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Heterotic twistor–string theory

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Abstract

We reformulate twistor–string theory as a heterotic string based on a twisted $(0,2)$ model. The path integral localizes on holomorphic maps, while the $(0,2)$ moduli naturally correspond to the states of $\mathcal{N}=4$ super-Yang–Mills and conformal supergravity under the Penrose transform. We show how the standard twistor–string formulae of scattering amplitudes as integrals over the space of curves in supertwistor space may be obtained from our model. The corresponding string field theory gives rise to a twistor action for $\mathcal{N}=4$ conformal supergravity coupled to super-Yang–Mills. The model helps to explain how the twistor–strings of Witten and Berkovits are related and clarifies various aspects of each of these models.

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1. Introduction

The twistor–string theories of Witten [1] and Berkovits [2] combine topological string theory with the Penrose transform [3] to describe field theories in four-dimensional spacetime. The models appear to be equivalent to each other and to $\mathcal{N}=4$ super-Yang–Mills theory coupled to a non-minimal conformal supergravity [4]. The mechanism is completely different from the usual string paradigm: spacetime is not introduced *ab initio* as a target, but emerges as the space of degree 1 worldsheet instantons in the twistor space target. It therefore provides a new way for both string theory and twistor theory to make contact with spacetime physics. As far as string theory is concerned, it does so without the extra spacetime dimensions and further infinite towers

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of massive modes of conventional string theory. As far as twistor theory is concerned, it resolves (albeit perturbatively) the most serious outstanding questions in the twistor programme. Firstly, it provides a solution to the 'googly problem' of encoding both the selfdual and anti-selfdual parts of Yang–Mills and gravitational fields on twistor space in such a way that interactions can be naturally incorporated. Classical twistor constructions have previously only been able to cope with anti-selfdual interactions. Secondly, twistor–string theory also provides a natural way to incorporate quantum field theory into twistor theory. Moreover the associated twistor–string field theory is closely related to the twistor actions constructed in [5–7]. These actions provide generating principles for all the amplitudes in the theories. Insight from the twistor–string has also led to a number of powerful new approaches to calculating scattering amplitudes in perturbative gauge theory, both directly in string theory [8–10], and indirectly through spacetime unitarity methods inspired by the twistor–string [11–17].

There remain a number of difficulties in making sense of twistor–string theory, and in exploiting it as a calculational tool. In particular, the presence of conformal supergravity limits ones ability to use twistor–string theory to calculate pure Yang–Mills amplitudes to tree level, since supergravity modes will propagate in any loops [1,18]. Conformal supergravity is thought neither to be unitary, nor to possess a stable vacuum [19] and so is widely viewed as an unwelcome feature of twistor–string theory. However, because conformal supergravity contains Poincaré supergravity as a subsector, one might more optimistically view it as an opportunity. Indeed, twistor–string theories with the spectrum of Poincaré supergravity have been constructed in [20], although these theories remain tentative as it has not yet been determined whether they lead to the correct interactions. If they do, and are consistent, they will provide a new approach to quantum gravity. Furthermore, for applications to loop calculations in gauge theories, one might then decouple gravity in the limit that the Planck mass becomes infinite while the gauge coupling stays finite.

This paper will not attempt to make further progress on these issues, but will provide a new model for twistor–string theory that goes some way towards resolving other puzzles arising from the original models. Witten's original twistor–string [1] is based on a topological string theory, the B-model, of maps from a Riemann surface into the twistor superspace $\mathbb{P}^{3|4}$, the projectivization of $\mathbb{C}^{4|4}$ with four bosonic coordinates and four fermionic. While one can always construct a topological string theory on a standard (bosonic) Calabi–Yau threefold [21,22], it is not obvious that the same construction works on a supermanifold such as $\mathbb{P}^{3|4}$ even if it is formally Calabi–Yau. Proceeding heuristically, Witten showed that the open string sector would successfully provide the anti-selfdual[1] interactions of $\mathcal{N}=4$ super-Yang–Mills. However, to include selfdual interactions requires the introduction of D1 branes wrapping holomorphic curves in projective supertwistor space. The full Yang–Mills perturbation theory then arises from strings stretched between these D1 branes and a stack of (almost) space-filling D5 branes, together with the holomorphic Chern–Simons theory of the D5–D5 strings. However, one would also expect to find open D1–D1 strings and the role of these in spacetime was left unclear. Gravitational modes decouple from the open B-model at the perturbative level, so conformal supergravity arises through the dynamics of the D1 branes in a manner that was not made entirely transparent. These D branes are non-perturbative features of the B-model and thus to fully understand the presence of conformal supergravity in Witten's model (perhaps so as to explore related theories with Einstein gravity), one would appear to have to understand the full non-perturbative topolog-

[1] Our conventions are those of Penrose and Rindler [23], whereby an on-shell massless field of helicity h is represented on twistor space $\mathbb{PT}'$ by an element of $H^1(\mathbb{PT}', \mathcal{O}(-2h-2))$; these conventions differ from those of Witten [1].

ical string, a rather daunting task. In the B-model, one expects Kodaira–Spencer theory to give rise to the gravitational story, but in the twistor–string context this does not seem to play a role.

Berkovits' model [2] is rather simpler: the worldsheet path integral localizes on holomorphic (rather than constant) maps, and worldsheet instantons of degree $d \geqslant 1$ play the role of the D1 branes in Witten's model. Berkovits' strings have boundaries on a totally real (and hence Lagrangian) submanifold $\mathbb{RP}^{3|4} \subset \mathbb{CP}^{3|4}$ which may be reminiscent of the open A-model. However, spacetime Yang–Mills interactions arise not from D branes wrapping $\mathbb{RP}^{3|4}$, but via a worldsheet current algebra, while gravitational modes are generated by vertex operators on the same footing as those of Yang–Mills in the sense that both are inserted on the worldsheet boundary. Moreover, $\mathbb{RP}^3$ corresponds to a spacetime metric of signature $(++--)$ and it is not clear that scattering theory makes sense in such a signature, because the lightcone is connected and there appears to be no consistent $\mathrm{i}\epsilon$ prescription.

In this paper we recast twistor–string theory as a heterotic string. The first reason to suspect that a heterotic perspective is relevant to the twistor–string is Nair's original observation [24] that Yang–Mills MHV amplitudes may be obtained from a current algebra on a $\mathbb{P}^1$ linearly embedded in twistor space; such a current algebra arises naturally in a heterotic model. Secondly, heterotic sigma models with complex manifolds such as twistor space as a target automatically have $(0,2)$ worldsheet supersymmetry. This supersymmetry may be twisted so that correlation functions of operators representing cohomology classes of the scalar supercharge localize on holomorphic maps to twistor space. So holomorphic curves in twistor space are naturally incorporated as worldsheet instantons, as in Berkovits' model, and no D branes are necessary (or even possible). Thirdly, the twisted theory depends only on the global complex structure of the target X and of a holomorphic bundle $E \to X$, as well as a certain complex analytic cohomology class on X. At the perturbative level, infinitesimal deformations of these structures correspond to elements of the cohomology groups $H^1(X, T_X)$, $H^1(X, \mathrm{End}\, E)$ and $H^1(X, \Omega^2_{\mathrm{cl}})$, where Ω^2_{cl} is the sheaf of closed holomorphic 2-forms on X. In the twistor context, this dovetails very naturally with the Penrose transform which gives an isomorphism between these cohomology groups (together with their supersymmetric extensions) and the on-shell states of linearized conformal supergravity and super-Yang–Mills. Thus the ingredients of twistor–string theory combine very naturally in a heterotic picture.

While our heterotic picture is closest in spirit to Witten's model, in particular representing target space cohomology groups via Dolbeault cohomology, twisted $(0,2)$ models have recently been understood to be very close cousins of $\beta\gamma$-systems through a quantum field theoretic version of the Čech–Dolbeault isomorphism (see [25], a paper that provided much of the stimulus for this one). This relationship provides the link between the heterotic and Berkovits' twistor–strings, with the latter becoming freed from its dependence on split signature spacetime. The open-string boundary conditions of Berkovits' model are shown to correspond to a choice of contour in the moduli space of curves in twistor space. Moreover, like the Berkovits string, the heterotic twistor–string is free from all sigma model and Virasoro worldsheet anomalies provided it is coupled to a holomorphic current algebra of central charge 28. The same constraint on the current algebra will also be seen to arise in the B-model picture. Despite these successes, it is not yet clear whether or not the heterotic (or indeed any) twistor–string makes sense as a fully consistent string theory; the main outstanding issue is whether the theory is modular invariant.

The paper is structured as follows. In Section 2 we review the theory of twisted $(0,2)$ sigma models. In Section 3, we introduce the twistor–string model that we will study. The target space of our model is (a region in) the non-supersymmetric twistor space $\mathbb{P}^3$, but we also include fermions which are worldsheet scalars with values in a non-trivial vector bundle $\mathcal{V} \to \mathbb{P}^3$. The fact that these fermions are worldsheet scalars means that vertex operators can have arbitrary dependence on them and so they play the role of the anticommuting coordinates on supertwistor space $\mathbb{P}^{3|4}$. In this section we show that the sigma model anomalies cancel, and study the moduli space of worldsheet instantons. In Section 4 we introduce the basic vertex operators of the model, paying particular attention to those which correspond to deformations of the complex structure or a NS B-field on the twistor space. These correspond on spacetime to the conformal supergravity degrees of freedom. In Section 5 we introduce further fermions (now spinors on the worldsheet) with values in another bundle $E \to \mathbb{P}^3$, and these provide a coupling to Yang–Mills fields on spacetime. In Section 6 we promote the previously studied sigma models to a string theory by coupling in a 'bc system', and study the associated conformal anomaly. In Section 7 we give a more detailed discussion of the deformed supertwistor spaces, in particular discussing the way in which the googly data is encoded. In Section 8 we show how this model relates to both the Berkovits model and the original Witten model, in particular clarifying the role of the D1–D1 strings in Witten's picture. In Section 9 we discuss the string field theory of the disconnected prescription and derive the corresponding twistor action. We conclude with a discussion in Section 10.

2. A review of the twisted (0, 2) sigma model

Let us begin by briefly reviewing the construction of a $(0,2)$ non-linear sigma model describing maps $\phi : \Sigma \to X$ from a compact Riemann surface Σ to a complex manifold X (see also [25,28] for recent work in a similar context). The basic fields in the model are worldsheet scalars ϕ, representing the pullback to Σ of coordinates on a local patch of X. Twisted $(0,2)$ supersymmetry requires that we pick a complex structure on Σ and introduce fields

$$\rho \in \Gamma\big(\Sigma, \bar{K} \otimes \phi^* T_X\big), \qquad \bar{\rho} \in \Gamma\big(\Sigma, \phi^* \bar{T}_X\big), \tag{1}$$

where $\bar{K}$ is the anticanonical bundle on Σ and T_X is the holomorphic tangent bundle on X. These fields are related to the ϕs by the supersymmetry transformations

$$\begin{aligned} &\delta\phi^i = \epsilon_2 \rho^i, && \delta\phi^{\bar{j}} = \epsilon_1 \bar{\rho}^{\bar{j}}, \\ &\delta\rho^i = \epsilon_1 \bar{\partial}\phi^i, && \delta\bar{\rho}^{\bar{j}} = \epsilon_2 \bar{\partial}\phi^{\bar{j}}, \end{aligned} \tag{2}$$

where ϵ_i are constant anticommuting parameters with ϵ_1 a scalar and ϵ_2 a section of $\bar{T}_\Sigma$. The transformation parameterized by ϵ_1 may be defined globally on Σ, whilst constant antiholomorphic vector fields only exist locally on Σ (except at genus 1), so ϵ_2 may only be defined within a local patch on Σ, with coordinates $(z, \bar{z})$. Let these transformations be generated by supercharges $\bar{Q}$ and $\bar{Q}^\dagger$, so that for a generic field Φ

$$\delta\Phi = \big[\epsilon_1 \bar{Q} + \epsilon_2 \bar{Q}^\dagger, \Phi\big] \tag{3}$$

with $\bar{Q}$ a scalar operator. It is straightforward to check that $\bar{Q}^2 = 0$ and, on our local patch, also $(\bar{Q}^\dagger)^2 = 0$ and $\{\bar{Q}, \bar{Q}^\dagger\} = \bar{\partial}$. These relations characterize $(0,2)$ (twisted) supersymmetry.

To write an action we pick a Hermitian metric g on X. The basic action for a non-linear sigma model is then

$$\begin{aligned} S_1 &= \int_\Sigma \big|\mathrm{d}^2 z\big| \frac{1}{2} g_{i\bar{j}} \big(\partial_{\bar{z}}\phi^i \partial_z \phi^{\bar{j}} + \partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}}\big) - \rho^i_{\bar{z}} \nabla_z \bar{\rho}^{\bar{j}} \\ &= \Big\{\bar{Q}, \int_\Sigma \big|\mathrm{d}^2 z\big| g_{i\bar{j}} \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}}\Big\} + \int_\Sigma \phi^* \omega, \end{aligned} \tag{4}$$

where $\nabla : \Gamma(\Sigma, \phi^*\bar{T}_X) \to \Gamma(\Sigma, K \otimes \phi^*\bar{T}_X)$ is the pullback to Σ of the Hermitian connection on $\bar{T}_X$ and $\omega = \mathrm{i} g_{i\bar{j}}\, \mathrm{d}\phi^i \wedge \mathrm{d}\phi^{\bar{j}}$. If $\mathrm{d}\omega = 0$ so that X is Kähler, the action is invariant under the $(0,2)$ transformations (2) and the connection ∇ is Levi-Civita. Because the action is $\bar{Q}$-exact upto the topological term $\int_\Sigma \phi^*\omega$, correlation functions of operators in the $\bar{Q}$-cohomology will not depend on the choice of Hermitian metric g. They do depend on the Kähler class of ω together with the complex structures on X and Σ, which were used to define the transformations (2).

There are various generalizations beyond this basic picture [25–27]. Firstly, by introducing a ∂-closed $(2,0)$ form t we may deform (4) by

$$\begin{aligned}\delta S_1 &= \mathrm{i}\int_\Sigma \left|\mathrm{d}^2 z\right| \partial_{\bar{k}} t_{ij}\, \bar{\rho}^{\bar{k}} \rho^i_{\bar{z}} \partial_z \phi^j + t_{ij}\, \partial_{\bar{z}}\phi^i \partial_z \phi^j \\ &= \mathrm{i}\left\{ \bar{Q}, \int_\Sigma \left|\mathrm{d}^2 z\right| t_{ij}\, \rho^i_{\bar{z}} \partial_z \phi^j \right\}. \end{aligned} \tag{5}$$

If t is globally defined on X, then this deformation is $\bar{Q}$-trivial and t does not affect correlators of operators representing $\bar{Q}$-cohomology classes. More interesting is the case where t is defined only on the local patches of some cover $\{U_\alpha\}$ of X, where α indexes the cover. If the differences $t^{(\alpha)} - t^{(\beta)}$ are holomorphic on each overlap $U_\alpha \cap U_\beta$, then they piece together to form an element $\mathcal{H}$ of the cohomology group $H^{0,1}(X, \Omega^{2,0}_{\mathrm{cl}})$ where $\Omega^{2,0}_{\mathrm{cl}}$ is the sheaf of ∂-closed $(2,0)$-forms on X. The correlation functions are then sensitive to this class. We can also think of $\mathcal{H}$ in terms of a Dolbeault representative, a global $(2,1)$-form satisfying $\partial\mathcal{H} = \bar{\partial}\mathcal{H} = 0$ obtained as $\mathcal{H} = \bar{\partial} t^\alpha$. Whilst the second line of (5) makes it clear that this modification is invariant under $\bar{Q}$ transformations, δS_1 is invariant under the full $(0,2)$ supersymmetry if and only if $\mathcal{H}$ satisfies $\mathcal{H} = 2\mathrm{i}\partial\omega$. Correspondingly, in the presence of $\mathcal{H}$ the Hermitian metric connection ∇ has torsion $T^i{}_{jk} = g^{i\bar{n}} \mathcal{H}_{\bar{n}jk}$.

