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On the Oscillation of Solutions and Existence of Positive Solutions of Neutral Difference Equations

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We obtain sufficient conditions for the oscillation of all solutions and existence of positive solutions of the neutral difference equation

$$\Delta(x_n + cx_{n-m}) + p_n x_{n-k} = 0, \qquad n = 0, 1, 2, \ldots,$$

where c and p_n are real numbers, m and k are integers, and p_n, m and k are nonnegative.

INTRODUCTION

In the past 10 years the oscillation and nonoscillation of solutions of difference equations have been extensively investigated [3, 5–7]. It turns out that many (but not all, see [6, 7]) of the substantial criteria for differential equations have discrete analogues. Further, criteria have also been obtained for the oscillatory and nonoscillatory behavior of discrete analogues of delay differential equations [3, 5]. Especially, the oscillations of all solutions of the neutral difference equation

$$\Delta(y_n + py_{n-k}) + qy_{n-l} = 0, \qquad n = 0, 1, 2, \ldots \tag{1.1}$$

have been investigated in [5], where Δ denotes the forward difference operator $\Delta y_n = y_{n+1} - y_n$.

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A nontrivial solution $\{y_n\}$ of Eq. (1.1) is said to be oscillatory if for every $N>0$ there exists an $n \geqslant N$ such that $y_n y_{n+1} \leqslant 0$. Otherwise it is nonoscillatory.

In this paper we consider the first order neutral difference equation of the form

$$\Delta(x_n + cx_{n-m}) + p_n x_{n-k} = 0, \qquad n = 0, 1, 2, ..., \tag{1.2}$$

and the forced equation of the form

$$\Delta(x_n + cx_{n-m}) + p_n x_{n-k} = F_n, \qquad n = 0, 1, 2, ... \tag{1.3}$$

Equation (1.2) has also been considered in the numerical analysis of functional differential equations (see [1]).

Let $M = \max\{m, k\}$, where m and k are nonnegative integers. Then by a solution of Eq. (1.2) we mean a sequence $\{x_n\}$ which is defined for $n \geqslant -M$ and which satisfies Eq. (1.2) for $n = 0, 1, 2, ...$ Clearly, if

$$x_n = A_n \qquad \text{for} \quad n = -M, ..., -1, 0 \tag{1.4}$$

are given, then Eq. (1.2) has a unique solution satisfying the initial conditions (1.4). We assume throughout that p_n cannot be eventually identically zero.

Our purpose in this paper is to obtain sufficient conditions for oscillation an nonoscillation of Eqs. (1.2) and (1.3). Some of the sufficient conditions are sharp, and most of the results in [5] are special cases of our results.

It is obvious that the behavior of the solutions of Eq. (1.2) depends on the parameter c. We establish results for Eq. (1.2) in Sections 2, 3, 4, and 5 according to the values of c. In Section 6 we present sufficient conditions for the oscillations of solutions of Eq. (1.3).

The following lemmas will be needed for the study of Eq. (1.2).

LEMMA 1.1 [3]. *Assume that*

$$\liminf_{n \to \infty} p_n = \alpha > 0$$

and

$$\liminf_{n \to \infty} p_n > 1 - \alpha.$$

Then

(i) $x_{n+1} - x_n + p_n x_{n-m} \leqslant 0$ *has no eventually positive solution;*

(ii) $x_{n+1} - x_n + p_n x_{n-m} \geqslant 0$ *has no eventually negative solution.*

LEMMA 1.2 [3]. *Assume that*

$$\liminf_{n \to \infty} p_n > \frac{m^m}{(m+1)^{m+1}}.$$

Then the conclusion of Lemma 1.1 *holds.*

LEMMA 1.3 [3]. *Assume that* $p_n \geqslant 0$ *and*

$$\limsup_{n \to \infty} \sum_{k=n-m}^{n} p_k > 1.$$

Then the conclusion of Lemma 1.1 *holds.*

2. CASE WHEN $c = -1$.

LEMMA 2.1. *Assume that* $c = -1$ *and* $p_n \geqslant 0$ *for* $n = 1, 2, \ldots$. *Let* $\{x_n\}$ *be an eventually positive solution of* (1.2), *setting*

$$z_n = x_n + c x_{n-m}; \tag{2.1}$$

then $z_n > 0$ *and* $\Delta z_n \leqslant 0$ *eventually.*

Proof. From (1.2) we have

$$\Delta z_n = -p_n x_{n-k} \leqslant 0$$

eventually since $p_n \not\equiv 0$, so z_n cannot be eventually identically zero. It follows that $\{z_n\}$ is eventually positive or eventually negative.

If $z_n < 0$ eventually, then

$$z_n \leqslant z_N < 0 \qquad \text{for} \quad n \geqslant N.$$

Hence

$$x_{N+mn} \leqslant z_N + x_{N+(n-1)m} \leqslant \cdots \leqslant n z_N + x_N.$$

By letting $n \to \infty$, we note that x_{N+nm} will be negative which is a contradiction with $x_n > 0$. The proof is complete.

THEOREM 2.1. *Assume that*

(i) $c = -1$

(ii) $p_n \geqslant 0$ *for* $n = 1, 2, \ldots$ *and* $\sum_{n=N}^{\infty} p_n = \infty$, *where* N *is a positive integer.*

Then every solution of (1.2) *is oscillatory.*

Proof. Suppose the contrary. Without loss of generality let $\{x_n\}$ be an eventually positive solution of (1.2). By Lemma 2.1 $z_n > 0$ and $\Delta z_n \leqslant 0$ eventually, which implies that $\lim_{n\to\infty} z_n = \alpha \geqslant 0$ exists.

