Cohomogeneity one Riemannian manifolds of non-positive curvature

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1

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Abstract

We study a G-manifold M which admits a G-invariant Riemannian metric g of non-positive curvature. We describe all such Riemannian G-manifolds (M, g) of non-positive curvature with a semisimple Lie group G which acts on M regularly and classify cohomogeneity one G-manifolds M of a semisimple Lie group G which admit an invariant metric of non-positive curvature. Some results on non-existence of invariant metric of negative curvature on cohomogeneity one G-manifolds of a semisimple Lie group G are given.

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Introduction

Homogeneous Riemannian manifolds of non-positive curvature have been studied in details in [1,3,9]. In [15] F. Podesta and A. Spiro initiated an investigation of cohomogeneity one Riemannian manifolds of negative curvature and got some interesting results. Some of them were generalized in [12] to the case of non-positive curvature.

The aim of this paper is to study cohomogeneity one complete Riemannian G-manifolds of non-positive curvature of a connected semisimple Lie group G.

Let (M, g) be a simply connected, cohomogeneity one complete Riemannian G-manifold of non-positive curvature. If the action of G on M is regular, that is all orbits are isomorphic to G/K where K is a maximal compact...
subgroup of $G$ then one can identify $M = \mathbb{R} \times G/K$ and $g = dt^2 + g_t$ where $g_t$ is a one parameter family of invariant Riemannian metrics on the homogeneous manifold $G/K$. If the action of $G$ is not regular then $Gp = G/K$ is the only singular orbit and all other orbits are isomorphic to $G/H$ where $H$ is a proper subgroup of $K$. In this case we can identify $M$ with a twisted product

$$M = G \times K V_\rho = (G \times V_\rho)/K$$

(1)

where $\rho : K \rightarrow GL(V_\rho)$ is a representation of the compact group $K$ into a vector space $V_\rho$ which is sphere transitive (i.e. has a codimension one orbit) and the action of $K$ on $M$ is given by

$$K \ni k : M \ni [a, v] \mapsto [ak^{-1}, \rho(k)] \in M.$$  

All sphere transitive compact connected linear Lie groups $\rho(K)$ were classified by A. Borel. A $G$-manifold (1) is called an admissible $G$-manifold. A natural question arises:

Which admissible $G$-manifold has an invariant metric of non-positive or negative curvature? One of the main results of the paper can be stated as follows.

**Theorem.** An admissible $G$-manifold $M$ of a semisimple connected Lie group $G$ which acts effectively admits an invariant metric of non-positive curvature if and only if $G$ has a finite center.

An explicit construction of such metrics is given.

Now we indicate the structure of the paper. In Section 1 we state some known results about non-positive curved Riemannian manifolds and $G$-manifolds and fix notations. In Section 2 we describe the structure of a Riemannian $G$-manifold $(M, g)$ of non-positive curvature, where $G$ is a connected semisimple Lie group of isometries with finite center which acts on $M$ regularly. In particular, it gives a description of all regular cohomogeneity one Riemannian $G$-manifolds of a semisimple Lie group $G$. These results easily follows from classical results by R.L. Bishop and B. O’Neill [6] about metrics of negative curvature.

In Section 3, a classification of admissible $G$-manifolds of a connected semisimple Lie group $G$ (Theorems 3.4, 3.5, 3.6) are given. The problem reduces to the case when $G$ is simple or a product of two simple Lie groups.

We calculate the curvature tensor of an invariant metric $g$ on an admissible $G$-manifold $M$ in Section 4. Using it, we construct explicitly a family of invariant metrics $g$ of non-positive curvature on $M$ in the case when $G$ is a semisimple Lie group with finite center (Theorem 4.10). The metrics $g$ are diagonal in the sense of Grove–Ziller [8].

In the case when the singular orbit $G/K$ is a rank one symmetric space, our construction gives a metric of negative curvature. We prove some results about non-existence of invariant metrics of negative curvature on some admissible $G$-manifolds of semisimple group. For this, we use a simple sufficient condition that the singular orbit is totally geodesic. We construct also an invariant metric of negative curvature on a direct product $M = \mathbb{R} \times G/K$ where $K$ is a maximal compact subgroup of a semisimple Lie group $G$ with infinite center. Note that the homogeneous manifold $G/K$ does not admit an invariant metric of non-positive curvature.

In the last section, we give a description of non-simply connected cohomogeneity one Riemannian $G$-manifold $M$ of non-positive curvature where $G$ is a semisimple Lie group. We prove that the universal covering manifold $\tilde{M}$ of $M$ is a regular $\tilde{G}$-manifold of the form $\tilde{M} = \mathbb{R} \times \tilde{G}/\tilde{K}$ where $\tilde{G}$ is a covering group of $G$ and $\tilde{K}$ is its maximal compact subgroup. Moreover, if $\tilde{G}$ has finite center then $M$ is a Riemannian direct product $M = \mathbb{R} \times G/K$ where $K$ is a maximal compact subgroup of $G$.