Hull and Witten [25,27] observed that locally this geometric structure can be derived from a smooth 1-form $K(\phi, \bar{\phi})$ which serves as a potential for both t and ω by $\mathrm{i}t = 2\partial K$ and $\omega = 2\,\mathrm{Re}\,\bar{\partial}K$ (and so also $\mathcal{H} = \partial\bar{\partial}K$). The action is then given by

$$\begin{aligned} S_1 &= \int \left|\mathrm{d}^2 z\right| \big(K_{i,\bar{j}}\, \partial_{\bar{z}}\phi^{\bar{j}} \partial_z \phi^i + \bar{K}_{\bar{\imath},j}\, \partial_{\bar{z}}\phi^j \partial_z \phi^{\bar{\imath}} \\ &\quad - \big(K_{i,\bar{j}}\, \bar{\rho}^{\bar{j}} \partial_z \rho^i + \bar{K}_{\bar{\imath},j}\, \rho^j_{\bar{z}} \partial_z \bar{\rho}^{\bar{\imath}} \big) + \big(K_{i,\bar{j}k}\, \bar{\rho}^{\bar{j}} \rho^k_{\bar{z}} \partial_z \phi^i - \bar{K}_{\bar{\imath},j\bar{l}}\, \rho^j_{\bar{z}} \bar{\rho}^{\bar{l}} \partial_z \phi^{\bar{\imath}} \big) \big) \\ &= \left\{ \bar{Q}, \int \left|\mathrm{d}^2 z\right| \big((K_{i,\bar{j}} + \bar{K}_{\bar{j},i}) \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}} - (K_{i,j} - K_{j,i}) \rho^i_{\bar{z}} \partial_z \phi^j \big) \right\}. \end{aligned} \tag{6}$$

It will also be useful to introduce a $(1,1)$-form b as $b = \bar{\partial}K$. Then $b = B + \mathrm{i}\omega$ where B is the usual B-field of string theory and $\mathcal{H} = \partial b$. See [25] for a fuller discussion of the geometry underlying these models.

The most important feature of twisted $(0,2)$ models is that the action is $\bar{Q}$-exact (except for topological terms) so the path integral localizes on $\bar{Q}$-invariant solutions to the equations of motion. In particular, the transformation $\{\bar{Q}, \rho^i_{\bar{z}}\} = \partial_{\bar{z}}\phi^i$ shows that such invariant configurations are holomorphic maps, or worldsheet instantons. The full action evaluated on such invariant solutions is $\int_\Sigma \phi^* b$. If b is not globally defined, one can only make sense of this expression provided the underlying de Rham cohomology class of $\mathcal{H}$ is integral.

2.1. Coupling to bundles

We can also incorporate holomorphic bundles over X: let $\mathcal{V}\to X$ be a holomorphic vector bundle and introduce fields

$$\begin{aligned} &\psi\in\Gamma\big(\Sigma,K^s\otimes\phi^*\mathcal{V}\big), &&\bar\psi\in\Gamma\big(\Sigma,K^{1-s}\otimes\phi^*\mathcal{V}^\vee\big),\\ &r\in\Gamma\big(\Sigma,\bar K\otimes K^s\otimes\phi^*\mathcal{V}\big), &&\bar r\in\Gamma\big(\Sigma,K^{1-s}\otimes\phi^*\mathcal{V}^\vee\big), \end{aligned}\tag{7}$$

where $\mathcal{V}^\vee$ is the dual bundle to $\mathcal{V}$. Note that classically, twisted $(0,2)$ supersymmetry does not fix the spin of these left-moving fields and at present we allow them to be sections of K^s for any half-integer s. For what follows, it will be convenient to choose the fields in (7) to behave equivariantly under $\bar Q$ transformations and gauge transformations on $\mathcal{V}$ (as in [28]), obtaining

$$\begin{aligned} &\delta\psi^a=\epsilon_2\big(r^a+A_i{}^a{}_b\psi^b\rho^i\big), &&\delta\bar\psi_a=\epsilon_1\bar r_a,\\ &\delta r^a=\epsilon_1\big(\bar D\psi^a+F_{i\bar j}{}^a{}_b\psi^b\rho^i\bar\rho^{\bar j}\big)+\epsilon_2A_i{}^a{}_br^b\rho^i, &&\delta\bar r_a=\epsilon_2\bar\partial\bar\psi_a, \end{aligned}\tag{8}$$

where $\bar D:\Gamma(\Sigma,K^s\otimes\phi^*\mathcal{V})\to\Gamma(\Sigma,\bar K\otimes K^s\otimes\phi^*\mathcal{V})$ is a connection on $K^s\otimes\phi^*\mathcal{V}$. One can check that the $(0,2)$ algebra is satisfied provided $\mathcal{V}$ is holomorphic so that $F_{ij}=F_{\bar i\bar j}=0$. The action for these bundle-valued fields is taken to be

$$\begin{aligned} S_2&=\int_\Sigma\big|d^2z\big|\bar\psi_aD_{\bar z}\psi^a+F_{i\bar j}{}^a{}_b\bar\psi_a\psi^b\rho^i_{\bar z}\bar\rho^{\bar j}+\bar r_ar^a\\ &=\Big\{\bar Q,\int_\Sigma\big|d^2z\big|\bar\psi_ar^a_{\bar z}\Big\}. \end{aligned}\tag{9}$$

In particular, this shows that r and $\bar r$ are auxiliary and decouple.

Classically, the stress-energy of S_1+S_2 has non-vanishing components

$$\begin{aligned} &T_{zz}=g_{i\bar j}\partial_z\phi^i\partial_z\phi^{\bar j}+\bar\psi_aD_z\psi^a,\\ &T_{\bar z\bar z}=g_{i\bar j}\big(\partial_{\bar z}\phi^i\partial_{\bar z}\phi^{\bar j}+\rho^i_{\bar z}\nabla_{\bar z}\bar\rho^{\bar j}\big)=\big\{\bar Q,g_{i\bar j}\rho^i_{\bar z}\partial_{\bar z}\phi^{\bar j}\big\}. \end{aligned}\tag{10}$$

Since $T_{\bar z\bar z}=\{\bar Q,\cdot\}$, as discussed in [25] all the Laurent coefficients $\bar L_n$ of $T_{\bar z\bar z}$ are also $\bar Q$-exact. In particular, $\bar L_0=\{\bar Q,\bar G_0\}$ for some $\bar G_0$, so that $\bar L_0$ maps $\bar Q$-closed states to $\bar Q$-exact ones and is thus zero in cohomology. But for any state of antiholomorphic weight $\bar h\neq 0$, $\bar L_0/\bar h$ is the identity, so the $\bar Q$-cohomology vanishes except at $\bar h=0$. Furthermore, the fact that $T_{\bar z\bar z}$ is $\bar Q$-exact means that correlation functions $\langle\prod_{i=1}^n\mathcal{O}_i(z_i)\rangle$ of $\bar Q$-closed operators depend only holomorphically on the insertion points $\{z_i\}\in\Sigma$. Were we studying a model with twisted $(2,2)$ supersymmetry, exactly the same argument for the left-movers would lead us to conclude that operators in the BRST cohomology must also have $h=0$, and that correlation functions are actually independent of the insertion points. However, here $T_{zz}\neq\{\bar Q,\cdot\}$ and so there is an infinite tower of $\bar Q$-cohomology classes depending on $h\in\mathbb{Z}_{\geqslant 0}$, and the twisted $(0,2)$ model is a conformal, rather than topological, field theory.

If we choose $\mathcal{V}\cong T_X$ and set $s=0$ the total action S_1+S_2 in fact has twisted $(2,2)$ worldsheet supersymmetry and is the action of the A-model, while choosing $\mathcal{V}=T_X$ but keeping $s=1/2$ gives a half-twisted version of this $(2,2)$ theory. $(0,2)$ models allow for more general choices of $\mathcal{V}$, as is familiar from compactifications of the physical heterotic string where $\mathcal{V}$ is a subbundle of the $E_8\times E_8$ or $Spin(32)/\mathbb{Z}_2$ gauge bundles in ten dimensions (where, in the physical string, $s=1/2$). In that context, setting $\mathcal{V}=T_X$ corresponds to the 'standard embedding' of the gauge connection in the spin connection of the compactification manifold. For recent work on twisted $(0,2)$ models related to heterotic compactification, see [28–32].

3. The twistor target space

In this paper, we will reformulate twistor–string theory as a $(0,2)$ model. One might anticipate that we should take X to be a region in $\mathbb{P}^{3|4}$ as in [1,2] but, while this may well be a reasonable way to proceed, in its most naïve form a $(0,2)$ model with $\mathbb{P}^{3|4}$ target leads to difficulties both in understanding the role of the bosonic worldsheet superpartners of the fermionic directions, and in handling the antiholomorphic fermionic directions without the possibility of appealing to a 'D brane at $\bar{\psi}=0$', since heterotic models do not possess D branes.

We therefore adopt a different strategy in which the basic target space is $\mathbb{P}^3$, the non-supersymmetric, projective twistor space of flat spacetime. The fermionic directions of $\mathbb{P}^{3|4}$ are incorporated by coupling to a bundle $\mathcal{V}\equiv\mathcal{O}(1)^{\oplus 4}$ as in (7)–(9) with $s=0$. With this choice of s, the ψ^a are anticommuting worldsheet scalars and so provide the fields that were used in the original twistor–string theories [1,2] to describe holomorphic coordinates on the fermionic directions of $\mathbb{P}^{3|4}$. The vertex operators will be seen to correspond to perturbations of both the complex structure and of the NS flux $\mathcal{H}$, and these perturbations can also have arbitrary dependence on ψ^a. With $s=0$, $\bar{\psi}_a$ are sections of $K\otimes\phi^*((\mathcal{O}(1)^{\oplus 4})^\vee)$ and are thus worldsheet $(1,0)$ forms, so ψ and $\bar{\psi}$ are naturally on a different footing. Correspondingly, we will see that the dependence of the vertex operators on $\bar{\psi}^a$ can be at most linear. Thus our model is equivalent to working on a $\mathbb{P}^{3|4}$ target, at least at the linearized level determined by the vertex operators. In order to incorporate Yang–Mills, in Section 5 we will also couple to a bundle with action (9), but where $s=1/2$. In this case the allowed vertex operators are different and will correspond to twistor data for super-Yang–Mills fields.

Initially, to consider the quantum theory we will take the action to be $S=S_1+S_2$ as in (4) and (9), with target $\mathbb{P}^3-\mathbb{P}^1$ and bundle $\mathcal{V}=\mathcal{O}(1)^{\oplus 4}$ with associated fermions $\psi^a\in\Gamma(\Sigma,\phi^*\mathcal{V})$ and $\bar{\psi}_{az}\in\Gamma(\Sigma,K\otimes\phi^*\mathcal{V}^\vee)$. The Kähler structure is given by the Fubini–Study metric which induces a metric and compatible connection on $\mathcal{O}(1)$. We postpone the coupling to Yang–Mills until Section 5. Note that the first-order action for the $\psi\bar{\psi}$-system is reminiscent of Berkovits' model [2]; we will make the relationship more precise in Section 8.1.

3.1. Anomalies

With these choices of X, $\mathcal{V}$ and s we must show that the classical action S_1+S_2 of Eqs. (4) and (9) defines a sensible quantum theory.

3.1.1. Sigma model anomalies

Field theories containing chiral fermions may fail to define a quantum theory because of the presence of sigma model anomalies: integrating out the fermions gives a one-loop determinant which must be treated as a function of the bosonic fields. However this determinant is really a section of a line bundle $\mathcal{L}\to\mathrm{Maps}(\Sigma,X)$ over the space of maps and we can only make a canonical identification of this section with a function if the determinant line bundle is flat [33]. In twisted $(0,2)$ models, integrating out the non-zero-modes of ρ and ψ gives a factor $\det'\nabla\det'\bar{D}$ which depends on the map ϕ through the pullback of $\bar{T}_X$ in ∇ and the pullback of $\mathcal{V}$ in $\bar{D}$. Since $\det'\nabla=\det'\Delta/\det'\bar{\partial}_{\phi^*T_X}$ and the ζ-regularized determinant of the self-adjoint Laplacian Δ is always well defined, the anomaly is governed by the virtual bundle $\mathcal{V}\ominus T_X$.

The geometric index theorem of Bismut and Freed [34,35] states that the curvature of the Quillen connection [36] on $\mathcal{L}$ is given by

$$F^{(\mathcal{L})} = \int_\Sigma \mathrm{Td}(T_\Sigma)\phi^* \mathrm{ch}(\mathcal{V} \ominus T_X)\big|_{(4)}$$
$$= \int_\Sigma \frac{\mathrm{c}_1(T_\Sigma)}{2}\phi^*\big(\mathrm{c}_1(\mathcal{V}) - \mathrm{c}_1(T_X)\big) + \int_\Sigma \phi^*\big(\mathrm{ch}_2(\mathcal{V}) - \mathrm{ch}_2(T_X)\big). \tag{11}$$

The first term in (11) is not present in the physical heterotic string and arises here because the worldsheet fermions ρ, ψ and their duals are scalars and 1-forms. This term depends on the genus of Σ and so it must vanish separately if the sigma model is to be well defined on an arbitrary genus worldsheet. Requiring that the second term also vanishes is then familiar as a consistency condition for the Green–Schwarz mechanism[2]

$$dH = \mathrm{ch}_2(T_X) - \mathrm{ch}_2(\mathcal{V}). \tag{12}$$

When $\mathcal{V} = T_X$ as in the A-model, $F^{(\mathcal{L})}$ vanishes trivially. In the B-model, $\mathcal{V} = T_X^\vee$ so $F^{(\mathcal{L})} = 0$ if and only if $\mathrm{c}_1(T_X) = 0$. For more general $(0, 2)$ models, the condition that (11) should vanish highly constrains the admissible choices of $\mathcal{V}$.

In the twistor–string case at hand, $X = \mathbb{P}^3$ and $\mathcal{V} = \mathcal{O}(1)^{\oplus 4}$. The bundle $\mathcal{O}(1)^{\oplus 4}$ appears in the Euler sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 4} \to T_{\mathbb{P}^3} \to 0 \tag{13}$$

in which the first map is multiplication by the homogeneous coordinates Z^α on $\mathbb{P}^3$, and the second map is $V^\alpha \to V^\alpha \partial/\partial Z^\alpha$ which defines the tangent bundle of projective space as a quotient of that on the non-projective space. Since (13) is exact,

$$\mathrm{c}\big(\mathcal{O}(1)^{\oplus 4}\big) = \mathrm{c}(\mathcal{O})\mathrm{c}(T_{\mathbb{P}^3}) = \mathrm{c}(T_{\mathbb{P}^3}) \tag{14}$$

so *all* the Chern classes of $\mathcal{O}(1)^{\oplus 4}$ agree with those of $T_{\mathbb{P}^3}$, ensuring that (11) vanishes. By comparison, for $\mathbb{P}^{3|4}$ the Euler sequence reads

$$0 \to \mathcal{O} \to \mathbb{C}^{4|4} \otimes \mathcal{O}(1) \to T_{\mathbb{P}^{3|4}} \to 0 \tag{15}$$

so that

$$\mathrm{ch}(T_{\mathbb{P}^{3|4}}) = \mathrm{ch}\big(\mathbb{C}^{4|4} \otimes \mathcal{O}(1)\big) - \mathrm{ch}(\mathcal{O}) = \mathrm{sdim}\,\mathbb{C}^{4|4}\,\mathrm{ch}\big(\mathcal{O}(1)\big) - 1 = -1 \tag{16}$$

showing that (formally) $\mathrm{sdim}\,\mathbb{P}^{3|4} = -1$ while all its Chern classes vanish. Note in particular that triviality of the Berezinian of $\mathbb{P}^{3|4}$ is equivalent to the statement that $K_{\mathbb{P}^3} \simeq \bigwedge^{\mathrm{top}}(\mathcal{O}(1)^{\oplus 4})^\vee$, while $\mathrm{sdim}\,\mathbb{P}^{3|4} = -1$ is equivalent to the fact that the vanishing locus of a generic section of $\mathcal{O}(1)^{\oplus 4}$ has virtual dimension -1. We now wish to show that a similar relationship holds at the level of the instanton moduli space.

