gamma(\alpha_i)} \frac{\Gamma(\theta + \gamma)}{\Gamma(\theta + \gamma) - \lambda \Gamma(\theta) \Gamma(\gamma)}, \quad \alpha := \sum_{i=1}^d \alpha_i, \quad \theta := \alpha + \delta - 1,$$

where $1_F$ is the confluent hypergeometric function of the first kind (also known as Kummer’s function of the first kind). $1_F$ has the following expansion

$$1_F(a; b; x) = 1 + \frac{a}{b} x + \frac{a(a+1)}{b(b+1)} x^2 + \cdots = \sum_{k=0}^{\infty} (\frac{a}{b})^k \frac{x^k}{k!}, \quad a \in \mathbb{R}, b > 0, x \in \mathbb{R},$$

where $(a)_k, (b)_k$ are the Pochhammer symbols. In view of the amalgamation property of LpGSD random vectors (Theorem 3.10 in [23]) the random vector $S$ is a beta-independent random vector for any $I, J$ partitions of $\{1, \ldots, d\}$, and furthermore

$$AS \sim BI \left( p, A, \sum_{i \in I} \alpha_i, \sum_{i \in J} \alpha_i, 1, F, U_I, U_J \right), \quad (2.11)$$

with $A \in \mathbb{R}^{d \times d}$ being a matrix as in Definition 2.4, $F$ the distribution function of the associated random radius $R$ of $S$, and $U_I, U_J$ two independent $L_p$-norm symmetrised Dirichlet random vectors with parameters $\alpha_i, i \in I$ and $\alpha_i, i \in J$, respectively. Furthermore, $R^p$ possesses the density function

$$\frac{\Gamma(\theta + \gamma)}{1_F(\theta; \theta + \gamma; -\lambda \Gamma(\theta) \Gamma(\gamma))} - x^{\theta - 1} (1 - x)^{\gamma - 1} \exp(-\lambda x), \quad \forall x \in (0, 1). \quad (2.12)$$

**Example 2.3 (Kotz Type I LpGSD).** Let $S$ be a Kotz Type I LpGSD random vector in $\mathbb{R}^d$, $d \geq 2$, with density function $h$ given by

$$h(x) := \left( \frac{p}{2} \right)^k \frac{r^{(N+\alpha)/s}}{s \Gamma((N+\alpha)/s)} \frac{\Gamma(\alpha)}{\prod_{i=1}^k \Gamma(\alpha_i)} \|x\|^{pN} \exp(-r \|x\|^p) \prod_{i=1}^d |x_i|^{\alpha_{1,i} - 1}, \quad \forall x \in \mathbb{R}^d,$$

where $\alpha_i > 0$, $1 \leq i \leq d$, $N > -\alpha := -\sum_{i=1}^d \alpha_i$, $r > 0$, $s > 0$, are given constants. The random vector $S$ has stochastic representation

$$S \overset{d}{=} R \tilde{U}, \quad R^{ps} \sim \Gamma(((N+\alpha)/s, r), \quad (2.13)$$

with $R > 0$ (almost surely) independent of $\tilde{U}$, which is an $L_p$-norm symmetrised Dirichlet random vector with parameters $\alpha_i > 0$, $1 \leq i \leq d$ (see [10, 23]). Since $S$ is a LpGSD random vector the random vector $X$ in $\mathbb{R}^d$ with stochastic representation

$$X \overset{d}{=} AR \tilde{U} \quad (2.14)$$

$\tilde{U}$. The proof follows from the fact that $S$ is a Kotz Type I LpGSD random vector, and the associated random radius $R$ of $S$ possesses the density function

$$\frac{\Gamma(\theta + \gamma)}{1_F(\theta; \theta + \gamma; -\lambda \Gamma(\theta) \Gamma(\gamma))} - x^{\theta - 1} (1 - x)^{\gamma - 1} \exp(-\lambda x), \quad \forall x \in (0, 1).$$
is a beta-independent one. If (2.13) holds with \( N = 0, p = 1/r = 2/s = 2 \), then \( X \) is a Gaussian random vector with covariance matrix \( \Sigma := AA^\top \).

As mentioned in the Introduction the Gaussian approximation of the conditional distribution of elliptical random vectors is of both theoretical and practical interest. In the context of beta-independent random vectors we show that a more general distribution, namely the Kotz Type I polar distribution should be utilised (instead of the Gaussian one) arriving at the following definition:

**Definition 2.4** (Kotz Type I Polar Random Vector). Let \( p > 0, A \in \mathbb{R}^{d \times d}, d \geq 2 \), and let \( \tilde{U} \) be a random vector in \( \mathbb{R}^d \) such that \( \|\tilde{U}\| = 1 \) almost surely. If \( R \) is a positive random variable independent of \( \tilde{U} \), then we refer to the random vector \( X \) in \( \mathbb{R}^d \) with stochastic representation (2.14) as a Kotz Type I polar random vector if additionally the associated random radius satisfies \( R^p \sim \Gamma(\alpha, r) \), where \( \alpha, r \) are some positive constants.

Clearly, a Kotz Type I LpGSD random vector \( X \) with stochastic representation (2.14) is a Kotz Type I polar random vector. We show in Section 3 that Kotz Type I polar random vectors play a crucial role in the asymptotic approximation of conditional distributions of beta-independent random vectors. Such approximations are referred to – in parallel to the Gaussian one – as the Kotz approximations.

If \( F \) is a distribution function with upper endpoint \( x_F \in (0, \infty) \), and \( \tau, a, b, p, q \) are some positive constants, then we define another distribution function \( Q_{\tau,a,b,p,q,F}(z) \) by

\[
Q_{\tau,a,b,p,q,F}(z) := 1 - \frac{\int_{(\tau + a b - p,q)}^{x_F}(s^q - \tau^q)^{a-1} s^{-q(a+b-1)} dF(s)}{\int_{\tau}^{x_F}(s^q - \tau^q)^{a-1} s^{-q(a+b-1)} dF(s)},
\]

\[\forall z \in (0, (x_F^p - \tau^p)^{1/p}), \tau < x_F, \quad (2.15)\]

which determines the conditional distribution \( X_I | X_J = x_J \) as will be shown in the next theorem.

**Theorem 2.5.** Let \( X \sim Bi(p, A, \alpha, \beta, \delta, F, U_I, U_J) \) be a beta-independent random vector in \( \mathbb{R}^d, d \geq 2 \). If \( a \in \mathbb{R}^d \) is such that \( F(|a_J|_A) \in (0, 1) \), then we have the stochastic representation

\[
(X_I | X_J = a_J) \overset{d}{=} A_{II} R|a_J|_A, \alpha, \beta, \delta U_I + A_{IJ} A_j^{-1} a_J, \quad (2.16)
\]

where \( R|a_J|_A, \alpha, \beta, \delta \) has distribution function \( Q|a_J|_A, \alpha, \beta, p, p, F, \) and is independent of the random vector \( U_I \).