By a $G$-manifold we mean an $n$-dimensional manifold $M$ with an almost effective action of a connected Lie group $G$ i.e. such that the kernel of effectiveness $\Gamma' = \{ g \in G, gx = x, \forall x \in M \}$ is a discrete subgroup of $G$. We say that a $G$-manifold is effective if $G$ acts on $M$ effectively. We denote by capital Latin letters $A, G, K, H, \ldots$ Lie groups and by corresponding Gothic letters $a, g, t, h, \ldots$ their Lie algebras.

### 1. Preliminaries

In this section we fix notations and recall some basic facts about Riemannian manifolds of non-positive curvature and Riemannian $G$-manifolds.
1.1. Basic properties of Riemannian manifolds of non-positive curvature

Let $(M^n, g)$ be a simply connected complete Riemannian manifold of non-positive curvature. Then

1.1. The exponential map $\exp_o : T_o M \rightarrow M$ of the tangent space $T_o M$ at a point $o \in M$ is a diffeomorphism. In particular, any two points $x, y \in M$ are joined by a unique geodesic.

1.2. The fixed point set $M^K$ of a compact subgroup $K$ of the isometry group $\text{Iso}(M, g)$ is a non-empty connected totally geodesic submanifold of $M$. In particular, if $M = G/K$, $G \subseteq \text{Iso}(M, g)$, is a homogeneous manifold, then $K$ is a maximal compact subgroup of $G$ (Cartan Theorem).

Recall that a (smooth) function $f$ on a Riemannian manifold $(M, g)$ is called convex (resp., strictly convex) if its Hessian $H_f(x) = \nabla^2 f|_x$ is non-negatively definite (resp., positively definite) quadratic form on $T_x M$ for any $x \in M$. We denote by $d_\varphi(x) = d(x, \varphi x)$ the distance function of an isometry $\varphi$ of $M$ where $d$ is the Riemannian distance of $M$.

1.3. (a) For any isometry $\varphi \in \text{Iso}(M, g)$ the function $d_\varphi^2$ is smooth and convex ([17] or [6]).

(b) The set $\text{crit}(\varphi)$ of critical points of $d_\varphi^2$ is a totally geodesic connected submanifold possibly with boundary [14].

(c) If $\min d_\varphi = a > 0$, then $a^2$ is the only critical value of $d_\varphi^2$ and $\text{crit}(\varphi)$ may be decomposed into a Riemannian direct product $W \times \mathbb{R}$ where $W$ is a closed convex subset of $M$ and $\varphi$ leaves this decomposition invariant. More precisely, $\varphi|_W \times \mathbb{R} = (1_W, T_a)$, where $T_a$ is the parallel translation $t \mapsto t + a$ in $\mathbb{R}$ [17, Prop. V.4.16].

1.4. If $(M, g)$ has no flat factor in its De Rham decomposition, then the only isometry $\varphi$ of $M$ with bounded distance function $d_\varphi$ is the identity [19].

1.5. Let $M$ be a connected Riemannian homogeneous manifold of non-positive curvature. Then $M$ is isometric to the product of a flat torus and a simply connected homogeneous Riemannian manifold. [19].

1.1.1. Examples of Riemannian manifolds of non-positive curvature

The simplest examples of Riemannian manifolds of non-positive curvature are symmetric spaces of non-compact type. They can be described as follows:

1.6. Let $G$ be a connected semisimple Lie group without compact factor and with finite center, and $K$ its maximal compact subgroup. Then $M = G/K$ equipped with a $G$-invariant Riemannian metric $g$ is a non-compact Riemannian symmetric space of non-positive curvature (in fact if $G/K$ is a symmetric space of non-compact type, then $K$ is compact iff $G$ has finite center, see [10]). Let

$$G = G_1 \cdots G_l$$

be the decomposition of $G$ into a product of simple factors, $K_i = K \cap G_i, \ i = 1, \ldots, l$, the maximal compact subgroup of $G_i$ and $g_i$ the invariant metric on the irreducible symmetric space $G_i/K_i$ (uniquely defined up to scaling). Then

$$M = G/K = G_1/K_1 \times \cdots \times G_l/K_l$$

is the De Rham decomposition of the Riemannian manifold $(M = G/K, g)$ into a product of irreducible symmetric spaces $M_i = G_i/K_i, \ i = 1, \ldots, l$, and

$$g = c_1 g_1 + \cdots + c_l g_l$$

where $c_i > 0$ are constants.

Moreover, this example exhausts all homogeneous Riemannian manifolds of non-positive curvature of semisimple Lie group $G$ as the following proposition shows.

**Proposition 1.7.** Let $M = G/K$ be a homogeneous Riemannian manifold of non-positive curvature of a semisimple connected Lie group $G$ which acts effectively on $M$. Then $M = G/K$ is a symmetric space of non-compact type and $K$ is a maximal compact subgroup of the group $G$ and $G$ has trivial center.