[2] On $\mathbb{P}^3$, the background Neveu–Schwarz fieldstrength H vanishes, so the left-hand side of (12) is zero as a form, and not just in cohomology. Consequently the Quillen connection must be flat, rather than merely have vanishing first Chern class, and so $\Omega^{(\mathcal{L})}$ itself must vanish. For target spaces with torsion, a flat connection on $\mathcal{L}$ may be constructed by modifying the Quillen connection by a term involving H [35].

3.1.2. Anomalous symmetries and the instanton moduli space

The action $S_1 + S_2$ is invariant under a global $U(1)_F \times U(1)_R$ symmetry, where $U(1)_R$ is the automorphism group of the (0, 2) superalgebra and $U(1)_F$ is a left-moving 'flavour symmetry' associated to the bundle-valued fermions. As in [28], we take ρ and $\bar\rho$ to have respective charges $(0, -1)$ and $(0, 1)$ under $U(1)_F \times U(1)_R$, while ψ and $\bar\psi$ have charges $(1, 0)$ and $(-1, 0)$; ϕ is uncharged. These symmetries are violated by the path integral measure because the fermion kinetic operators have non-zero index. The violation is tied directly to the geometry of the instanton moduli space and restricts the combinations of vertex operators that can contribute to a non-vanishing amplitude.

The anomalies arise from the index theorem applied to the fermion kinetic terms. The kinetic term $g_{i\bar j}\rho^i_{\bar z}\nabla_z\bar\rho^{\bar j}$ implies that a $\bar\rho$ zero-mode is an *antiholomorphic* section of $\phi^*\bar T_{\mathbb{P}^3}$ and so is complex conjugate to an element of $H^0(\Sigma, \phi^* T_{\mathbb{P}^3})$. Similarly, zero-modes of $g_{i\bar j}\rho^i_{\bar z}$ are complex conjugate to elements of $H^0(\Sigma, K \otimes \phi^* T^\vee_{\mathbb{P}^3}) \simeq H^1(\Sigma, \phi^* T_{\mathbb{P}^3})$, by Serre duality. The Hirzebruch–Riemann–Roch theorem then says that the difference in the complex dimensions of the spaces of such zero-modes on a worldsheet of genus g is

$$\begin{aligned} h^0\big(\Sigma, \phi^* T_{\mathbb{P}^3}\big) - h^1\big(\Sigma, \phi^* T_{\mathbb{P}^3}\big) &= \int_\Sigma \phi^* \mathrm{c}_1(T_{\mathbb{P}^3}) + \dim\big(\mathbb{P}^3\big)\frac{\mathrm{c}_1(T_\Sigma)}{2} \\ &= 4d + 3(1-g) \end{aligned} \tag{17}$$

for a degree d map to twistor space.

Given a holomorphic map ϕ, a nearby map $\phi + \delta\phi$ is also holomorphic provided $\delta\phi \in H^0(\Sigma, \phi^* T_X)$. Consequently, the holomorphic tangent bundle $T_{\mathcal{M}}$ to instanton moduli space $\mathcal{M}$ has fibre $T_{\mathcal{M}}|_\phi = H^0(\Sigma, \phi^* T_X)$. The $\bar\rho$ zero-modes are anticommuting elements of $\overline{H^0(\Sigma, \phi^* T_X)}$ and thus represent (0, 1)-forms on $\mathcal{M}$. Maps ϕ at which $h^1(\Sigma, \phi^* T_X) = 0$ are non-singular points of the instanton moduli space and the tangent space there has dimension equal to the above index. In the twistor–string case, either at genus zero or when the degree is sufficiently larger than the genus, such points form a dense open set of the instanton moduli space. So our model has no ρ^i zero-modes and $4d + 3$ $\bar\rho^{\bar j}$ zero-modes at genus zero. In the rational case with target $\mathbb{P}^3$, a degree d map can be expressed as a polynomial of degree d in the homogeneous coordinates Z^α, as $Z^\alpha(\sigma) = \sum_{i=0}^d A^\alpha{}_i \sigma^i$. The coefficients $A^\alpha{}_i$ are therefore homogeneous coordinates on the moduli space $\mathcal{M}$ and one can identify[3] $\mathcal{M} \cong \mathbb{P}^{4d+3}$ for genus zero maps to $\mathbb{P}^3$.

Turning now to the ψ fields, the kinetic term $\bar\psi_a \bar D \psi^a$ shows that a ψ zero-mode represents an element of $H^0(\Sigma, \phi^*\mathcal{V})$ while a $\bar\psi$ zero-mode represents an element of $H^0(\Sigma, K \otimes \phi^*\mathcal{V}^\vee) \cong H^1(\Sigma, \phi^*\mathcal{V})^\vee$, again by Serre duality. Hence the difference in the number of zero-modes is

$$\begin{aligned} h^0\big(\Sigma, \phi^*\mathcal{V}\big) - h^1\big(\Sigma, \phi^*\mathcal{V}\big) &= \int_\Sigma \phi^* \mathrm{c}_1(\mathcal{V}) + \mathrm{rk}(\mathcal{V})\frac{\mathrm{c}_1(T_\Sigma)}{2} \\ &= 4(d + 1 - g), \end{aligned} \tag{18}$$

for $\mathcal{V} = \mathcal{O}(1)^{\oplus 4}$. This anomaly is familiar in the twistor–string story. It says that correlation functions on a degree d, genus g curve vanish unless the path integral contains an insertion of net

[3] More accurately, the moduli space of instantons in the non-linear sigma model at genus zero is a dense open subset in $\mathbb{P}^{3+4d}$, noncompact because of 'bubbling'. A linear sigma model presentation provides a natural compactification [37] of $\mathcal{M}$ to $\mathbb{P}^{4d+3}$ and we will henceforth work over this compact moduli space.

$U(1)_F$ number $4(d+1-g)$. We will see that, just as in the Witten and Berkovits twistor–strings, the vertex operators naturally form spacetime $\mathcal{N}=4$ multiplets by depending polynomially on ψ, but not $\bar{\psi}$. In particular, a correlation function involving n external gluons of positive[4] helicity and arbitrary gluons of negative helicity is supported on a worldsheet instanton of degree

$$d = n - 1 + g, \tag{19}$$

as in [1]. More generally, scattering amplitudes of n_h external SYM states of helicity h are supported on curves of degree

$$d = g - 1 + \sum_{h=-1}^{1} \frac{h+1}{2} n_h \tag{20}$$

and must necessarily vanish unless $d \in \mathbb{Z}_{\geqslant 0}$.

As discussed by Katz and Sharpe in [32], just as for the $\bar{\rho}$ zero-modes, the ψ zero-modes may be interpreted geometrically in terms of a bundle (really, a sheaf) over $\mathcal{M}$. Consider the diagram

$$\begin{array}{ccc} \mathcal{M} \times \Sigma & \xrightarrow{\Phi} & X \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{M} & & \end{array} \tag{21}$$

where Φ is the universal instanton and π the obvious projection. Given a sheaf $\mathcal{V}$ on X we can construct a sheaf $\mathcal{W}$ over $\mathcal{M}$ by pulling back $\mathcal{V}$ to $\mathcal{M} \times \Sigma$ via the universal instanton, and then taking its direct image under the projection map, i.e., $\mathcal{W} \equiv \pi_* \Phi^* \mathcal{V}$. The direct image sheaf is defined so that its sections over an open set $U \subset \mathcal{M}$ are

$$\mathcal{W}(U) = (\pi_* \Phi^* \mathcal{V})(U) = (\Phi^* \mathcal{V})(\pi^{-1} U) = H^0(U \times \Sigma, \Phi^* \mathcal{V}), \tag{22}$$

so that over a generic instanton, $\mathcal{W}|_\phi = H^0(\Sigma, \phi^* \mathcal{V})$ with dimension $4(d+1-g)$. Consequently, we may generically interpret a ψ zero-mode as a point in the fibre $\mathcal{W}|_\phi$.

For families of instantons for which there are no ρ or $\bar{\psi}$ zero-modes (i.e., whenever the higher direct image sheaves $R^1 \pi_* \Phi^* T_X$ and $R^1 \pi_* \Phi^* \mathcal{V}$ vanish), the definition of $\mathcal{W}$ shows that it has first Chern class [32]

$$c_1(\mathcal{W}) = \int_\Sigma \mathrm{Td}(T_\Sigma) \Phi^* \mathrm{ch}(\mathcal{V})\big|_{(4)} \tag{23}$$

so the condition $\mathrm{ch}(\mathcal{V}) = \mathrm{ch}(T_X)$ ensures that $c_1(\mathcal{W}) = c_1(T_\mathcal{M})$, or

$$\bigwedge^{\mathrm{top}} \mathcal{W}^\vee \simeq K_\mathcal{M}. \tag{24}$$

This isomorphism is important in computing correlation functions: operationally, to integrate out the ψ zero-modes one merely extracts the coefficient of the ψs in the vertex operators, restricting ones attention to instantons whose degree is determined by (20). This coefficient is a section of $\bigwedge^{\mathrm{top}} \mathcal{W}^\vee$, so by (24) we may interpret it as a top holomorphic form on instanton moduli space.

[4] In our conventions, elements of the cohomology group $H^1(\mathbb{PT}', \mathcal{O}(-2h-2))$ correspond via the Penrose transform to spacetime fields of helicity h, so that in particular a negative helicity gluon corresponds to a twistor wavefunction of weight 0.

Again, this story has a familiar counterpart in the original construction of twistor–strings [1] as a theory with target space $\mathbb{P}^{3|4}$. Assuming that there is a dense open subset of the moduli space over which there are no $\bar{\psi}$ zero-modes, (24) shows that the total space of the bundle $\mathcal{W}$, parity reversed on the fibres, can be thought of as a Calabi–Yau supermanifold with a canonically[5] defined holomorphic volume form (or Berezinian). In particular, at genus zero there are no $\bar{\psi}$ or ρ zero-modes, and (24) simply states the isomorphism $K_{\mathbb{P}^{4d+3}} \simeq \mathcal{O}(-4-4d)$. This is the $(0,2)$ analogue of the statement that the moduli space of rational maps to $\mathbb{P}^{3|4}$ is the supermanifold $\mathbb{P}^{4d+3|4d+4}$ with trivial Berezinian.

Beyond genus zero, there can be zero-modes of both ρ and $\bar{\psi}$, and the dimension of $\mathcal{M}$ and rank of $\mathcal{W}$ may jump as we move around in instanton moduli space. However, the indices (17) and (18) remain constant and so the selection rule (20) is not affected by such excess zero-modes. To obtain non-zero correlation functions we must now expand the action in powers of the four-Fermi term $F_{i\bar{j}}{}^{a}{}_{b}\bar{\psi}_a\psi^b\rho^i\bar{\rho}^{\bar{j}}$ until the excess zero-modes are soaked up. This is analogous to the way $(2,2)$ models construct the Euler class of the obstruction sheaf [38], but $(0,2)$ models have the added complication that $h^1(\Sigma,\phi^*T_X)$ may not equal $h^1(\Sigma,\phi^*\mathcal{V})$, so that it may be necessary to absorb some of the factors of $\rho\bar{\rho}$ or $\bar{\psi}\psi$ using their respective propagators [32]. Generically, when d is much larger than g there are no excess zero-modes and (24) again tells us that the moduli space of instantons from a fixed worldsheet behaves as a Calabi–Yau supermanifold.[6]

Incidentally, had we started with an untwisted model involving worldsheet fermions that are sections of the square roots of the canonical or anticanonical bundles, the anomaly in both the $U(1)_F$ and $U(1)_R$ symmetries would be $4d$, independent of the genus. A diagonal subgroup of $U(1)_F \times U(1)_R$ would be anomaly free and could be used to twist the spins of the fermions. One might compare this to a $(2,2)$ model on a Kähler manifold. There, a diagonal subgroup of the $U(1)\times U(1)$ R-symmetry group is guaranteed to be anomaly free simply because the left- and right-moving fermions take values in the same bundle. Twisting by this subgroup leads to the A-model. Even though the left- and right-moving fermions of our $(0,2)$ model are valued in different bundles, the same subgroup is still anomaly free, again because of (14).

3.2. *Worldsheet perturbative corrections*

Because $T_{zz} \neq \{\bar{Q},\cdot\}$, twisted $(0,2)$ models are conformal rather than topological field theories and we must examine the effect of worldsheet perturbative corrections on the $\bar{Q}$-cohomology. $(0,2)$ supersymmetry ensures[7] that quantum corrections to the action will always be of the form $\{\bar{Q},\int_\Sigma\ldots\}$ so $T_{\bar{z}\bar{z}}$ will always remain $\bar{Q}$-exact. Likewise [25,28], although quantum corrections may lead to a violation of scale invariance, since $T_{z\bar{z}}$ has antiholomorphic weight $\bar{h}=1$, any such violation is always $\bar{Q}$-exact and worldsheet perturbative corrections will not affect correlators representing $\bar{Q}$-cohomology classes. One-loop corrections to worldsheet instantons also have the effect of modifying the classical weighting by $\int_\Sigma \phi^*\omega$ by the pullbacks of $\mathrm{c}_1(T_X)$ and $\mathrm{c}_1(\mathcal{V})$ [40,41]; these corrections cancel in the twistor–string.

5 The holomorphic volume form is defined upto scale, as is the isomorphism (24).

6 See also work by Movshev [39].

7 In terms of superfields, the most general action with $(0,2)$ supersymmetry may be written as $\int \mathrm{d}^2\bar{\theta}\,\mathcal{D} + \int \mathrm{d}\bar{\theta}\,\Gamma + \int \mathrm{d}\bar{\theta}^\dagger\,\Gamma'$. The first two terms are $\bar{Q}$-exact, while the third is not generated by quantum corrections if it is not present at the classical level.

The only remaining issue is the correction to T_{zz}. Classically, as in Eq. (10) we have

$$T_{zz} = g_{i\bar{j}}\partial_z\phi^i\partial_z\phi^{\bar{j}} + \bar{\psi}_{a\,z}D_z\psi^a \tag{25}$$

which is not $\bar{Q}$-exact, and obeys $\{\bar{Q}, T_{zz}\} = 0$ only once one enforces the ρ equation of motion and vanishing of the auxiliary fields r. Consequently, loop corrections to the worldsheet effective action can easily upset $\bar{Q}$-closure of T_{zz}. At 1-loop, the action receives a correction

$$\Delta S^{1\,\mathrm{loop}} \propto \left\{\bar{Q}, \int_\Sigma \left|\mathrm{d}^2z\right| R_{i\bar{j}}\rho^i_{\bar{z}}\partial_z\phi^{\bar{j}} + g^{i\bar{j}}F_{i\bar{j}}{}^a{}_b\bar{\psi}_{a\,z}r^b_{\bar{z}}\right\} \tag{26}$$

and generically $T^{1\,\mathrm{loop}}_{zz}$ is not $\bar{Q}$-closed unless the target metric is Ricci-flat and the background connection on $\mathcal{V}$ obeys the Hermitian Yang–Mills equations so that this correction vanishes. Neither of these conditions hold when $X \cong \mathbb{P}^3$ and $\mathcal{V} \cong \mathcal{O}(1)^{\oplus 4}$. However, if g is the Fubini–Study metric then $SU(4)$ symmetry constrains $R_{i\bar{j}} = 4g_{i\bar{j}}$, while the curvature of $\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}$ obeys $F_{i\bar{j}}{}^a{}_b = g_{i\bar{j}}\delta^a{}_b$ so that the 1-loop correction (26) is proportional to the classical action. Consequently, the field equations are unaltered and $\{\bar{Q}, T^{1\,\mathrm{loop}}_{zz}\} = 0$ still holds. Similar results presumably hold for higher loops in the worldsheet theory.