**Remarks.** (i) If we simply assume that the random variable \( Z_{\alpha,\beta}^{2/p} \) in (2.4) possesses a positive density function \( g \), as shown in the proof of Theorem 2.5 (see (5.1)) the stochastic representation (2.16) holds where \( R|a_J|_A, \alpha, \beta, \delta \) possesses the survivor function

\[
\frac{\int_{(\tau + a \beta - p,q)}^{x_F} (|a_J|_A/r)^{r-1} dF(r)}{\int_{|a_J|_A}^{x_F} (|a_J|_A/r)^{r-1} dF(r)}, \quad \forall z \in (0, (x_F^p - |a_J|_A^{p})^{1/p}).
\]

(ii) Let \( S \) be an \( L_p \)-norm \( (p > 0) \) spherical random vector in \( \mathbb{R}^d, d \geq 2 \), and let \( I, J \) be two non-empty disjoint index sets such that \( I \cup J = \{1, \ldots, d\} \). Since \( S \) belongs to the class of beta-independent random vectors the conditional distribution \( X_I | X_J = x_J \) obeys the stochastic representation (2.16), which in fact holds for the larger class of LpGSD random vectors (see [23]).
Example 2.6. Let $I_1, I_2$ be two independent discrete random variables taking values $±1$ with equal probability $1/2$, and let $W_{α, β} ∈ (0, 1), α, β > 0$, be another random variable satisfying (2.4) being independent of $I_1, I_2$. Let $A$ be a square matrix given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 1 \end{pmatrix}, \quad a_{11}, a_{12} ∈ ℝ.$$  

If $R > 0$ (almost surely) is another random variable with distribution function $F$ and $R, I_1, I_2, W_{α, β}$ are mutually independent, then a bivariate random vector $X$ with stochastic representation

$$X \overset{d}{=} AS = (a_{11}S_1 + a_{12}S_2, S_2)^\top,$$

where

$$(S_1, S_2) \overset{d}{=} \left( RW_{α, β}I_1, R(1 - W_{α, β}^p)^{1/p}I_2 \right),$$

is a beta-independent random vector with parameters $p, A, α, β, δ, F$. In view of Theorem 2.5 for any $y ∈ (0, x_F)$

$$(X_1|X_2 = y) \overset{d}{=} a_{11}R_γα, β, δI_1 + a_{12}y$$

holds where $R_γα, β, δ$ is independent of $I_1$ with distribution function $Q_{γ, α, β, p, p/δ, F}$.

3. Main results

Let $X ∼ Bi(p, A, α, β, δ, F, U_1, U_2)$ be a beta-independent random vector as defined in the previous section. Denote throughout the paper by $x_F := \sup\{s : F(s) < 1\}$ the upper endpoint of the distribution function $F$. For notational simplicity write in the following $x_k \in ℝ^d, k ∈ ℕ$, and $L$ a non-empty index set of $\{1, \ldots, d\}$. In the following let $Z_n, n ≥ 1$ and $Z_n^*, n ≥ 1$ be two sequences of random vectors defined on the same probability space such that

$$Z_n \overset{d}{=} (X_J - A_{JJ}^{-1}u_{n,J})|X_J = u_{n,J}, \quad (3.1)$$

$$Z_n^* \overset{d}{=} ((X_J - A_{JJ}u_{n,J}), (X_J - u_{n,J}))|X_J > u_{n,J} \quad (3.2)$$

hold for any $n ≥ 1$, and set

$$χ_n := \|u_{n,J}\|_A, \quad n ≥ 1, \quad (3.3)$$

with $u_n, n ≥ 1$, being constants in $ℝ^d$ where $χ_n ∈ (0, x_F), n ≥ 1$.

Our main interest is in asymptotic approximations of the distribution function of $Z_n,J$ and $Z_n^*$ when $χ_n$ tends to $x_F$. The only asymptotic condition imposed on $F$ is that it belongs to the max-domain of attraction of a univariate extreme value distribution function $H$. Explicitly, we suppose that

$$\lim_{n → ∞} \sup_{x ∈ ℝ} \left| F^n(a_nx + b_n) - H(x) \right| = 0 \quad (3.4)$$

holds for some $a_n > 0, b_n ∈ ℝ, n ≥ 1$. For more information on extreme value theory we refer the reader to the following standard monographs: [12,30,28,9,27], or [7]. The univariate distributions satisfying (3.4) are only the Gumbel, Weibull, or the Fréchet distributions. We
consider each case separately, starting next with the Gumbel one. We derive two types of limiting results which can be seen as multivariate generalisations of the conditional limiting results for both $Z_n$ and $Z_n^*$ mentioned in the Introduction.

### 3.1. Kotz approximation

The Gaussian approximation of the conditional distribution of elliptical random vectors has several statistical applications. For the larger class of beta-independent random vectors we show below that the Kotz approximation of the conditional distribution of interest is an adequate one, provided that the distribution function $F$ of the associated random radius of the beta-independent random vector is in the Gumbel max-domain of attraction. Denote by $A(x) = \exp(-\exp(-x))$, $x \in \mathbb{R}$, the unit Gumbel distribution. Explicitly, we assume that there exists a positive scaling function $w$ (see e.g., [30] or [27]) such that

\[
\lim_{u \uparrow x_F} \frac{1 - F(u + x/w(u))}{1 - F(u)} = \exp(-x), \quad \forall x \in \mathbb{R}
\]

is valid with $x_F$ being the upper endpoint of $F$. The scaling function $w$ satisfies

\[
\lim_{u \uparrow x_F} uw(u) = \infty \quad \text{and} \quad \lim_{u \uparrow x_F} w(u)(x_F - u) = \infty \quad \text{if} \quad x_F < \infty.
\]

If (3.5) holds, then we write in the following $F \in \text{MDA}(A, w)$. Canonical examples for the Gumbel max-domain of attraction are the univariate Gaussian and the gamma distributions. In both mentioned cases the scaling function is explicitly known (see Example 3.3).

We state next the Kotz approximation of $Z_{n,1}$.

**Theorem 3.1.** Let $X \sim Bi(p, A, \alpha, \beta, \delta, F, U_I, U_J)$ be a beta-independent random vector in $\mathbb{R}^d$, $d \geq 2$, and let $\mathbf{u}_n$, $n \geq 1$, be constants in $\mathbb{R}^d$ defining $Z_{n,1}$, $n \geq 1$, and $\chi_n$, $n \geq 1$, as in (3.1) and (3.3), respectively. If $F \in \text{MDA}(A, w)$, and for large $n$, we have $\chi_n \in (0, x_F)$ such that

\[
\lim_{n \to \infty} \chi_n = x_F,
\]

then the convergence in distribution

\[
h_n Z_{n,1} \overset{d}{\to} A_{11}/\mathcal{R}_\alpha U_I, \quad n \to \infty
\]

is valid with $h_n := (w(\chi_n)/\chi_n^{p-1})^{1/p}$, $n > 1$, and $\mathcal{R}_\alpha$ being a positive random variable independent of $U_I$ such that $\mathcal{R}_\alpha^p \sim \text{Gamma}(\alpha, 1/p)$.