**Proof.** By 1.5, the manifold $M$ is simply connected and by 1.2 $K$ is a maximal compact subgroup of $G$. It is sufficient to prove that $G$ has finite center. Then by effectiveness of the action the center of $G$ is trivial and the result follows from 1.6. Assume that $G$ has infinite center and choose an element $\sigma$ of its center which generate a subgroup $Z = \langle \sigma \rangle \cong \mathbb{Z}$ (such a $\sigma$ exists, since a simple Lie group with infinite center has a torsion free central subgroup isomorphic...
to $\mathbb{Z}$ (see [13]). Then $Z \cap K = \{e\}$ (since $Z$ is a discrete cyclic group of infinite order and $K$ is compact). Hence, $\sigma$ acts non-trivially on $M = G/K$. On the other hand $d_\sigma(gK) = d(gK, \sigma gK) = d(gK, g\sigma K) = d(eK, \sigma K)$ is constant which contradicts to 1.4. \hfill \Box

Using the warped product, one can construct non-homogeneous Riemannian manifold of non-positive curvature due to the following result by R.L. Bishop and B. O’Neill [6].

**Definition 1.8.** Let $(B, g_B)$ and $(F, g_F)$ be Riemannian manifolds and $f$ a positive smooth function on $B$. The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the metric

$$g_M = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F).$$

where $\pi : B \times F \longrightarrow B$ and $\sigma : B \times F \longrightarrow F$ are projections.

We simplify the notation and write $h$ for $\pi^* h = h \circ \pi$.

The natural projection $\pi : M = B \times F \longrightarrow B$ is a Riemannian submersion of $(M, g_M)$ onto $(B, g_B)$ and that the metric $g_M$ is complete if and only if $g_B$ and $g_F$ are complete.

**Proposition 1.9.** (See [6].) Let $(B, g_B), (M_1, g_1), \ldots, (M_l, g_l)$ be complete Riemannian manifolds and $f_1, \ldots, f_l$ positive functions on $B$. Consider the manifold $M = B \times M_1 \times \cdots \times M_l$ with the metric $g = g_B + f_1^2 g_1 + \cdots + f_l^2 g_l$. Then

(i) $(M, g)$ has negative curvature if and only if the following conditions hold:

1. $\dim B = 1$ or $B$ has negative curvature.
2. $(M_1, g_1), \ldots, (M_l, g_l)$ have non-positive curvature.
3. Each function $f_j$ is strictly convex on $B$ and $g_B(\nabla f_i, \nabla f_j) > 0$, $1 \leq i, j \leq l$.

(ii) $(M, g)$ has non-positive curvature if and only if the following conditions hold:

1. $(B, g_B), (M_1, g_1), \ldots, (M_l, g_l)$ have non-positive curvature.
2. The functions $f_i$ are convex and $g_B(\nabla f_i, \nabla f_j) \geq 0 \forall i, j$.

**Proof.** For $l = 2$, it is proved in [6]. The general case follows by induction, using the following lemma.

**Lemma 1.10.** (See [6].) A convex (resp., strictly convex) function $h$ on $B$ lifts to a convex (resp., strictly convex) function $\pi^* h$ on a warped product $M = B \times_f F$ if and only if $g_B(\nabla f, \nabla h) \geq 0$ (resp., $g_B(\nabla f, \nabla h) > 0$).

In particular $h = \pi^* f$ is strictly convex on $M$ if and only if $f$ is strictly convex on $B$ and has no critical point.

Note also that any critical point of a convex function is a minimum point and $\pi^* f$ has a minimum on $M$ if and only if $f$ has a minimum on $B$. \hfill \Box

1.2. Riemannian $G$-manifolds

A Riemannian $G$-manifold is a Riemannian manifold $(M, g)$ together with an isometric action $\phi : G \longrightarrow \text{Iso}(M, g)$ of a Lie group $G$. We will always assume that the action of $G$ on $M$ is almost effective and that $\phi(G)$ is a connected and closed subgroup of the full group of isometries of $M$. If $\text{Ker}\phi$ is finite (or if the action is effective) then the stabilizer $G_x$ of any point $x \in M$ is a compact subgroup of $G$. It is known [7] that there exists an open dense and connected $G$-invariant submanifold $M_{\text{reg}}$ of $M$ which consists of the points $x$, whose stabilizer $G_x$ is conjugate to a fixed subgroup $K$ of $G$, and such that for any point $y \in M \setminus M_{\text{reg}}$ the stabilizer $G_y$ is conjugate to a subgroup which properly contains $K$. The points of $M_{\text{reg}}$ (resp. $M \setminus M_{\text{reg}}$) and their orbits are called regular (resp, singular). We say that a group $G$ acts on $M$ regularly or $M$ is a regular $G$-manifold if all orbits of $G$ are regular, i.e. $M = M_{\text{reg}}$. If $(M, g)$ is a connected Riemannian $G$-manifold, then $(M_{\text{reg}}, g|_{M_{\text{reg}}})$ is an open connected regular $G$-manifold.