Summing (1.2) from N to n, we have

$$z_{n+1} - z_N + \sum_{i=N}^{n} p_i x_{i-k} = 0. \tag{2.2}$$

Letting $n \to \infty$, we get

$$z_N \geqslant \sum_{i=N}^{\infty} p_i x_{i-k}. \tag{2.3}$$

Setting $\min_{N \leqslant i \leqslant N+m} x_{i-k} = s > 0$, we know that

$$z_n = x_n - x_{n-m} > 0 \qquad \text{for} \quad n \geqslant N.$$

Hence $x_n \geqslant s$ for $n \geqslant N$. From (2.2) we have

$$\infty > z_{N+k} \geqslant \sum_{i=N+k}^{\infty} p_i x_{i-k} \geqslant s \sum_{i=N+k}^{\infty} p_i,$$

which contradicts condition (ii). The proof is complete.

Example 2.1. Consider

$$\Delta(x_n - x_{n-m}) + p_n x_{n-m'} = 0, \qquad n = 0, 1, 2, ..., \tag{2.4}$$

where $m > 0$, $m' > 0$, and

$$p_n = \frac{m}{(n+1)(n-m+1)} \left(\sum_{i=1}^{n-m'} \frac{1}{i} \right)^{-1}. \tag{2.5}$$

It is obvious that $\sum_{i=N}^{\infty} p_i < \infty$ for $N \geqslant m$. Equation (2.4) does not satisfy assumption (ii) in Theorem 2.1. In fact, (2.4) has a nonoscillatory solution $x_n = \sum_{i=1}^{n} 1/i$, $n = 1, 2, ...$.

Example 2.2. Consider

$$\Delta(x_n - x_{n-m}) + 4x_{n-k} = 0, \qquad n = 0, 1, 2, ..., \tag{2.6}$$

where m is odd, k is even, and k, m are positive integers. Equation (2.6) satisfies the assumptions of Theorem 2.1, therefore every solution of (2.6) is oscillatory. In fact $x_n = (-1)^{n+1}$, $n = 1, 2, ...$ is a solution of (2.6).

Remark 2.1. Theorem 2.1 includes Theorem 1 (i) in [5] as a special case.

3. CASE WHEN $-1<c<0$

LEMMA 3.1. *Assume that* $-1<c<0$ *and* $p_n \geqslant 0$. *Let* $\{x_n\}$ *be an eventually positive solution of* (1.2). *Then* $z_n>0$ *and* $\Delta z_n<0$ *eventually, where* z_n *is defined by* (2.1).

The proof of this lemma is similar to that of Lemma 2.1 and hence is omitted.

THEOREM 3.1. *Assume that*

(i) $-1<c<0$, $k>m$;

(ii) $p_n \geqslant p_{n-m}$ *for all large* n;

(iii) $(1/(1+c)) \liminf_{n\to\infty} p_n > k^k/(k+1)^{k+1}$.

Then every solution of (1.2) *oscillates.*

Proof. If not, we assume that $\{x_n\}$ is an eventually positive solution. Set

$$z_n = x_n + cx_{n-m}, \qquad w_n = z_n + cz_{n-m}.$$

By Lemma 3.1, we know that $z_n>0$, $\Delta z_n<0$ and $w_n>0$, $\Delta w_n<0$. In fact,

$$\begin{aligned} \Delta w_n &= \Delta z_n + c\,\Delta z_{n-m} \\ &= -p_n x_{n-k} - cp_{n-m}x_{n-m-k} \\ &\leqslant -p_n(x_{n-k}+cx_{n-m-k}) \\ &\leqslant -p_n z_{n-k} \\ &\leqslant 0. \end{aligned} \tag{3.1}$$

Since $\lim_{n\to\infty} z_n = l \geqslant 0$ exists, we get

$$\lim_{n\to\infty} w_n = l + cl = (1+c)l \geqslant 0.$$

Therefore $w_n>0$ for all large n. On the other hand,

$$w_n = z_n + cz_{n-m} \leqslant (1+c)z_n$$

or

$$z_n \geqslant \frac{w_n}{1+c}. \tag{3.2}$$

From (3.1) and (3.2), we have

$$\Delta w_n \leqslant -p_n z_{n-k} \leqslant -\frac{p_n}{1+c} w_{n-k}. \tag{3.3}$$

By Lemma 1.2 and under condition (iii) Eq. (3.3) has no eventually positive solution, a contradiction.

Remark 3.1. Theorem 3.1 includes Theorem 2 (i) and (iii) in [5].

EXAMPLE 3.1. Consider

$$\Delta\left(x_n - \frac{1}{2}x_{n-1}\right) + \frac{(n-2)(3n-1)}{2n(n+1)(n-1)}x_{n-2} = 0, \qquad n = 0, 1, 2, ..., \tag{3.4}$$

where $p_n = (n-2)(3n-1)/2n(n+1)(n-1)$. It is easy to see that $p_n > p_{n-1}$ for all large n and that conditions (i) and (iii) are satisfied. Therefore by Theorem 3.1 every solution of (3.4) oscillates. In fact, $x_n = (-1)^n 1/n$ is such a solution.

THEOREM 3.2. *Assume that*

(i) $-1 < c < 0$;

(ii) $\liminf_{n\to\infty} p_n = \Delta > 0$ *and* $\limsup_{n\to\infty} p_n > 1 - \Delta$ *or*

(iii) $\liminf_{n\to\infty} p_n > k^k/(k+1)^{k+1}$.