The random vector $Y_I := \mathcal{R}_\alpha A_{11} U_I$ appearing in (3.8) is a Kotz Type I polar random vector. Therefore we refer to the distribution approximation in (3.8) as the Kotz approximation. It is well known (see e.g., [26]) that $Y_I$ is a Gaussian random in $\mathbb{R}^m$, $m := |I|$ with covariance matrix $A_{11} A_{11}^\top$, provided that $p = 2$, $\alpha = m/2$, and $U_I$ is uniformly distributed on the unit sphere $S_m^{m-1}$. This Gaussian approximation is first shown for elliptical random vectors in [16]. It is an interesting fact that if $I := \{1, \ldots, d - 1\}$, $A_{JJ} := 1$, then the random sequence $Z_{n,1}, n \geq 1$, defined in (3.2) can be approximated by the same limiting distribution function as the random vector $Z_{n,1}$. The Gaussian approximation of $Z_{n,1}^*$ dates back to Gale (1980), [3]. The general result (assuming only the Gumbel max-domain of attraction of $F$) is derived for bivariate elliptical random vectors in [5]. Note in passing that the Gaussian approximation of $Z_{n,1}^*$ does not in general imply the Gaussian approximation of $Z_{n,1}$. 


In the recent article [1] statistical applications of the Gaussian approximation of \( Z_{n,l}^* \) are presented. See also [14] for several related theoretical results. We show next that for the larger class of beta-independent random vectors the approximation of \( Z_{n,l}^* \) by a Kotz Type I polar distribution (when \( p \neq 1 \)) is possible. Given \( p \in (0, \infty) \), we set in the following \( 1_p := 1 \) if \( p = 1 \), and \( 1_p := 0 \) otherwise.

**Theorem 3.2.** Under the assumptions of Theorem 3.1 we suppose further that \( I := \{1, \ldots, d-1\}, \) \( J := \{d\} \) and \( A_{I J} = 1 \). Assume that \( A_{I J} \) has all entries equal to 0 if \( p \in (0, 1) \). Then for any sequence \( u_n := u_n I, u_n \in (0, x_F) \), \( n \geq 1 \), such that \( \lim_{n \to \infty} u_n = x_F \) we have the convergence in distribution
\[
(h_n Z_{n,l}^*, w(u_n)Z_{n,l}^*) \overset{d}{\to} \left( A_{I I} \mathcal{R}_\alpha U_I + \mathcal{E} 1_p A_{I J}, \mathcal{E} \right), \quad n \to \infty,
\]
with \( h_n := (w(u_n)/u_n^{p-1})^{1/p}, \mathcal{E} \sim \text{Gamma}(1, 1) \), and \( \mathcal{R}_\alpha \) as defined in Theorem 3.1. Furthermore, \( \mathcal{R}_\alpha, U_I \) and \( \mathcal{E} \) are mutually independent.

**Remarks.** (i) For LpGSD random vectors with associated random radius \( R \sim F \) such that \( F \) is in the Gumbel max-domain of attraction the results of Theorems 3.1 and 3.2 are shown in Theorem 5.1 of [23] and Theorem 4.3 in [24], respectively.

(ii) In view of (3.6), condition (3.7) implies that the sequence \( h_n, n \geq 1 \), in (3.8) satisfies \( \lim_{n \to \infty} h_n = \infty \).

We give next two illustrative examples.

**Example 3.3** (Kotz Type III Beta-Independent). Let \( X \sim Bi(p, \alpha, \beta, \delta, F, U_I, U_J) \) be a beta-independent random vector in \( \mathbb{R}^d, d \geq 2 \). Assume that the distribution function \( F \) satisfies \((u \to \infty)\)
\[
1 - F(u) = (1 + o(1))K u^N \exp(-ru^\gamma), \quad K > 0, r > 0, \gamma \in \mathbb{R}, N \in \mathbb{R}.
\]
If \( \gamma \leq 0 \), then we require \( N \) to be negative. We refer to \( X \) as a Kotz Type III beta-independent random vector.

Since for any \( \gamma > 0 \)
\[
\frac{1 - F(u + xu^{1-\gamma}/(r\gamma))}{1 - F(u)} = (1 + o(1)) \left( 1 + \frac{x}{ru^\gamma} \right)^N \times \exp \left( -ru^\gamma \left[ \left( 1 + \frac{x}{ru^\gamma} \right)^\gamma - 1 \right] \right) \to \exp(-x), \quad u \to \infty
\]
we deduce that \( F \in \text{MDA}(A, w) \) with the scaling function \( w \) defined by
\[
w(u) := ru^\gamma - 1, \quad u > 0.
\]
Consequently, (3.8) holds with \( h_n := (r\gamma)^{1/p}x_n^{\gamma/p - 1}, n \geq 1 \). If \( \gamma = p \), then \( h_n, n \geq 1 \), does not depend on \( n \), hence (3.8) can be written as
\[
Z_{n,l} \overset{d}{\to} (rp)^{-1/p} A_{I I} \mathcal{R}_\alpha U_I, \quad n \to \infty.
\]
If \( X \) is a Gaussian random vector the above is obviously true since \( p = \gamma = 1/r = 2 \) and
\[
(X | X = u_{n,l}) \overset{d}{\to} A_{I I} \mathcal{R}_\alpha U_I, \quad \forall n \in \mathbb{N}.
\]
Example 3.4. We continue Example 2.6 allowing here the matrix \( A \) to depend on \( n \) as follows

\[
A := A_n = \begin{pmatrix} \lambda_n & \rho_n \\ 0 & 1 \end{pmatrix}, \quad \lambda_n, \rho_n \in [-1, 1], n \in \mathbb{N},
\]

with \( \lim_{n \to \infty} \lambda_n = \lambda \neq 0 \). Let \( (X_n, Y_n), n \geq 1, \) be a bivariate sequence of beta-independent random vectors defined by

\[
X_n := S_2, \quad Y_n := \lambda_n S_1 + \rho_n S_2, \quad S_1 \overset{d}{=} RW_{\alpha, \beta} \mathcal{I}_1, \quad S_2 \overset{d}{=} R(1 - W_{\alpha, \beta}^p)^{1/p} \mathcal{I}_2, \quad i = 1, 2,
\]

(3.12)

where \( \mathcal{I}_1, \mathcal{I}_2, R, W_{\alpha, \beta}, \alpha, \beta > 0, \) are as in Example 2.6. Let \( u_n, n \geq 1, \) be such that \( |u_n| < x_F, n \in \mathbb{N}, \) and \( \lim_{n \to \infty} u_n = x_F. \) Under the assumptions of Theorem 3.2 if \( p > 1, \) or \( p \in (0, 1] \) and \( \rho_n = 0, \forall n \geq 1, \) then for any \( x > 0, y \in \mathbb{R} \) we obtain

\[
\lim_{n \to \infty} P \{ h_n(Y_n - \rho_n u_n) \leq y, w(u_n)(X_n - u_n) \leq x | X_n > u_n \} = P \{ \lambda \mathcal{R}_\alpha \mathcal{I}_1 \leq y, \mathcal{E} \leq x \}, \quad n \to \infty,
\]

(3.13)

where \( h_n := (w(u_n)/u_n^{p-1})^{1/p}, \) and \( \mathcal{R}_\alpha > 0, \mathcal{R}_\alpha^p \sim \text{Gamma}(\alpha, 1/p), \mathcal{E} \sim \text{Gamma}(1, 1). \) Furthermore, \( \mathcal{I}_1, \mathcal{R}_\alpha, \mathcal{E} \) are mutually independent.