The structure of a Riemannian $G$-manifold in a neighborhood of an orbit $Gx \cong G/H$ is described by the following slice theorem.
Slice Theorem 1.11. Let \((M, g)\) be a Riemannian G-manifold, \(\rho : H \rightarrow O(V) \subseteq G(V)\) the natural orthogonal representation of the stabilizer \(H = G_x\) in the normal space \(V = T^\perp_x(Gx)\) and \(B \subseteq V\) the ball of sufficiently small radius with center at the origin. Then the exponential map
\[
\exp_{g_x} : T^\perp_x(Gx) \ni g_x v \mapsto \exp_{g_x}(g_x v) \in M, \quad g \in G, \quad v \in B
\]
defines a G-equivariant diffeomorphism of the tubular neighborhood \(GB = G \times_H B = (G \times B)/H\) of the zero section of the normal bundle \(T^\perp_x(Gx)\) onto a G-invariant neighborhood \(M(x)\) of the orbit \(Gx\).

This allows to identify the G-invariant neighborhood \(M(x)\) of the orbit \(Gx\) with the twisted product \(G \times_H B = (G \times B)/H\) where the action of \(H\) on \(G \times B\) is given by \(h(g, b) = (gh^{-1}, \rho(h)b)\). A point \(x \in M\) is regular, if and only if the representation of \(H = G_x\) in the normal space \(V\) is trivial. This means that the G-invariant neighborhood \(M(x)\) can be identified with a direct product \(M(x) = G/H \times B\).

2. Regular Riemannian G-manifolds of non-positive curvature

In this section we describe the structure of a simply connected complete regular Riemannian G-manifold of non-positive curvature \((M, g)\) of a connected semisimple Lie group \(G\). Let \(B = M/G\) be the orbit space. Then the natural projection \(\pi : M \rightarrow B = M/G\) is a locally trivial bundle with typical fiber \(F = G/K\) (see 9.3. of [4]) and the metric \(g\) induces a Riemannian metric \(g_B\) on the base manifold \(B\) such that \(\pi : M \rightarrow B\) is a Riemannian submersion.

The following theorem describes the structure of a regular simply connected Riemannian G-manifold of non-positive curvature where \(G\) is a semisimple Lie group with finite center.

Theorem 2.1. (i) Let \((M_1 = G_1/K_1, g_1), \ldots, (M_l = G_l/K_l, g_l)\) be irreducible non-flat Riemannian symmetric spaces of non-positive curvature, \((B, g_B)\) a complete simply connected Riemannian manifold of non-positive curvature and \(f_1, \ldots, f_l\) positive convex smooth functions on \(B\) such that \(g_B(\text{grad } f_i, \text{grad } f_j) \geq 0, i, j = 1, \ldots, l\). Then the simply connected Riemannian manifold \((M, g)\) where \(M = B \times M_1 \times \cdots \times M_l\) and \(g = g_B + f_1^2 g_1 + \cdots + f_l^2 g_l\) with the natural isometric action of the group \(G = G_1 \times \cdots \times G_l\) is a simply connected regular Riemannian G-manifold of non-positive curvature.

(ii) The manifold \((M, g)\) has negative curvature if and only if the following conditions hold:
1) \(\text{dim } B = 1\) or \((B, g_B)\) has negative curvature.
2) Each function \(f_i\) is strictly convex on \(B\) and \(g_B(\text{grad } f_i, \text{grad } f_j) > 0, 1 \leq i, j \leq l\).

(iii) Conversely, let \(G\) be a semisimple Lie group with a finite center and without compact factor. Then any simply connected regular Riemannian G-manifold \(M\) of non-positive curvature can be obtained by the above construction. Moreover if \(G\) acts effectively on \(M\) then it has no center.

To prove this theorem we need the following lemma.

Lemma 2.2. Let \((M, g)\) be a regular simply connected complete Riemannian G-manifold of non-positive curvature and \(K\) the stabilizer of a point \(x \in M\). Assume that the normalizer \(N_G(K) = K\). Then \(M\) is isometric to \(B \times G/K, B = M/G\), equipped with a metric of the form \(g = g_B + \hat{g}_b\) where \(g_B\) is a Riemannian metric on \(B\) and \(\hat{g}_b\) is a family of G-invariant metrics on \(G/K\) parametrized by \(b \in B\).