In a model with $\mathbb{P}^{3|4}$ target space, these issues are more straightforward: since $\mathrm{c}_1(T_{\mathbb{P}^{3|4}}) = 0$ one can find a Ricci-flat metric (the Fubini–Study metric on the superspace [1]) in which all one-loop corrections vanish and there is always a metric in the same Kähler class in which loop corrections vanish to any order. We have not taken this route for the reasons discussed previously.

4. Vertex operators and (0, 2) moduli

We now wish to determine the vertex operators representing $\bar{Q}$-cohomology classes. Since the action is $\bar{Q}$-exact (upto the topological term), correlation functions of such operators localize on a first-order neighbourhood of the instanton moduli space $\mathcal{M} \subset \mathrm{Maps}(\Sigma, X)$, just as for the A-model. Consequently, the one-loop approximation is exact for directions normal to $\mathcal{M}$ in $\mathrm{Maps}(\Sigma, X)$.

To construct these vertex operators [25,28], we first note that they must all be independent of $\rho^i_{\bar{z}}$, since this field has antiholomorphic weight 1 (and the (0, 2) theory does not contain any fields with $\bar{h} < 0$). Similarly, they must be independent of antiholomorphic worldsheet derivatives of any of the fields. However, (0, 2) supersymmetry does not impose any constraints on the holomorphic conformal weight, so *a priori* vertex operators may be arbitrary functions of the remaining fields $\{\phi, \bar{\phi}, \bar{\rho}, \psi, \bar{\psi}\}$ together with arbitrary powers of their holomorphic derivatives (except that holomorphic derivatives of $\bar{\rho}$ may be always be exchanged for other fields using the ρ equation of motion). The entire infinite family of vertex operators is certainly of great interest, interpreted in [25] as providing a sheaf of chiral algebras over the target space X, while the operators of conformal weight $(h, \bar{h}) = (0, 0)$ form an interesting generalization of the chiral ring of (2, 2) theories [28,32].

Not all of these vertex operators will survive when we extend the sigma model to a string theory in Section 6. For string theory, the key vertex operators are those which generate deformations of the (0, 2) moduli. These deformations are in one-to-one correspondence with $\bar{Q}$-closed operators $\mathcal{O}^{(1,0)}$ of conformal weight $(h, \bar{h}) = (1, 0)$ and charge $+1$ under $U(1)_R$, since given such an operator we can construct an descendant $\int_\Sigma \mathcal{O}^{(1,1)} \equiv \int_\Sigma \{\bar{Q}^\dagger, \mathcal{O}^{(1,0)}\}$ which satisfies

$$\left[\bar{Q}, \int_\Sigma \{\bar{Q}^\dagger, \mathcal{O}^{(1,0)}\}\right] = \int_\Sigma \left[\{\bar{Q}, \bar{Q}^\dagger\}, \mathcal{O}^{(1,0)}\right] = \int_\Sigma \bar{\partial}\mathcal{O}^{(1,0)} = 0, \tag{27}$$

because $\bar{\partial} = \mathrm{d}$ when acting on sections of the canonical bundle. Thus by its construction $\int_\Sigma \mathcal{O}^{(1,1)}$ is invariant under $(0,2)$ supersymmetry, and if $\mathcal{O}^{(1,0)}$ has $U(1)_R$ charge $+1$ then $\mathcal{O}^{(1,1)}$ will be uncharged, so that it provides a marginal deformation of the worldsheet action.[8] As usual, these marginal deformations are best interpreted as tangent vectors on the moduli space of $(0,2)$ models (at the base-point defined by the model in question). We will have more to say on the role of finite deformations in the twistor context in Section 7.

Because $\bar{\psi}$ is a worldsheet $(1,0)$-form, operators of weight $(h,\bar{h}) = (1,0)$ must be linear in either $\bar{\psi}_z$, $\partial_z\phi$, $\partial_z\bar{\phi}$ or $\partial_z\psi$. These fields are all uncharged under $U(1)_R$, so if we want $\mathcal{O}^{(1,0)}$ to have charge $+1$ it must also be linear in $\bar{\rho}$. Then the only such operators are

$$\begin{aligned} &g_{i\bar{k}}\delta J(\phi,\bar{\phi},\psi)^i_{\bar{j}}\bar{\rho}^{\bar{j}}\partial_z\phi^{\bar{k}}, \qquad &&\bar{\psi}_{az}\delta j(\phi,\bar{\phi},\psi)^a_{\bar{j}}\bar{\rho}^{\bar{j}}, \\ &\delta b(\phi,\bar{\phi},\psi)_{i\bar{j}}\bar{\rho}^{\bar{j}}\partial_z\phi^i, &&\delta\beta(\phi,\bar{\phi},\psi)_{a\bar{j}}\bar{\rho}^{\bar{j}}\partial_z\psi^a. \end{aligned} \tag{28}$$

Note that δJ, δj, δb and $\delta\beta$ may depend arbitrarily on ψ since it has $(h,\bar{h}) = (0,0)$, although since ψ is fermionic such dependence will be polynomial. On the other hand, they must be independent of $\bar{\psi}$ since this is a section of K_Σ. Each vertex operator thus has a Taylor expansion in powers of ψ and the pth coefficient of this expansion represents a section of $\bigwedge^p \mathcal{V}^\vee$. In particular, we can interpret the $U(1)_F$ quantum number as giving the transformation properties of the fields under automorphisms of the line bundle $(\det\mathcal{V})^{1/\mathrm{rk}\mathcal{V}}$, whereupon the coefficients of the ψ expansion have $U(1)_F$ charge while the vertex operators as a whole are uncharged. Geometrically, the fact that the ψs are included in the vertex operators in this way corresponds to the fact that the external states should be thought of as wavefunctions on the supermanifold $\mathbb{P}^{3|4}$ that are holomorphic in the ψs and may be expanded as

$$f = \sum_{p=0}^{4} f_{i_1\cdots i_p}\psi^{i_1}\cdots\psi^{i_p}, \tag{29}$$

where $f_{i_1\cdots i_k}$ is a section of $\bigwedge^p \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4}$. More abstractly, our presentation of $\mathbb{P}^{3|4}$ is as the space $\mathbb{P}^3$ together with the structure sheaf of superalgebras

$$\mathcal{O}_{\mathbb{P}^{3|4}} = \mathcal{O}\left(\bigoplus_{p=0}^{4}\bigwedge^{p}\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 4}\right), \tag{30}$$

as in the standard abstract definition of a supermanifold (see, e.g., [42,44]). The quantities δJ and δj in the vertex operators (28) can, according to this interpretation, be identified with a perturbation of the almost complex structure of the supermanifold $\mathbb{P}^{3|4}$ while δb and $\delta\beta$ describe perturbations of the B-field and Hermitian structure on $\mathbb{P}^{3|4}$.

[8] In the A- or B-models the descent procedure may be taken one stage further, relating deformations of the action to scalar operators of vanishing conformal weight. But in $(0,2)$ models there is only an antiholomorphic supersymmetry so the descent procedure only affects the antiholomorphic weight, mapping sections of $K^p \otimes \bar{K}^q$ to sections of $K^p \otimes \bar{K}^{q+1}$.

The transformations (2) and (8) show that $\bar{Q}$ acts on (28) as

$$\bar{Q} = \bar{\rho}^{\bar{j}} \frac{\delta}{\delta \phi^{\bar{j}}}, \tag{31}$$

in other words $\bar{Q}$ acts as the $\bar{\partial}$-operator on Maps(Σ, X) (and restricts to the $\bar{\partial}$-operator on instanton moduli space). Therefore, if (28) are to be non-trivial in $\bar{Q}$-cohomology, δJ, δj, δb and $\delta\beta$ must represent (pullbacks to Σ of) non-trivial elements

$$[\delta J] \in \bigoplus_{p=0}^{4} H^{0,1}\Big(X, T_X \otimes \bigwedge^p \mathcal{V}^\vee\Big), \qquad [\delta j] \in \bigoplus_{p=0}^{4} H^{0,1}\Big(X, \mathcal{V} \otimes \bigwedge^p \mathcal{V}^\vee\Big),$$
$$[\delta b] \in \bigoplus_{p=0}^{4} H^{0,1}\Big(X, T_X^\vee \otimes \bigwedge^p \mathcal{V}^\vee\Big), \qquad [\delta \beta] \in \bigoplus_{p=0}^{4} H^{0,1}\Big(X, \mathcal{V}^\vee \otimes \bigwedge^p \mathcal{V}^\vee\Big). \tag{32}$$

In fact, the interpretation of δb is slightly more subtle. δb is defined upto the equivalence relation

$$\delta b \sim \delta b + \bar{\partial}\Lambda + \partial M, \tag{33}$$

where $\Lambda \in \Omega^{1,0}(X)$ and $M \in \Omega^{0,1}(X)$. While the freedom to add $\bar{\partial}\Lambda$ is the usual freedom in choice of representative for a Dolbeault cohomology class, here we are also free to add ∂M since $\partial_i M_{\bar{j}} \bar{\rho}^{\bar{j}} \partial_z \phi^i = \partial_z (M_{\bar{j}} \bar{\rho}^{\bar{j}})$ using the $\bar{\rho}$ equations of motion, and so this term is a total derivative. This corresponds to the fact that only the cohomology class of $\mathcal{H} = \partial b \in H^1_{\bar{\partial}}(X, \Omega^2_{\mathrm{cl}})$ contributes to the moduli of a twisted (0, 2) model.

If we take $X = \mathbb{P}^3$, then because the Dolbeault complex is elliptic and $\mathbb{P}^3$ is compact, the above cohomology groups are at most finite dimensional. Such cohomology corresponds via the Penrose transform to fields on spacetime that extend over S^4 in the Euclidean context (and indeed over the full compactified complexification of Minkowski space, $\mathrm{Gr}_2(\mathbb{C}^4)$). To obtain fields on some subset of spacetime, we should take the target space to be the noncompact region in twistor space swept out by the corresponding lines. In the context of scattering theory, momentum eigenstates extend holomorphically over affine complexified Minkowski space $\mathbb{C}^4 \subset \mathrm{Gr}_2(\mathbb{C}^4)$, the complement of the lightcone at infinity. A suitable corresponding choice of target subspace of twistor space is then $\mathbb{PT}' \equiv \mathbb{P}^3 - \mathbb{P}^1$, and $\mathbb{PT}'$ is isomorphic to the total space of the normal bundle $\mathcal{O}(1) + \mathcal{O}(1) \to \mathbb{P}^1$ of a line in $\mathbb{P}^3$. More generally, one could simply choose a tubular neighbourhood $\hat{U}$ of some fixed line $L_p \subset \mathbb{P}^3$, corresponding to a region U around a chosen spacetime point p. A particularly natural, conformally invariant case is when U is the *future tube*: the points of complexified Lorentzian Minkowski space for which the imaginary part is timelike and future pointing, as this is the maximal domain of extension of positive frequency functions. In this case, $\hat{U}$ is the region $\mathbb{PT}^+$ on which the natural $SU(2,2)$-invariant inner product is positive.

It is easy to see that the theory with noncompact target will remain anomaly-free: we can naturally restrict the determinant line bundle $\mathcal{L} \to \mathrm{Maps}(\Sigma, \mathbb{P}^3)$ to a line bundle over $\mathrm{Maps}(\Sigma, \mathbb{PT}')$, say, and the restricted bundle will be flat since $\mathcal{L}$ itself is. With this target space understood, via the Penrose transform δJ describes an anti-selfdual $\mathcal{N} = 4$ conformal supergravity multiplet with helicities -2 to 0 (and containing, in effect, two fields of helicity -2), δj describes four gravitino multiplets containing helicities $-\frac{3}{2}$ to $+\frac{1}{2}$, while δb and $\delta\beta$ are the CPT conjugates of δJ and δj. From the supermanifold point of view, δJ and δj combine to describe deformations of the complex structure of $\mathbb{P}^{3|4}$, while δb and $\delta\beta$ together represent deformations of the cohomology class of the Kähler structure and NS flux on the supermanifold, as detailed in [4].

5. Coupling to Yang–Mills

We can incorporate Yang–Mills fields into the model by introducing a worldsheet current algebra. This could be represented by adding in further left-moving fermionic fields as in standard heterotic constructions, or by a gauged WZW model, fibred over twistor space as in [45]. For definiteness we will consider here the simplest case of left-moving fermions

$$\lambda^\alpha \in \Gamma\big(\Sigma, \sqrt{K} \otimes \phi^* E\big), \qquad \bar{\lambda}_\alpha \in \Gamma\big(\Sigma, \sqrt{K} \otimes \phi^* E^\vee\big) \tag{34}$$

together with their (auxiliary) (0, 2) superpartners. Here E is a rank r holomorphic vector bundle over $\mathbb{P}^3$ and, in contrast to the ψ fields, we have taken the λs to be spinors on Σ. The $(0,2)$ transformations and action of these fields take exactly the same form as in Eqs. (8)–(9), although the connection $\bar{D}$ acts now on sections of $\sqrt{K} \otimes \phi^* E$, rather than just $\phi^* E$.

There are restrictions on E arising from the requirement that this coupling to E does not disturb the anomaly cancellation in Section 3.1. All components of the quantum stress-tensor will remain $\bar{Q}$-closed provided that the curvature $F^{(E)}$ of the background connection on E satisfies the Hermitian–Yang–Mills equations $g^{i\bar{j}} F^{(E)}_{i\bar{j}} = 0$. It is possible to find such a connection [46] if E is stable and

$$\int_X \mathrm{c}_1(E) \wedge \omega \wedge \omega = 0, \tag{35}$$

which for $X \simeq \mathbb{P}^3$ implies that $\mathrm{c}_1(E) = 0$ as $H^{1,1}(\mathbb{P}^3)$ is one-dimensional. Thus correlators in the $\bar{Q}$-cohomology will conformally invariant at the quantum level if $\mathrm{c}_1(E) = 0$ and E is stable. Vanishing first Chern class of the gauge bundle is a familiar condition in heterotic string compactification, but it also plays a role in the Penrose–Ward transform. A point in spacetime corresponds to a $\mathbb{P}^1$ in twistor space, so any twistor bundle that is the pullback of a spacetime bundle must be trivial on every holomorphic twistor line, and this will generically be the case provided $c_1(E) = 0$.

In addition, $\mathrm{c}_1(E) = 0$ ensures that there is an anomaly-free $U(1)_{F'}$ global symmetry under which λ and $\bar{\lambda}$ have equal and opposite charges and all other (dynamical) fields are uncharged. Since this $U(1)_{F'}$ is conserved at the quantum level, all correlation functions will vanish unless they involve equal numbers of λ and $\bar{\lambda}$ insertions.

5.1. NS branes and Yang–Mills instantons

Integrating out the non-zero-modes of λ and $\bar{\lambda}$ provides a factor of $\det'(\bar{\partial}_{K^{1/2} \otimes \phi^* E})$ which affects the sigma model anomaly, modifying the Green–Schwarz condition to

$$0 = \mathrm{ch}_2(T_{\mathbb{P}^3}) - \mathrm{ch}_2\big(\mathcal{O}(1)^{\oplus 4}\big) - \mathrm{ch}_2(E). \tag{36}$$

Since $\mathrm{ch}_2(T_{\mathbb{P}^3}) = \mathrm{ch}_2(\mathcal{O}(1)^{\oplus 4})$, we must require that $\mathrm{ch}_2(E)$ is trivial in $H^4(\mathbb{P}^3, \mathbb{Z})$. Given that $\mathrm{c}_1(E) = 0$ for E to be pulled back from a bundle over spacetime, (36) requires further that E is the pullback of a Yang–Mills bundle with zero instanton number. Whilst it is interesting to see how this well-known limitation of twistor–string theory arises (which was not transparent in the original models), it would be disappointing if twistor–string theory were truly restricted to studying perturbative aspects of gauge theories. Fortunately, the heterotic approach furnishes us with a mechanism to avoid this constraint. At the non-perturbative level, heterotic strings contain Neveu–Schwarz branes: magnetic sources for the NS B-field. In the physical, ten-dimensional model, B has a six-form magnetic dual potential and the NS brane worldvolume is six-dimensional. However, in our six-dimensional twisted theory the magnetic dual of the B-field is again a two-form, so the twisted theory contains NS branes with two-dimensional worldvolumes, wrapping curves $C \subset \mathbb{P}^3$ that are holomorphic if the NS brane does not break supersymmetry. If $[C] \in H^4(\mathbb{P}^3, \mathbb{Z})$ is the Poincaré dual of C, then the presence of an NS brane gives a further contribution to the Green–Schwarz condition [47] which in our case reads

$$\mathrm{ch}_2(E) = [C], \tag{37}$$

so that including NS branes wrapping holomorphic curves corresponds to studying twistor–string theory in an instanton background.