Then every solution of Eq. (1.2) *oscillates.*

Proof. Suppose the contrary. Without loss of generality let $\{x_n\}$ be an eventually positive solution of (1.2). By Lemma 3.1 $z_n > 0$, $\Delta z_n < 0$ eventually. We note that $0 < z_n < x_n$ for $n = 1, 2, ...$ So Eq. (1.2) becomes

$$\Delta z_n + p_n z_{n+k} \leqslant 0. \tag{3.5}$$

From Lemmas 1.1 and 1.2, condition (ii) or (iii) implies that (3.5) has no eventually positive solution, a contradiction.

THEOREM 3.3. *Assume that*

(i) $-1 < c < 0$, $m > k$;

(ii) $p_n \geqslant 0$ *and* $p_n \geqslant p_{n-m}$ *for all large* n.

Set

$$\liminf_{n\to\infty} p_n > F(\bar{l}) \equiv \frac{(\bar{l}-1)(1+c\bar{l}^m)}{\bar{l}^{k+1}}, \tag{3.6}$$

where $\bar{l} \in (1, (-c)^{1/m})$ *is a unique real root of the equation*

$$1 + cl^m = (l-1)(k + kcl^m - cml^m). \tag{3.7}$$

Then every solution of (1.2) *oscillates.*

Proof. Suppose the contrary, and let $\{x_n\}$ be an eventually positive solution of (1.2). By Lemma 3.1. $z_n > 0$ $\Delta z_n < 0$ eventually. From (3.1), we have

$$\Delta(z_n + cz_{n-m}) + p_n z_{n-k} \leqslant 0. \tag{3.8}$$

Set $r_n = z_{n-1}/z_n$; then $r_n \geqslant 1$ for all large n. Dividing (3.8) by z_n, we get

$$\begin{aligned}\frac{1}{r_{n+1}} &\leqslant 1 + c\left(\frac{z_{n-m}}{z_n} - \frac{z_{n-m+1}}{z_n}\right) - p_n \frac{z_{n-k}}{z_n} \\ &\leqslant 1 + c(r_{n-m+1} \cdots r_{n-1} r_n - r_{n-m+2} \cdots r_{n-1} r_n) \\ &\quad - p_n r_{n-k+1} \cdots r_n.\end{aligned} \tag{3.9}$$

From (3.9), r_n is bounded above. We set

$$\liminf_{n \to \infty} r_n = l \geqslant 1.$$

Then l is finite. From (3.9), we have

$$\limsup_{n \to \infty} \frac{1}{r_{n+1}} = \frac{1}{l} \leqslant 1 + cl^{m-1}(l-1) - l^k \liminf_{n \to \infty} p_n.$$

Hence

$$\liminf_{n \to \infty} p_n \leqslant \frac{(1 + cl^m)(l-1)}{l^{k+1}}. \tag{3.10}$$

Set

$$F(l) = \frac{(1 + cl^m)(l-1)}{l^{k+1}}. \tag{3.11}$$

From $F'(l) = 0$, we get the equation

$$1 + cl^m + (l-1)[cml^m - k(1 + cl^m)] = 0. \tag{$*$}$$

Equation (3.12) has a unique real root $\bar{l}$ on $[1, \infty]$. It is easy to see that $F(\bar{l})$ is a maximum value of $F(l)$ on $[1, \infty)$. Thus we have

$$\liminf_{n \to \infty} p_n \leqslant F(\bar{l}),$$

which contradicts condition (3.6). The proof is complete.

Remark 3.2. Theorem 3.3 is a discrete analogue of a result in [9].

THEOREM 3.4. *Assume that*

(i) $-1 < c < 0$.

(ii) $p_n \geqslant 0$, $n = 1, 2, \ldots, \sum_{n=1}^{\infty} p_i = \infty$ *and for any subsequence* $\{n_i\} \subseteq \{n\}$, $\sum_{i=1}^{\infty} p_{n_i} = \infty$. *Then every nonoscillatory solution of* (1.2) *tends to zero as* $n \to \infty$.

Proof. If not, let $\{x_n\}$ be an eventually positive solution of (1.2). By Lemma 3.1, $z_n > 0$ and $\Delta z_n < 0$ eventually. Then $\lim_{n \to \infty} z_n = l \geqslant 0$ exists.

Summing (1.2) from N to n, we have

$$z_{n+1} - z_N = -\sum_{i=N}^{n} p_i x_{i-k},$$

which implies that

$$\sum_{i=N}^{\infty} p_i x_{i-k} < \infty.$$

On the other hand, if $\limsup_{n \to \infty} x_n > 0$, then there exists a subsequence $\{n_i\} \subseteq \{n\}$ such that $\lim_{i \to \infty} x_{n_i} = s > 0$.

Then we have

$$\infty > \sum_{i=N_1}^{\infty} p_{n_i+k} x_{n_i} \geqslant \frac{s}{2} \sum_{i=N_1}^{\infty} p_{n_i+k} \tag{3.12}$$

where N_1 is a sufficiently large number such that $x_{n_i} \geqslant s/2$ for $i \geqslant N_1$. The inequality (3.12) contradicts condition (ii). The proof is complete.

EXAMPLE 3.2. Consider

$$\Delta\left(x_n - \frac{1}{2e} x_{n-1}\right) + \frac{1}{2e^2}\left(1 - \frac{1}{e}\right) x_{n-2} = 0. \tag{3.13}$$

By Theorem 3.4, every solution of (3.13) tends to zero as $n \to \infty$. In fact, $\{x_n\}$ with $x_n = e^{-n}$ is such a solution.