Proof. We set \(S = \{x \in M: G_x = K\}\). Then \(S\) is a \(K\)-invariant section of \(\pi\) and it defines a trivialization of \(M\) as follows
\[
\tau : B \times G/K \ni (b, aK) \mapsto ax \in M
\]
where \(\{x\} = \pi^{-1}(b) \cap S\) is the unique element of the orbit \(\pi^{-1}(b)\) with stabilizer \(K\). By Theorem 3.2. of [16] the horizontal distribution \(\mathcal{H}\) is integrable and therefore by 9.26. of [4] \(M\) is isometric to \(B \times G/K\) with a Riemannian metric of the form \(g_B + \hat{g}_b\). \(\Box\)

Proof of Theorem 2.1. The claims (i) and (ii) follow from 1.9. For (iii), let \(K\) be a maximal compact subgroup of \(G\). By 1.2. \(K\) is the stabilizer of some point \(x \in M\). Then \(Gx = G/K\) is a symmetric space of non-compact type by 1.5.
It is known that $N_G(K) = K$ [10]. By Lemma 2.2, we can write $M = B \times G/K, \ g = g_B + \hat{g}_B$. Since the center $Z(G)$ is finite, it is contained in $K$ and acts trivially on $M$. Now the claim follows from 1.5 and 1.9. □

Applying Theorem 2.1 to a cohomogeneity one manifold, we get

**Corollary 2.3.** Let $(M, g)$ be a simply connected non-positively curved cohomogeneity one regular Riemannian $G$-manifold of a connected semisimple Lie group $G$ with finite center and without compact factor. Then $G = G_1 \cdot G_2 \cdots G_l$ where $G_i$ are simple factors of $G$ and $(M, g)$ is isometric to the Riemannian manifold $M = \mathbb{R} \times G/K = \mathbb{R} \times G_1/K_1 \times \cdots \times G_l/K_l,$

$g = dt^2 + f_1^2 \bar{g}_1 + \cdots + f_l^2 \bar{g}_l$

where $K = K_1 \cdot K_2 \cdots K_l$ is a maximal compact subgroup of $G$, $K_i = K \cap G_i$, $g_i$ is the unique (up to a scaling) invariant metric of the non-compact irreducible symmetric space $G_i/K_i$ and $f_i$ is a smooth, positive and convex function on $\mathbb{R}$ such that $f_i f_j > 0$, $1 \leq i, j \leq l$. Moreover $(M, g)$ has negative curvature if and only if each $f_i$ is strictly convex and $f_i f_j > 0$.

If $G$ is a semisimple Lie group with infinite center, then by 1.7 the quotient space $G/K$ of $G$ by a maximal compact subgroup $K$ has no invariant metric of non-positive curvature. However, we get the following corollary from Proposition 4.11 (Section 4).

**Corollary 2.4.** Let $G$ be a semisimple Lie group with infinite center and $K$ a maximal compact subgroup of $G$. Then the cohomogeneity one regular $G$-manifold $M = \mathbb{R} \times G/K$ admits a $G$-invariant metric of negative curvature.

The following result shows that any connected semisimple Lie group $G$ of isometries of the hyperbolic space $H^n$ which has no compact factor acts on $H^n$ regularly. We denote by $SO_0(1, n)$ the connected group of isometries of $H^n$ such that $H^n = SO_0(1, n)/SO(n)$.

**Proposition 2.5.** Any connected semisimple Lie group $G \subseteq SO_0(1, n)$ of isometries of the hyperbolic space $H^n$ which has no compact factor is conjugated to the standard embedding of the group $SO_0(1, m)$ in $SO_0(1, n)$ for some $m \leq n$ and it has no singular orbit on $H^n$.

**Proof.** Let $V = \mathbb{R}^{1,n}$ be the Minkovski space and $G \subseteq SO_0(1, n)$ a connected semisimple subgroup without compact factor. Then the real rank of $G$ is positive, see [13, page 270]. This means that the (linear) Lie algebra $\mathfrak{g} \subseteq \mathfrak{so}(1, n) \cong \bigwedge^2 V$ of $G$ contains a semisimple element $a$ with real eigenvalues. Without loss of generality, we may assume that $a = p \wedge q$, where $p, q \in V = \mathbb{R}^{1,n}$ are isotropic vectors with the scalar product $\langle p, q \rangle = 1$. Let $E = (\mathbb{R} p + \mathbb{R} q)^\perp$ be the orthogonal complement to the hyperbolic plane $\mathbb{R} p + \mathbb{R} q$. Then $\mathfrak{so}(1, n) = \mathfrak{so}(V) = q \wedge E + \mathbb{R} p \wedge q + \bigwedge^2 E + p \wedge E$ is a gradation of $\mathfrak{so}(V)$ by eigenspaces of $ad_{p \wedge q}$. Since $p \wedge q \in \mathfrak{g}$, $\mathfrak{g}$ is a graded subalgebra, i.e. $\mathfrak{g} = q \wedge E_1 + \mathbb{R} p \wedge q + \mathfrak{k} + p \wedge E'_1$ where $\mathfrak{k} \subseteq \mathfrak{so}(V)$. On the other hand, $\mathfrak{g}$ is semisimple, so $E_1 = E'_1$. Hence $\mathfrak{k}$ contains $[q \wedge E_1, p \wedge E_1] = \bigwedge^2 E_1 = \mathfrak{so}(E_1)$. This shows that