3.2. Regularly varying case

We consider next the Weibull case, i.e., we assume that (3.4) holds with \( H \) being the Weibull distribution function \( \Psi_\gamma(x) = \exp(-|x|^{\gamma}), x < 0, \gamma \in (0, \infty). \) The upper endpoint \( x_F \) of the distribution function \( F \) is necessarily finite. We assume without loss of generality that \( x_F = 1. \) The case \( x_F \neq 1, x_F \in (0, \infty), \) follows easily and is therefore omitted here. It is well known (see e.g., [30,27]) that \( F \) is in the max-domain of attraction of the unit Weibull distribution \( \Psi_\gamma, \gamma \in (0, \infty), \) is equivalent with the fact that \( 1 - F \) is regularly varying at 1 with index \( \gamma, \) i.e.,

\[
\lim_{u \to \infty} \frac{1 - F(1 - x/u)}{1 - F(1 - 1/u)} = x^\gamma, \quad \forall x > 0.
\]

(3.14)

Hashorva [20] shows that when considering elliptical random vectors the regular variation of \( 1 - F \) at the upper endpoint of \( F \) is a crucial assumption for deriving an asymptotic approximation of the distribution function of \( Z_{n,1}. \) In the next theorem we show that this assumption implies an asymptotic approximation for the distribution function of \( Z_{n,1} \) when considering the class of beta-independent random vectors.

Theorem 3.5. Let \( X \sim Bi(p, A, \alpha, \beta, \delta, F, U_1, U_f) \) and \( u_n, \chi_n, Z_{n,1}, n \geq 1, \) be as in Theorem 3.1. Assume that the distribution function \( F \) has upper endpoint \( x_F = 1 \) satisfying (3.14) with some \( \gamma \in (0, \infty). \) Let \( a_n, n \geq 1, \) be positive constants such \( \lim_{n \to \infty} a_n = 0, \) and further \( \chi_n = 1 - a_n \) is satisfied for all large \( n. \) Then we have

\[
h_n Z_{n,1} \overset{d}{\to} A_{11}/\mathcal{R}_{\alpha, \gamma} U_1, \quad n \to \infty
\]

(3.15)

with \( h_n := (pa_n)^{-1/p}, n \geq 1, \) and \( \mathcal{R}_{\alpha, \gamma} \) being a positive random variable independent of \( U_1 \) such that \( \mathcal{R}_{\alpha, \gamma}^p \sim \text{Beta}(\alpha, \gamma). \)
As mentioned in the Introduction approximation of the distribution function of $Z^*_n$ is of both theoretical and practical interest. Such approximations for spherical random vectors when additionally $F$ possesses a density function can be found in Gale (1980), and Eddy and Gale (1981). Elliptical random vectors are dealt with in [5,20] where no restriction on the existence of the density function of $F$ is imposed. Next we obtain a similar result to that presented in Theorem 3.2.

**Theorem 3.6.** Under the assumptions of Theorem 3.5 we assume further that $I := \{1, \ldots, d-1\}$ and $A_{jj} = 1$. Suppose that the matrix $A_{ij}$ has all entries equal to 0 if $p \in (0, 1)$. If $a_n, n \geq 1$, are positive constants such that $\lim_{n \to \infty} a_n = 0$, then for any $x \in \mathbb{R}^d$ with $x_d \in (-\infty, 0)$ we have

$$
\lim_{n \to \infty} \mathbb{P}\left\{ (pan)^{-1/p}(X_I - A_{jj}) \leq (X_d - 1)/a_n \leq x_d|X_d > 1 - a_n \right\} = \mathbb{P}\{|\mathcal{E}|^{1/p}(A_{jj} R_{\alpha, \gamma} U_I - 1_p) \leq x_d, \mathcal{E} \leq x_d\},
$$

(3.16)

where $\mathcal{E}$ is a negative random variable with distribution function $1 - |s|^{-\alpha+\gamma}, s \in (-1, 0)$, and $R_{\alpha, \gamma}, U_I$ are as in Theorem 3.5 being independent of $\mathcal{E}$.

We present next two illustrative examples.

**Example 3.7 (Kummer-Beta-Independent).** Let $X \sim Bi(p, A, \alpha, \beta, \delta, F, U_I, U_J)$ be a beta-independent random vector in $\mathbb{R}^d, d \geq 2$. If the associated random radius $R$ is such that $R^p$ possesses the density function given in (2.12), then we call $X$ a Kummer-beta-independent random vector. It follows easily that the distribution function $F$ of $R$ is in the max-domain of attraction of $\Psi_{\gamma}$, hence for this instance both the above theorems hold.

**Example 3.8.** Let $F, I_1, I_2, S_1, S_2, A_n, u_n, n \geq 1$, be as in Example 3.4, and let $a_n, n \geq 1$, be a positive sequence converging to 0 as $n \to \infty$. Suppose that $F$ with upper endpoint $x_F = 1$ is in the max-domain of attraction of $\Psi_{\gamma}, \gamma \in (0, \infty)$, and set $I := \{1\}, J := \{2\}$. In view of (3.15) if further $p > 1$, then for any $x \in \mathbb{R}, y < 0$ we obtain

$$
\lim_{n \to \infty} \mathbb{P}\left\{ \lambda_n S_1 + \rho_n (S_2 - 1) \leq x(pan)^{1/p}, S_2 \leq 1 + a_n y|S_2 > 1 - a_n \right\} = \mathbb{P}\{\lambda |\mathcal{E}|^{1/p} R_{\alpha, \gamma} I_1 \leq x, \mathcal{E} \leq y\},
$$

(3.17)

with $\mathcal{E}$ a negative random variable with distribution function $1 - |x|^{-\alpha+\gamma}, x \in (-1, 0)$, being independent of $I_1$. If $p \in (0, 1]$ and $\rho_n = 0, n \geq 1$, then (3.17) still holds.

Finally, we treat the case where the distribution function $F$ of the associated random radius $R$ is in the Fréchet max-domain of attraction. Explicitly, we suppose that (3.4) holds with $H$ being the Fréchet distribution function $\Phi_{\gamma}(x) = \exp(-x^{-\gamma}), x > 0, \gamma \in (0, \infty)$. Necessarily we have $x_F = \infty$, and furthermore (see e.g., [27])

$$
\lim_{u \to \infty} \frac{1 - F(xu)}{1 - F(u)} = x^{-\gamma}, \quad \forall x > 0.
$$

(3.18)

The above asymptotics mean that $1 - F$ is regularly varying at infinity with index $-\gamma < 0$. Conditional limiting results for regularly varying $1 - F$ are obtained in [18,23]. In our last theorem below we extend our previous results considering beta-independent random vectors.
Theorem 3.9. Let $X \sim Bi(p, \alpha, \beta, \delta, F, U_1, U_f)$ and $u_n, Z_{n,1}, n \geq 1$, be as in Theorem 3.1, and let $u_n, n \geq 1$, be a positive sequence such that $\lim_{n \to \infty} u_n = \infty$. If $F$ is in the max-domain of attraction of $\Phi_\gamma$, with $\gamma > 0$, then for any $a \in \mathbb{R}^d$ such that $\|a\|_A > 0$ we have

$$\frac{1}{u_n} Z_{n,1} \xrightarrow{d} A_{11} R U_1, \quad n \to \infty,$$

where $R$ is independent of $U_1$, and

$$R \sim Q_{\|a\|_A, \alpha, \beta, p, p/\delta, G}, \quad \text{with } G(s) := 1 - \|a\|_A s^{-\gamma}, \forall s \geq \|a\|_A.$$

If $F$ satisfies (3.18) and $A \in \mathbb{R}^{d \times d}$ is a non-singular matrix, utilising the stochastic representation (2.6) along the lines of Theorem 12.6.1 of [5] (see also Proposition 3.6 in [15]) we obtain

$$\lim_{t \to \infty} \frac{P[X/t \in B]}{P[R > t]} = \int_0^\infty P[s A \tilde{U} \in B] d(s^{-\gamma}) < \infty,$$

with $B$ a given Borel set away from the origin of $\mathbb{R}^d$, and random vector $\tilde{U}$ as in (2.6). A corresponding result to (3.16) is thus immediate.