THEOREM 3.5. *Assume that*

(i) $-1 \leqslant c \leqslant 0$ *and* $p_n \geqslant p > 0$

(ii)

$$\inf_{\substack{n \geqslant N \\ \mu > 1}} \left\{\frac{1}{\mu} + p_n \mu^k - c \frac{p_n}{p_{n-m}} \mu^{m-1}(\mu - 1)\right\} > 1. \tag{3.14}$$

Then every solution of Eq. (1.2) *oscillates.*

Proof. If not, assume that there is a solution $\{x_n\}$ of (1.2) with $x_n > 0$ for $n \geqslant N-m$, where N is a sufficiently large integer. Define

$$z_n = x_n + cx_{n-m}.$$

By Lemma 3.1, we have $z_n > 0$, $\Delta z_n < 0$ for $n \geqslant N$. Define

$$w_n = -\frac{\Delta z_n}{z_n} > 0, \qquad n \geqslant N. \tag{3.15}$$

It is easy to see that $w_n < 1$ for $n \geqslant N$. From (1.2) we have

$$\Delta z_n + p_n z_{n-k} - c\frac{p_n}{p_{n-m}} p_{n-m} x_{n-k-m} = 0. \tag{3.16}$$

Dividing (3.16) by z_n we have

$$-\frac{\Delta z_n}{z_n} - p_n \frac{z_{n-k}}{z_n} + c\frac{p_n}{p_{n-m}}\left(-\frac{\Delta z_{n-m}}{z_n}\right) = 0. \tag{3.17}$$

From (3.15), we have $z_{n+1}/z_n = 1 - w_n$ and $z_n/z_{n-k} = \prod_{i=n-k}^{n-1}(1-w_i)$. Thus (3.17) becomes

$$w_n - p_n \prod_{i=n-k}^{n-1} (1-w_i)^{-1} + c\frac{p_n}{p_{n-m}} w_{n-m} \prod_{i=n-m}^{n-1} (1-w_i)^{-1} = 0. \tag{3.18}$$

Set

$$v_i = \frac{1}{1-w_i}, \qquad i = N, \quad N+1, ..., \tag{3.19}$$

or

$$w_i = \frac{v_i - 1}{v_i}, \qquad i = N, \quad N+1, ... \tag{3.20}$$

Hence (3.18) becomes

$$\frac{v_n - 1}{v_n} = p_n \prod_{i=n-k}^{n-1} v_i - c\frac{p_n}{p_{n-m}} \frac{v_{n-m}-1}{v_{n-m}} \prod_{i=n-m}^{n} v_i, \tag{3.21}$$

or

$$\begin{aligned} v_n &= 1 + p_n \prod_{i=n-k}^{n} v_i - c\frac{p_n}{p_{n-m}} (1 - v_{n-m}^{-1}) \textstyle\prod_{i=n-m}^{n} v_i \\ &= 1 + p_n \prod_{i=n-k}^{n} v_i - c\frac{p_n}{p_{n-m}} \prod_{i=n-m+1}^{n} v_i (v_{n-m} - 1). \end{aligned} \tag{3.22}$$

Define sequence $\{\lambda_n^{(l)}\}$, $n=N, N+1, ..., l=1, 2, ...$ as follows:

$$\{\lambda_n^{(1)}\}=\{1\}, \qquad n=N, N+1, ...,$$

$$\{\lambda_n^{(2)}\}=\begin{cases} 1+p_n \prod\limits_{i=n-k}^{n} \lambda_i^{(1)}-c\dfrac{p_n}{p_{n-m}} \prod\limits_{i=n-m+1}^{n} \lambda_i^{(1)}(\lambda_{n-m}^{(1)}-1), \\ \quad n\geqslant N+M \\ \lambda_{N+M}^{(2)}, \qquad N\leqslant n\leqslant N+M \end{cases}$$

Here $M=\max(m, k)$. In general,

$$\{\lambda_n^{l+1}\}=\begin{cases} 1+p_n \prod\limits_{i=n-k}^{n} \lambda_i^{(l)}-c\dfrac{p_n}{p_{n-m}} \prod\limits_{i=n-m+1}^{n} \lambda_i^{(l)}(\lambda_{n-m}^{(l)}-1), \\ \quad n\geqslant N+M \\ \lambda_{N+M}^{(l)}, \qquad N\leqslant n\leqslant N+M. \end{cases}$$

Define a sequence $\{\mu^l\}$ as follows:

$$\mu^1=1$$

$$\mu^{l+1}=\inf_{n\geqslant N}\left\{1+p_n(\mu^l)^{k+1}-c\frac{p_n}{p_{n-m}}(\mu^l)^m(\mu^l-1)\right\}$$

$$l=1, 2, ...$$

It is easy to see that

$$\mu^2=\inf_{n\geqslant N}\{1+p_n\}>\mu^1=1.$$

In general, by condition (3.14), we have

$$\begin{aligned} 1&<\inf_{\substack{n\geqslant N\\ \mu>1}}\left\{\frac{1}{\mu}+p_n\mu^k-c\frac{p_n}{p_{n-m}}\mu^{m-1}(\mu-1)\right\} \\ &\leqslant \inf_{n\geqslant N}\left\{\frac{1}{\mu^l}+p_n(\mu^l)^k-c\frac{p_n}{p_{n-m}}(\mu^l)^{m-1}(\mu^l-1)\right\} \\ &=\frac{1}{\mu^l}\inf_{n\geqslant N}\left\{1+p_n(\mu^l)^{k+1}-c\frac{p_n}{P_{n-m}}(\mu^l)^m(\mu^l-1)\right\} \\ &=\frac{\mu^{l+1}}{\mu^l}, \end{aligned}$$

i.e., $\{\mu^l\}$ is increasing.

