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Spectral stability of the Neumann Laplacian ☆

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Abstract

We prove the equivalence of Hardy- and Sobolev-type inequalities, certain uniform bounds on the heat kernel and some spectral regularity properties of the Neumann Laplacian associated with an arbitrary region of finite measure in Euclidean space. We also prove that if one perturbs the boundary of the region within a uniform Hölder category, then the eigenvalues of the Neumann Laplacian change by a small and explicitly estimated amount.

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1. Introduction

Let Ω be an arbitrary region in $\mathbf{R}^N$ and let us define the Neumann Laplacian to be the non-negative self-adjoint operator $H = -\Delta_N$ acting in

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$L^2(\Omega)$ and associated with the quadratic form

$$Q(f) = \begin{cases} \int_\Omega |\nabla f|^2 \, \mathrm{d}^N x & \text{if } f \in W^{1,2}(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{1}$$

as described in [4, Section 4.4]. It is well known that if Ω is bounded with continuous boundary $\partial\Omega$, then H has compact resolvent since the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compact.[1] However, in general the spectrum of H may be quite wild, even for bounded regions in $\mathbf{R}^2$ [8]. These phenomena are not well understood, with the result that the Neumann Laplacian is far less studied than the Dirichlet Laplacian.

In this paper, we prove a number of general results concerning the spectral behaviour of the Neumann Laplacian. We start by investigating the relationship between Hardy- and Sobolev-type inequalities for arbitrary regions of finite inradius. We then establish the equivalence of Sobolev-type inequalities to some spectral properties of the Neumann Laplacian. The results apply in particular to bounded regions with Hölder continuous boundaries.

Even if one knows that the spectrum is discrete, the numerical computation of the eigenvalues by the finite element or other methods depends upon the assumption that if one replaces a very irregular boundary by a suitable polygonal or piecewise smooth approximation, then the eigenvalues are very little affected. This continuous dependence of the spectrum on the boundary holds in great generality for Dirichlet boundary conditions, but is much less obvious for Neumann boundary conditions.

In the last part of the paper, we investigate the effect of perturbing the boundary. We first prove a quasi-monotonicity property of the eigenvalues when the region decreases, under suitable regularity hypotheses on the larger region. We then apply a scaling trick to prove that the eigenvalues vary continuously with the region provided the boundaries of the regions concerned satisfy a uniform Hölder condition. Moreover, the change in the eigenvalues of the Neumann Laplacian is explicitly estimated.

Many of the results of this paper apply not only to the Neumann Laplacian but to general strictly elliptic second-order operators or Schrödinger operators whose quadratic form domains are contained in $W^{1,2}(\Omega)$. The proofs need almost no alterations.

[1] See for example [2]. The definition of a region with continuous boundary may be obtained from the definition of a region with Lip γ-boundary in Section 2 by replacing in part (ii) the Lip γ-condition for φ_j by the assumption that φ_j is continuous on $\overline{W_j}$. Necessary and sufficient conditions for the compactness of this embedding in terms of capacities were obtained in [12]. See also recent paper [7] in which sufficient conditions for the compactness in geometric terms have been established weaker than the assumption of the continuity of the boundary.

2. Relationship between the Sobolev- and Hardy-type inequalities

Let $\Omega \subset \mathbf{R}^N$ be a region, for $x \in \Omega$, $d(x)$ be the distance of the point x from the boundary $\partial\Omega$ of Ω and, for $\varepsilon > 0$,

$$\Omega^\varepsilon = \{x \in \mathbf{R}^N : \operatorname{dist}(x, \Omega) < \varepsilon\},$$

$$\Omega_\varepsilon = \{x \in \Omega : d(x) > \varepsilon\}$$

and

$$\partial_\varepsilon \Omega = \Omega \backslash \Omega_\varepsilon = \{x \in \Omega : d(x) \leqslant \varepsilon\}.$$

For a region with a finite measure $|\Omega|$, the *Minkowski dimension of* $\partial\Omega$ *relative to* Ω (briefly, the *Minkowski dimension of* $\partial\Omega$) is the following quantity:

$$M(\partial\Omega) = \inf\{\lambda > 0 : M^\lambda(\partial\Omega) < \infty\},$$

where

$$M^\lambda(\partial\Omega) = \limsup_{\varepsilon \to 0+} \frac{|\partial_\varepsilon \Omega|}{\varepsilon^{N-\lambda}}.$$

Obviously, $M(\partial\Omega) \leqslant N$. However, there exist Ω such that $M^\lambda(\partial\Omega) = \infty$ for all $\lambda \in (0, N)$ [5]. It can be proved that $M(\partial\Omega) \geqslant N - 1$ [9]. If Ω satisfies the cone condition, then $M(\partial\Omega) = N - 1$ [9].

Recall that a *Whitney covering* $\mathscr{W}$ of an open set Ω is a family of closed cubes Q each having edge length $L_Q = 2^{-k}, k = 1, 2, \ldots$, such that

(i) $\Omega = \bigcup_{Q \in \mathscr{W}} Q$;
(ii) the interiors of distinct cubes are disjoint;
(iii) $\operatorname{diam}(Q) \leqslant \operatorname{dist}(Q, \partial\Omega) \leqslant 4 \operatorname{diam}(Q)$;
(iv) $\frac{1}{4} \operatorname{diam}(Q_2) \leqslant \operatorname{diam}(Q_1) \leqslant 4 \operatorname{diam}(Q_2)$ if $Q_1 \cap Q_2 \neq \emptyset$;
(v) at most 12^N other cubes in $\mathscr{W}$ can touch a fixed $Q \in \mathscr{W}$, and for a fixed $t \in (1, 5/4)$ each $x \in \Omega$ lies in at most 12^N of the dilated cubes tQ, $Q \in \mathscr{W}$.

It is known (see, for example, [14, Chapter VI]) that such a covering exists for any Ω. Note that condition (iii) implies that $\operatorname{diam}(Q) \leqslant d(x) \leqslant 5 \operatorname{diam}(Q)$ for any $x \in Q$.

Let, for a positive integer k, $n(k)$ denote the number of cubes in $\mathscr{W}_k = \{Q \in \mathscr{W} : L_Q = 2^{-k}\}$. If Ω has finite measure, then $n(k) \leqslant c_1 2^{Nk}$, where $c_1 > 0$ is independent of k. Moreover, $M^\lambda(\partial\Omega) < \infty$ if, and only if, $n(k) \leqslant c_2 2^{\lambda k}$, where $c_2 > 0$ is independent of k [10].

Let $0 < \gamma \leqslant 1, M, \delta > 0, s \geqslant 1$ be an integer, and let $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{\lambda_j\}_{j=1}^s$ be a family of rotations. We say that, for a bounded region $\Omega \subset \mathbf{R}^n$, its boundary $\partial\Omega \in \operatorname{Lip}(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$ if

(i) $\Omega \subset \bigcup_{j=1}^s (V_j)_\delta$ and $(V_j)_\delta \cap \Omega \neq \emptyset$;

(ii) for $j = 1, \ldots, s$

$$\lambda_j(V_j) = \{ x \in \mathbf{R}^N : a_{ij} < x_i < b_{ij},\ i = 1, \ldots, N\},$$

$$\lambda_j(\Omega \cap V_j) = \{x \in \mathbf{R}^N : a_{Nj} < x_N < \varphi_j(\bar{x}), \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, \ldots, x_{N-1})$, $W_j = \{\bar{x} \in \mathbf{R}^{N-1} : a_{ij} < x_i < b_{ij},\ i = 1, \ldots, N-1\}$ and

$$|\varphi_j(\bar{x}) - \varphi_j(\bar{y})| \leqslant M\,|\bar{x} - \bar{y}|^{\gamma}, \quad \bar{x}, \bar{y} \in \bar{W}_j;$$

(iii) if $V_j \cap \partial\Omega \neq \emptyset$, then

$$a_{Nj} + \delta \leqslant \varphi_j(\bar{x}) \leqslant b_{Nj} - \delta, \quad \bar{x} \in W_j.$$

However, if $V_j \subset \Omega$, then $\varphi_j(\bar{x}) \equiv b_{Nj}$.

We also say that, for a bounded region Ω and $0 < \gamma \leqslant 1$, $\partial\Omega \in \mathrm{Lip}\,\gamma$ if there exist $M, \delta > 0$, an integer $s \geqslant 1$, a family of bounded open cuboids $\{V_j\}_{j=1}^s$, and a family of rotations $\{\lambda_j\}_{j=1}^s$ such that $\partial\Omega \in \mathrm{Lip}(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$.

If Ω is a bounded region and $\partial\Omega \in \mathrm{Lip}\,\gamma$, then $M(\partial\Omega) \leqslant N - \gamma$. Moreover, if $\partial\Omega \in \mathrm{Lip}(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$, there exist $\varepsilon_0, a_0 > 0$, depending only on $N, \gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s$, such that for all $0 < \varepsilon \leqslant \varepsilon_0$

$$|\partial_\varepsilon \Omega| \leqslant a_0 \varepsilon^{\gamma}. \tag{2}$$

Theorem 1. *Let $\Omega \subset \mathbf{R}^N$ be a region with a finite inradius, i.e.,* $\sup_{x \in \Omega} d(x) < \infty$, *and let* $1 \leqslant p < \infty$.

1. *If for some* $\alpha > 0, c_1 > 0$

$$\|d^{-\alpha} f\|_{L^p(\Omega)} \leqslant c_1 \|f\|_{W^{1,p}(\Omega)} \tag{3}$$

for all $f \in W^{1,p}(\Omega)$, *then there exists* $c_2 > 0$ *such that*

$$\|f\|_{L^q(\Omega)} \leqslant c_2 \|f\|_{W^{1,p}(\Omega)} \tag{4}$$

for all $f \in W^{1,p}(\Omega)$, *where* $q = \frac{Mp}{M-p}, M = N(1+\alpha)$ *if* $N > p$ *and* q *is any number such that* $p < q < p(1 + \frac{\alpha p}{N})$ *if* $N \leqslant p$.

2. *If for some* $\sigma > 0$

$$\int_\Omega d(x)^{-\sigma}\, \mathrm{d}^N x < \infty \tag{5}$$

and for some $q > p$ *and* $c_2 > 0$ *inequality* (4) *holds, then there exists* $c_1 > 0$ *such that inequality* (3) *holds with* $\alpha = \sigma(\frac{1}{p} - \frac{1}{q})$.