4. Applications of Kotz approximation

The conditional limiting results above have several applications such as the approximation of the maxima of beta-independent triangular arrays, approximation of the concomitant of order statistics, tail approximation, and estimation of the conditional survivor and quantile function. The first two applications are shown for the class of elliptical distributions in [17,22]. In the following we provide few details for the last two applications.

4.1. Tail approximation

Consider a sequence of beta-independent bivariate random vectors $(X_n, Y_n), n \geq 1$. We show next that the Kotz approximation in Theorem 3.2 implies a new asymptotic result ($n \to \infty$) for the joint survivor probability $P[X_n > x_n, Y_n > y_n]$, with $x_n, y_n, n \geq 1$, two sequences of constants.

Let $F, I_1, I_2, \lambda, \lambda_n, \rho_n, u_n, n \geq 1$, and $(X_n, Y_n), n \geq 1$, with stochastic representation (3.12) be as in Example 3.4. If $F \in MDA(\Lambda, \omega)$, then for any $x, y$ positive and $p > 1$ we may write (recall (3.13))

$$P[X_n > x/w(u_n) + u_n, Y_n > y(u_n^{p-1}/w(u_n))^{1/p} + \rho_n u_n] = (1 + o(1)) \frac{1}{2} \exp(-x) P[R^*_\alpha > (y/\lambda)^p] P[X_n > u_n], \quad n \to \infty,$$

with $R^*_\alpha \sim \text{Gamma}(\alpha, 1/p)$. Applying Theorem 6.2 in [19] we obtain the tail approximation

$$P[X_n > x_n, Y_n > y_n] = (1 + o(1)) \frac{1}{4} \exp(-x)(p/\delta)^\alpha \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)} P[R^*_\alpha > (y/\lambda)^p]$$

$$\times (u_n w(u_n))^{-\alpha}[1 - F(u_n)], \quad n \to \infty,$$

where

$$x_n := x/w(u_n) + u_n, \quad y_n := y(u_n^{p-1}/w(u_n))^{1/p} + \rho_n u_n, \quad n \geq 1.$$
The above asymptotics hold for any \( p > 0 \), provided that \( \rho_n = 0, \forall n \geq 1 \). Tail asymptotics of joint survivor functions are of certain theoretical interest. See [21,22] for results on tail asymptotic expansions of elliptical random vector.

### 4.2. Estimation of conditional survivor and quantile function

Let \((X_n, Y_n), n \geq 1\), be as above and suppose that the distribution function \( F \) has an infinite upper endpoint such that (3.5) holds with some scaling function \( w \). Define the conditional survivor function \( \Psi_x(y) \) by

\[
\Psi_x(y) := P[Y_1 > y|X_1 > x], \quad x, y \in \mathbb{R}.
\]

Estimation of \( 1 - \Psi_x(y) \) (elliptical setup) when \( x \) is large is discussed in detail in [1]. As noted in the aforementioned paper for \( x \) large enough no observation might be available for estimating \( \Psi_x(y) \). Utilising the Gaussian approximation two different estimators for \( \Psi_x \) are introduced in the aforementioned paper. In our model with beta-independent random vectors we utilise the Kotz approximation in order to estimate \( \Psi_x \) for \( x \) large. Explicitly, suppose that

\[
\lambda_n := (1 - b^p)^{1/p}, \quad \rho_n := b \in (-1, 1), \quad \forall n \geq 1,
\]

implying that \((X_1, Y_1), \ldots, (X_n, Y_n)\) are identically distributed, and let \( \hat{b}_n, n \geq 1 \), be an estimator of \( b \) which can be easily constructed provided that \( E(X_1^2) \in (0, \infty) \). Further, let \( \hat{w}_n(x), n \geq 1 \), be an estimator of the scaling function \( w \). Such an estimator is in general not easy to construct, see [1]. In view of Theorem 6.2 in [19] both \( X_1 \) and \( Y_1 \) have distribution functions in the Gumbel max-domain of attraction with the same scaling function \( w \) as \( F \). If \((X_1, Y_1)\) is a Kotz Type III beta-independent random vector (see Example 3.3), then the estimation of \( w \) (see (3.11)) can be performed by estimating the constants \( r \) and \( \gamma \) based on the observations of \( X_1, \ldots, X_n \). In view of the Kotz approximation in (3.13) we can estimate \( \Psi_x(y) \) for any \( y \) positive and \( x \) large by

\[
\hat{\Psi}_{x,n}(y) := T_\alpha \left( \hat{h}_n(y - \hat{b}_nx)/(1 - \hat{b}_n^p)^{1/p} \right), \quad n > 1,
\]

with \( T_\alpha \) the survivor function of \( (1 - b^p)^{1/p} R_\alpha T_1 \), and

\[
\hat{h}_n(x) := (\hat{w}_n(x)/x^{p-1})^{1/p}, \quad x > 0, n > 1.
\]

If the parameters \( \alpha, p \) are unknown, we need to estimate them. As communicated in [1] estimation of the conditional quantile function is also of some interest. In view of (4.2) such an estimator can be easily constructed.

### 5. Proofs and further results

**Proof of Theorem 2.5.** Let \( x_F \in (0, \infty) \), be the upper endpoint of \( F \). We closely follow the proof of Theorem 5 of [6]. By the assumption \( \|U_I\| = \|U_J\| = 1 \), hence

\[
\|RW_{\alpha,\beta}U_I\|^p + \|R(1 - W_{\alpha,\beta}^p)^{1/p}U_J\|^p = R^p.
\]

Since further \( R, U_I, U_J, W_{\alpha,\beta} \) are mutually independent for any \( x \in \mathbb{R}^d \) such that \( \|x_J\|_A \in (0, x_F) \) we have

\[
(S_I|S_J = x_J) \overset{d}{=} \left( RW_{\alpha,\beta}U_I|R(1 - W_{\alpha,\beta}^p)^{1/p}U_J = x_J \right)
\]
\[
\begin{align*}
&= \left( U_I (R^P - \| x_J \|^P_A)^{1/P} | R^P (1 - W_{\alpha, \beta}^P) = \| x_J \|^P_A \right) \\
&= R_{\| x_J \|_A, \alpha, \beta} U_I,
\end{align*}
\]

where

\[
R_{\| x_J \|_A, \alpha, \beta} \overset{d}{=} (R^P - \| x_J \|^P_A)^{1/P} | R^P (1 - W_{\alpha, \beta}^P) = \| x_J \|^P_A,
\]

with \( R_{\| x_J \|_A, \alpha, \beta} \) being independent of \( U_I \). Since \( A_{IJ} \) has all elements equal to 0, we have

\[
X_I \overset{d}{=} (AS)_I = A_{II} S_I + A_{IJ} S_J, \quad X_J \overset{d}{=} (AS)_J = A_{JJ} S_J.
\]