$\mathfrak{g} = q \wedge E_1 + \mathbb{R} p \wedge q + \mathfrak{so}(E_1) + p \wedge E_1 + \mathfrak{k}' = \mathfrak{so}(V_1) + \mathfrak{k}'$

where $V_1 = \mathbb{R} p + \mathbb{R} q + E_1$ and $\mathfrak{k}' \subseteq \mathfrak{so}(E_1^{\perp})$. Since $G$ is semisimple and has no compact factor, $\mathfrak{k}' = 0$ and therefore $G$ is conjugate to $SO_0(1, m)$ where $m = \dim E_1 + 1$. We may assume that $G = SO_0(1, m) = \{ \text{diag}(A, \text{id}) \}$ with respect to the decomposition $\mathbb{R}^{1,n} = \mathbb{R}^{1,m} + \mathbb{R}^{n-m}$. Let $t = (1, 0, \ldots, 0) \in \mathbb{R}^{1,m}$ so $\langle t, t \rangle = -1$, then each $x \in H^n$ is $G$-equivalent to a point of the form $x = (\cosh \alpha) t + (\sinh \alpha) y$ where $\alpha \geq 0$ and $y \in \mathbb{R}^{n-m}$ is a unit vector. If $g = \text{diag}(A, \text{id}) \in G$ then $g \cdot x = (\cosh \alpha) A t + (\sinh \alpha) y$ and therefore the stabilizer of any point of $H^n$ is isomorphic to $SO(m) = G_t$. □

3. Admissible $G$-manifolds

3.1. Simply connected cohomogeneity one Riemannian $G$-manifolds of non-positive curvature with a singular orbit and admissible $G$-manifolds

**Definition 3.1.** (i) A representation $\rho : K \rightarrow GL(V)$ of a compact Lie group $K$ is called sphere transitive if $\rho(K)$ acts transitively on the unit sphere in $V$ defined by a $\rho(K)$-invariant metric.
(ii) A connected Lie group $G$ is called admissible if its maximal compact subgroup $K$ admits a sphere transitive representation $\rho$.

(iii) A simply connected $G$-manifold of the form $M = G \times_K V\rho$, where $K$ is a closed subgroup of an admissible Lie group $G$ and $\rho : K \rightarrow SO(V\rho)$ is a sphere transitive representation of $K$ in a vector space $V\rho$ is called an admissible $G$-manifold if $K/\Gamma$ is a maximal compact subgroup of the quotient group $G/\Gamma$ where $\Gamma$ is the kernel of effectivity of $G$ on $M$.

Note that $G/\Gamma$ is an admissible group and $M$ is an effective cohomogeneity one $G/\Gamma$-manifold. Moreover, if $G$ has finite center $Z(G)$, then $\Gamma \subset Z(G)$ is a finite group and $K$ is a maximal compact subgroup of $G$. Obviously, a covering of an admissible Lie group $G$ is an admissible Lie group. The following proposition (see, for example, [2]) shows that any simply connected Riemannian $G$-manifold of non-positive curvature can be identified with an admissible $G$-manifold with an invariant metric.

**Proposition 3.2.** Let $(M, g)$ be an effective simply connected complete non-positively curved cohomogeneity one Riemannian $G$-manifold of a connected Lie group $G$ with a singular orbit $P = Go = G/\Gamma$. Then $M$ is $G$-diffeomorphic to an admissible $G$-manifold $M = G \times_K V\rho$ where $K = G_\rho$ is a maximal compact subgroup of $G$ and $\rho : K \rightarrow O(V\rho)$ is the normal isotropy representation of $K$ into the normal space $V\rho = T_o^\perp P$ of the orbit $P$. In particular, the group $G$ is admissible. Moreover, if $G$ is a semisimple Lie group, the center of $G$ is finite.

The last claim is proved in Corollary 5.5 below.

This proposition reduces the description of simply connected complete Riemannian $G$-manifolds of non-positive curvature of semisimple Lie groups $G$ to the classification of admissible $G$-manifolds $M = G \times_K V\rho$ of semisimple Lie groups with finite center and the description of invariant metrics of non-positive curvature on $M$. In the next subsection we describe the structure of admissible $G$-manifolds of a connected semisimple Lie group $G$ with a finite center.

### 3.2. Structure of admissible $G$-manifolds of a semisimple Lie group $G$ with finite center

Let $M = G \times_K V\rho$ be an admissible $G$-manifold. We will denote by $N$ the kernel of the representation $\rho : K \rightarrow SO(V\rho)$ and by $A = \rho(K)$ its image. Then $A \subset GL(V\rho)$ is a compact connected sphere transitive linear group. A. Borel [5] classified all such linear groups and got the following list.