Proof. (1) First, we note that there exists $c_3 > 0$ such that

$$\|d^{N(\frac{1}{p} - \frac{1}{r})} f\|_{L^r(\Omega)} \leqslant c_3 \|f\|_{W^{1,p}(\Omega)}$$

for all $f \in W^{1,p}(\Omega)$, where $r = \frac{Np}{N-p}$ if $N > p$, and $p < r < \infty$ if $N \leqslant p$. This inequality follows by scaling the standard Sobolev inequality for cubes and by using the Whitney decomposition of Ω. (It is contained in a more general statement of such type proved by Brown [1, Theorem 3.1].)

Let $0 < \lambda < 1$ and $q \in (p, r)$ be such that

$$\frac{1}{q} = \frac{1-\lambda}{p} + \frac{\lambda}{r}.$$

Bearing in mind that

$$|f| = (d^{-N(\frac{1}{p}-\frac{1}{r})\frac{\lambda}{1-\lambda}}|f|)^{1-\lambda}(d^{N(\frac{1}{p}-\frac{1}{r})}|f|)^{\lambda},$$

we choose λ so that $N(\frac{1}{p}-\frac{1}{r})\frac{\lambda}{1-\lambda} = \alpha$. Then

$$\lambda = \frac{\alpha}{N}\left(\frac{\alpha}{N} + \frac{1}{p} - \frac{1}{r}\right)^{-1},$$

$$q = \left(\frac{\alpha}{N} + \frac{1}{p} - \frac{1}{r}\right)\left[\frac{1}{p}\left(\frac{\alpha}{N} + \frac{1}{p} - \frac{1}{r}\right) - \frac{\alpha}{N}\left(\frac{1}{p} - \frac{1}{r}\right)\right]^{-1}.$$

By applying Hölder's inequality with the exponents $\frac{p}{1-\lambda}$ and $\frac{r}{\lambda}$, we have

$$\begin{aligned}\|f\|_{L^q(\Omega)} &= \|(d^{-\alpha}|f|)^{1-\lambda}(d^{N(\frac{1}{p}-\frac{1}{r})}|f|)^{\lambda}\|_{L^q(\Omega)}\\ &\leqslant \|d^{-\alpha}f\|_{L^p(\Omega)}^{1-\lambda}\|d^{N(\frac{1}{p}-\frac{1}{r})}f\|_{L^r(\Omega)}^{\lambda} \leqslant c_1^{1-\lambda}c_3^{\lambda}\|f\|_{W^{1,p}(\Omega)}.\end{aligned}$$

If $N > p$, then $r = \frac{Np}{N-p}$ and hence $q = \frac{Mp}{M-p}$. If $N \leqslant p$, then by passing to the limit as $r \to \infty$ we see that q can be any real number satisfying $p < q < p(1 + \frac{\alpha p}{N})$.

(2) The second statement follows immediately by Hölder's inequality with the exponents $\frac{qp}{q-p}$ and q:

$$\|d^{-\alpha}f\|_{L^p(\Omega)} \leqslant \|d^{-\alpha}\|_{L^{\frac{qp}{q-p}}(\Omega)}\|f\|_{L^q(\Omega)} \leqslant \left(\int_{\Omega} d(x)^{-\sigma}\, \mathrm{d}^N x\right)^{\frac{1}{p}-\frac{1}{q}} c_2\|f\|_{W^{1,p}(\Omega)}.$$

□

Corollary 2. *Let $\Omega \subset \mathbf{R}^N$ be a region of finite measure and $1 \leqslant p < \infty$. Then the following conditions are equivalent*:

(a) *For some $\alpha, c_1 > 0$ inequality* (3) *holds for all $f \in W^{1,p}(\Omega)$.*
(b) *For some $\sigma > 0$ condition* (5) *is satisfied, and for some $q > p$ and $c_2 > 0$ inequality* (4) *holds for all $f \in W^{1,p}(\Omega)$.*
(c) *$M(\partial\Omega) < N$ and for some $q > p$ and $c_2 > 0$ inequality* (4) *holds for all $f \in W^{1,p}(\Omega)$.*

Proof. Inequality (3) implies, by putting $f \equiv 1$, that

$$\int_\Omega d(x)^{-\alpha p}\, \mathrm{d}^N x \leqslant c_1^p |\Omega|.$$

Now it suffices to recall that, for regions Ω of finite measure, the inequality $M(\partial\Omega) < N$ is equivalent to the existence of $\mu \in (0,1)$ such that $\int_\Omega d(x)^{-\mu}\, \mathrm{d}^N x < \infty$ [1]. □

3. Equivalence of the Sobolev-type inequalities to some spectral properties of Neumann Laplacian

In this section we assume that Ω is a region in $\mathbf{R}^N$ and suppose that $H = -\Delta_N$ acts in $L^2(\Omega)$ subject to Neumann boundary conditions.

Proposition 3. *Assume that $N \geqslant 2$ and that $\Omega \subset \mathbf{R}^N$ is any region.*

1. *Let $2 < q \leqslant \frac{2N}{N-2}$ if $N \geqslant 3$, $2 < q < \infty$ if $N = 2$ and $M = \frac{2q}{q-2}$. Then the following conditions are equivalent*:
   (a) *There exists $c_4 > 0$ such that*
   $$\|f\|_{L^q(\Omega)} \leqslant c_4 \|f\|_{W^{1,2}(\Omega)} \tag{6}$$
   for all $f \in W^{1,2}(\Omega)$.
   (b) *There exists $c_5 > 0$ such that*
   $$\|\mathrm{e}^{-Ht} f\|_{L^\infty(\Omega)} \leqslant c_5 t^{-M/4} \|f\|_{L^2(\Omega)} \tag{7}$$
   for all $f \in L^2(\Omega)$ and all $0 < t \leqslant 1$.
   (c) *The semigroup e^{-Ht} has a continuous integral kernel $K(t,x,y)$, $t > 0$, $x, y \in \Omega$ and there exists $c_6 > 0$ such that*
   $$0 < K(t,x,y) \leqslant c_6 t^{-M/2} \tag{8}$$
   for all $x, y \in \Omega$ and $0 < t \leqslant 1$.
2. *Let $0 < \gamma \leqslant 1$, Ω be bounded and $\partial\Omega \in \mathrm{Lip}\,\gamma$. If $\gamma = 1$, then* (a) *is satisfied with $q = \frac{2N}{N-2}$ for $N \geqslant 3$* (*hence in* (7) *and* (8) *$M = N$*) *and with any $2 < q < \infty$ for $N = 2$* (*hence in* (7) *and* (8) *any $M > 2$*). *If $0 < \gamma < 1$, then* (a) *is satisfied with $q = \frac{2(\gamma+n-1)}{N-1-\gamma}$* (*hence in* (7) *and* (8) *$M = \frac{\gamma+N-1}{\gamma}$*).

The first statement is proved, for example, in [3, Corollary 2.4.3, Lemma 2.1.2]. (One needs to take into account that $\mathrm{Quad}(H) = W^{1,2}(\Omega)$.) The second statement is proved in [6,11,13].

Remark 4. Each of the constants c_4, c_5, c_6 can be estimated from any of the others, given q, or equivalently M.

Remark 5. If $N = 1$, then (a) is satisfied with $q = \infty$ and (b) and (c) are satisfied with $M = 1$.

If $N > 1$, there exists a region, say a region with exponentially degenerate boundary [13], such that (a) is not valid for any $q > 2$. However, if such $q > 2$ exists, it must satisfy the assumptions of Proposition 3. The appropriate range for $M = \frac{2q}{q-2}$ is $N \leqslant M < \infty$ for $N \geqslant 3$ and $2 < M < \infty$ for $N = 2$.

On the other hand (b) and (c), which are always equivalent [2, Lemma 2.1.2], could be also invalid for some region Ω for all $M > 0$. However, if there exist $M > 0$ for which (b) and (c) are valid, then $N \leqslant M < \infty$ for any $N \geqslant 1$.

Hence, if $N \leqslant M < \infty$ for $N \geqslant 3$ and $2 < M < \infty$ for $N = 2$, then (b) and (c) are equivalent to (a) where $q = \frac{2M}{M-2}$. However, if $M = 2$ for $N = 2$, then (b) and (c) are not equivalent to (a) for any $q > 2$. (In this case (b) and (c) are equivalent to a certain logarithmic Sobolev inequality or to a certain Nash inequality [3, Example 2.3.1, Corollary 2.4.7].)

Example 6. Let $N \geqslant 3$ and $0 < \gamma \leqslant 1$ or $N = 2$ and $0 < \gamma < 1$. The following well-known example shows that in this case, the exponents q and M in the second statement of Proposition 3 are the best possible, i.e., q cannot be replaced by a larger one and M cannot be replaced by a smaller one. Let $\Omega = \{(x, y) : y \in \mathbf{R}^{N-1}, |y| < 1, |y|^\gamma < x < 1\}$. Then $\partial\Omega \in \mathrm{Lip}\,\gamma$. A direct computation shows that $x^{-\delta} \in W^{1,2}(\Omega)$ if, and only if, $\delta < -1 + \frac{1}{2}(1 + \frac{N-1}{\gamma})$ and $x^{-\delta} \in L^q(\Omega)$ if, and only if, $\delta < \frac{1}{q}(1 + \frac{N-1}{\gamma})$. If (a) holds, then

$$\frac{1}{q}\left(1 + \frac{N-1}{\gamma}\right) \geqslant -1 + \frac{1}{2}\left(1 + \frac{N-1}{\gamma}\right) \Leftrightarrow q \leqslant \frac{2(\gamma + N - 1)}{N - 1 - \gamma}.$$

Since in the case under consideration (b) and (c) are equivalent to (a) it follows also that $M \geqslant \frac{\gamma + N - 1}{\gamma}$.

Theorem 7. *Assume that $\Omega \subset \mathbf{R}^N$ is a region of finite measure.*

1. *The following conditions are equivalent*:
   (a) *For some $q > 2$ and $c_4 > 0$, the inequality*

   $$\|f\|_{L^q(\Omega)} \leqslant c_4 \|f\|_{W^{1,2}(\Omega)}$$

   is satisfied for all $f \in W^{1,2}(\Omega)$.