Hence using the fact that \( A_{JJ}^{-1} \) exists (see also the proof of Theorem 2.18 in [10]) for any \( a \in \mathbb{R}^d \) such that \( F(\| A_{JJ}^{-1} a \|) \in (0, 1) \) we obtain

\[
(X_I | X_J = a_J) \overset{d}{=} A_{II} S_I + A_{IJ} S_J | A_{JJ} S_J = a_J
\]

\[
= A_{II} S_I + A_{IJ} S_J | A_{JJ} S_J = A_{JJ}^{-1} a_J
\]

\[
= R_{\| A_{JJ}^{-1} a_J \|_A, \alpha, \beta} A_{II} U_I + A_{IJ} A_{JJ}^{-1} a_J
\]

\[
= R_{\| a_J \|_A, \alpha, \beta} A_{II} U_I + A_{IJ} A_{JJ}^{-1} a_J.
\]

Utilising Lemma 3 of [6] and (2.4) for any \( y > 0 \) such that \( y^P < x_F^P - \| a_J \|^P_A \) we have

\[
P(R_{\| a_J \|_A, \alpha, \beta} \leq y) = P((R^P - \| a_J \|^P_A)^{1/P} \leq y | R^P (1 - W_{\alpha, \beta}^P) = \| a_J \|^P_A)
\]

\[
= P(R^P \leq y^P + \| a_J \|^P_A | R^P (1 - W_{\alpha, \beta}^P) = \| a_J \|^P_A)
\]

\[
= P(R \leq (y^P + \| a_J \|^P_A)^{1/P} | R z_{\alpha, \beta}^\delta = \| a_J \|^P_A)
\]

\[
= \int_{\| a_J \|^P_A}^{x_F} \frac{g(\| a_J \|^P_A / r) r^{-1} dF(r)}{\int_{\| a_J \|^P_A}^{x_F} g(\| a_J \|^P_A / r) r^{-1} dF(r)},
\]

(5.1)

with \( g \) being the density function of \( Z_{\alpha, \beta}^\delta \). Straightforward calculations yield that \( R_{\| a_J \|_A, \alpha, \beta} \) possesses the survivor function

\[
\frac{\int_{\| a_J \|^P_A + z}^{x_F} (r^P / \delta - z^P / \delta)^{(\alpha - 1)r - p(\alpha + \beta - 1)/\delta} dF(r)}{\int_{\| a_J \|^P_A}^{x_F} (r^P / \delta - z^P / \delta)^{(\alpha - 1)r - p(\alpha + \beta - 1)/\delta} dF(r)}, \quad \forall z \in (0, (x_F^P - \| a_J \|^P_A)^{1/P}),
\]

thus the result follows. \( \square \)

**Lemma 5.1.** Let \( F \) be a univariate distribution function with upper endpoint \( x_F \in (0, \infty) \), and let \( y_u \geq 0, z_u \in \mathbb{R}, u \in (0, x_F) \), be such that

\[
\lim_{u \uparrow x_F} y_u = y \in [0, \infty), \quad \lim_{u \uparrow x_F} z_u = z \in \mathbb{R}.
\]

(5.2)

If \( F \in MDA(\Lambda, w) \) with some positive scaling function \( w \), then for any \( \alpha > -1, \beta \leq 0, \) and \( \delta > 0 \) we have

\[
\int_{u + (y_u + z_u)/w(u)}^{x_F} [x^\delta - (u + z_u / w(u))^\delta]^{\alpha} x^\beta dF(x)
\]
\[
(1 + o(1)) \exp(-z) \left( \frac{\delta u^{\delta-1}}{w(u)} \right)^{\alpha} u^\beta [1 - F(u)] \int_0^\infty s^{\alpha} \exp(-s) ds, \quad u \uparrow x_F. \tag{5.3}
\]

**Proof of Lemma 5.1.** The proof follows with similar arguments as those in the proof of Lemma 3.5 in [16] where the case \( \delta = 2 \) is shown. We provide a shorter proof showing that the aforementioned lemma can be applied also if \( \delta > 0 \) is an arbitrary constant. Assume for simplicity that \( x_F = \infty \) or \( x_F = 1 \). By the assumption on \( F \) and (3.6) it follows that \( F_\lambda(x) := F(x^{1/\lambda}), \lambda := 2/\delta \) is in the max-domain of attraction of \( A \) with the scaling function \( w_\lambda \) defined by
\[
 w_\lambda(x) := \lambda x^{1-\lambda} w(x^{1/\lambda}), \quad x \in (0, x_F).
\]

Hence, utilising Lemma 3.5 in [16] we have (set \( u_\lambda := u^{1/\lambda} \))
\[
\int_{u + (y_n + z_n)/w(u)}^{x_F} [x^\delta - (u + z_n/w(u))^\delta] ^{\alpha} x^\beta \, dF(x)
\]
\[
= \int_{u_\lambda + (1 + o(1)) (y_n + z_n)/w_\lambda(u_\lambda)}^{x_F} [y^2 - (u_\lambda + (1 + o(1)) z_n/w_\lambda(u_\lambda))] ^{\alpha} y^2 \, dF_\lambda(y), \quad u_\lambda \uparrow x_F
\]
\[
= (1 + o(1)) \exp(-z) \left( \frac{2u_\lambda}{w_\lambda(u_\lambda)} \right)^{\alpha} u_\lambda^2 [1 - F_\lambda(u_\lambda)] \int_0^\infty s^{\alpha} \exp(-s) ds, \quad u_\lambda \uparrow x_F
\]
\[
= (1 + o(1)) \exp(-z) \left( \frac{\delta u^{\delta-1}}{w(u)} \right)^{\alpha} u^\beta [1 - F(u)] \int_0^\infty s^{\alpha} \exp(-s) ds, \quad u \uparrow x_F,
\]
thus the result follows. \( \Box \)

**Proof of Theorem 3.1.** For any \( n \geq 1 \) set
\[
\chi_n := \|u_{n,j}\|_A = \|A_j^{-1} u_{n,j}\|_p, \quad h_n := \left( \frac{w(\chi_n)}{\chi_n^{\alpha-1}} \right) ^{1/p}
\]
and denote by \( x_F \) the upper endpoint of the distribution function \( F \). By the assumptions, \( F(\chi_n) \in (0, 1) \), for large \( n \). Hence, Theorem 2.5 implies for all large \( n \)
\[
Z_{n,j} \overset{d}{=} (X_j - A_{I,j} A_j^{-1_j} u_{n,j})|X_j = u_{n,j}, \overset{d}{=} A_{I,j} R_n U_1, \tag{5.4}
\]
where \( R_n, n \geq 1, \) is a random variable independent of \( U_j \) with the survivor function
\[
P[R_n > z] := \frac{\int_{x_F}^{x_F} (s^q - \chi_n^q)^{\alpha - 1} s^{-r} \, dF(s)}{\int_{x_n}^{x_F} (s^q - \chi_n^q)^{\alpha - 1} s^{-r} \, dF(s)}, \quad \forall z \in (0, (x_F)^p - \chi_n^{p\alpha})^{1/p}),
\]
where \( q := p/\delta > 0, r := p(\alpha + \beta - 1)/\delta \).