In fact, the relation between Yang–Mills instantons and curves in $\mathbb{P}^3$ has long been known, and indeed was one of the earliest applications of algebraic geometry to theoretical physics [48,49]. For example, to construct the simplest case of an $SU(2)$ k-instanton described by the 't Hooft ansätz[9] [50]

$$A(x) = \mathrm{i}\,\mathrm{d}x^\mu\,\sigma_{\mu\nu}\partial^\nu \log\Phi, \qquad \Phi(x) = \sum_{i=0}^{k} \frac{\lambda_j}{(x-x_i)^2}, \tag{38}$$

one wraps NS branes on the $k+1$ lines $L_i \subset \mathbb{P}^3$ corresponding to the points x_i (with x_0 the point 'at infinity'). More specifically, each summand[10] in $\Phi(x)$ is represented on twistor space by $\tilde{\Phi}_i \in H^1(\mathbb{P}^3 - L_i, \mathcal{O}(-2))$ via the inverse Penrose transform. Similar considerations hold for generic $SU(2)$ instantons [48,49], although it is less clear how to extend the approach to higher rank gauge groups.

5.2. Yang–Mills vertex operators

For the remainder of this paper, we will concentrate on Yang–Mills perturbations around the zero-instanton vacuum. In a gauge in which the background connection on E vanishes, the $(0,2)$ transformations of λ simplify to become

$$\delta\lambda^\alpha = \epsilon_2 r^\alpha, \qquad \delta\bar{\lambda}_\alpha = \epsilon_1 \bar{r}_\alpha, \qquad \delta r^\alpha = \epsilon_1 \bar{\partial}\lambda^\alpha, \qquad \delta\bar{r}_\alpha = \epsilon_2 \bar{\partial}\bar{\lambda}_\alpha, \tag{39}$$

so that the action is

$$\left\{\bar{Q}, \int_\Sigma \left|\mathrm{d}^2 z\right| \bar{\lambda}_\alpha r^\alpha\right\} = \int_\Sigma \left|\mathrm{d}^2 z\right| \bar{\lambda}_\alpha \partial_{\bar{z}} \lambda^\alpha + \bar{r}_\alpha r^\alpha. \tag{40}$$

Thus the level one current algebra is represented as usual by free fermions with propagator $\delta^\alpha{}_\beta/2\pi\mathrm{i}(z_1 - z_2)$ in local coordinates z on Σ. It is this current algebra which is the natural heterotic realization of the current algebra on the worldsheet of Berkovits' twistor–string, or the current algebra of the D1–D5 strings in Witten's B-model twistor–string.

The coupling to E provides new vertex operators of conformal weight $(h, \bar{h}) = (1, 0)$ and $U(1)_R$ charge 1, given by

$$\mathcal{O}_{\mathcal{A}}^{(1,0)} = \mathcal{A}(\phi, \bar{\phi}, \psi)_{\bar{j}}{}^\alpha{}_\beta \bar{\rho}^{\bar{j}} \bar{\lambda}_\alpha \lambda^\beta, \tag{41}$$

[9] Here, $\sigma_{\mu\nu}$ is the $su(2)$-valued anti-selfdual two form defined by $\sigma_{ij} \equiv \frac{\mathrm{i}}{4}[\sigma_i, \sigma_j]$, $\sigma_{0i} \equiv -\frac{1}{2}\sigma_i$.

[10] The summands $\Phi_i(x)$ are Green's functions for the scalar Laplacian on spacetime, and are examples of twistor 'elementary states'.

where again we allow $\mathcal{A}$ to depend on ψ but not $\bar{\psi}$. This operator is non-trivial in $\bar{Q}$-cohomology provided $\mathcal{A}$ represents a non-trivial element of $\bigoplus_{p=0}^{4} H^{0,1}(\mathbb{PT}', \mathrm{End}\, E \otimes \bigwedge^p \mathcal{V}^\vee)$ and represents a deformation of the complex structure of $E \to \mathbb{P}^3$, together with the $\mathcal{N}=4$ completion. The integrated vertex operator corresponding to (41) is

$$\begin{aligned}\mathcal{O}^{(1,1)}_{\mathcal{A}} &= \mathrm{tr}\,\bar{\lambda}\left(\mathcal{A}_{\bar{j}}\bar{\partial}\phi^{\bar{j}} + \partial_i\mathcal{A}_{\bar{j}}\rho^i\bar{\rho}^{\bar{j}} + \frac{\delta\mathcal{A}_{\bar{j}}}{\delta\psi^a}A_i{}^a{}_b\psi^b\rho^i\bar{\rho}^{\bar{j}}\right)\lambda \\ &= \mathrm{tr}\,\bar{\lambda}\left(\phi^*\mathcal{A} + D_i\mathcal{A}_{\bar{j}}\rho^i\bar{\rho}^{\bar{j}}\right)\lambda \qquad (42)\end{aligned}$$

up to terms proportional to the auxiliary fields, and where the trace is over the Yang–Mills indices. The third term in the first line arises through the ψ dependence of $\mathcal{A}$ and involves the background connection A on $\phi^*\mathcal{V}$. Because $\mathcal{V} = \mathcal{O}(1)^{\oplus 4}$ is a sum of line bundles, we can always choose this connection to be diagonal $A_i{}^a{}_b = A_i\delta^a{}_b$. The second line, with D the holomorphic exterior derivative on sections of $\bigoplus_{p=0}^{4}\phi^*(\mathrm{End}\, E \otimes \bigwedge^p \mathcal{V})$, then follows since $\mathcal{A}$ can depend only polynomially on the fermions ψ. As expected, comparing (42) to (9) shows that $\int_\Sigma \mathcal{O}^{(1,1)}_{\mathcal{A}}$ provides an infinitesimal deformation of the worldsheet action corresponding to an infinitesimal change in background super-Yang–Mills connection.

To summarize, we have found a twisted (0, 2) sigma model whose path integral localizes on holomorphic maps to twistor space. Under the Penrose transform, the tangent space to the moduli space of such a (0, 2) model corresponds to states in $\mathcal{N}=4$ conformal SUGRA and SYM, linearized around a flat background. For the SYM states, introducing NS branes allows us also to discuss linearized perturbations around an instanton background. However, the model really contains an infinite number of other vertex operators that we have not discussed, and at present there is no fully satisfactory descent procedure relating deformations of the action to scalar vertex operators. We will see that these issues are resolved when we promote the sigma model to a string theory in the next section. Moreover, whilst we were free to include an additional left-moving current algebra to describe a SYM multiplet, nothing in the formalism has yet forced us to make a specific choice.

6. Promotion to a string theory

The (0, 2) sigma model of the previous section depends on the choice of a complex structure on Σ. This entered right at the beginning in the definition of the (0, 2) supersymmetry transformations (2) and (8). A choice of complex structure on Σ, together with n marked points to attach vertex operators, is a choice of a point in the moduli space of stable[11] curves $\bar{\mathcal{M}}_{g,n}$ and to promote the sigma model to a string theory, we should integrate over this space also.

In a twisted (0, 2) model, as in the A- or B-models [21,22], right-moving worldsheet supersymmetry allows us to construct a top antiholomorphic form on $\bar{\mathcal{M}}_{g,n}$. Specifically, at genus $\geqslant 2$ we choose $3g-3+n$ antiholomorphic Beltrami differentials $\bar{\mu}^{(i)} \in \overline{H^{0,1}(\Sigma, T_\Sigma)}$ and construct a fermionic operator via the natural pairing $(\bar{\mu}^{(i)}, \bar{G}) \equiv \int_\Sigma \bar{\mu}^{(i)} \lrcorner\, \bar{G}$ with the (0, 2) supercurrent $\bar{G} = g_{i\bar{j}}\rho^i\bar{\partial}\phi^{\bar{j}} \in \Gamma(\Sigma, \bar{K}\otimes\bar{K})$. Inserting the product of $3g-3+n$ such operators into the correlation function then provides a top antiholomorphic form on $\bar{\mathcal{M}}_{g,n}$.

In a twisted (2, 2) model, the same procedure may also be used to construct a top holomorphic form from the left-movers, but in our (0, 2) model we have no holomorphic supercurrent. Instead,

[11] We allow the abstract worldsheet to have nodes.

we introduce a holomorphic bc ghost system (with apologies for possible confusion with the $b = b_{i\bar{j}}$ field introduced earlier), with

$$b \in \Gamma(\Sigma, K \otimes K), \qquad c \in \Gamma(\Sigma, T_\Sigma) \tag{43}$$

having the natural action $S_{bc} = \int_\Sigma b\bar{\partial}c$. We will take both b and c to be annihilated by the (0, 2) supercharges $\bar{Q}$ and $\bar{Q}^\dagger$. As in the bosonic (or left-moving sector of the heterotic) string, including holomorphic bc ghosts provides us with a holomorphic BRST operator Q such that the holomorphic stress-energy tensor $T + T^{bc}$ of the sigma-model plus bc system is Q-exact, $T_{zz} + T^{bc}_{zz} = \{Q, b_{zz}\}$. In parallel to the discussion above, a top holomorphic form on $\bar{\mathcal{M}}_{g,n}$ may be constructed from the b antighosts by inserting the product of $3g - 3 + n$ operators $(\mu^{(i)}, b) = \int_\Sigma \mu^{(i)} \lrcorner\, b$ into the path integral. Of course, a proper treatment of a twisted (0, 2) string theory really requires an understanding of twisted versions of the worldsheet (0, 2) supergravity of [51,52], just as the A- and B-model topological strings may be understood from twisted (2, 2) supergravity [53,54].

6.1. Constraints on the gauge group

The holomorphic BRST operator is nilpotent provided the left-moving fields have vanishing net central charge. As in Berkovits' model [2], this requires that the Yang–Mills current algebra contributes $c = 28$. This constraint arises from integrating out the non-zero modes of $\{\phi, \rho, \psi, b, c\}$ and the current algebra fields. If we represent the current algebra in terms of left-moving fermions λ as in Section 5, we obtain a ratio of determinants[12]

$$\frac{\det{}' \bar{\partial}_{\phi^*\mathcal{V}} \det{}' \bar{\partial}_{\sqrt{K}\otimes\phi^*E} \det{}' \bar{\partial}_{T_\Sigma}}{\det{}' \bar{\partial}_{\phi^*T_X}} \tag{44}$$

in the genus g partition function. As in Section 3.1, for $X = \mathbb{P}^3$ and $\mathcal{V} = \mathcal{O}(1)^{\oplus 4}$, the Quillen connection on this determinant line bundle has curvature[13] [34–36]

$$\begin{aligned} F &= \int_\Sigma \mathrm{Td}(T_\Sigma)\,\mathrm{ch}(T_\Sigma) + \mathrm{Td}(T_\Sigma)\phi^*\,\mathrm{ch}\big(\mathcal{O}(1)^{\oplus 4} \ominus T_{\mathbb{P}^3}\big) + \hat{A}(T_\Sigma)\,\mathrm{ch}(\phi^* E)\Big|_{(4)} \\ &= \int_\Sigma \left(1 + \frac{x}{2} + \frac{x^2}{12}\right)\left(2 + x + \frac{x^2}{2}\right) - \frac{x^2}{24}\,\mathrm{rk}\,E\Big|_{(4)} \\ &= (28 - \mathrm{rk}\,E)\int_\Sigma \frac{x^2}{24}, \end{aligned} \tag{45}$$

where $x = \mathrm{c}_1(T_\Sigma)$. So for a current algebra at level one we would require that E has rank (30) as a complex vector bundle in order to ensure that the determinant line bundle is flat and the section (44) may be taken as constant. More generally, a current algebra at level k contributes

[12] The presence of this ratio is really a feature of (0, 2) models; in a twisted (2, 2) model $\mathcal{V} = T_X$ while there is no extra gauge bundle E or bc system, so (44) would automatically be unity. (0, 2) supersymmetry is sufficient to ensure that the ratio depends only holomorphically on the moduli (as it ensures we only have determinants of $\bar{\partial}$-operators), but the condition that (44) be a section of a flat line bundle becomes a non-trivial requirement.

[13] The second line in (45) follows if E is trivial. In the presence of a Yang–Mills instanton, the Quillen connection is not flat, but there is a modification constructed from the NS field H which is [35].

a central charge $c = k\,\mathrm{rk}\,G/(k + h(G))$ for each semisimple factor G of the Yang–Mills gauge group, where $h(G)$ is the dual Coxeter number of G.

We have recovered the same constraint on the central charge of the current algebra as in Berkovits' model [2]. As pointed out in [4], this is a rather puzzling result. In conformal supergravity an $SU(4)$ subgroup of the $U(4)$ R-symmetry group is gauged.[14] Spacetime field theory calculations by Römer and van Nieuwenhuizen [55] show that this gauged $SU(4)_R$ is anomalous unless the conformal supergravity is coupled to an $\mathcal{N} = 4$ SYM multiplet with gauge group either $U(1)^4$ or $U(2)$. We may view this result as analogous to the statement [56] that $\mathcal{N} = 1$ Poincaré supergravity in ten dimensions is anomalous unless coupled to $\mathcal{N} = 1$ SYM with gauge group either $U(1)^{496}$, $E_8 \times U(1)^{248}$, $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$. However, the small admissible gauge groups $U(1)^4$ and $SU(2) \times U(1)$ in the conformal theory do not sit well with the requirement that the Yang–Mills current algebra contributes central charge (30), irrespective of the level k. In contrast, for the physical heterotic string the required bundle contribute central charge of (18) is perfectly tailored to the rank of $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$. Possible resolutions discussed in [4] include changing the level of the current algebra or trying to include additional worldsheet fields without changing the BRST cohomology.[15]

In the physical heterotic string, the requirement that the determinant line bundle (45) has trivial holonomy over the moduli space of complex structures on Σ fixes the gauge group [57,58]. (At genus 1, this amounts to checking that the string partition function is invariant under modular transformations of Σ.) We anticipate that modular invariance will play a similarly important role in the context of twistor–strings, and will likely rule out many solutions of the central charge condition.

6.2. *Vertex operators in the string theory*

When $Q^2 = 0$, there is a left-moving BRST complex graded by ghost number, where b and c have ghost numbers -1 and $+1$, respectively. As in Section 4, the relation $\{Q, b_0\} = L_0$ shows that the Q-cohomology vanishes except for states of holomorphic conformal weight $h = 0$. Moreover, as in the bosonic string, physical states are created by vertex operators of ghost number $+1$. Since $c \in \Gamma(\Sigma, T_\Sigma)$, to construct a (reparametrization invariant) vertex operator with $h = 0$ we must couple c to a sigma-model vertex operator of conformal weight $(h, \bar{h}) = (1, 0)$. These are the operators of Eqs. (28) and (41). The fact that, when coupled to the bc system, only these vertex operators remain out of the entire sheaf of chiral algebras is the real reason for having singled them out in the first place.

The relation $\{Q, b_{-1}\} = L_{-1}$ now enables us to complete the descent procedure: given an operator $\mathcal{O}^{(p,q)}$ obeying $\{Q, \mathcal{O}^{(p,q)}\} = 0$ we find that $\{b_{-1}, \mathcal{O}^{(p,q)}\}$ has conformal weight $(p + 1, q)$ and is Q-closed upto a total holomorphic derivative. Consequently, there is now a complete descent procedure between scalar vertex operators and deformations of the worldsheet action.

[14] The remaining $U(1)$ factor is the $U(1)_F$ symmetry acting on ψ and $\bar{\psi}$, responsible for the 'helicity *vs* degree' selection rule (20).