   (b) *H has discrete spectrum and if all its eigenvalues λ_n, $n = 0, 1, 2, \ldots$, which are non-negative and of finite multiplicity, are written in increasing order and repeated according to multiplicity and f_n is the corresponding orthonormal basis of eigenvectors, then there exist $\alpha_1, c_7, \alpha_2, c_8 > 0$ and an integer*

$n_0 \geqslant 1$ such that

$$\lambda_n \geqslant c_7 n^{\alpha_1}, \quad \|f_n\|_{L^\infty(\Omega)} \leqslant c_8 \lambda_n^{\alpha_2} \tag{9}$$

for all $n \geqslant n_0$.

2. *If $N = 1$ or $N \geqslant 2$, Ω is bounded and $\partial\Omega \in \mathrm{Lip}\,\gamma$ where $0 < \gamma \leqslant 1$, then conditions* (a) *and* (b) *are satisfied.*

Remark 8. One could also assume that conditions (9) were valid for all $n \geqslant 1$. However, the first few eigenvalues may be extremely small if Ω is nearly disconnected and it is not easy to provide explicit bounds on the constants c_7, c_8 which apply for all $n \geqslant 1$.

Remark 9. The relationship between q and (α_1, α_2) is not symmetrical and we do not expect that a symmetrical relationship can be obtained.

The proof of this theorem will be based on the following lemmas containing additional information.

Lemma 10. *Let $M, c_5 > 0$ and let $\Omega \subset \mathbf{R}^N$ be a region of finite measure such that inequality* (7) *is satisfied for all $0 < t \leqslant 1$. Then*

$$\|f_n\|_{L^\infty(\Omega)} \leqslant c_9 \begin{cases} 1 & \text{if } 0 < \lambda_n \leqslant 1, \\ \lambda_n^{\frac{M}{4}} & \text{if } \lambda_n > 1, \end{cases} \tag{10}$$

where $c_9 = \mathrm{e} c_5$.

Proof. If $0 < \lambda_n \leqslant 1$ put $t = 1$ in (7) to get $\mathrm{e}^{-\lambda_n} \|f_n\|_{L^\infty(\Omega)} \leqslant c_5$. So $\|f_n\|_{L^\infty(\Omega)} \leqslant c_5 \mathrm{e}^{\lambda_n} \leqslant c_5 \mathrm{e}$. If $\lambda_n > 1$ put $t = 1/\lambda_n$ in (7) to get $\mathrm{e}^{-1} \|f_n\|_{L^\infty(\Omega)} \leqslant c_5 \lambda_n^{(M/4)}$. □

Lemma 11. *Let $M, c_6, c_{10} > 0$ and let $\Omega \subset \mathbf{R}^N$ be a region such that inequality* (8) *is satisfied for all $x, y \in \Omega, 0 < t \leqslant 1$ and $|\Omega| \leqslant c_{10}$. Then there exists an integer $n_0 \geqslant 1$, depending only on c_6, c_{10}, such that*

$$\lambda_n \geqslant \left(\frac{n}{n_0}\right)^{\frac{2}{M}}, \quad n \geqslant n_0. \tag{11}$$

Proof. By integrating (8) with $x = y$ over Ω we get

$$n\mathrm{e}^{-\lambda_n t} \leqslant \sum_{k=0}^{n} \mathrm{e}^{-\lambda_k t} \leqslant \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda_k t} = \int_\Omega K(t, x, x)\, \mathrm{d}^N x \leqslant c_6 |\Omega| t^{-\frac{M}{2}},$$

hence

$$e^{\lambda_n t} \geqslant \frac{nt^{\frac{M}{2}}}{c_6|\Omega|} \geqslant \frac{nt^{\frac{M}{2}}}{c_6 c_{10}}, \quad 0 < t \leqslant 1.$$

By putting here $t = 1$ it follows that

$$\lambda_n \geqslant 1, \quad n \geqslant n_0 = [ec_6 c_{10}] + 1.$$

Finally, for $n \geqslant n_0$ we put $t = \lambda_n^{-1}$ to get

$$\lambda_n \geqslant \left(\frac{n}{ec_6 c_{10}}\right)^{\frac{2}{M}} \geqslant \left(\frac{n}{n_0}\right)^{\frac{2}{M}}. \qquad \square$$

Lemma 12. *Let $M, c_9 > 0$ and let $n_0 \geqslant 1$ be an integer. Moreover, let $\Omega \subset \mathbf{R}^N$ be a region of finite measure such that inequalities* (10) *and* (11) *are satisfied. Then there exist $c_5, c_6 > 0$, depending only on M, c_9 and n_0, such that the inequalities* (7) *and* (8) *are satisfied with $2M$ replacing M.*

Proof. If $0 < t \leqslant 1$ and $x, y \in \Omega$, then

$$\begin{aligned} 0 < K(t,x,y) &= \sum_{n=0}^{\infty} e^{-\lambda_n t} f_n(x) f_n(y) \leqslant \sum_{n=0}^{\infty} e^{-\lambda_n t} \|f_n\|^2_{L^\infty(\Omega)} \\ &\leqslant c_9^2 \left(\sum_{n=0}^{n_0-1} 1 + \sum_{n=n_0}^{\infty} e^{-\lambda_n t} \lambda_n^{\frac{M}{2}} \right). \end{aligned}$$

Since

$$e^{-\lambda_n t/2} \lambda_n^{\frac{M}{2}} = \left(\frac{t}{2}\right)^{-\frac{M}{2}} e^{-\lambda_n t/2} \left(\frac{\lambda_n t}{2}\right)^{\frac{M}{2}} \leqslant \left(\frac{t}{2}\right)^{-\frac{M}{2}} \left(\frac{M}{2e}\right)^{\frac{M}{2}} = c_{11} t^{-\frac{M}{2}},$$

it follows that

$$\begin{aligned} \sum_{n=n_0}^{\infty} e^{-\lambda_n t} \lambda_n^{\frac{M}{2}} &\leqslant c_{11} t^{-\frac{M}{2}} \sum_{n=n_0}^{\infty} e^{-\frac{\lambda_n t}{2}} \\ &\leqslant c_{11} n_0 t^{-\frac{M}{2}} \frac{1}{n_0} \sum_{n=n_0}^{\infty} \exp\left(-\frac{t}{2} \left(\frac{n}{n_0}\right)^{\frac{2}{M}} \right) \\ &\leqslant c_{11} n_0 t^{-\frac{M}{2}} \int_0^{\infty} \exp\left(-\frac{t}{2} x^{\frac{2}{M}} \right) dx \\ &= c_{11} n_0 t^{-M} \int_0^{\infty} \exp\left(-\frac{1}{2} s^{\frac{2}{M}} \right) ds = c_{12} t^{-M}. \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} e^{-\lambda_n t} \|f_n\|^2_{L^\infty(\Omega)} \leqslant c_6 t^{-M}, \tag{12}$$

where $c_6 = c_9^2(n_0 + c_{12})$, and (8) follows with the exponent $-\frac{M}{2}$ replaced by $-M$.

Furthermore,

$$\begin{aligned} \|e^{-Ht}f\|_{L^\infty(\Omega)} &= \left\| \sum_{n=0}^{\infty} e^{-\lambda_n t} f_n(x)(f, f_n) \right\|_{L^\infty(\Omega)} \\ &\leqslant \sum_{n=0}^{\infty} e^{-\lambda_n t} \|f_n\|_{L^\infty(\Omega)} |(f, f_n)| \\ &\leqslant \left(\sum_{n=0}^{\infty} e^{-2\lambda_n t} \|f_n\|^2_{L^\infty(\Omega)} \right)^{\frac{1}{2}} \|f\|_{L^2(\Omega)}. \end{aligned}$$

Hence by (12)

$$\|e^{-Ht}f\|_{L^\infty(\Omega)} \leqslant (2^{-M}c_6)^{\frac{1}{2}} t^{-\frac{M}{2}} \|f\|_{L^2(\Omega)}$$

and (7) follows with $c_5 = (2^{-M}c_6)^{\frac{1}{2}}$ and the exponent $-\frac{M}{4}$ replaced by $-\frac{M}{2}$. □

Proof of Theorem 7. As is well known, condition (a) implies that the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compact. Hence H has compact resolvent, its spectrum is discrete, and all its eigenvalues, which are non-negative, are of finite multiplicity. Furthermore, by Proposition 3 and Lemmas 10, 11 (a) implies (b) with $\alpha_1 = \frac{2}{M}$ and $\alpha_2 = \frac{M}{4}$ where $M = \frac{2q}{q-2}$. Conversely by the proof of Lemma 12, it follows that (b) implies inequality (7) with $M = 2(2\alpha_2 + \frac{1}{\alpha_1})$ which in its turn by Proposition 3 implies (a) with $q = \frac{2M}{M-2}$. □

4. Perturbations of the domain

In this section we compare the spectrum of $H_i = -\Delta_N$ acting in $L^2(\Omega_i)$ when Ω_1 and Ω_2 are very close to each other in a suitable sense. We will also need to assume regularity, since it is known that even if Ω_1 has smooth boundary and Ω_2 only differs from it in an arbitrarily small neighbourhood of a single point of $\partial\Omega_1$, the spectrum of H_2 need not be discrete.

We start with the more general argument. Following [4, Chapter 4] we define the variational quantities $\mu_{n,i}$ for all non-negative integers n by

$$\mu_{n,i} = \inf\{\mu(L) : \dim(L) = n + 1\},$$

where $\mu(L)$ is defined for every finite-dimensional subspace L of $L^2(\Omega_i)$ by

$$\mu(L) = \sup\{Q_i(f)/\|f\|^2_{L^2(\Omega)} : 0 \neq f \in L\}$$

and Q_i are defined as in (1). Note that $\mu_{0,i} = 0$ since 0 is an eigenvalue of H_i for $i = 1, 2$. It is known that $\mu_{n,i}$ are equal to the eigenvalues $\lambda_{n,i}$ of H_i written in increasing order and repeated according to multiplicity in case H_i has compact resolvent.