**Borel list of connected compact sphere transitive linear Lie groups**

<table>
<thead>
<tr>
<th>$A$</th>
<th>$SO(n)$</th>
<th>$SU(n)$</th>
<th>$Sp(n)$</th>
<th>$G_2$</th>
<th>Spin(7)</th>
<th>Spin(9)</th>
<th>$U(n)$</th>
<th>$Sp(1) \cdot Sp(n)$</th>
<th>$T^1 \cdot Sp(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>$\mathbb{R}^n$</td>
<td>$\mathbb{R}^{2n}$</td>
<td>$\mathbb{R}^{4n}$</td>
<td>$\mathbb{R}^7$</td>
<td>$\mathbb{R}^8$</td>
<td>$\mathbb{R}^{16}$</td>
<td>$\mathbb{R}^{2n}$</td>
<td>$\mathbb{R}^{4n}$</td>
<td>$\mathbb{R}^{4n}$</td>
</tr>
</tbody>
</table>

When $G$ is compact we have the following obvious proposition.

**Proposition 3.3.** Let $M$ be an effective admissible $G$-manifolds of a compact connected Lie group $G$. Then $G$ is isomorphic to one of the groups of the Borel list and the $G$-manifold $M$ is isomorphic to the vector space $V$ with sphere transitive linear action of $G$. Any such manifold $M$ admits an invariant metric of non-positive curvature, e.g. a flat metric.

Now we describe admissible $G$-manifolds $M = G \times_K V\rho$ where $G$ is a semisimple Lie group with finite center. Changing $G$ to an appropriate finite covering if necessary, we may assume that $G$ is a direct product of simple factors. Note that $K$ is a maximally compact subgroup of $G$ and it has the form $K = T^k \cdot K'$ where $K'$ is a semisimple connected subgroup. The quotient $G/K$ is a non-compact symmetric space [10].

We will denote by $\mathfrak{a}$ and $n$ the Lie algebras of the groups $A = \rho(K)$ and $N = \ker \rho$. Note that $A$ is a connected compact Lie group. We can identify $\mathfrak{a}$ with an ideal of $\mathfrak{k}$ such that the following direct sum decomposition holds

$$\mathfrak{k} = \mathfrak{a} \oplus n$$
and the subgroup $A \subset K$ generated by $a$ is compact. Checking the Borel list, we conclude that there are two possibilities:

(I) $a \subseteq g_1$ where $g_1$ is a simple ideal of $g$;

(II) $a$ is a sum of two ideals $a = a_1 \oplus a_2$ such that $a_i \subseteq g_i$, where $g_i$, $i = 1, 2$, are two different simple ideals of $g$.

(This may happen only when $A = T^1 \cdot SU(m)$, $T^1 \cdot Sp(m)$ or $Sp(1) \cdot Sp(n)$.)

The following theorem describes the structure of an admissible $G$-manifold $M$ in both cases. Here by a symmetric space of non-compact type we mean a quotient space $G/K$ of a non-compact simply connected semisimple Lie group $G$ with finite center by a maximal compact subgroup $K$ of $G$, see [10].

**Theorem 3.4.** Let $M = G \times_K V_\rho$ be an admissible $G$-manifold of a semisimple Lie group $G$ with finite center which is a direct product of simple factors. Let $\mathfrak{g}$ and $\mathfrak{a}$ be the Lie algebras of $G$ and $A = \rho(K)$, respectively.

(I) If $a \subseteq g_1$ where $g_1$ is a simple ideal of $g$, such that $\mathfrak{g} = g_1 \oplus g'$, $G = G_1 \times G'$, then $M$ is $G$-diffeomorphic to a direct product

$$M \approx \tilde{M} \times M' = (G_1 \times K_1 V_\rho) \times G' / K'$$

where $\tilde{M} = G_1 \times K_1 V_\rho$ is an admissible $G_1$-manifold of the simply connected normal Lie subgroup $G_1 \subseteq G$ generated by the subalgebra $g_1$, $K_1 = K \cap G_1$ and $M' = G' / K'$ is a symmetric space of non-compact type, where $G'$ is the simply connected normal subgroup of $G$ generated by $g'$ and $K' = K \cap G'$.

(II) If $a$ is a sum of two ideals $a = a_1 \oplus a_2$ which are contained in two different simple ideals $g_i$ of $g$ such that $\mathfrak{g} = g_1 \oplus g_2 \oplus g'$, $G = (G_1 \times G_2) \times G'$, where $G_1, G_2, G'$ are normal subgroups generated by $g_1, g_2, g'$, then $M$ is $G$-diffeomorphic to a direct product

$$M = \tilde{M} \times M' = ((G_1 \times G_2) \times (K_1 \times K_2) V_\rho) \times G' / K'$$

where $M' = G' / K'$ is a symmetric space of non-compact type and $\tilde{M} = \tilde{G} \times \mathfrak{g} V_\rho$ is an admissible $(\tilde{G} = G_1 \times G_2)$-manifold where $\tilde{K} = K \cap \tilde{G} = K_1 \times K_2, K_i = K \cap G_i$ and $\rho : \tilde{K} \rightarrow SO(V_\rho)$ is the restriction of $\rho$ to $\tilde{K}$ with $\rho(\tilde{K}) = A$.