For each $n \geqslant N$,

$$\lambda_n^{(1)} = \mu^1 = 1.$$

Assume that

$$\lambda_n^{(l)} \geqslant \mu^l \qquad \text{for} \quad n \geqslant N;$$

we see that

$$\begin{aligned} \lambda_n^{(l+1)} &= 1 + p_n \prod_{i=n-k}^{n} \lambda_i^{(l)} - c\frac{p_n}{p_{n-m}} \prod_{i=n-m+1}^{n} \lambda_i^{(l)}(\lambda_{n-m}^{(l)} - 1) \\ &\geqslant 1 + p_n(\mu^l)^{k+1} - c\frac{p_n}{p_m}(\mu^l)^m (\mu^l - 1) \\ &= \mu^{l+1}. \end{aligned}$$

By induction, we get that

$$\lambda_n^{(l)} \geqslant \mu^l, \qquad \text{for} \quad n \geqslant N, \quad l = 1, 2, \ldots.$$

Now we shall show that

$$v_n \geqslant \lambda_n^{(l)} \qquad \text{for} \quad n \geqslant N + Ml. \tag{3.23}$$

In fact, $v_n > 1 = \lambda_n^{(1)}$, $n \geqslant N + M$.

Assume that

$$v_n \geqslant \lambda_n^{(l)}, \qquad n \geqslant N + Ml.$$

Then

$$\begin{aligned} v_n &= 1 + P_n \prod_{i=n-k}^{n} v_i - c\frac{p_n}{p_{n-m}} \prod_{i=n-m+1}^{n} v_i(v_{n-m} - 1) \\ &\geqslant 1 + p_n \prod_{i=n-k}^{n} \lambda_i^{(l)} - c\frac{p_n}{p_{n-m}} \prod_{i=n-m+1}^{n} \lambda_i^{(l)}(\lambda_{n-m}^{(l)} - 1) \\ &= \lambda_n^{(l+1)}, \qquad \text{for} \quad n \geqslant N + (l+1)M. \end{aligned}$$

By induction we have proved (3.23).

Let $\mu^* = \lim_{l \to \infty} \mu^l$. We shall discuss two possible cases for μ^*.

First, we assume that μ^* is finite. It is obvious that $\mu^* > 1$ and

$$\mu^* = \inf_{n \geqslant N} \left\{ 1 + p_n(\mu^*)^{k+1} - c\frac{p_n}{p_{n-m}}(\mu^*)^m (\mu^* - 1) \right\} \tag{3.24}$$

by definition of $\{\mu^l\}$.

From (3.24) we obtain

$$\inf_{\substack{n \geq N \\ \mu > 1}} \left\{ \frac{1}{\mu} + p_n \mu^k - c \frac{p_n}{p_{n-m}} \mu^{m-1}(\mu - 1) \right\} \leq 1,$$

which contradicts condition (3.14).

Now in case $\mu^* = +\infty$, then $\lim_{n \to \infty} v_n = +\infty$. Consequently, $\lim_{n \to \infty} w_n = 1$.

From Eq. (1.2) and the fact that $x_n \geq z_n$, we have

$$\Delta z_n + p z_{n-k} \leq 0.$$

Hence

$$\frac{z_{n+1}}{z_n} - 1 + p \frac{z_{n-k}}{z_n} \leq 0. \tag{3.25}$$

Now $\lim_{n \to \infty} w_n = 1$ implies that $\lim_{n \to \infty} z_{n+1}/z_n = 0$ and $\lim_{n \to \infty} z_{n-k}/z_n = +\infty$. Therefore (3.25) is impossible and hence the proof of Theorem 3.5 is complete.

THEOREM 3.6. *Assume that*

(i) $-1 \leq c \leq 0$;

(ii) $p_n > 0$ *for all large* $n \geq N - m$;

(iii) *there exists a constant* $\mu^* > 1$ *such that*

$$\sup_{n \geq N} \left\{ \frac{1}{\mu^*} + p_n (\mu^*)^k - c \frac{p_n}{p_{n-m}} (\mu^*)^{m-1} (\mu^* - 1) \right\} \leq 1; \tag{3.26}$$

then Eq. (1.2) *has a positive solution.*

Proof. Let

$$v_{N-M} = \cdots = v_{N-1} = q, \tag{3.27}$$

where q is a constant and $q \in (1, \mu^*)$. Define

$$v_n = 1 + p_n \prod_{i=n-k}^{n} v_i - c \frac{p_n}{p_{n-m}} \prod_{i=n-m+1}^{n} v_i [v_{n-m} - 1] \qquad n = N, N+1, ... \tag{3.28}$$