Theorem 13. *Let $\Omega_1 \subset \mathbf{R}^N$ be a region of finite measure.*

1. *If for some $q > 2, c_{13} > 0$*

$$\|f\|_{L^q(\Omega_1)} \leqslant c_{13} \|f\|_{W^{1,2}(\Omega_1)} \tag{13}$$

for all $f \in W^{1,2}(\Omega_1)$, then for every integer $n \geqslant 1$ there exist $b_{n,1} = b_{n,1}(\Omega_1), \varepsilon_{n,1} = \varepsilon_{n,1}(\Omega_1) > 0$ such that for all regions $\Omega_2 \subset \Omega_1$, satisfying $|\Omega_1 \backslash \Omega_2| \leqslant \varepsilon_{n,1}$, the inequality

$$\mu_{n,2} \leqslant (1 + b_{n,1} |\Omega_1 \backslash \Omega_2|) \lambda_{n,1} \tag{14}$$

holds.

2. *If, in addition, $M(\partial\Omega_1) < N$, then for every $\sigma \in (0, N - M(\partial\Omega_1))$ and for every integer $n \geqslant 1$ there exist $b_{n,2} = b_{n,2}(\Omega_1), \varepsilon_{n,2} = \varepsilon_{n,2}(\Omega_1) > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_{n,2}$ and for all regions Ω_2, satisfying $\Omega_1 \backslash \partial_\varepsilon \Omega_1 \subset \Omega_2 \subset \Omega_1$, the inequality*

$$\mu_{n,2} \leqslant (1 + b_{n,2} \varepsilon^\sigma) \lambda_{n,1} \tag{15}$$

holds.

Proof. (1) Let $L = \mathrm{lin}\{\phi_0, \ldots, \phi_n\}$ where ϕ_i are the eigenfunctions of H_1 associated with $\lambda_{i,1}$ and let $M = PL$ where P is the restriction map from $L^2(\Omega_1)$ to $L^2(\Omega_2)$. By Proposition 3 it follows that there exist $M > 2$ and $c_5 > 0$ such that inequality (7) is satisfied with Ω_1 and H_1 replacing Ω and H for all $0 < t \leqslant 1$ and $f \in L^2(\Omega_1)$. Hence if $f = \sum_{k=0}^n \alpha_k \phi_k \in L$, $\|f\|_{L^2(\Omega)} = 1$, then by (7) where $t = 1$ applied to $\mathrm{e}^{H_1} f$ we have

$$\|f\|_{L^\infty(\Omega)} \leqslant c_5 \|\mathrm{e}^{H_1} f\|_{L^2(\Omega)} = c_5 \left\| \sum_{k=0}^n \alpha_k \mathrm{e}^{\lambda_{k,1}} \phi_k \right\|_{L^2(\Omega)}$$
$$\leqslant c_5 \left(\sum_{k=0}^n |\alpha_k|^2 \mathrm{e}^{2\lambda_{k,1}} \right)^{\frac{1}{2}} \leqslant c_5 \mathrm{e}^{\lambda_{n,1}}.$$

Furthermore,

$$\|Pf\|^2_{L^2(\Omega_1)} = \|f\|^2_{L^2(\Omega_2)} = \|f\|^2_{L^2(\Omega_1)} - \|f\|^2_{L^2(\Omega_1 \backslash \Omega_2)}$$
$$\geqslant 1 - |\Omega_1 \backslash \Omega_2| \, \|f\|^2_{L^\infty(\Omega_1 \backslash \Omega_2)} \geqslant 1 - c_5^2 \mathrm{e}^{2\lambda_{n,1}} |\Omega_1 \backslash \Omega_2|.$$

Assume that $|\Omega_1 \backslash \Omega_2| \leqslant (2 c_5^2 \mathrm{e}^{2\lambda_{n,1}})^{-1}$. Then

$$\|Pf\|^{-2}_{L^2(\Omega_1)} \leqslant 1 + 2 c_5^2 \mathrm{e}^{2\lambda_{n,1}} |\Omega_1 \backslash \Omega_2|.$$

If $g = Pf$ where $f \in L$ and $\|f\|_{L^2(\Omega_1)} = 1$, then

$$\frac{Q_2(g)}{\|g\|^2_{L^2(\Omega_2)}} = \frac{Q_2(f)}{\|Pf\|^2_{L^2(\Omega_1)}} \leqslant (1 + 2c_5^2 e^{2\lambda_{n,1}} |\Omega_1 \backslash \Omega_2|) Q_1(f).$$

Since $\dim(M) \leqslant \dim(L) = n + 1$, the first statement of the theorem with $b_{n,1} = 2c_5^2 e^{2\lambda_{n,1}}$ and $\varepsilon_{n,1} = b_{n,1}^{-1}$ follows by using the variational definitions of $\mu_{n,2}$ and $\lambda_{n,1}$.

(2) If $M(\partial\Omega_1) < N$, then for every $\sigma \in (0, N - M(\partial\Omega))$ there exist $\varepsilon_1 = \varepsilon_1(\sigma), a_1 = a_1(\sigma) > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_1$

$$|\partial_\varepsilon \Omega_1| \leqslant a_1 \varepsilon^\sigma. \tag{16}$$

Hence the second statement of the theorem with $b_{n,2} = b_{n,1} a_1$ and $0 < \varepsilon \leqslant \varepsilon_{n,2} \equiv \min\{b_{n,2}^{-\frac{1}{\sigma}}, \varepsilon_1\}$ follows from the first one. $\square$

Remark 14. The size of $b_{n,1}(\Omega)$ and $\varepsilon_{n,1}(\Omega)$ depends upon $\lambda_{n,1}$. An upper bound to $\lambda_{n,1}$ can be given in terms of the inradius $r = \max\{d(x) : x \in \Omega\}$ as follows. If $B(a,r) \subset \Omega$, then $\lambda_{n,1} \leqslant \gamma_{n,a,r}$ where $\gamma_{n,a,r}$ is the nth eigenvalue of $-\Delta$ in $L^2(B(a,r))$ subject to Dirichlet boundary conditions. By scaling one also has $\gamma_{n,a,r} = \gamma_{n,0,1} r^{-2}$.

Remark 15. If for some $\alpha, c_{14} > 0$

$$\|d^{-\alpha} f\|_{L^2(\Omega_1)} \leqslant c_{14} \|f\|_{W^{1,2}(\Omega_1)}$$

for all $f \in W^{1,2}(\Omega_1)$, then both statements of Theorem 13 are valid by Corollary 2.

The conditions of Theorem 13 are not sufficient to establish that H_2 has a compact resolvent, since $\partial\Omega_2$ may have arbitrarily bad local singularities subject to the above conditions. In order to obtain an inequality in the reverse direction we make further assumptions.

Corollary 16. *Assume that* Ω_1 *satisfies the conditions of the first part of Theorem* 13 *and for some* $\sigma > 0$ *inequality* (16) *holds. Moreover, let regions* $\Omega_3(\varepsilon), \varepsilon > 0$, *be such that*

(i) $\Omega_3(\varepsilon) \subset \Omega_1 \backslash \partial_\varepsilon \Omega_1$,

(ii) *there exist* $\varepsilon_2, a_2 > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_2$

$$|\Omega_1 \backslash \Omega_3(\varepsilon)| \leqslant a_2 \varepsilon^\sigma,$$

(iii) *for every integer* $n \geqslant 1$ *there exist* $b_{n,3} = b_{n,3}(\Omega_1), \varepsilon_{n,3} = \varepsilon_{n,3}(\Omega_1) > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_{n,3}$

$$\mu_{n,3} \geqslant \lambda_{n,1}(1 - b_{n,3}\varepsilon^\sigma).$$

Then for every integer $n \geqslant 1$ *there exist* $b_{n,4} = b_{n,4}(\Omega_1), \varepsilon_{n,4} = \varepsilon_{n,4}(\Omega_1) > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_{n,4}$ *and for every region* Ω_2, *for which inequalities* (13) *and* (16) *holds with* Ω_2 *replacing* Ω_1 (*with the same* $q, c_{13}, \sigma, a_1, \varepsilon_1$) *and* $\Omega_1 \backslash \partial_\varepsilon \Omega_1 \subset \Omega_2 \subset \Omega_1$, *the inequality*

$$(1 - b_{n,4}\varepsilon^\sigma)\lambda_{n,1} \leqslant \lambda_{n,2} \leqslant (1 + b_{n,4}\varepsilon^\sigma)\lambda_{n,1} \tag{17}$$

holds.

Proof. An application of Theorem 13 to the pair Ω_2, Ω_1 yields

$$\lambda_{n,2} \leqslant (1 + b_{n,2}\varepsilon^\sigma)\lambda_{n,1}$$

for $0 < \varepsilon \leqslant \varepsilon_{n,2}$. In particular,

$$\lambda_{n,2} \leqslant (1 + b_{n,2}\varepsilon_{n,2}^\sigma)\lambda_{n,1}.$$

An application of Theorem 13 to the pair $\Omega_3(\varepsilon), \Omega_2$ yields

$$\mu_{n,3} \leqslant (1 + b_{n,5}\varepsilon^\sigma)\lambda_{n,2}$$

for $0 < \varepsilon \leqslant \varepsilon_{n,5}$, where $b_{n,5} = 2c_5^2 e^{2\lambda_{n,2}}$ and $\varepsilon_{n,5} = \min\{b_{n,5}^{-(1/\sigma)}, \varepsilon_1\}$. Note that $b_{n,5} \leqslant b_{n,6}$ and $\varepsilon_{n,5} \geqslant \varepsilon_{n,6}$ where $b_{n,6} = 2c_5^2 \exp(2(1 + b_{n,2}\varepsilon_{n,2}^\sigma)\lambda_{n,1})$ and $\varepsilon_{n,2} = \min\{b_{n,6}^{-(1/\sigma)}, \varepsilon_1\}$ depend only on n, σ and Ω_1. Together with assumption (iii) this yields (17). □

The above theorem may be applied to regions with Lip γ boundaries. We start with the simplest example. Let $0 < \gamma \leqslant 1, M, k > 0$ and let

$$\Omega_i = \{x \in \mathbf{R}^N : 0 < x_N < \phi_i(\bar{x}), \bar{x} \in G\}, \quad i = 1, 2,$$

where G is a bounded region in $\mathbf{R}^{N-1}$ with a smooth boundary. We assume that

$$|\phi_i(\bar{x}) - \phi_i(\bar{y})| \leqslant M|\bar{x} - \bar{y}|^\gamma, \quad \bar{x}, \bar{y} \in \bar{G}, \quad i = 1, 2$$

and

$$k^{-1} \leqslant \phi_i(\bar{x}) \leqslant k, \quad \bar{x} \in \bar{G}, \quad i = 1, 2.$$

We do not assume any relationship between the directions of normals of Ω_1 and Ω_2, or even that these normal directions exist.