[15] It is perhaps worth noting that, if it is possible to promote the sigma model to a string theory without including a bc system (as in the antiholomorphic sector), then the net holomorphic central charge vanishes provided the current algebra contributes $c = 2$. This would be in better accord with the required gauge groups. However, we do not know how to do this.

6.3. Contour integration on $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^3, d)$

To compute scattering amplitudes involving n external states, we pick n marked points on Σ and attach a fixed vertex operator for the appropriate external state to each. As usual, there is an anomaly in the ghost number of the bc system, given by the excess of c over b zero-modes

$$h^0(\Sigma, T_\Sigma) - h^1(\Sigma, T_\Sigma) = 3 - 3g. \tag{46}$$

This anomaly is completely absorbed by the n vertex operators and $3g-3+n$ factors of $(\mu^{(i)}, b)$.

In the antiholomorphic sector however, the anomaly calculation (17) showed that correlation functions vanish unless they contain net $U(1)_R$ charge

$$h^0\big(\Sigma, \phi^* T_{\mathbb{P}^3}\big) - h^1\big(\Sigma, \phi^* T_{\mathbb{P}^3}\big) = 4d + 3(1-g). \tag{47}$$

Since $\bar{G}_{\bar{z}\bar{z}}$ and the vertex operators have $U(1)_R$ charges -1 and $+1$ respectively, the insertion $\prod^{3g-3+n}(\bar{\mu}^{(i)}, \bar{G})$ together with the n vertex operators contribute net $U(1)_R$ charge $3(1-g)$, but an anomaly of $4d$ still remains.[16] This residual anomaly—arising from an excess of $\bar{\rho}$ zero-modes—has a simple interpretation. Upon transforming the fixed vertex operators to integrated ones using the $(\bar{\mu}^{(i)}, \bar{G})$ insertions we are left with an integral over the moduli space $\bar{\mathcal{M}}_{g,0}(\mathbb{P}^3, d)$ of degree d stable maps to $\mathbb{P}^3$. This space has virtual dimension

$$\operatorname{vdim} \bar{\mathcal{M}}_{g,0}\big(\mathbb{P}^3, d\big) = \int_\beta \mathrm{c}_1(T_{\mathbb{P}^3}) + \big(\dim^3_{\mathbb{CP}} - 3\big)(1-g) = 4d. \tag{48}$$

Consequently, the twistor–string path integral reduces to an integral over a $4d$-dimensional moduli space (when the map is unobstructed and $d > 0$) in contrast to the case of a Calabi–Yau target where the moduli space is (virtually) a discrete set of points. This positive dimension is of course fully expected; in particular $\bar{\mathcal{M}}_{0,0}(\mathbb{P}^3, 1) = \mathrm{Gr}_2(\mathbb{C}^4)$, the conformal compactification of complexified flat spacetime. Integrating out all the fermion zero-modes, except the $4d$ 'excess' $\bar{\rho}$ zero modes, provides us not with a top form on $\bar{\mathcal{M}}_{g,0}(\mathbb{P}^3, d)$, but instead a section of the canonical bundle[17] $\Omega^{4d,0}$. Such a form is most naturally integrated over a real slice of $\bar{\mathcal{M}}_{g,0}(\mathbb{P}^3, d)$, which at $g = 0$ and $d = 1$ is just a real slice of complexified spacetime. Indeed, on physical grounds it is entirely appropriate that amplitudes should arise from integrals over the real slice of spacetime rather than its complexification.

A natural way to find a contour is to choose real structures, i.e., antiholomorphic involutions $\tau_{\mathbb{P}^3} : \mathbb{P}^3 \to \mathbb{P}^3$ and $\tau_\Sigma : \Sigma \to \Sigma$ obeying $\tau_{\mathbb{P}^3}^2 = 1 = \tau_\Sigma^2$. These induce a real structure τ on $\bar{\mathcal{M}}_{g,0}(\mathbb{P}^3, d)$ by $\tau(\phi) = \tau_{\mathbb{P}^3} \circ \phi \circ \tau_\Sigma$. The contour is then the locus of maps invariant under τ, so that $\tau\phi = \phi$. This method was used by Berkovits in [2] to define twistor strings for spacetime of signature $(++--)$, where $\tau_{\mathbb{P}^3}$ and τ_Σ act by standard complex conjugation on the homogeneous coordinates of the target space and worldsheet. These choices of real structure leave fixed an $\mathbb{RP}^3$ submanifold of twistor space and an equatorial $S^1 \subset \Sigma$ at genus zero. In this case, real maps (i.e., those left fixed by τ) must take marked points of Σ to the fixed slice in twistor space

[16] Note that this issue is not resolved merely by moving to a model with $\mathbb{P}^{3|4}$ target; one then finds $h^0(\Sigma, \phi^* T_{\mathbb{P}^{3|4}}) - h^1(\Sigma, \phi^* T_{\mathbb{P}^{3|4}}) = -(1-g)$.

[17] This section is constructed from the ψ zero-modes, representing a section of the canonical bundle of instanton moduli space as in Section 3.1.2, and the bc zero-modes, furnishing a section of the canonical bundle of the moduli space of curves.

so that vertex operators are inserted on this fixed slice, as in Berkovits' model. The same contour was used in the explicit calculations of Roiban, Spradlin and Volovich [9].

It would be desirable not to be reliant on split signature. Calculations in split signature give satisfactory answers at tree level, but it is thought that they will not straightforwardly extend to loop amplitudes because the $i\epsilon$ prescription for the Feynman propagator will not be properly incorporated. Euclidean spacetime signature corresponds to the real structure on $\mathbb{P}^3$ given by quaternionic conjugation of the homogeneous coordinates. At genus zero, one can combine this conjugation with the antipodal map on the Riemann surface[18] to give a real structure on $\bar{\mathcal{M}}_{g,n}(\mathbb{P}^3, d)$. When $g = 0$ and $d = 2k + 1$ this method works well, but when $d = 2k$ the fixed locus is empty.

For Lorentz signature, the reality conditions map twistor space to dual twistor space and so do not define a real structure on $\mathbb{P}^3$ in the same way as above, but instead give a pseudo-Hermitian structure of signature $(2, 2)$ on the non-projective twistor space. The real points of Lorentz signature spacetime correspond to those degree one rational curves in twistor space that lie in the zero-set $\mathbb{PN}$ of the Hermitian form. However, connected curves of higher degree are not likely to lie in $\mathbb{PN}$. Thus, in neither of these physically more useful signatures are we able to obtain a canonically defined real slice of the moduli space of stable maps.

One can avoid these problems if one is allowed to consider disconnected curves, as, in the Euclidean case, a curve of even degree can be represented as the union of two real curves of odd degree, while in the Lorentzian case, one can simply make up a degree d curve as a union of d degree 1 lines in $\mathbb{PN}$. Allowing disconnected curves essentially entails moving to string field theory, and this is discussed in Section 9.

However, to make sense of twistor–string amplitudes in Euclidean and Lorentzian signature, one does not need to go into string field theory. The key point is that the contour only needs to be defined as a homology cycle supported in an appropriate subset of the moduli space. According to the philosophy given in [10], it is natural to think of the moduli space of instantons of fixed degrees, but with different numbers of components as being joined across spaces of nodal curves, and it is natural to allow the contour to pass through these loci of singular curves. Although the integrands have simple poles at such singular loci, the residues are the same from both sides. Thus we can define the contour canonically at degree d as the appropriate d-fold product of real spacetime in the space of d-component degree one curves. Then we deform this contour into the space of connected, degree d curves through nodal curves. Although such a deformed contour will be non-canonical, it is reasonable to hope that its homology class will be.

However the contour is chosen, we must implement it in the path integral. To do so, suppose first of all that the contour has Poincaré dual $\Gamma \in \Omega^{4d}(\bar{\mathcal{M}}_{g,0}(\mathbb{P}^3, d))$, and let $\{t^A\}$ be a set of coordinates on a local patch of instanton moduli space $\mathcal{M}$, where $A = 1, \ldots, h^0(\Sigma, \phi^* T_X)$. Then for any stable holomorphic map ϕ, we may expand a $\bar\rho$ zero-mode as

$$\bar\rho^{\bar J} = \bar\rho^{\bar A} \frac{\partial \phi^{\bar J}}{\partial \bar t^{\bar A}} \tag{49}$$

so that $\{\bar\rho^{\bar A}\}$ correspond to a basis of (0, 1)-forms on $\mathcal{M}$. Projecting Γ onto its $(0, 4d)$-form part (as usual for contour integrals) we insert the operator $\mathcal{O}_\Gamma = \Gamma_{\bar A_1 \cdots \bar A_{4d}} \bar\rho^{\bar A_1} \cdots \bar\rho^{\bar A_{4d}}$ at degree d, so

[18] The real structure also extends beyond genus zero, as is most easily seen by considering the higher genus Riemann surface as a branched cover over $\mathbb{P}^1$, with pairs of branch points at mutually antipodal points.

that we compute

$$\left\langle \mathcal{O}_\Gamma \prod_{i=1}^{3g-3+n} \big(\mu^{(i)}, b\big)\big(\bar\mu^{(i)}, \bar G\big) \prod_{j=1}^{n} \mathcal{O}_j^{(0,0)} \right\rangle, \tag{50}$$

where $\mathcal{O}_j^{(0,0)}$ is a fixed vertex operator, formed from the contraction of a c ghost with one of the sigma model vertex operators in (28) or (41) for external states in the conformal supergravity or super-Yang–Mills multiplets, respectively. The $\mathcal{O}_\Gamma$ insertion is to be thought of as part of the definition of the degree d heterotic path integral measure.

At $g=0$ there are no zero-modes of b, ρ or $\bar\psi$, so as usual the bc and $\rho\bar\rho$ OPEs may be used to replace $n-3$ of the fixed vertex operators and all the $(\mu^{(i)}, b)(\bar\mu^{(i)}, \bar G)$ insertions in (50) by $n-3$ integrated vertex operators $\int_\Sigma \mathcal{O}^{(1,1)}$, leaving us with

$$\left\langle \prod_{i=1}^{3} \mathcal{O}_i^{(0,0)} \prod_{j=4}^{n} \int_\Sigma \mathcal{O}_j^{(1,1)} \right\rangle_\Gamma, \tag{51}$$

where the subscript Γ indicates the choice of contour.

Let us assume that the external states are all from the Yang–Mills supermultiplet. We now integrate out the $\bar\lambda\lambda$ current algebra. There are no holomorphic sections of $K^{1/2} \otimes \mathbb{C}^r$ at genus zero, so we must take account of the $\lambda\bar\lambda$ insertions when integrating out their non-zero-modes. A standard approach is to introduce a coupling $\int_\Sigma \operatorname{tr} \bar\lambda J \lambda$ to an arbitrary source J, and then replace the $\bar\lambda\lambda$ factors in the vertex operators by functional derivatives with respect to J. The path integral over λs then gives $\delta^n/\delta J^n \det(\bar\partial_{\sqrt{K}\otimes\phi^* E} + J)$, evaluated at $J=0$. We have

$$\delta \det(\bar\partial_{\sqrt{K}\otimes\phi^* E} + J) = \frac{\det(\bar\partial_{\sqrt{K}\otimes\phi^* E} + J)}{2\pi \mathrm{i}} \int_\Sigma \operatorname{tr} G'_J(u,u)\,\delta J(u), \tag{52}$$

where u are homogeneous coordinates on the $\mathbb{P}^1$ worldsheet and $G'_J = G_J - G_0$ is the regulated Green's function for the $\bar\partial + J$ operator, with

$$G_J|_{J=0} = \frac{1}{2\pi \mathrm{i}} \frac{\langle u_2\,\mathrm{d}u_2\rangle}{\langle u_1 u_2\rangle}, \tag{53}$$

where $\langle uv\rangle = \epsilon_{ab} u^a v^b$ is the $SL(2,\mathbb{C})$-invariant inner product. (Regularing by subtracting the singular part $G_0(u,u)$ does not affect higher variations, which do not require regularization.) This procedure gives multi-trace contributions to the genus zero amplitudes, as in all the known twistor–string theories: further variations can either act on G'_J (leading to a single-trace contribution) or else act on the determinant producing multi-trace terms. In [1,4] these multi-trace terms were attributed to conformal supergravity, formed from a number of pure Yang–Mills interactions strung together with propagators associated to fields in the conformal supergravity multiplet. From the heterotic perspective also, such interactions are inevitable since upon cutting the worldsheet between the fixed Yang–Mills vertex operators, unitarity demands that all the states in the BRST cohomology,[19] including the conformal supergravity modes, appear in the cut. Note that, after turning off the external current, both the single-trace and multi-trace terms are accompanied by a factor of $\det(\bar\partial_{\sqrt{K}\otimes\phi^* E})$. This factor combines with the integral over the

[19] Subject to the usual selection rules.

non-zero-modes of ϕ, ρ, ψ and the bc system to yield the ratio (44), which as discussed before may be taken as a constant due to anomaly cancellation.

Identifying the tree-level SYM amplitude with the leading-trace term and integrating out the three c zero-modes one obtains

$$\int [\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\bar\rho]_0 \Gamma\, \mathrm{e}^{-S_{\text{inst}}} \operatorname{tr}\left\{ \mathcal{A}_{\bar\jmath_1}\bar\rho^{\bar\jmath_1}\mathcal{A}_{\bar\jmath_2}\bar\rho^{\bar\jmath_2}\mathcal{A}_{\bar\jmath_3}\bar\rho^{\bar\jmath_3}\prod_{p=4}^{n}\int_\Sigma \frac{\langle u_p\,\mathrm{d}u_p\rangle}{\langle u_p u_{p+1}\rangle}\mathcal{A}_{\bar\jmath_p}\bar\partial\phi^{\bar\jmath_p}\right\}, \tag{54}$$

plus non-cyclic permutations, where $u_{n+1} \equiv u_4$ and the trace is over the Yang–Mills indices. Finally, integrating out the $3 + 4d$ $\bar\rho$ zero-modes from the vertex operators and the contour insertion reduces this to the same integral that was the starting point for the amplitude calculations in [1,9]. We have thus shown that the leading-trace contribution to the amplitudes of heterotic twistor–strings coincide with those of Witten's B-model.

7. The geometry of supertwistor spaces and googly data

We have quantized on a region in a homogeneous twistor space $\mathbb{P}^3$, coupled in different ways to bundles $\mathcal{V} = \mathcal{O}(1)^{\oplus 4}$ and a trivial bundle E. The vertex operators correspond via the descent procedure to perturbations of the action that correspond to deformations of the geometric structures on this space. In particular, the operators in the first line of (28) were seen to correspond to integrable deformations of the complex structure $\mathcal{J} = (J, j)$ on the supermanifold $\mathbb{P}^{3|4}$ and the second line to $\bar\partial$-closed deformations of an NS field $\mathcal{B} := (b, \beta)$. Thus, as reviewed in Sections 4 and 6, the physical states of (heterotic) twistor–string theory are in one-to-one correspondence with elements of the cohomology groups $H^1(\mathbb{PT}'^{3|4}, T_{\mathbb{PT}'^{3|4}})$, $H^1(\mathbb{PT}'^{3|4}, \Omega^2_{\text{cl}})$ and $H^1(\mathbb{PT}'^{3|4}, \operatorname{End} E)$. In turn, these groups correspond via the Penrose transform to supermultiplets in $\mathcal{N} = 4$ conformal supergravity and super-Yang–Mills, but it is important to note that they represent only *linearized perturbations* around some fixed background. For example, in the gravitational sector the group $H^1(\mathbb{PT}'^{3|4}, T_{\mathbb{PT}'^{3|4}})$ contains states describing fluctuations of helicities -2 upto $+1/2$ that constitute the anti-selfdual half of the spectrum of linearized $\mathcal{N} = 4$ conformal supergravity. Going beyond perturbation theory, one first identifies $H^1(\mathbb{PT}', T_{\mathbb{PT}'})$ as the tangent space to the moduli space of complex structures on twistor space, and then Penrose's non-linear graviton construction [59] states that a finite deformation of the complex structure on $\mathbb{PT}'$ corresponds to a four-dimensional spacetime with vanishing selfdual Weyl tensor $W^+ = 0$.