We see that, from (3.28),

$$v_N = 1 + p_N v_N q^k - c \frac{p_N}{p_{N-m}} v_N q^m (q - 1)$$

or

$$v_N\left(1-p_Nq^k+c\frac{p_N}{p_{N-m}}q^{m-1}(q-1)\right)=1.$$

From (3.26), we know that

$$1-p_Nq^k+c\frac{p_N}{P_{N-m}}q^{m-1}(q-1)\geqslant\frac{1}{q}$$

so

$$v_N\cdot\frac{1}{q}\leqslant 1.$$

Hence

$$1<v_N\leqslant q<\mu^*.$$

By induction, we can prove that $\{v_n\}$ is well defined by (3.28).
Define

$$w_i=\frac{v_i-1}{v_i},\qquad i=N-M, N-M+1, \ldots.$$

From (3.28), we have

$$\frac{1}{1-w_n}=1+p_n\prod_{i=n-k}^{n}(1-w_i)^{-1}-c\frac{p_n}{p_{n-m}}\prod_{i=n-m+1}^{n}\times\frac{1}{1-w_i}w_{n-m}\left(\frac{1}{1-w_{n-m}}\right)$$

or

$$w_n=p_n\prod_{i=n-k}^{n-1}(1-w_i)^{-1}-c\frac{p_n}{p_{n-m}}w_{n-m}\prod_{i=n-m}^{n-1}(1-w_i)^{-1}, \qquad (3.29)$$

where $n=N, N+1, \ldots$ and $1>w_n>0$.
Define

$$z_{N-M}=1$$

and

$$z_{n+1}=z_n(1-w_n),\qquad n>N-M.$$

Hence

$$z_n = \prod_{i=N-M}^{n-1} (1 - w_i), \qquad z_n > 0.$$

We see that

$$\begin{aligned} \Delta z_n &= z_{n+1} - z_n \\ &= -w_n z_n < 0, \end{aligned}$$

i.e.,

$$w_n = -\frac{\Delta z_n}{z_n}. \tag{3.30}$$

Substituting (3.30) into (3.29), we have

$$-\frac{\Delta z_n}{z_n} = p_n \frac{z_{n-k}}{z_n} - c\,\frac{p_n}{p_{n-m}} \left(1 - \frac{z_{n-m+1}}{z_{n-m}}\right) \frac{z_{n-m}}{z_n}$$

so

$$-\Delta z_n = p_n z_{n-k} - c\,\frac{p_n}{p_{n-m}} \left(-\Delta z_{n-m}\right)$$

or

$$-\frac{\Delta z_n}{p_n} = z_{n-k} + \frac{c}{p_{n-m}} \Delta z_{n-m}. \tag{3.31}$$

Now we define

$$y_n = -\frac{\Delta z_{n+k}}{p_{n+k}} > 0, \qquad n = N - M, N - M + 1, \dots. \tag{3.32}$$

From (3.31) and (3.32) we have

$$y_{n-k} = z_{n-k} - c y_{n-m-k}$$

or

$$z_{n-k} = y_{n-k} + c y_{n-m-k}. \tag{3.33}$$

Combining (3.33) and (3.32), we see that $\{y_n\}$ defined by (3.32) is a solution of Eq. (1.2). The proof is complete.

COROLLARY 3.1. *If $p_n \equiv p > 0$ in Eq. (1.2), then every solution of (1.2) oscillates if and only if*

$$\inf_{\mu > 1} \left\{ \frac{1}{\mu} + p\mu^k - c\mu^{m-1}(\mu - 1) \right\} > 1. \tag{3.34}$$

Remarks 3.3. If $m = 0$, $k \geqslant 1$, then (3.34) is equivalent to

$$\frac{p}{1+l} > \frac{k^k}{(k+1)^{k+1}} \tag{3.35}$$

and hene (3.35) is a sufficient and necessary condition for every soluton of (1.2) to be oscillatory.

Remark 3.4. If $m > 0$, $k \geqslant 1$, then (3.35) implies that (3.34) holds. Therefore (3.35) is a sufficient condition for every solution of (1.2) to be oscillatory. This is a known result (Theorem 2, (iii) in [5]).

Remark 3.5. If $k = 0$, $m \geqslant 1$, then (3.34) becomes

$$\inf_{\mu > 1} \left\{ \frac{1}{\mu} + p - c\mu^{m-1}(\mu - 1) \right\} > 1. \tag{3.36}$$

It is sufficient to have

$$\inf_{\mu > 1} \left\{ \frac{1}{\mu} + p - c(\mu - 1) \right\} > 1. \tag{3.37}$$

From (3.37) we get the sufficient condition

$$p > 1 - c - 2\sqrt{-c} \tag{3.38}$$

for every solution of (1.2) to be oscillatory, which is better than the condition

$$p \geqslant 1 + c$$

(Theorem 2, (i) in [5]).

4. CASE WHEN $c < -1$

LEMMA 4.1. *Assume that $p_n \geqslant 0$ and $\sum_{n=1}^{\infty} p_n = \infty$.*
Let $\{x_n\}$ be an eventually positive solution of (1.2).
Set $z_n = x_n + cx_{n-m}$. Then $z_n < 0$ and $\Delta z_n \leqslant 0$ eventually.

Proof. In fact, from (1.2), $\Delta z_n = -p_n x_{n-k} \leqslant 0$ for all large n. We shall prove that $z_n < 0$ eventually.