Lemma 17. *Under the conditions of the last paragraph for every integer* $n \geqslant 1$ *there exist* $b_{n,7} = b_{n,7}(\Omega_1), \varepsilon_{n,7} = \varepsilon_{n,7}(\Omega_1) > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_{n,7}$ *and all* ϕ_2, *satisfying*

$$(1 - \varepsilon)\phi_1(\bar{x}) \leqslant \phi_2(\bar{x}) \leqslant \phi_1(\bar{x}), \quad \bar{x} \in \bar{G},$$

the inequality

$$(1 - b_{n,7}\varepsilon)\lambda_{n,1} \leqslant \lambda_{n,2} \leqslant (1 + b_{n,7}\varepsilon)\lambda_{n,1}$$

holds.

Proof. Since $\partial\Omega_1 \in \mathrm{Lip}\,\gamma$ it follows, as noted in Proposition 3, that inequality (13) is valid for some $q, c_{13} > 0$. Moreover, it also holds with Ω_2 replacing Ω_1 (with the same $q, c_{13} > 0$). Since the operator H_2 has compact resolvent and $|\Omega_1 \backslash \Omega_2| \leqslant k\varepsilon|G|$, inequality (14) yields

$$\lambda_{n,2} \leqslant (1 + b_{n,1}k|G|\varepsilon)\lambda_{n,1}.$$

Similarly for

$$\Omega_3(\varepsilon) = \{x \in \mathbf{R}^N : 0 < x_N < (1 - \varepsilon)\phi_1(\bar{x}), \bar{x} \in G\},$$

where $0 < \varepsilon < 1/2$ we have

$$\lambda_{n,3} \leqslant (1 + b_{n,1}k|G|\varepsilon)\lambda_{n,2}.$$

To derive the estimate below for $\lambda_{n,3}$ we transfer the quadratic form Q_3 to $L^2(\Omega_1)$ by means of the unitary map $U_\varepsilon : L^2(\Omega_1) \to L^2(\Omega_3(\varepsilon))$ defined by

$$(U_\varepsilon f)(\bar{x}, x_N) = (1 - \varepsilon)^{-1/2} f(\bar{x}, (1 - \varepsilon)^{-1} x_N).$$

The inequality

$$Q_1(f) \leqslant Q_3(U_\varepsilon f)$$

valid for all $f \in W^{1,2}(\Omega_1)$, yields the inequality $\lambda_{n,1} \leqslant \lambda_{n,3}$ by the variational method. $\square$

We now turn to the application of Theorem 13 to a general region of Hölder type. The proof of our main result, Theorem 21, depends upon the construction of mappings T_ε of Ω into itself satisfying the properties (20), (22) and (25) below.

Other definitions of regions of Hölder type are possible but Theorem 21 is still valid for such definitions provided similar mappings can be constructed. The underlying idea of that theorem can also be applied to uniformly elliptic operators of the form

$$Hf = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left\{a_{i,j}(x)\frac{\partial f}{\partial x_j}\right\}$$

subject to Neumann boundary conditions provided the coefficients are Hölder continuous in some neighbourhood of the boundary.

Let a bounded region $\Omega \subset \mathbf{R}^N$ be such that $\partial\Omega \in \mathrm{Lip}(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$. Then also $\partial\Omega \in \mathrm{Lip}(\gamma, M, \frac{\delta}{4}, s, \{(V_j)_{\frac{\delta}{2}}\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$. Note that $(V_j)_{\frac{\delta}{2}} = \lambda_j^{-1}((W_j)_{\frac{\delta}{2}}(\tilde{a}_{Nj}, \tilde{b}_{Nj}))$ where $\tilde{a}_{Nj} = a_{Nj} + \frac{\delta}{2}$, $\tilde{b}_{Nj} = b_{Nj} - \frac{\delta}{2}$, and, in addition to conditions (i)–(iii) of the appropriate definition in Section 2, also the following condition is satisfied:

$$\partial\Omega \cap \lambda_j^{-1}\left((W_j)_{\frac{\delta}{2}} \times \left[\tilde{a}_{Nj} - \frac{\delta}{4}, \tilde{a}_{Nj}\right]\right) = \emptyset, \quad j = 1, \dots, s. \tag{18}$$

Moreover, let functions $\psi_j \in C^\infty(\mathbf{R}^N)$ satisfy $0 \leqslant \psi_j \leqslant 1$, supp $\psi_j \subset (V_j)_{\frac{3}{4}\delta}$, $|\nabla\psi_j| \leqslant \frac{b}{\delta}$, where $b > 0$ is a constant, $j = 1, \dots, s$, and $\sum_{j=1}^{s} \psi_j(x) = 1$ for $x \in \Omega$.

Let $e_N = (0, \dots, 0, 1)$ and $\xi_j = \lambda_j^{-1}(e_N)$, $j = 1, \dots, s$. For $x \in \mathbf{R}^N$ and $\varepsilon \in (0, \frac{\delta}{4}]$ define

$$T_\varepsilon(x) = x - \varepsilon \sum_{j=1}^{s} \xi_j \psi_j(x). \tag{19}$$

Lemma 18. *Let a bounded region $\Omega \subset \mathbf{R}^N$ be such that $\partial\Omega \in \mathrm{Lip}(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$. Then there exist $A_1, A_2, A_3, \varepsilon_3 > 0$, depending only on $N, \gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s$, such that for all $0 < \varepsilon \leqslant \varepsilon_3$*

$$\left|\frac{\partial T_{\varepsilon i}}{\partial x_j}(x) - \delta_{ij}\right| \leqslant A_1 \varepsilon, \quad x \in \mathbf{R}^N, \tag{20}$$

in particular, the Jacobian determinant $\mathrm{Jac}(T_\varepsilon, x)$ satisfies the inequality

$$\tfrac{1}{2} \leqslant 1 - A_2\varepsilon \leqslant \mathrm{Jac}(T_\varepsilon, x) \leqslant 1 + A_2\varepsilon, \quad x \in \mathbf{R}^N. \tag{21}$$

Moreover,

$$\Omega \backslash \partial_\varepsilon \Omega \subset T_\varepsilon(\Omega) \subset \Omega \backslash \partial_{\frac{1}{A_3 \varepsilon^{\gamma}}} \Omega. \tag{22}$$

Proof. (1) Since the Jacobi matrix of the map T_ε has the form $I + \varepsilon B$ where I is the identity matrix and B is a matrix whose elements b_{ij} are independent of ε and bounded: $|b_{ij}| \leqslant \frac{bs}{\delta}$, it follows that there exists $A_1 > 0$, depending only on N, δ and s such that inequality (20) is satisfied for all $0 < \varepsilon \leqslant 1$. Hence, for all sufficiently small $\varepsilon > 0$ and for all $x \in \mathbf{R}^N$ inequality (21) is satisfied. Consequently, for all those ε the map $T_\varepsilon : \mathbf{R}^N \to \mathbf{R}^N$ is one-to-one. Indeed, it is locally one-to-one since $\mathrm{Jac}(T_\varepsilon, x) \geqslant \frac{1}{2}$ and it is also globally one-to-one since $|x - y| > 2\varepsilon$ implies $T_\varepsilon(x) \neq T_\varepsilon(y)$. Also $T_\varepsilon(\Omega)$ is a region and $T_\varepsilon(\partial\Omega) = \partial T_\varepsilon(\Omega)$.

(2) For $x \in \mathbf{R}^N$ let

$$J(x) = \{j \in \{1, \dots, s\} : x \in (V_j)_{\frac{3}{4}\delta}\}.$$

The inclusion supp $\psi_j \subset (V_j)_{\frac{3}{4}\delta}$ implies that $\psi_j(x) = 0$ for $j \notin J(x)$ and

$$T_\varepsilon(x) = x - \varepsilon \sum_{j \in J(x)} \xi_j \psi_j(x).$$

Let

$$C(x) = \left\{ \sum_{j \in J(x)} \alpha_j \xi_j, \ \ \alpha_j > 0 \right\}$$

and

$$C(x,\varepsilon) = \left\{ \varepsilon \sum_{j \in J(x)} \alpha_j \xi_j, \ \ \alpha_j > 0, \ \ \sum_{j \in J(x)} \alpha_j < 1 \right\}.$$

We claim that

$$(x - C(x)) \cap \left(\bigcap_{j \in J(x)} V_j \right) \subset \Omega, \quad x \in \bar{\Omega} \tag{23}$$

and

$$(x - C(x,\varepsilon)) \subset \Omega \cap \left(\bigcap_{j \in J(x)} V_j \right), \quad x \in \bar{\Omega}, \quad 0 < \varepsilon \leqslant \tfrac{\delta}{4}. \tag{24}$$

Also,

$$(x - \overline{C(x)}) \cap \left(\bigcap_{j \in J(x)} V_j \right) \subset \Omega, \quad x \in \Omega \tag{25}$$

and

$$(x - \overline{C(x,\varepsilon)}) \subset \Omega \cap \left(\bigcap_{j \in J(x)} V_j \right), \quad x \in \Omega, \quad 0 < \varepsilon \leqslant \frac{\delta}{4}, \tag{26}$$

which implies, in particular, that $T_\varepsilon(\Omega) \subset \Omega$.