The fact that perturbations of $\mathcal{J}$ and $\mathcal{B}$ only have holomorphic dependence on ψ^a is not a restriction because a general complex supermanifold $\mathcal{M}_s$ can be expressed as the parity reverse of a holomorphic vector bundle $\mathcal{V}$ over the body $\mathcal{M}$ but with $\bar\partial$-operator deformed by terms that depend holomorphically on the anticommuting fibre coordinates ψ^a of $\mathcal{V}$. Thus we require that the antiholomorphic tangent bundle of $\mathcal{M}_s$ be spanned by vectors of the form

$$\left\{\frac{\partial}{\partial\phi^{\bar\imath}} + J_{\bar\imath}{}^{j}\frac{\partial}{\partial\phi^{j}} + j_{\bar\imath}^{a}\frac{\partial}{\partial\psi^a}, \frac{\partial}{\partial\bar\psi^{\bar a}}\right\}, \tag{55}$$

where $\mathcal{J} = (J, j)$ depends only on $(\phi^i, \bar\phi^{\bar\jmath}, \psi^a)$ with ψ^a taken to be anticommuting; we never need to have non-trivial functional dependence on $\bar\psi^a$. That this is no restriction on the class of supermanifolds considered follows from the details of the classification of complex supermanifolds in terms of cohomology on the body [42,43]. The above representation corresponds to the situation in which the cohomology classes are to be Dolbeault.

Similar considerations apply to the second line of (28), which corresponds to deformations of a supersymmetric extension $\mathcal{K} = (K_i\,\mathrm{d}\phi^i, \kappa_a\,\mathrm{d}\psi^a)$ of the form K required to write the action and its derivative

$$\mathcal{B} = (b, \beta) = \left(K_{i,\bar{j}}\,\mathrm{d}\phi^i \wedge \mathrm{d}\phi^{\bar{j}}, \kappa_{a,\bar{j}}\,\mathrm{d}\psi^a \wedge \mathrm{d}\phi^{\bar{j}}\right). \tag{56}$$

In the simplest case, b and $\mathcal{B}$ can be chosen to be global (note that K is not generally globally unless $\mathcal{H}$ is trivial).[20]

One remarkable feature of twistor–string theory is that it gives a partial resolution of the 'googly problem'. As far as non-linear constructions are concerned, this is the problem that while anti-selfdual fields are understood fully non-linearly in terms of deformations of the complex structure of twistor space, it has not been possible to understand fully nonlinear selfdual fields (one can only incorporate them linearly).

Twistor-string theory only resolves the issue of the non-linearities associated to selfdual fields perturbatively, at least in a holomorphic manner. In the case of Yang–Mills, the $\mathcal{N} = 4$ supersymmetry incorporates the selfdual part of the field into the same multiplet as the anti-selfdual part described by the deformation $\mathcal{A}$ of the $\bar{\partial}$-operator $\bar{\partial}_E$ on E. In the case of conformal supergravity, the anti-selfdual part of the field and the selfdual part form two distinct super-multiplets, with twistor data $\mathcal{J}$ and $\mathcal{B}$. These were shown to give rise respectively to the anti-selfdual and selfdual parts of the standard $\mathcal{N} = 4$ conformal supergravity multiplets in linear theory by Berkovits and Witten [4]. The novel part as far as twistor theory is concerned is in the encoding of the selfdual part into $\mathcal{B}$ which at the perturbative level, as discussed earlier, should really be thought of as defining a class $\partial\mathcal{B}$ in $H^1(\mathbb{PT}'^{3|4}, \Omega^2_{\mathrm{cl}})$. Thus the googly problem in this context is to understand how to similarly exponentiate this cohomology group. In the string theory, a vertex operator representing a class in $H^1(\mathbb{PT}', \Omega^2_{\mathrm{cl}})$ has the interpretation of deforming the target space by turning on flux of the NS B-field. The appropriate framework for studying target spaces with B-field flux, and thus twistor spaces of general four-manifolds, would then appear to be the twisted generalized geometry of Hitchin and Gualtieri [60,61], in which holomorphic objects $\{X + \xi, Y + \eta\} \in T\mathcal{M} \oplus T\mathcal{M}^\vee$ are closed with respect to the twisted Courant bracket

$$[X + \xi, Y + \eta]_{\mathrm{TC}} \equiv [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}\mathrm{d}(\iota_X\eta - \iota_Y\xi) + \iota_X\iota_Y H \tag{57}$$

rather than the Lie bracket. It is fascinating that generalized geometry, of interest in compactifying physical string theory, also appears to be an important ingredient in solving the googly problem in twistor theory.

[20] The long exact sequence in cohomology that follows from the short exact sheaf sequence

$$0 \to \mathcal{O}/\mathbb{C} \xrightarrow{\partial} \Omega^{(1,0)} \xrightarrow{\partial} \Omega^{(2,0)}_{\mathrm{cl}} \to 0$$

gives an obstruction in $H^2(\mathcal{O}/\mathbb{C})$ for $\mathcal{H} \in H^1(\Omega^2_{\mathrm{cl}})$ to be written as $\mathcal{H} = \partial b$ for $b \in H^1(\Omega^{(1,0)})$. However, it can be seen that $H^2(\mathcal{O}/\mathbb{C}) = 0$ in the twistor context: this follows from the long exact sequence in cohomology arising from the sheaf sequence

$$0 \to \mathbb{C} \to \mathcal{O} \to \mathcal{O}/\mathbb{C} \to 0$$

together with the vanishing of $H^3(\mathbb{C})$ and $H^2(\mathcal{O})$. The first of these vanishes because the twistor spaces for topologically trivial spacetimes have topology $S^2 \times \mathbb{R}^4$ which has no third cohomology. The second follows for twistor spaces for Stein regions in spacetime by the Penrose transform.

8. Relation to other twistor–string models

We would now like to explain the relation of the heterotic twistor–string constructed above to the twistor–string models of Berkovits [2] and Witten [1].

8.1. The Čech–Dolbeault isomorphism and Berkovits' twistor–string

Berkovits' twistor–string has a first-order worldsheet action and is usually viewed as a theory of open strings with boundary mapped to a real slice of the target space. We will see that this real slice arises as a specific choice of contour in the path integral of a closed string theory, appropriate when the spacetime signature is $(++--)$. In particular, this real slice is not to be thought of as supporting a D brane. In fact, the relation of general twisted $(0,2)$ models to $\beta\gamma$-systems with a first-order action has been explored already in [25] and we need do little more here than apply these ideas to the case when the target space is twistor space.

Consider a $(0,2)$ model with its standard action

$$S=\int_{\Sigma}\left|\mathrm{d}^2z\right|g_{i\bar{j}}\left(\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}+\rho_{\bar{z}}^{i}\nabla_z\bar{\rho}^{\bar{j}}\right)+\bar{\psi}_{a\,z}D_{\bar{z}}\psi^a+F_{i\bar{j}}{}^{a}{}_{b}\bar{\psi}_{a\,z}\psi^a\rho_{\bar{z}}^{i}\bar{\rho}^{\bar{j}},\tag{58}$$

but where the target space is now taken to be a patch $U\subset\mathbb{P}^3$ that is homeomorphic to an open ball in $\mathbb{C}^3$. Because U is contractible, the topological term $\int_\Sigma\phi^*(\omega-\mathrm{i}B)$ necessarily vanishes. Also, U admits a flat metric and since the $\bar{Q}$ cohomology is not sensitive to the choice of metric, we are free to set $g_{i\bar{j}}=\delta_{i\bar{j}}$. Likewise, since $\mathcal{V}\to U$ is necessarily trivial, the background connection A on $\mathcal{V}$ may also be chosen to be flat. Thus the $(0,2)$ model over U reduces to the free theory

$$S=\int_{\Sigma}\left|\mathrm{d}^2z\right|\delta_{i\bar{j}}\left(\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}+\rho_{\bar{z}}^{i}\partial_z\bar{\rho}^{\bar{j}}\right)+\bar{\psi}_{a\,z}\partial_{\bar{z}}\psi^a.\tag{59}$$

Non-trivial vertex operators correspond to elements of the Dolbeault cohomology groups $H^{0,p}(U,\mathcal{S})$ where $\mathcal{S}$ is the sheaf of chiral algebras, but since U is contractible these cohomology groups vanish if $p>0$. Consequently, the only non-trivial vertex operators are holomorphic sections of $\mathcal{S}$ over U, represented in the sigma model by operators which have the form[21]

$$\mathcal{O}\left(\phi^i,\partial_z\phi^i,\partial_z^2\phi^i,\ldots;\partial_z\phi^{\bar{j}},\partial_z^2\phi^{\bar{j}},\ldots;\psi^a,\partial_z\psi^a,\partial_z^2\psi^a,\ldots,\bar{\psi}_{az},\partial_z\bar{\psi}_{az},\ldots\right).$$

These vertex operators are independent of ρ and $\bar{\rho}$, and must depend holomorphically on ϕ so that they involve $\phi^{\bar{j}}$ only through its first and higher derivatives. Therefore we may equally well obtain them from the $\beta\gamma$-system

$$S_{\beta\gamma}=\int_{\Sigma}\left|\mathrm{d}^2z\right|\left(\beta_{i\,z}\partial_{\bar{z}}\gamma^i+\bar{\psi}_{a\,z}\partial_{\bar{z}}\psi^a\right),\tag{60}$$

where $\gamma^i:=\phi^i$ and $\beta_{iz}:=\delta_{i\bar{j}}\partial_z\phi^{\bar{j}}$. Note that the interpretation of (ϕ^i,ψ^a) as holomorphic coordinates on a supermanifold is once again manifest in this $\beta\gamma$ picture.

[21] Recall that the vertex operator must be independent of $\rho_{\bar{z}}^{i}$ and antiholomorphic derivatives of the fields since it must have weight $\bar{h}=0$. Also, the ρ equation of motion may always be used to eliminate dependence on holomorphic derivatives of $\bar{\rho}$.

To recover the higher cohomology groups $H^p(X, \mathcal{S})$ from this $\beta\gamma$ system, we work with a quantum field theoretic implementation of Čech cohomology. Let $\{U_\alpha\}$ be a good[22] cover for X, where α indexes the covering set. On each open set U_α we may construct a free $\beta\gamma$-system as in (60), but to recover the sigma model globally we must ensure that these free field theories glue together compatibly on overlaps $U_\alpha \cap U_\beta$: as explained in e.g. [25,66], this entails that the target space X and bundle $\mathcal{V} \to X$ obeys the same anomaly conditions as found in Section 3.1. If $\mathcal{O}_{\alpha_0\alpha_1\cdots\alpha_p}$ is a vertex operator which is holomorphic in γ when restricted to the p-fold overlap $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$, the Čech cohomology group $H^p(X, \mathcal{S})$ is represented by a collection of vertex operators that obey the cocycle condition $\rho_{[\alpha_0}\mathcal{O}_{\alpha_1\alpha_2\cdots\alpha_{p+1}]} = 0$ on $p+1$-fold overlaps, where ρ_α restricts a vertex operator defined on U_β to the intersection $U_\alpha \cap U_\beta$, and the square brackets denote antisymmetrization. This collection is defined modulo the equivalence relation

$$\mathcal{O}_{\alpha_0\alpha_1\cdots\alpha_p} \sim \mathcal{O}_{\alpha_0\alpha_1\cdots\alpha_p} + \sum_{k=0}^{p} (-1)^k \mathcal{O}_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_p} \tag{61}$$

for coboundaries, where $\mathcal{O}_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_p}$ is holomorphic on the $(p-1)$-fold overlap $U_{\alpha_0} \cap \cdots \cap U_{\alpha_{k-1}} \cap U_{\alpha_{k+1}} \cap \cdots \cap U_{\alpha_p}$ with U_{α_k} omitted.

Rather than working with a covering of the projective twistor space, we could equally well consider a 'gauged $\beta\gamma$ system' of maps $Z : \Sigma \to \mathbb{C}^{4|4}$ with action

$$S = \int_\Sigma Y_I \bar{D} Z^I, \tag{62}$$

where $I = (\alpha, a)$ runs over the four bosonic and four fermionic directions, while the kinetic operator $\bar{D}Z^I = (\bar\partial + A)Z^I$ gauges the $\mathbb{C}^*$ symmetry so as to carry out the quotient $\mathbb{P}^{3|4} = (\mathbb{C}^{4|4} - \{0\})/\mathbb{C}^*$. It is straightforward to see how these approaches are related: integrating out A yields the constraint $Y_I Z^I = 0$ which may be solved on the patch $Z^0 \neq 0$ by setting $Y_0 = -(Y_i Z^i + Y_a Z^a)/Z^0$, where i runs over the three remaining bosonic directions. Substituting this into (62) gives

$$\begin{aligned} S_{\{Z^0\neq 0\}} &= \int_\Sigma Y_i \bar\partial Z^i + Y_a \bar\partial Z^a - \left(Y_i Z^i + Y_a Z^a\right)\left(\frac{\bar\partial Z^0}{Z^0}\right) \\ &= \int_\Sigma \beta_i \bar\partial \gamma^i + \bar\psi_a \bar\partial \psi^a, \end{aligned} \tag{63}$$

where $\gamma^i = Z^i/Z^0$ and $\psi^a = Z^a/Z^0$ are affine coordinates on the patch $Z^0 \neq 0$, whereas $\beta_i = Z^0 Y_i$ and $\bar\psi_a = Z^0 Y_a$. For more general twistor spaces, the non-projective twistor space is not flat and cannot be covered by a single coordinate patch.

In order to promote either (60) or (62) to a string theory, one must again introduce a holomorphic bc system and a worldsheet current algebra to ensure that the total central charge vanishes. The associated BRST operator restricts the interesting vertex operators to those formed from a c ghost contracted with a $\beta\gamma$ vertex operator of weight $h = 0$, just as in Section 6. The path integral now involves only the holomorphic coordinates Z^I and is naturally treated as a contour integral.

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[22] I.e., the covering $\{U_\alpha\}$ must be a Leray cover of X, meaning roughly that nothing new arises on choosing a finer subcover. See e.g. [62,63] for introductions to Čech cohomology, [64,65] for introductions in the context of twistor theory and [25] for a discussion in the context of (0, 2) models and $\beta\gamma$ systems.

Berkovits' model [2] is usually presented as a theory of open strings with the boundary $\partial\Sigma$ of the worldsheet being mapped to the real slice $\mathbb{RP}^{3|4}$ of supertwistor space. His action is

$$S = \mathrm{Re}\left\{\int_{\Sigma} Y_I \bar{D} Z^I + b\partial c\right\} \tag{64}$$

together with a current algebra contributing central charge (28) to both the left- and right-moving sectors. The fields obey the boundary conditions

$$Z^I = \bar{Z}^{\bar{I}}, \qquad Y_I = \bar{Y}_{\bar{I}}, \qquad b = \bar{b}, \qquad c = \bar{c} \tag{65}$$

on $\partial\Sigma$. This action and boundary conditions can be turned into a closed string theory by gluing Σ to its complex conjugate $\bar{\Sigma}$ along the boundary to form a compact Riemann surface Σ_D: the 'double' of Σ. By construction, we have a complex conjugation $\Sigma_D \to \Sigma_D$ which interchanges Σ with $\bar{\Sigma}$ and fixes the boundary $\partial\Sigma$. To go in the reverse direction, start with an action

$$S = \frac{1}{2}\int_{\Sigma_D} Y_I \bar{D} Z^I + b\bar{\partial}c + S_{\mathrm{YM}} \tag{66}$$

on the closed Riemann surface Σ_D, where S_{YM} here is a holomorphic current algebra. Upon restricting the path integral to maps for which $Z^{\bar{I}}(\sigma) = Z^I(\bar{\sigma})$ (i.e., taking an orientifold projection) and decomposing $\Sigma_D = \Sigma \cup \bar{\Sigma}$, this action reduces to Berkovits' model (64)–(65). Thus from our perspective, viewing the Berkovits model as an open string is really a way of 'hardwiring' in a choice of contour. Starting from a closed string picture enables one to choose other contours relevant for other spacetime signatures, at least in principle. Nonetheless, it is remarkable that the original Berkovits model automatically takes care of this issue and provides a practical way of evaluating scattering amplitudes on a real spacetime slice, even if this comes at the cost of the flexibility one expects in a contour picture.