Let \( R_\alpha \) be a positive random variable independent of \( U_I \) such that \( R_\alpha^p \sim \varGamma(\alpha, 1/p) \). Then, (3.6) and Lemma 5.1 imply for any \( z > 0 \)
\[
\lim_{n \to \infty} P[h_n R_n > z] = \lim_{n \to \infty} \frac{\int_{x_F}^{x_F} (s^q - \chi_n^q)^{\alpha - 1} s^{-r} \, dF(s)}{\int_{x_n}^{x_F} (s^q - \chi_n^q)^{\alpha - 1} s^{-r} \, dF(s)}
\]
\[
= \lim_{n \to \infty} \frac{\int_{x_n}^{x_F} (1 + o(1)) z^p/(\chi_n w(\chi_n)) (s^q - \chi_n^q)^{\alpha - 1} s^{-r} \, dF(s)}{\int_{x_n}^{x_F} (s^q - \chi_n^q)^{\alpha - 1} s^{-r} \, dF(s)}
\]
Consequently, we obtain the convergence in distribution
\[ h_n \xrightarrow{d} R_\alpha, \quad n \to \infty. \]
Since \( U_I \) and \( R_n, n \geq 1, \) are independent
\[ h_n A_{1I} R_n U_I \xrightarrow{d} A_{1I} R_\alpha U_I, \quad n \to \infty. \]
Hence the proof is established. □

**Proof of Theorem 3.2.** Let \( H \) denote the distribution function of \( X_d. \) By the assumptions, we have
\[ X_d \overset{d}{=} R(1 - V)^{\delta/p} \mathcal{I}_1, \quad \text{and} \quad |X_d| \overset{d}{=} R(1 - V)^{\delta/p}, \] (5.5)
where \( \mathcal{I}_1 \in \{-1, 1\}, P[\mathcal{I}_1 = 1] \in (0, 1], \) and \( V \sim \text{Beta}(\alpha, \beta). \) Furthermore, \( R, V \) and \( \mathcal{I}_1 \) are mutually independent. Utilising Theorem 6.2 of [19] we obtain
\[ P[|X_d| > u] = (1 + o(1))(p/\delta)^\alpha \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)} (uw(u))^{-\alpha}[1 - F(u)], \quad u \uparrow x_F. \] (5.6)
The self-neglecting property of the scaling function \( w \) (see e.g., [30]), i.e.,
\[ \lim_{u \uparrow x_F} \frac{w(u + z/w(u))}{w(u)} = 1 \] (5.7)
uniformly for \( z \) in compact sets of \( \mathbb{R} \) and (3.6) (recall \( \lim_{u \to \infty}(u + x/w(u)) = \infty, \forall x \in \mathbb{R} \)) imply
\[ \lim_{u \uparrow x_F} \frac{P[X_d > u + z/w(u)]}{P[X_d > u]} = \lim_{u \uparrow x_F} \frac{1 - F(u + z/w(u))}{1 - F(u)} = \exp(-z), \quad \forall z \in \mathbb{R}. \]
Consequently, \( H \in MDA(\Lambda, w), \) hence
\[ \lim_{n \to \infty} \frac{1 - H(u_n + z/w(u_n))}{1 - H(u_n)} = \exp(-z), \quad \forall z \in \mathbb{R}. \]

Let \( x \in \mathbb{R}^d \) and \( t \in \mathbb{R} \) be given and set
\[ h_n := (u_n w(u_n))^{(1-p)/p} w(u_n), \quad u_n^* := u_n + t/w(u_n), \]
\[ h_n^* := (u_n^* w(u_n^*))^{(1-p)/p} w(u_n^*), \quad n \geq 1. \]

For any \( p > 1 \) we have (recall (3.6))
\[ \lim_{n \to \infty} \frac{h_n}{w(u_n)} = \lim_{n \to \infty} (u_n w(u_n))^{(1-p)/p} = 0. \]
Hence if \( p \geq 1, \) then **Theorem 3.1** implies
\[ \lim_{n \to \infty} P \{h_n X_I - u_n A_{1I} \leq x_I | X_d = u_n + t/w(u_n)\} = \lim_{n \to \infty} P \{h_n X_I - (u_n + t/w(u_n)) A_{1I} + th_n/w(u_n) A_{1I} \leq x_I | X_d = u_n + t/w(u_n)\} = P \{A_{1I} R_\alpha U_I + 1_p t A_{1I} \leq x_I\}. \]
Theorem 3.1

Theorem 3.1

(5.7) corresponding to independent of (5.4) we arrive at

Let we may further write \( \lim_{n \to \infty} \frac{1}{w(u_n)} \) defined by

The proof for \( R \) in [16] we obtain for any \( x \in \mathbb{R}^d \) such that \( x_d > 0 \)

\[
\begin{align*}
\mathbb{P}\{h_n(X_I - u_n A_{IJ}) \leq x_I, w(u_n)(X_d - u_n) \leq x_d | X_d > u_n\} &= \frac{1}{1 - H(u_n)} \int_{0}^{x_f} \mathbb{P}\{h_n(X_I - u_n A_{IJ}) \leq x_I, X_d \leq u_n + x_d/w(u_n)|X_d = s\} dH(s) \\
&= \int_{0}^{x_d} \mathbb{P}\{h_n(X_I - u_n A_{IJ}) \leq x_I | X_d = u_n + t/w(u_n)\} \\
&\times dH(u_n + t/w(u_n))/(1 - H(u_n)) \\
&\to \int_{0}^{x_d} \mathbb{P}\{A_{IJ} \mathcal{R}_\alpha U_I + 1_p t A_{IJ} \leq x_I\} d(\exp(-t)), \quad n \to \infty,
\end{align*}
\]

hence the result follows. \( \Box \)

Lemma 5.2. Let \( F \) be a univariate distribution function with upper endpoint 1, and let \( y, z, y_u, z_u, u > 0 \), be non-negative constants such that

\[
\lim_{u \downarrow 0} y_u/u = y, \quad \lim_{u \downarrow 0} z_u/u = z \geq y. \tag{5.8}
\]

If \( F \) is in the max-domain of attraction of \( \Psi_\gamma, \gamma \in (0, \infty) \), then for given constants \( \alpha > -1, \beta \in \mathbb{R}, \delta > 0 \), we have

\[
\begin{align*}
\int_{1+y_u-z_u}^{1} (x^\delta - (1 - z_u)^\beta)^\alpha x^\beta dF(x) &= (1 + o(1))[1 - F(1 - u)](\delta u)^\alpha \\
&\times \int_{0}^{z-y} (z - t)^\alpha d(t^\gamma), \quad u \downarrow 0. \tag{5.9}
\end{align*}
\]

The proof for \( \delta = 2 \) is given in Lemma 4.3 of [20]. Since the distribution function \( F_\tau, \tau > 0 \) defined by \( F_\tau(x) := F(x^\tau), x \in \mathbb{R} \) is also in the Weibull max-domain of attraction, the proof for any \( \delta > 0 \) follows easily by applying the aforementioned lemma. \( \Box \)