Indeed, let $x \in \bar{\Omega}$ and $j_1 \in J(x)$. Since $x \in V_{j_1} \cap \Omega$ and $\lambda_{j_1}(V_{j_1} \cap \Omega)$ is a subgraph it follows that $\{x - \alpha_{j_1} \xi_{j_1}, \alpha_{j_1} > 0\} \cap V_{j_1} \subset \Omega$. Next, let $j_2 \in J(x)$, $j_2 \neq j_1$, and for some $\alpha_{j_1} > 0$, $x - \alpha_{j_1} \xi_{j_1} \in V_{j_1} \cap V_{j_2}$. For similar reasons it follows that $\{x - \alpha_{j_1} \xi_{j_1} - \alpha_{j_2} \xi_{j_2}, \alpha_{j_2} > 0\} \cap V_{j_2} \subset \Omega$, hence

$$\{x - \alpha_{j_1} \xi_{j_1} - \alpha_{j_2} \xi_{j_2}, \alpha_{j_1}, \alpha_{j_2} > 0\} \cap V_{j_1} \cap V_{j_2} \subset \Omega$$

and so on. Since for $y \in C(x,\varepsilon)$

$$|y| = \varepsilon \sum_{j \in J(x)} \alpha_j \xi_j \leqslant \varepsilon \sum_{j \in J(x)} \alpha_j \leqslant \varepsilon \leqslant \frac{\delta}{4},$$

condition (18) implies that for all $j \in J(x)$

$$x - C(x,\varepsilon) \subset ((V_j)_{\frac{3\delta}{4}})^{\frac{\delta}{4}} \subset V_j.$$

Hence $x - C(x,\varepsilon) \subset \bigcap_{j \in J(x)} V_j$ and, by (23), (24) follows. If $x \in \Omega$, then in the argument above one may assume that $\alpha_j \geqslant 0$, $j \in J(x)$, hence (25) and (26) follow.

(3) Assume that $0<\varepsilon\leqslant\min\{\frac{A_3}{2},\frac{\delta}{4}\}$ and $x\in\partial\Omega$. Then

$$d(T_\varepsilon(x))\leqslant|T_\varepsilon x-x|=\varepsilon\left|\sum_{j=1}^{s}\xi_j\psi_j(x)\right|\leqslant\varepsilon\sum_{j=1}^{s}\psi_j(x)=\varepsilon. \tag{27}$$

Given $x\in\Omega$ there exists $\sigma>0$ such that $x\in T_\varepsilon(\Omega)$ for all $0\leqslant\varepsilon<\sigma$ and $x\in\partial(T_\sigma(\Omega))=T_\sigma(\partial\Omega)$. Hence $d(x)\leqslant\sigma$. So $d(x)>\varepsilon$ implies $x\in T_\varepsilon(\Omega)$ and $\Omega\backslash\partial_\varepsilon\Omega\subset T_\varepsilon(\Omega)$.

(4) Let $d_j(x)$ be the distance of $x\in\Omega\cap V_j$ from the boundary $\partial\Omega$ in the direction of ξ_j, i.e., $d_j(x)=\phi_j(\overline{\lambda_j(x)})-(\lambda_j(x))_N$. Then it is known that there exist $A_4>0$ such that for all $j=1,\dots,s$ and all $x\in\Omega\cap V_j$

$$A_4 d_j(x)^{\frac{1}{\gamma}}\leqslant d(x)\leqslant d_j(x).$$

(5) Let $x\in\Omega$ and $j\in J(x)$. Then, for $0<\varepsilon\leqslant\frac{\delta}{4}$, $T_\varepsilon(x)\in\Omega\cap(V_j)_{\frac{\delta}{2}}$ and

$$d(T_\varepsilon(x))^\gamma\geqslant A_4^\gamma d_j(T_\varepsilon(x)).$$

By Step 2

$$\begin{aligned}[T_\varepsilon(x),T_\varepsilon(x)+\varepsilon\psi_j(x)\xi_j]&\subset\{T_\varepsilon(x)+\varepsilon(\psi_j(x)-\alpha_j)\xi_j,\ 0\leqslant\alpha_j\leqslant\psi_j(x)\}\\&=\left\{x-\varepsilon\alpha_j\xi_j-\varepsilon\sum_{i\in J(x),i\neq j}\xi_i\psi_i(x),\ 0\leqslant\alpha_j\leqslant\psi_j(x)\right\}\subset x-\overline{C(x,\varepsilon)}\subset\Omega,\end{aligned}$$

hence

$$d_j(T_\varepsilon(x))\geqslant\varepsilon\psi_j(x).$$

Thus for all $j\in J(x)$

$$d(T_\varepsilon(x))^\gamma\geqslant A_4^\gamma\varepsilon\psi_j(x)$$

and

$$sd(T_\varepsilon(x))^\gamma\geqslant|J(x)|d(T_\varepsilon(x))^\gamma\geqslant A_4^\gamma\varepsilon\sum_{j\in J(x)}\psi_j(x)=A_4^\gamma\varepsilon.$$

Consequently,

$$d(T_\varepsilon(x))\geqslant A_4 s^{-\frac{1}{\gamma}}\varepsilon^{\frac{1}{\gamma}},$$

which implies that $T_\varepsilon(\Omega)\subset\Omega\backslash\partial_{A_1\varepsilon^{\frac{1}{\gamma}}}\Omega$ where $A_1=A_4 s^{-\frac{1}{\gamma}}$. □

Lemma 19. *Under the conditions of Lemma* 18 *there exist* $A_5,\varepsilon_4>0$, *depending only on* $N,\gamma,M,\delta,s,\{V_j\}_{j=1}^s,\{\lambda_j\}_{j=1}^s$, *such that for all* $0<\varepsilon\leqslant\varepsilon_4$

$$|\Omega\backslash T_\varepsilon(\Omega)|\leqslant A_5\varepsilon. \tag{28}$$

By (2) $|\Omega\backslash\partial_\varepsilon\Omega|\leqslant A_6\varepsilon^\gamma$ and $|\Omega\backslash\partial_{A_1\varepsilon^{\frac{1}{\gamma}}}\Omega|\leqslant A_7\varepsilon$. Therefore, the left inclusion of (22) immediately implies that $|\Omega\backslash T_\varepsilon(\Omega)|\leqslant A_5\varepsilon^\gamma$ but (28) makes a stronger claim: estimate (28) has the same order in ε as the estimate for $|\Omega\backslash\partial_{A_1\varepsilon^{\frac{1}{\gamma}}}\Omega|$.

In the proof of Lemma 19, the following property of regions satisfying the cone condition will be used. We say that C is a cone of size $\delta > 0$ if

$$C = \{(\bar{x}, x_D) \in \mathbf{R}^D : 0 < x_D < \delta, \bar{x} \in x_D K\},$$

where K is an open convex set in $\mathbf{R}^{D-1}$ such that $0 \in K \subset B(0,1)$.

Lemma 20. *Let* $\delta > 0$, U, U' *be bounded regions in* $\mathbf{R}^{D-1}$ *and* $U' \subset U_\delta$. *Moreover, let* $\phi, \psi : U \to \mathbf{R}$, $V = \{(\bar{x}, x_D) \in \mathbf{R}^D : \bar{x} \in U, \psi(\bar{x}) < x_D < \phi(\bar{x})\}$ *and let* $C \subset \mathbf{R}^D$ *be a cone of size* δ. *If*

$$(\bar{x}, \phi(\bar{x})) - C \subset V \quad \text{for all } x \in U',$$

then ϕ *satisfies the Lipschitz condition on* U'. *Moreover, the Lipschitz constant depends only on* K.

The proof uses that $\{(\bar{x}, \phi(\bar{x}) + C\} \cap V = \emptyset$ because otherwise $(\bar{x}, \phi(\bar{x})) \in V$.

Proof of Lemma 19. (1) For any subset $J \subset \{1, \ldots, s\}$ put

$$\tilde{V}_J = \{x \in \mathbf{R}^N : J(x) = J\},$$

so $\mathbf{R}^N$ is the disjoint union of all $\tilde{V}_J$: $\mathbf{R}^N = \bigcap_J \tilde{V}_J$, and

$$\Omega \backslash T_\varepsilon(\Omega) = \bigcup_J (\tilde{V}_J \cap (\Omega \backslash T_\varepsilon(\Omega))). \tag{29}$$

We will prove that

$$\tilde{V}_J \cap (\Omega \backslash T_\varepsilon(\Omega)) \subset (\partial_{2\varepsilon} \tilde{V}_J) \cup \tilde{V}_J^{(\varepsilon)}, \tag{30}$$

where

$$\tilde{V}_J^{(\varepsilon)} = \{\tilde{V}_J \cap \partial\Omega - A_J \varepsilon \alpha \xi_J, \ 0 \leqslant \alpha \leqslant 1\}, \tag{31}$$

A_J is a certain positive number and ξ_J is a certain unit vector in $\mathbf{R}^N$.

(2) First let $J = \{i\}$. Then for all $x \in \tilde{V}_J$ we have $J(x) = \{i\}$ and $\psi_i(x) = \sum_{j=1}^s \psi_j(x) = 1$. Hence for all $x \in \tilde{V}_J$

$$T_\varepsilon(x) = x - \varepsilon \xi_i$$

and for all $z \in T_\varepsilon(\tilde{V}_J)$

$$T_\varepsilon^{-1}(z) = z + \varepsilon \xi_i.$$

Assume that $x \in \tilde{V}_J \cap (\Omega \backslash T_\varepsilon(\Omega))$. Since $x \notin T_\varepsilon(\Omega)$ and $T_\varepsilon(x) \in T_\varepsilon(\Omega)$, there is a point $z \in \partial T_\varepsilon(\Omega) = T_\varepsilon(\partial\Omega)$ which lies in the interval $(x - \varepsilon\xi_i, x)$. Hence $y = T_\varepsilon^{-1}(z) = z + \varepsilon\xi_i \in \partial\Omega$, $x \in (y, z) = (y, y - \varepsilon\xi_i) \subset \tilde{V}_J^{(\varepsilon)}$ and (30) follows. (In this case, the first entry of the union in the right-hand side of (30) can be omitted.)