If not, then

$$z_n = x_n + c x_{n-m} \geqslant 0 \quad \text{for} \quad n \geqslant N;$$

i.e.,

$$x_n \geqslant -c x_{n-m} \qquad \text{for} \quad n \geqslant N,$$

which implies

$$0 < x_{N-m} \leqslant \left(-\frac{1}{c}\right) x_N \leqslant \cdots \leqslant \left(-\frac{1}{c}\right)^j x_{N+(j-1)m}, \tag{4.1}$$

$j = 1, 2, ...,$. Letting $j \to \infty$ in (4.1), we get

$$x_n \to \infty \qquad \text{as} \quad n \to \infty.$$

But

$$\Delta z_n = -p_n x_{n-m} \leqslant -M p_n \qquad \text{for large } n, \tag{4.2}$$

where M is a positive number. Summing (4.2) we get that

$$z_{n+1} - z_N \leqslant -M \sum_{i=N}^{n} p_i$$

which implies that

$$z_n \to -\infty \qquad \text{as} \quad n \to \infty.$$

This contradicts the fact that $z_n \geqslant 0$, for $n \geqslant N$. The proof is complete.

THEOREM 4.1. *Assume that*

(i) $c < -1$

(ii) $m > k$

(iii) $p_n \leqslant p_{n-m}$ *for all large* n

(iv) $-(1/(1+c)) \liminf p_n > (m-k-1)^{m-k-1}/(m-k)^{m-k}$.

Then every solution of Eq. (1.2) *oscillates.*

Proof. Otherwise, without loss of generality, let $\{x_n\}$ be an eventually positive solution of Eq. (1.2). By Lemma 4.1, we have

$$z_n < 0, \qquad \Delta z_n \leqslant 0.$$

Set $w_n = z_n + cz_{n-m}$; then we have $w_n > 0$, $\Delta w_n \geqslant 0$ eventually. Note that

$$w_n \leqslant (1+c) z_{n-m};$$

then

$$z_{n-m} \leqslant \frac{w_n}{1+c}.$$

Hence

$$\begin{aligned}\Delta w_n = \Delta z_n + c\, \Delta z_{n-m} &= -p_n x_{n-k} - c p_{n-m} x_{n-k+m}\\ &\geqslant -p_n z_{n-k}\\ &\geqslant 0\end{aligned}$$

or

$$0 \leqslant \Delta w_n + p_n z_{n-k} \leqslant \Delta w_n + \frac{p_n}{1+c} w_{n-k+m}. \tag{4.3}$$

Set $\gamma_n = w_{n+1}/w_n \geqslant 1$. The inequality (4.3) implies that

$$\gamma_n \geqslant 1 - \frac{p_n}{1+c} \gamma_n \cdots \gamma_{n+(m-k)-1}. \tag{4.4}$$

From condition (iv), we know that γ_n is bounded above.

Taking limit inferior on both sides of (4.4) we have

$$l \geqslant 1 - \frac{1}{1+c} \liminf p_n l^{m-k}, \tag{4.5}$$

where $l = \liminf_{n\to\infty} \gamma_n$. From (4.5) we have

$$\frac{1}{1+c} \liminf_{n\to\infty} p_n \leqslant \frac{l-1}{l^{m-k}} \leqslant \frac{(m-k-1)^{m-k-1}}{(m-k)^{m-k}}$$

which contradicts condition (iv). The proof is complete.

If condition (iii) does not hold, then we have the following criterion.

THEOREM 4.2. *Assume that the assumptions* (i) *and* (ii) *in Theorem* 4.1 *hold. Further, assume that*

$$\frac{1}{c} \liminf_{n\to\infty} p_n > \frac{(m-k-1)^{m-k-1}}{(m-k)^{m-k}}. \tag{4.6}$$

Then every solution of (1.2) *oscillates.*

Proof. If not, let $\{x_n\}$ be an eventualy positive solution. Note that $z_n = x_n + cx_{n-m} > cx_{n-m}$. From (1.2), we have

$$\Delta z_n = -p_n x_{n-k} \leqslant -\frac{p_n}{c} z_{n+(m-k)}. \tag{4.7}$$

The rest of the proof is similar to that of Theorem 4.1. We know that under assumption (4.6) the inequality (4.7) has no eventualy negative solution. But by Lemma 4.1, z_n is eventually negative. This contradiction proves the Theorem.

THEOREM 4.3. *Assume that the assumptions of Lemma* 4.1 *hold. Then every nonoscillatory solution of* (1.2) *tends to* $+\infty$ *or* $-\infty$ *as* $n \to \infty$.

Proof. Let $\{x_n\}$ be an eventually positive solution of (1.2). By Lemma 4.1 we have

$$z_n < 0, \qquad \Delta z_n \leqslant 0 \quad \text{eventually}.$$

Therefore $0 > \lim_{n\to\infty} z_n = l \geqslant -\infty$. We shall show that $l = -\infty$.

Assume that $-\infty < l < 0$. Summing (1.2) from N to n we get

$$z_{n+1} - z_N + \sum_{i=N}^{n} p_i x_{i-k} = 0, \tag{4.8}$$

which implies that

$$\sum_{i=N}^{\infty} p_i x_{i-k} < \infty. \tag{4.9}$$

Since $\sum_{i=N}^{\infty} p_i = \infty$, we have

$$\liminf_{n\to\infty} x_n = 0,$$

i.e., there exists a subsequence $\{n_j\}$ such that

$$\lim_{z\to\infty} n_j = \infty \qquad \text{and} \qquad \lim_{j\to\infty} x_{n_j - m} = 0.$$

On the other hand, $z_{n_j} > cx_{n_j - m}$; thus

$$0 < \frac{z_{n_j}}{c} < x_{n_j - m}, \tag{4.10}$$

which implies that $\lim_{j\to\infty} z_{n_j} = 0$, a contradiction. Therefore $\lim_{n\to\infty} z_n = -\infty$. From (4.10) we have $\lim_{n\to\infty} x_n = +\infty$. The proof of eventually negative solution is similar.