8.2. *Witten's twistor–string: D5–D5, D5–D1 and D1–D1 strings*

Witten's model consists of an open string topological B-model coupled to D1 branes in supertwistor space $\mathbb{PT}_s$, a region in $\mathbb{P}^{3|4}$. The D1 branes wrap holomorphic curves $C \subset \mathbb{PT}_s$, and the D1–D5 open strings are modelled by a pair of fermionic worldsheet spinors

$$\alpha \in \Gamma(C, S_- \otimes E), \qquad \beta \in \Gamma\big(C, S_- \otimes E^\vee\big) \tag{67}$$

with action $\int_C \beta\bar{\partial}_E\alpha$ on the holomorphic curve C. Performing the $\alpha\beta$ path integral yields the determinant $\det\bar{\partial}_{E\otimes S_-}$ which depends on the complex structure of the bundle. In the original proposal [1,9], one seeks to obtain a generating functional for Yang–Mills scattering amplitudes by integrating this determinant over a contour in the moduli space of curves. Expanding $\det\bar{\partial}_{E\otimes S_-}$ in powers of a perturbation of the background connection on E leads to multi-trace terms which were the first hint of a coupling to conformal supergravity [1]. Welcome or not, if conformal supergravity is present one would expect to be able to describe scattering processes involving external conformal supergravity states, so it is clear that the above proposal cannot be the whole story.

What is lacking is a theory of the D1–D1 strings on the worldvolume of the D-instanton. This may be obtained by dimensional reduction from of the Abelian holomorphic Chern–Simons theory

$$S_{\mathrm{D5}} = \int_{\mathrm{D5}} \Omega^{3|4} \wedge \mathcal{A} \wedge \bar{\partial}\mathcal{A} \tag{68}$$

on the worldvolume of a single D5 brane, as in [67,68]. To dimensionally reduce this action, we take the D5 worldvolume to be the total space of the normal bundle $N_{C|\mathbb{PT}_s}$ to a fixed curve C, so that the tangent space to the D5 brane decomposes as $T_{\mathrm{D5}} = T_C \oplus N_{C|\mathbb{PT}}$. Similarly, the (0, 1)-form $\mathcal{A}$ decomposes as

$$\mathcal{A} \in \Gamma\big(\mathrm{D5}, \bar{T}_C^\vee\big) \oplus \Gamma\big(\mathrm{D5}, \bar{N}_{C|\mathbb{PT}_s}^\vee\big) \tag{69}$$

and only the components in $\Gamma(\mathrm{D5}, \bar{N}_{C|\mathbb{PT}_s}^\vee)$ survive in (68) under the assumption that $\mathcal{A}$ is constant along the normal bundle fibres. Integrating out these fibre directions then gives the action

$$S_{\mathrm{D1}} = \mathrm{vol}(N) \int_C \Phi_1 \bar{\partial} \Phi_0 \tag{70}$$

on the worldvolume of the D1 brane, where $\Phi_0 \in \Gamma(C, N)$ and $\Phi_1 \in \Gamma(C, K_C \otimes N^\vee)$.

Putting this together, integrating out the fluctuations of the D1–D1 and D1–D5 strings gives a net contribution

$$\frac{\det \bar{\partial}_{E\otimes S_-}(C)}{\det' \bar{\partial}_{N_{C|\mathbb{PT}_s}}(C)} \tag{71}$$

to the path integral for each curve C that the D-instantons wrap. We now compare this to the ratio (44) obtained by integrating out the non-zero modes of the heterotic string. Using the facts that $N_{\Sigma|\mathbb{P}^3} = T_{\mathbb{P}^3}/T_\Sigma$ and $T_{\mathbb{PT}_s} = T_{\mathbb{P}^3} \oplus \Pi\mathcal{V}$ shows that (44) and (71) coincide, at least when the heterotic map $\phi : \Sigma \to \mathbb{P}^3$ is an embedding. The full contribution of a degree d map in the heterotic string also involves the string action evaluated on a worldsheet instanton, and is

$$\int_{\mathcal{M}_d} \mathrm{d}\mu \exp\left(-\frac{A(C)}{2\pi} + \mathrm{i}\int_C B\right) \frac{\det \bar{\partial}_{E(-1)}}{\det' \bar{\partial}_{N_{C|\mathbb{PT}'_s}}}, \tag{72}$$

where $A(C)$ is the area of the curve as measured by the restriction of the Kähler form to C (one may rewrite this exponential in terms of $b = B + \mathrm{i}\omega$) and we have also integrated over the space of curves $\mathcal{M}_d$ in supertwistor space using the natural measure $\mathrm{d}\mu$ as described earlier. Expression of this type of familiar from 'physical gauge' calculations of corrections to the spacetime superpotential in heterotic compactifications due to worldsheet instantons [69–72]. Thus, the B-model and heterotic calculations agree so long as the D1 branes in the B-model couple electrically to the b-field. Precisely this coupling was assumed in [4] by an argument based on the Green–Schwarz mechanism, and has also arisen in the context of a conjectured S-duality in topological strings on a standard Calabi–Yau [73].

To summarize, we have seen that the D1–D1 strings of the B-model describe perturbative deformations of holomorphic curves in supertwistor space, and are thereby associated with (the anti-selfdual) half of the conformal supergravity multiplet in spacetime. The D1 branes themselves involve a coupling to the b-field which provides the selfdual half. The entire D1/D5 system, including all the strings stretched between them, is succinctly captured by the heterotic model. It would be fascinating to investigate this duality further in the context of standard topological strings.

9. String field theory and twistor actions

In this section we make contact with the twistor actions of [5–7]. The basic idea is that, with some reasonable assumptions, the complete string field theory can be shown to be equivalent to certain actions on twistor space, which can in turn be shown to reduce to versions of conformal supergravity coupled to Yang–Mills on spacetime. Modulo the assumptions that we have to make, this gives a proof of equivalence between our heterotic twistor–string and a particular version of $\mathcal{N}=4$ conformal supergravity coupled to super-Yang–Mills.

In order to simplify notation in this section we will work with supermanifolds. Thus supertwistor space $\mathbb{PT}_s$ will in the flat case be a region in $\mathbb{P}^{3|4}$. For our purposes $\mathbb{PT}_s$ is the total space of $\mathcal{V}$ with parity reversed fibres. In the context of string field theory, we must work off-shell which means that, at least initially, we consider almost complex structures $\mathcal{J}$ on supertwistor space $\mathbb{PT}_s$ that are not necessarily integrable. In the context of the earlier discussions of vertex operators, $\mathcal{J}=(J,j)$ and infinitesimal deformations of $\mathcal{J}$ correspond to the top family of vertex operators in (28). Similarly, the lower family corresponds to a variation of the complexified Kähler structure $\mathcal{B}=(\omega+\mathrm{i}B,\beta)$ on $\mathbb{PT}_s$. We first seek to formulate the theory when the geometric background is 'off-shell'. That is, the almost complex structure $\mathcal{J}$ is not necessarily taken to be integrable, while $\mathcal{B}$ and $\mathcal{A}$ are taken to be arbitrary (so that the $\bar\partial$-operator on E defined by $\mathcal{A}$ is not integrable). We will, however, take the almost complex structure (J,j) to define a Calabi–Yau almost complex structure on the manifold $\mathbb{PT}_s$ in which the vector fields $\partial/\partial\bar\psi^a$ are antiholomorphic. The Calabi–Yau condition in this non-integrable context is taken to mean that there is a canonical isomorphism between $\Omega^{3,0}$ and $(\det\mathcal{V})^\vee$ and this defines a $(3|4,0|0)$ integral superform Ω. In this almost complex situation, the form Ω cannot be closed, but $\mathrm{d}\Omega$ will have bosonic type $(2|4,2|0)\oplus(3|3,2|0)$ with no $(3|4,1|0)$ term.

We consider first the contribution of a single degree zero instanton. This reduces to an integral over constant maps to supertwistor space and zero-modes of the worldsheet fields $(c,\bar\rho)$. In principal, one should construct the contribution to the string field theory action by formulating the sigma model for an off-shell $(\mathcal{J},\mathcal{B},\mathcal{A})$ and integrating out the zero modes of c and ρ. An easier route, as followed in [4], is to calculate the cubic amplitudes as integrals of cubic expressions in $(\mathcal{J},\mathcal{B},\mathcal{A})$ and their derivatives, and then guess the quadratic terms required to make these contributions geometrically natural. This process led Berkovits and Witten to the following top degree form on supertwistor space $\mathbb{PT}'_s=\mathbb{CP}^{3|4}-\mathbb{P}^1$

$$\mathcal{L}_0(\mathcal{J},\mathcal{B},\mathcal{A})=\big(\mathrm{CS}(\mathcal{A})+N(\mathcal{J})\lrcorner\mathcal{B}+\mathrm{CS}(\partial\mathcal{J})\big)\wedge\Omega, \tag{73}$$

where $\mathrm{CS}(\mathcal{A})=\mathrm{tr}(\frac{1}{2}\mathcal{A}\wedge\mathrm{d}\mathcal{A}+\frac{1}{3}\mathcal{A}^3)$ is the Chern–Simons 3-form constructed from $\mathcal{A}$. $N(\mathcal{J})$ is the Nijenhuis tensor of the almost complex structure $\mathcal{J}$ on the supermanifold. It is a section of $T^{(1,0)}\otimes\Omega^{(0,2)}$ and may be thought of as $(\bar\partial)^2$. Then $N\lrcorner\mathcal{B}$ is the $(0,3)$-form obtained by contracting the vector field part of N into $\mathcal{B}$ and skewing over the antiholomorphic indices. Note that $(N\lrcorner\mathcal{B})\wedge\Omega$ may also be represented as $\mathcal{B}\wedge\mathrm{d}\Omega$. Finally, $\mathrm{CS}(\partial\mathcal{J})$ is the Chern–Simons $(0,3)$-form associated to the $\bar\partial$-operator naturally induced on the holomorphic tangent bundle of $\mathbb{PT}'_s$ by $\mathcal{J}$. The contribution of a single degree-zero instanton to the string field theory action is then $S_0[\mathcal{J},\mathcal{B},\mathcal{A}]=\int_{\mathbb{PT}_s}\mathcal{L}_0(\mathcal{J},\mathcal{B},\mathcal{A})$. Although (73) was originally arrived at via the Berkovits and Witten string theories, we have seen that the formulae for amplitudes is the same in our heterotic theory, so the procedure will lead to the same expression for our theory also.

With a rescaling b to fit in with earlier conventions, the contribution of the degree 1 instantons is given in Eq. (72) as

$$S_1[\mathcal{J},\mathcal{B},\mathcal{A}]=\int \mathrm{d}^{4|8}x\exp\biggl(\int_C b\biggr)\frac{\det\bar\partial_{E(-1)}}{\det'\bar\partial_{N_{C|\mathbb{PT}'^{3|4}}}}.$$

For worldsheet instantons of degree greater than or equal to one, as discussed in earlier sections we are reduced to a half-dimensional contour integral inside the moduli space of curves of degree d. Gukov, Motl and Neitzke [10] have argued that the contour can essentially be continued through the boundaries of the moduli spaces of Riemann surfaces of different degrees of connectedness, so long as propagators associated to the above degree zero action are allowed between points on the different components of the curve (these can be thought of as being associated to degenerations of a degree d curve with vanishingly thin necks connecting points on the different components). The contact terms between different components are therefore taken care of by the degree zero action and so the contribution of a degree-d instanton consisting of d degree 1 components is simply the product of d copies of the degree 1 contribution.

To see this we note that if $C=\bigcup_{i=1}^d \mathbb{P}^1_{x_i}$, the integrals over C behave additively, $\int_C=\sum_i\int_{\mathbb{P}^1_{x_i}}$ and so the exponentials behave multiplicatively; similarly the determinants behave multiplicatively. Since the d copies of $\mathbb{P}^1$ are indistinguishable, the degree d integral becomes

$$\frac{1}{d!}\int\prod_{i=1}^d \mathrm{d}^{4|8}x_i\,\mathcal{L}_1(x_i)=\frac{1}{d!}\biggl(\int \mathrm{d}^{4|8}x\,\mathcal{L}_1(x)\biggr)^d=\frac{(S_1)^d}{d!}. \tag{74}$$

The total contribution must also be summed over the number k of degree zero components, as well as over d. These contributions should be divided by the number $k!$ of permutations of the indistinguishable degree zero components. Thus the overall contribution of degree d instantons can be written as

$$\sum_d\biggl\{\sum_k\frac{1}{k!}\biggl(\int\mathcal{L}_0\biggr)^k\frac{(S_1)^d}{d!}\biggr\}=\exp(S_0+S_1). \tag{75}$$

In string field theory one considers disconnected string worldsheets, so the above argument shows that it is natural to take S_0+S_1 to be the string field theory action. These actions are also actions on twistor space, with S_0 being local, but S_1 non-local.

Parts of the action S_0+S_1 have been studied elsewhere. The truncation to spin one and spin two fields was studied in [5] and shown to provide twistor space actions that give rise to standard Yang–Mills theory and conformal gravity on spacetime. (In that analysis, the determinant factors in (74) were not incorporated. Presumably they do not change the truncated theory.) The fully supersymmetric case for Yang–Mills theory was studied in [6] (see also [74]) where it was shown that pure $\mathcal{N}=4$ super-Yang–Mills theory corresponds to the twistor action

$$\int_{\mathbb{PT}'_s}\Omega\wedge\mathrm{CS}(\mathcal{A})+\int \mathrm{d}^{4|8}x\log\det\bar\partial_{E(-1)}. \tag{76}$$

The non-local part of the action here involves $\log\det\bar\partial_{E(-1)}$, rather than $\det\bar\partial_{E(-1)}$ which would be the truncation of the above, but leads to multitrace terms in the action. We do not know how to obtain such a term from string theory. We have not yet followed through the full details of the Penrose transform (along the lines of [5,6]) to find the spacetime action that is equivalent to S_0+S_1 above and thereby check the conjectures of Berkovits and Witten [4].

10. Discussion

To date, twistor–string theory has mainly been used indirectly as a source of inspiration for calculating gauge and gravitational scattering amplitudes in spacetime [11,12,15,16]. However, we find it difficult to believe that these structures in gauge and gravity theories are simply coincidental, and would like to argue that their existence gives strong new support to Penrose's original twistor programme [75]. This programme seeks to reformulate all of fundamental physics in terms of complex analytic objects on twistor space, with the intention that twistor space be in some way the primary arena for physics, in which quantum gravity might make the most sense. The remarkable reformulation of anti-selfdual gravitational [59] and Yang–Mills [76] fields in terms of deformations of the complex structures of twistor space itself or of a bundle over twistor space provided impressive early successes which motivated this programme. As we discussed in Section 7, these twistor–string ideas have given new insight into the googly problem, as well as providing a new avenue towards incorporating quantum field theoretic ideas into the twistor programme.

Clearly, more work is required to discover what other twistor–string theories can be constructed. In particular, one would like to have twistor–string theories that give rise to Poincaré supergravities, or to pure super-Yang–Mills, or that incorporate other representations of the gauge and Lorentz groups. Some steps have been made in this direction [20,77]. It is clear from the calculations of Section 6 that enforcing modular invariance will play a key role in selecting the gauge group, and we would like to investigate this further. Finally, we saw that the heterotic string path integral is naturally treated as a contour integral. Such a contour integral interpretation is required to correctly derive the results of Roiban, Spradlin and Volovich [8] for scattering amplitudes in the 'connected prescription'. Witten has proposed that the equivalence between the connected and disconnected prescriptions might be understood in terms of a residue theorem [31] for a twisted heterotic string. We hope that the work in this paper will provide further tools for studying these questions.

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