(3) Next let $J = \{i,j\}$. Then, for all $x \in \tilde{V}_J$, $J(x) = \{i,j\}$,

$$T_\varepsilon(x) = x - \varepsilon(\psi_i(x)\xi_i + \psi_j(x)\xi_j).$$

Let

$$C_J = \{\alpha_1\xi_i + \alpha_2\xi_j,\ \alpha_1, \alpha_2 > 0\},$$

$$C_J(\varepsilon) = \{\varepsilon(\alpha_1\xi_i + \alpha_2\xi_j),\ \alpha_1, \alpha_2 > 0,\ \alpha_1 + \alpha_2 < 1\}$$

and let $x \in (V_i)_{\frac{3\delta}{4}} \cap (V_j)_{\frac{3\delta}{4}} \cap \Omega$. Since $J \subset J(x)$, by Step 2 of the proof of Lemma 18 it follows that

$$V_i \cap V_j \cap (x - C_J) \subset V_i \cap V_j \cap (x - \overline{C(x)}) \subset \Omega \tag{32}$$

and

$$x - C_J(x) \subset x - \overline{C(x, \varepsilon)} \subset V_i \cap V_j \cap \Omega. \tag{33}$$

(3.1) First assume that ξ_i and ξ_j are proportional. Let $\xi_i = -\xi_j$. If $x \in V_i \cap V_j \cap \partial\Omega$, then $x - \alpha\xi_i \in \Omega$ for small $\alpha > 0$ and $x - \alpha\xi_j \in \Omega$ for small $\alpha > 0$. So $x + \alpha\xi_i \in \Omega$ and $x - \alpha\xi_i \in \Omega$ for small $\alpha > 0$ which contradicts condition (ii) in the definition of a boundary of class Lip γ. Thus $\xi_i \neq -\xi_j$, hence $\xi_i = \xi_j$ and

$$T_\varepsilon(x) = x - \varepsilon(\psi_i(x) + \psi_j(x))\xi_i = x - \varepsilon\xi_i.$$

Similarly to Step 2 we obtain inclusion (30)–(31) where $\xi_J = \xi_i$ and $A_J = 1$.

(3.2) Next assume that ξ_i and ξ_j are not proportional and set $\xi_J = \frac{\xi_i + \xi_j}{|\xi_i + \xi_j|}$. Let λ_J be a rotation such that $\lambda_J(\xi_J) = e_N$ and the image of the plane spanned by ξ_i and ξ_j is the plane spanned by e_N, e_{N-1}. Inclusion (32) implies that

$$V_i \cap V_j \cap \{x - \alpha\xi_J,\ \alpha > 0\} \subset \Omega. \tag{34}$$

(3.3) Let

$$H_J = \lambda_J(V_i \cap V_j \cap \Omega), \quad H'_J = \lambda_J((V_i)_{\frac{3\delta}{4}} \cap (V_j)_{\frac{3\delta}{4}} \cap \Omega)$$

and G_J, G'_J be the projections of H_J, H'_J, respectively, onto the hyperplane $x_N = 0$. Condition (34) implies that for all $\bar{x} \in G_J$ there exist $\phi_J(\bar{x}), \psi_J(\bar{x})$ such that $(\bar{x}, \phi_J(\bar{x})) \in \Omega$, $\psi_J(\bar{x}) < \phi_J(\bar{x})$ and the intersection of the line, parallel to e_N and passing through $(\bar{x}, 0)$, and H_J is $(\psi_J(\bar{x}), \phi_J(\bar{x}))$. Hence

$$H_J = \{(\bar{x}, x_N) \in \mathbf{R}^N : \bar{x} \in G_J,\ \psi_J(\bar{x}) < x_N < \phi_J(\bar{x})\}.$$

Furthermore, by (33), for all $\bar{x} \in G'_J$

$$(\bar{x}, \phi(\bar{x})) - \lambda_J(C_J(\varepsilon)) \subset H_J.$$

(3.4) Next we apply, for fixed $x_1, \ldots, x_{N-2}$, Lemma 20 where $D = 2$,

$$C = \lambda_J(C_J(\varepsilon)) \subset \mathbf{R}^2_{e_{N-1}, e_N} = \{(x_{N-1}, x_N) : x_{N-1}, x_N \in \mathbf{R}\},$$

$\delta = \varepsilon, K = \{x_{N-1} < \sin\frac{\alpha_J}{2}\}$ (α_J is the angle between ξ_i and ξ_j), U and U' are the projections in $\mathbf{R}^2_{e_{N-1}, e_N}$ onto the line $x_N = 0$ of the cross-sections of G_J, G'_J respectively, in $\mathbf{R}^{N-1}_{e_1, \ldots, e_{N-1}}$ by the line parallel to e_{N-1} and passing through

$(x_1, \dots, x_{N-2}, 0)$, $\phi = \phi_J, \psi = \psi_J$. It follows that, for fixed $x_1, \dots, x_{N-2}$, the function ϕ_J satisfies the Lipschitz condition in x_{N-1}. Moreover, the Lipschitz constant L_J depends only on K, hence on α_J, therefore is independent of $x_1, \dots, x_{N-2}$. (In fact $L_J = \cot \frac{\alpha_J}{2}$.) Thus, the function ϕ_J satisfies the Lipschitz condition in x_{N-1} uniformly with respect to $x_1, \dots, x_{N-2}$:

$$|\phi_J(x_1, \dots, x_{N-2}, x_{N-1}) - \phi_J(x_1, \dots, x_{N-2}, y_{N-1})| \leqslant L_J |x_{N-1} - y_{N-1}|$$

for all $x_1, \dots, x_{N-2}, x_{N-1}, y_{N-1}$ satisfying $(x_1, \dots, x_{N-2}, x_{N-1}), (x_1, \dots, x_{N-2}, y_{N-1}) \in G'_J$.

(3.5) Assume that $x \in (\tilde{V}_J)_{2\varepsilon} \cap (\Omega \backslash T_\varepsilon(\Omega))$. Since $x \notin T_\varepsilon(\Omega)$ and $T_\varepsilon(x) \in T_\varepsilon(\Omega)$, there is a point $z \in \partial T_\varepsilon(\Omega) = T_\varepsilon(\partial\Omega)$ which lies in the interval $[x, T_\varepsilon(x))$. Let $y = T_\varepsilon^{-1}(z)$, then $y \in \partial\Omega$. If $y \notin \tilde{V}_J$, then by (27)

$$d(x, \partial\tilde{V}_J) \leqslant |x - y| \leqslant |x - z| + |z - y| < |x - T_\varepsilon(x)| + |y - T_\varepsilon(y)| \leqslant 2\varepsilon,$$

which contradicts the assumption $x \in (\tilde{V}_J)_{2\varepsilon}$. So $y \in \tilde{V}_J \cap \partial\Omega$, hence $z \in y - C_J(\varepsilon)$ where $C_J(\varepsilon)$.

Let $\lambda_J(x) = \beta, \lambda_J(y) = \eta, \lambda_J(z) = \zeta$ and let $d_J(x)$ denote the distance of $x \in \Omega$ from $\partial\Omega$ in the direction of the vector ξ_J, then

$$\beta = (\beta_1, \dots, \beta_{N-2}, \beta_{N-1}, \beta_N), \quad \eta = (\beta_1, \dots, \beta_{N-2}, \eta_{N-1}, \eta_N),$$

$$\eta_N = \phi_{ij}(\beta_1, \dots, \beta_{N-2}, \eta_{N-1}), \quad \zeta = (\beta_1, \dots, \beta_{N-2}, \zeta_{N-1}, \zeta_N),$$

$$d_J(x) \leqslant \phi_J(\beta_1, \dots, \beta_{N-2}, \beta_{N-1}) - \beta_N.$$

Since also $z \in [x, T_\varepsilon(x)) \subset x - \overline{C_J(\varepsilon)}$, it follows that

$$|\eta_{N-1} - \zeta_{N-1}|, |\eta_N - \zeta_N| \leqslant |\eta - \zeta| = |z - T_\varepsilon(z)| \leqslant \varepsilon,$$
$$|\zeta_{N-1} - \beta_{N-1}| \leqslant |z - x| \leqslant |x - T_\varepsilon(x)| \leqslant \varepsilon.$$

Also $\zeta_N < \beta_N$. Therefore,

$$\begin{aligned} d_J(x) &\leqslant \phi_J(\beta_1, \dots, \beta_{N-2}, \beta_{N-1}) - \phi_J(\beta_1, \dots, \beta_{N-2}, \eta_{N-1}) \\ &\quad + \phi_J(\beta_1, \dots, \beta_{N-2}, \eta_{N-1}) - \beta_N \\ &\leqslant L_J |\beta_{N-1} - \eta_{N-1}| + \eta_N - \zeta_N \\ &\leqslant L_J(|\beta_{N-1} - \zeta_{N-1}| + |\zeta_{N-1} - \eta_{N-1}|) + \varepsilon \\ &\leqslant (2L_{ij} + 1)\varepsilon. \end{aligned}$$

Hence (30)–(31) holds where $A_J = 2L_{ij} + 1$.

(4) The argument for the cases in which the number of elements in J is greater than 2 is similar. Let, for example, $J = \{i, j, k\}$. If $\dim \mathrm{Span}\{\xi_i, \xi_j, \xi_k\} = 1$, then $\xi_i = \xi_j = \xi_k$, and we set $\xi_J = \xi_i$ and argue as in Step 3.1. If $\dim \mathrm{Span}\{\xi_i, \xi_j, \xi_k\} = 2$, then we take any two linearly independent vectors, say ξ_i, ξ_j, set $\xi_J = \frac{\xi_i + \xi_j}{|\xi_i + \xi_j|}$ and argue as in Steps 3.2–3.5. If $\dim \mathrm{Span}\{\xi_i, \xi_j, \xi_k\} = 3$, then we set $\xi_{ijk} = \frac{\xi_i + \xi_j + \xi_k}{|\xi_i + \xi_j + \xi_k|}$ and argue as in Steps 3.2–3.5. In this case the appropriate function ϕ_J satisfies the Lipschitz

condition in x_{N-1}, x_{N-2} uniformly with respect to $x_1, \ldots, x_{N-3}$. In Lemma 20 one can take K to be the projection of $C_J(\varepsilon)$ onto $x_N = 0$ and $\delta = \inf\{x_N : (x_{N-3}, x_{N-2}, x_{N-1}) \in C_J(\varepsilon)\}$.

(5) By (29) and (30)

$$|\Omega \backslash T_\varepsilon(\Omega)| \leqslant \sum_{J \subset \{1,\ldots,s\}} (|\partial_{2\varepsilon} \tilde{V}_J| + |\tilde{V}_J^{(\varepsilon)}|).$$

Since each summand does not exceed ε multiplied by a constant depending only on $N, \gamma, M, \delta, s, \{V_j\}_{j=1}^s$ and $\{\lambda_j\}_{j=1}^s$, inequality (28) follows. $\square$

The proof of the following theorem can be adapted to any situation in which maps Λ_ε exist with properties (36)–(38).