5. CASE WHEN $c > 0$

THEOREM 5.1. *Assume that*

(i) $c > 0$, $k > m$

(ii) $p_n \geqslant p_{n-m}$ *and*

$$\frac{1}{1+c} \liminf_{n\to\infty} p_n > \frac{(k-m)^{k-m}}{(k-m+1)^{k-m+1}}.$$

Then every solution of (1.2) *oscillates.*

Proof. The proof of this theorem is essentially the same as the proof of Theorem 4.1, and hence is omitted.

Remark 5.1. Theorem 5.1 includes a part of Theorem 3 in [5].

6. NONHOMOGENEOUS DIFFERENCE EQUATIONS

Consider

$$\Delta(x_n + c_n x_{n-m}) + p_n x_{n-k} = F_n. \tag{6.1}$$

THEOREM 6.1. *Assume the following*:

(i) $c \geqslant c_n \geqslant 0$, *$c$ is a positive number, m and k are positive integers.*

(ii) $p_n \geqslant 0$ *and there exists a constant number* $M > 0$ *such that*

$$p_n \leqslant M p_{n-m}. \tag{6.2}$$

(iii) *Set* $\Delta f_n = F_n$ *and denote*

$$f_n^+ = \max(f_n, 0)$$

$$f_n^- = \max(-f_n, 0)$$

and

$$\sum_{n=N}^{\infty} p_{n+k} f_n^{\pm} = \infty. \tag{6.3}$$

Then every solution of (1.3) *oscillates.*

Proof. If not, without loss of generality, assume that $\{x_n\}$ is an eventually positive solution. Then

$$\Delta(z_n - f_n) = p_n x_{n-k} \leqslant 0, \tag{6.4}$$

where $z_n = x_n + c_n x_{n-m} > 0$ eventually.

From (6.4), $(z_n - f_n)$ is nonincreasing eventually. If $z_n - f_n \leqslant 0$ for $n \geqslant N$, then $z_n \leqslant f_n$ for $n \geqslant N$, which contradicts the positiveness of z_n. Therefore $z_n - f_n > 0$ for all $n \geqslant N$. Hence

$$z_n \geqslant f_n^+ \qquad \text{for} \quad n \geqslant N. \tag{6.5}$$

On the other hand, from (6.4) $\lim_{n \to \infty}(z_n - f_n) = l \geqslant 0$ exists. Consequently, $\sum_{i=N}^{\infty} p_i x_{i-k} < \infty$. We see that

$$\sum_{n=N}^{\infty} p_{n+k} z_n = \sum_{n=N}^{\infty} p_{n+k} x_n + \sum_{n=N}^{\infty} p_{n+k} c_n x_{n-m} < \infty \tag{6.6}$$

because of condition (6.2). Combining (6.6) and (6.5) we have

$$\sum_{n=N}^{\infty} p_{n+k} f_n^+ < \infty$$

which contradicts (6.3). The proof is complete.

THEOREM 6.2. *Assume that*

(i) $c_n \geqslant 0$, $p_n \geqslant 0$, $n = 1, 2, \ldots$;

(ii) *there exists* f_n *such that* $\Delta f_n = F_n$ *and*

$$\limsup_{n \to \infty} f_n = +\infty, \qquad \liminf_{n \to \infty} f_n = -\infty. \tag{6.7}$$

Then every solution of Eq. (1.3) *oscillates.*

Proof. Suppose the contrary; without loss of generality, let $\{x_n\}$ be an eventually positive solution of Eq. (1.3). As in the proof of Theorem 6.1, we have

$$z_n - f_n \geqslant 0 \qquad \text{for} \quad n \geqslant N$$

and

$$\Delta(z_n - f_n) \leqslant 0, \qquad \text{so} \qquad \lim_{n \to \infty} (z_n - f_n) = \alpha \geqslant 0.$$

From (6.7) there exists a sequence $\{n_k\}$ such that $\lim_{k \to \infty} f_{n_k} = -\infty$. We see that

$$\lim_{n \to \infty} (z_{n_k} - f_{n_k}) = \alpha \geqslant 0,$$

which implies that $\{z_{n_k}\}$ cannot be eventualy positive, a contradiction. The proof is complete.

EXAMPLE 6.1. Consider

$$\Delta(x_n + x_{n-2}) + \left[\frac{4n^3 - 6n^2 - 2n + 2}{n(n+1)(n-1)} - \frac{(n-2)(2n+1)}{n(n+1)}\right] x_{n-2}$$

$$= F_n = (-1)^{n+1} \frac{2n+1}{n(n+1)}. \tag{6.8}$$

It is easy to see that $f_n = (-1)^n (1/n)$, $p_n \to 2$ as $n \to \infty$, and

$$\sum_{n=}^{\infty} p_{n+k} f_n^{\pm} = \infty.$$

Therefore every solution of (6.8) oscillates by Theorem 6.1. In fact, $x_n = (-1)^n (1/n)$ is a solution of (6.8).

EXAMPLE 6.2. Consider

$$\Delta(x_n + x_{n-2}) + \frac{2n-3}{n-2} x_{n-2} = (-1)^{n+1} (2n+1). \tag{6.9}$$

We see that $F_n = (-1)^{n+1} (2n+1)$, $f_n = (-1)^n n$. Hence all assumptions of Theorem 6.2 are satisfied. Therefore every solution of (6.9) oscillates. In fact, $x_n = (-1)^n n$ is such a solution.

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