Proof of Theorem 3.5. Let \( R_n, n \geq 1 \), be positive random variables as in (5.4) corresponding to \( u_n, f = 1 - a_n \). Since \( \lim_{n \to \infty} a_n = 0 \) we have

\[
[(1 - a_n)^p + p a_n t^p]^{1/p} = 1 - a_n (1 - t^p)(1 + o(1)), \quad n \to \infty.
\]

Hence for any \( t \in (0, 1) \), and \( q := p/\delta, r := p(\alpha + \beta - 1)/\delta \) utilising Lemma 5.2 we obtain

\[
\lim_{n \to \infty} \mathbb{P}\left\{ \left( \frac{1}{p a_n} \right)^{1/p} R_n > t \right\} = \lim_{n \to \infty} \frac{\int_{[1-a_n]^p + p a_n t^p]^{1/p}[s^q - (1 - a_n)^q]|x - 1 - s - t|dF(s)}{\int_{1-a_n}^{1} |s^q - (1 - a_n)^q|\alpha^{-1} s^{-r}dF(s)}
\]
with \( B(s, \alpha, \gamma), s \in (0, 1) \), the beta distribution function with positive parameters \( \alpha, \gamma \). Consequently, the convergence in distribution

\[
\left( \frac{1}{p_{n}} \right)^{1/p} R_{n} \overset{d}{\to} \mathcal{R}_{\alpha, \gamma}, \quad n \to \infty
\]

holds with \( \mathcal{R}_{\alpha, \gamma} > 0 \) (almost surely) and \( \mathcal{R}_{\alpha, \gamma}^{p} \sim \text{Beta}(\alpha, \gamma) \). Since for any \( n \in \mathbb{N} \) the random variable \( R_{n} \) is independent of \( U_{I} \) we can assume further that \( \mathcal{R}_{\alpha, \gamma} \) is independent of \( U_{I} \), hence

\[
\left( \frac{1}{p_{n}} \right)^{1/p} A_{II} R_{n} U_{I} \overset{d}{\to} A_{II} \mathcal{R}_{\alpha, \gamma} U_{I}, \quad n \to \infty,
\]

thus (5.4) yields the result. \( \square \)

**Proof of Theorem 3.6.** As in the proof of Theorem 3.2 utilising Theorem 6.2 in [19] we obtain

\[
P(|X_{d}| > u) = (1 + o(1))(p/\delta)^{\alpha} \Gamma(\alpha + \beta) \Gamma(\gamma + 1) \Gamma(\beta) \Gamma(\gamma + \alpha + 1)(1 - u)^{\alpha}[1 - F(u)], \quad u \uparrow 1.
\]

By (3.14) and the fact that \( \lim_{n \to \infty} a_{n} = 0 \) for any \( x \in (0, \infty) \) we have

\[
\lim_{n \to \infty} \frac{P(X_{d} > 1 - a_{n} x)}{P(X_{d} > 1 - a_{n})} = x^{\alpha} \lim_{n \to \infty} \frac{1 - F(1 - a_{n} x)}{1 - F(1 - a_{n})} = x^{\alpha + \gamma},
\]

hence the distribution function \( H \) of \( X_{d} \) is in the max-domain of attraction of the Weibull distribution \( \Psi_{\alpha + \gamma} \). Let \( \mathcal{R}_{\alpha, \gamma} \) be as in Theorem 3.1 being independent of \( U_{I} \), and set \( h_{n} := (p_{n})^{-1/p}, n \geq 1 \). Since \( \lim_{n \to \infty} a_{n} = 0 \), then for any \( x \in \mathbb{R}^{d} \) and \( p \geq 1, t > 0 \), Theorem 3.5 and (5.4) imply

\[
P\{h_{n}(X_{I} - A_{IJ}) \leq x_{I}|X_{d} = 1 - a_{n} t\} = P\left\{(h_{n} t^{-1/p}) A_{II} R_{n} U_{I} - A_{n} h_{n} t^{-1/p} A_{II} \leq t^{-1/p} x_{I}\right\}
\]

\[
\overset{d}{=} P\left\{A_{II} \mathcal{R}_{\alpha, \gamma} U_{I} - p A_{II} \leq t^{-1/p} x_{I}\right\}, \quad n \to \infty,
\]

with \( R_{n} \) satisfying (5.4) for \( u_{n, I} = 1 - a_{n} t, n \geq 1 \), and \( 1_{p} := 1 \) if \( p = 1 \) and 0 otherwise. Similarly, if \( p \in (0, 1) \), and \( A_{IJ} \) has all components equal to 0, then we have

\[
\lim_{n \to \infty} P\{h_{n} X_{I} \leq x_{I}|X_{d} = 1 - a_{n} t\} = P\{A_{II} \mathcal{R}_{\alpha, \gamma} U_{I} \leq t^{-1/p} x_{I}\}
\]

\[
= P\{A_{II} \mathcal{R}_{\alpha, \gamma} U_{I} - p A_{II} \leq t^{-1/p} x_{I}\}.
\]
Let $F$ be a univariate distribution function in the Fréchet max-domain of attraction. Again let $c > 0$.

By the regular variation of $1 - F$, $\gamma$, and the assumption $\lambda < \gamma$, for any $c > 0$ (see Lemma 1 in [25]) we obtain

$$\int_c^{\infty} (s^{\delta - \delta^a b^b} a_s b F(s) = (1 + o(1))[1 - F(u)] y'$$

$$\times \int_c^{\infty} (1 - (z/s)^{\delta^a b^b} s^{\lambda - \gamma - 1} ds, \quad u \to \infty.$$  

Transforming the variables and applying the above result we have

$$\int_{(y_p + z_p)}^{\infty} (s^{\delta - \delta^a b^b} a_s b F(s)$$

$$= (1 + o(1))u^{\lambda^a b^b} \int_{(y_p + z_p)}^{\infty} (1 - (z/s)^{\delta^a b^b} s^{\lambda - \gamma - 1} ds, \quad u \to \infty,$$  

hence the result follows. □

**Proof of Theorem 3.9.** Again let $R_n, n \geq 1$, be defined as in (5.4) where we take $u_n, \gamma := u_n \|a_j\|_A, n \geq 1$. For any $z > 0$ applying Lemma 5.3 (set $c := \|a_j\|_A$ and $q := p/\delta > 0, r := \ldots$)
q(\alpha + \beta - 1)) we obtain
\[
\lim_{n \to \infty} P(R_n > u_n z) = \lim_{n \to \infty} \frac{\int_{u_n(c+p+z)} \cdots}{\int_{u_n(c+p)} \cdots}
\]
where \(R_{c, \alpha, \beta, \gamma, p, \delta} \sim Q_{\alpha, \beta, \gamma, p, q, G}\) with distribution function \(G\) defined by
\[
G(s) := 1 - c\gamma s^{-\gamma}, \quad \forall s \geq c > 0.
\]
Furthermore, we can take \(R_{c, \alpha, \beta, \gamma, p, \delta}\) to be independent of \(U_1\). Consequently
\[
\frac{1}{u_n} A_{11} R_n U_1 \overset{d}{\to} A_{11} R_{c, \alpha, \beta, \gamma, p, \delta} U_1, \quad n \to \infty,
\]
thus the proof is complete. \(\square\)

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References