Theorem 21. *Let $N \geqslant 2, 0 < \gamma \leqslant 1$, $M, \delta > 0$ and integer $s \geqslant 1$. Moreover, let $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{\lambda_j\}_{j=1}^s$ a family of rotations. Suppose that $\Omega_1 \subset \mathbf{R}^N$ is a bounded region such that $\partial\Omega_1 \in \mathrm{Lip}(\gamma, M, \delta, s, V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$.*

Then for every integer $n \geqslant 1$ there exist $b_{n,7} = b_{n,7}(\Omega_1), \varepsilon_{n,7} = \varepsilon_{n,7}(\Omega_1) > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_{n,7}$ and for all bounded regions Ω_2, for which $\partial\Omega_2 \in \mathrm{Lip}(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$ and $\Omega_1 \backslash \partial_\varepsilon \Omega_1 \subset \Omega_2 \subset \Omega_1$, the inequality

$$(1 - b_{n,7}\varepsilon^\gamma)\lambda_{n,1} \leqslant \lambda_{n,2} \leqslant (1 + b_{n,7}\varepsilon^\gamma)\lambda_{n,1} \tag{35}$$

holds.

Proof. (1) Since $\partial\Omega_1 \in \mathrm{Lip}(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$ it follows that inequality (13) holds, where $q > 2$ and $c_{13} > 0$ depend only on $N, \gamma, M, \delta, s, \{V_j\}_{j=1}^s$, $\{\lambda_j\}_{j=1}^s$ [6,11,13]. Since also $\partial\Omega_2 \in \mathrm{Lip}(\gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$ inequality (13) holds with Ω_2 replacing Ω_1 with the same q and c_{13}. Furthermore, by (2) there exist $A_8, \varepsilon_5 > 0$, depending only on $N, \gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s$ such that for all $0 < \varepsilon \leqslant \varepsilon_5$

$$|\partial_\varepsilon \Omega_1|, |\partial_\varepsilon \Omega_2| \leqslant A_8 \varepsilon^\gamma.$$

(2) Let

$$\Lambda_\varepsilon = T_{(\frac{\varepsilon}{A_1})^\gamma}.$$

Then by Lemmas 18 and 19 there exist $A_9, A_{10}, A_{11}, \varepsilon_6 > 0$, depending only on N, γ, M, δ, s, $\{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s$, such that for $0 < \varepsilon \leqslant \varepsilon_6$

$$\Omega_3(\varepsilon) = \Lambda_\varepsilon(\Omega_1) \subset \Omega_1 \backslash \partial_\varepsilon \Omega_1, \tag{36}$$

$$|\Omega_1 \backslash \Omega_3(\varepsilon)| \leqslant A_9 \varepsilon^\gamma \tag{37}$$

and

$$\left|\frac{\partial \Lambda_{\varepsilon i}}{\partial x_j}(x) - \delta_{ij}\right| \leqslant A_{10}\varepsilon^{\gamma}, \tag{38}$$

which implies that

$$\tfrac{1}{2} \leqslant 1 - A_{11}\varepsilon^{\gamma} \leqslant \mathrm{Jac}(\Lambda_\varepsilon, x) \leqslant 1 + A_{11}\varepsilon^{\gamma}, \quad x \in \Omega_1 \tag{39}$$

and the map $\Lambda_\varepsilon : \Omega_1 \to \Lambda_\varepsilon(\Omega_1)$ is one-to-one. Denote $U_\varepsilon = \Lambda_\varepsilon^{-1}$.

(3) Next we obtain a lower bound for $\mu_{n,3}$ by applying the idea used in the proof of Lemma 17 based on the variational method.

If L is an $(n+1)$-dimensional subspace of $L^2(\Omega_1)$, then $\tilde{L} = \{f(U_\varepsilon(x)),\ f \in L\}$ is an $(n+1)$-dimensional subspace of $L^2(\Lambda_\varepsilon(\Omega_1))$, and conversely, if $\tilde{L}$ is an $(n+1)$-dimensional subspace of $L^2(\Lambda_\varepsilon(\Omega_1))$, then $L = \{g(\Lambda_\varepsilon(x)), g \in \tilde{L}\}$ is an $(n+1)$-dimensional subspace of $L^2(\Omega_1)$. Therefore,

$$\mu_{n,3} = \inf_{\tilde{L}:\dim \tilde{L}=n+1} \ \sup_{g \in \tilde{L}} \frac{\int_{\Omega_3(\varepsilon)} |\nabla g|^2 \, \mathrm{d}^N y}{\int_{\Omega_3(\varepsilon)} |g|^2 \, \mathrm{d}^N y}$$
$$= \inf_{L:\dim L=n+1} \ \sup_{f \in L} \frac{\int_{\Lambda_\varepsilon(\Omega_1)} |\nabla(f(U_\varepsilon(y)))|^2 \, \mathrm{d}^N y}{\int_{\Lambda_\varepsilon(\Omega_1)} |f(U_\varepsilon(y))|^2 \, \mathrm{d}^N y}.$$

Note that

$$|\nabla(f(U_\varepsilon(y)))|^2 = \sum_{i=1}^{N} \left| \sum_{k=1}^{N} \left(\frac{\partial f}{\partial x_k}\right)(U_\varepsilon(y)) \frac{\partial (U_\varepsilon(y))_k}{\partial y_i} \right|^2$$
$$= \sum_{i=1}^{N} \left| \left(\frac{\partial f}{\partial x_i}\right)(U_\varepsilon(y)) \right|^2 \left| \frac{\partial (U_\varepsilon(y))_i}{\partial y_i} \right|^2 + \sum_{i=1}^{N} \sum_{k,l=1}^{N}{}^{i} \left(\frac{\partial f}{\partial x_k}\right)(U_\varepsilon(y)) \overline{\left(\frac{\partial f}{\partial x_l}\right)(U_\varepsilon(y))} \frac{\partial (U_\varepsilon(y))_k}{\partial y_i} \overline{\frac{\partial (U_\varepsilon(y))_l}{\partial y_i}},$$

where $\sum^i$ means that summation is taken with respect to such k, l that either $k \neq i$ or $l \neq i$.

Recall that

$$\frac{\partial (U_\varepsilon(y))_k}{\partial y_i} = \Delta_{ki} (\mathrm{Jac}(\Lambda_\varepsilon, U_\varepsilon(y)))^{-1},$$

where $(-1)^{k+i}\Delta_{ki}$ is the determinant obtained by deleting kth row and ith column in the Jacobian determinant $\mathrm{Jac}(\Lambda_\varepsilon, U_\varepsilon(y)))$. The Jacobi matrix of the map Λ_ε has the form $I + (\frac{\varepsilon}{A_1})^{\gamma} B$, where I and B are defined in the proof of Lemma 18. Hence $(-1)^{k+i}\Delta_{ki}$ is the determinant of the matrix $I_{ki} + (\frac{\varepsilon}{A_1})^{\gamma} B_{ki}$, where I_{ki} and B_{ki} are obtained by deleting kth rows and ith columns in matrices I, B respectively. Since $|I_{ii}| = 1$ and $|I_{ki}| = 0$ if $k \neq i$ and the elements of the matrix B are uniformly bounded, there exists $A_{12}, \varepsilon_7 > 0$, depending only on N, δ and s, such that for all $0 < \varepsilon \leqslant \varepsilon_7$ and $y \in \Lambda_\varepsilon(\Omega_1)$

$$\frac{1}{2} \leqslant 1 - A_{12}\varepsilon^\gamma \leqslant \left|\frac{\partial (U_\varepsilon(y))_i}{\partial y_i}\right| \leqslant 1 + A_{12}\varepsilon^\gamma, \quad i = 1, \dots, N$$

and

$$\left|\frac{\partial (U_\varepsilon(y))_k}{\partial y_i}\right| \leqslant A_{12}\varepsilon^\gamma, \quad i, k = 1, \dots, N, \quad k \neq i.$$

Consequently there exists $A_{13} > 0$, depending only on N, δ and s, such that for all $0 < \varepsilon \leqslant \varepsilon_7$ and $y \in \Lambda_\varepsilon(\Omega_1)$

$$(1 - A_{13}\varepsilon^\gamma)|(\nabla f)(U_\varepsilon(y))|^2 \leqslant |\nabla(f(U_\varepsilon(y)))|^2 \leqslant (1 + A_{13}\varepsilon^\gamma)|(\nabla f)(U_\varepsilon(y))|^2.$$

Therefore, by changing variables: $y = \Lambda_\varepsilon(x)$ and taking into account inequality (38), we have

$$\begin{aligned} \mu_{n,3} &\geqslant (1 - A_{13}\varepsilon^\gamma) \inf_{L:\dim L = n+1} \sup_{f \in L} \frac{\int_{\Lambda_\varepsilon(\Omega_1)} |(\nabla f)(U_\varepsilon(y))|^2 \, \mathrm{d}^N y}{\int_{\Lambda_\varepsilon(\Omega_1)} |f(U_\varepsilon(y))|^2 \, \mathrm{d}^N y} \\ &= (1 - A_{13}\varepsilon^\gamma) \inf_{L:\dim L = n+1} \sup_{f \in L} \frac{\int_{\Omega_1} |(\nabla f)(x)| \mathrm{Jac}(\Lambda_\varepsilon, x)| \, \mathrm{d}^N x}{\int_{\Omega_1} |f(x)|^2 |\mathrm{Jac}(\Lambda_\varepsilon, x)| \, \mathrm{d}^N x} \\ &\geqslant (1 - A_{13}\varepsilon^\gamma)(1 - A_{11}\varepsilon^\gamma)(1 + A_{11}\varepsilon^\gamma)^{-1} \inf_{L:\dim L = n+1} \sup_{f \in L} \frac{\int_{\Omega_1} |\nabla f|^2 \, \mathrm{d}^N x}{\int_{\Omega_1} |f|^2 \, \mathrm{d}^N x}. \end{aligned}$$

Hence, finally, there exist $b_9, \varepsilon_9 > 0$, depending only on $N, \gamma, M, \delta, s, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s$, such that for $0 < \varepsilon \leqslant \varepsilon_9$

$$\mu_{n,3} \geqslant (1 - \varepsilon^\gamma b_9)\lambda_{n,1}.$$

Now the theorem follows by taking into account (36) and (37), and applying Corollary 16. □

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