**Proof.** (I) We have $G = G_1 \times G'$ and $K = K_1 \times K'$. Since $\mathfrak{k} = \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{k}_1 \oplus \mathfrak{k}'$, $\mathfrak{a} \subset \mathfrak{k}_1$ and the group $K'$ is connected it belongs to the kernel of $\rho$. This implies that $(G = G_1 \times G')$-manifold $M$ is $G$-diffeomorphic to a direct product of $G_1$-manifold $\tilde{M}$ and $G'$-manifold $M' = G' / K'$. The proof of (II) is similar. □

**Remark.** Note that if the group $A = \rho(K)$ is simple, i.e. it is one of the groups 1–6 from the Borel list, then only the case I is possible.

Theorem 3.4 reduces the classification of admissible $G$-manifolds $M = G \times_K V_\rho$ of a semisimple Lie group $G$ with finite center to the case when $G$ is a simple Lie group (type I) or a direct product $G_1 \times G_2$ of two simple Lie groups (type II). In the second case $K = K_1 \times K_2$ and $\rho(K_1) = T^1$ or $Sp(1)$ and $\rho(K_2) = SU(m)$ or $Sp(n)$.

The following Theorem enumerates all such admissible $G$-manifolds in the case when the universal covering $\tilde{G}$ of $G$ has finite center. Then we may assume that $G = \tilde{G}$ is simply connected.

**Notations.** We will denote by $1$ the trivial representation of a Lie group $K$, by $\rho_n, \mu_n$ and $\nu_n$ the standard tautological representations of $SO(n)$, $SU(n)$ and $Sp(n)$ respectively. The standard representation of $U(n)$ is also denoted by $\mu_n$ (in fact it is $\mu_1 \circ C \mu_n$ where $\mu_1$ is the standard representation of $U(1) = T^1$), the orthogonal representation of the spinor group $Spin(n)$ is denoted by $\rho_n$ too and its spin representation is denoted by $\Delta_n$. The sphere transitive representation of $G_2$ in $\mathbb{R}^7$ is denoted by $\phi_1$. By $\rho_2^m : \mathbb{R} \rightarrow SO(2), a \mapsto e^{2\pi ai}$ we denote the 2-dimensional representation of $\mathbb{R}$ with the kernel $m \mathbb{Z}$. We denote the universal covering of a classical Lie group $G$ by $\tilde{G}$.

**Theorem 3.5.** (1) All admissible $G$-manifolds $M = G \times_K V_\rho$ of non-compact simply connected simple Lie groups $G$ with finite center are enumerated in Table 2, where the admissible groups $G$, their maximal compact subgroups $K$ and the sphere transitive representations $\rho : K \rightarrow GL(V)$ are given.

(2) All admissible $G$-manifolds $M = (G_1 \times G_2) \times_K V_\rho$ of type II where $G_1, G_2$ are simple simply connected non-compact Lie groups with finite center are described as follows:
Proof. The description of admissible $G$-manifolds $M = G \times_K V_\rho$ of a given simply connected semisimple Lie group $G$ with finite center reduces to a description of sphere transitive representations of a maximal compact subgroup $K$ of $G$, that is homomorphisms of $K$ onto one of the groups from the Borel list. It can be described directly using the list of simply connected simple non-compact Lie groups $G$ with finite center from [13] or [10] and their maximal compact subgroups $K$. We use the fact that the spinor group $K = Spin(m)$ for $m = 3, 4, 5, 6$ is isomorphic to $SU(2), SU(2) \times SU(2), Sp(2), SU(4)$, respectively. Due to this, besides the orthogonal representation $\rho_n$, the tautolog-

\begin{table}[h]
\centering
\caption{List of non-compact simply connected simple Lie groups $G$ with infinite center $Z(G)$ and the normalizer $K$ of a maximal compact subgroup $K'$}
\begin{tabular}{|c|c|c|c|}
\hline
$G$ & $K = K' \times \mathbb{R}$ & $Z(G)$ & Condition \\
\hline
1 & $SO_3(n, 2)$ & $Spin(n) \times \mathbb{R}$ & $\mathbb{Z}_2 \times \mathbb{Z}$ & $n \geq 5$ \\
2 & $SU(p, q)$ & $SU(p) \times SU(q) \times \mathbb{R}$ & $\mathbb{Z}_d \times \mathbb{Z}$ & $d = \gcd(p, q)$ \\
3 & $Sp(n, \mathbb{R})$ & $SU(n) \times \mathbb{R}$ & $\mathbb{Z}, n = 2l + 1$ & $l \geq 1$ \\
4 & $SO^\times(2n)$ & $SU(n) \times \mathbb{R}$ & $\mathbb{Z}_2 \times \mathbb{Z}$ & $n = 2l$ \\
5 & $E_6^{-14}$ & $Spin(10) \times \mathbb{R}$ & $\mathbb{Z}$ & \\
6 & $E_7^{-25}$ & $E_6 \times \mathbb{R}$ & $\mathbb{Z}$ & \\
7 & $SU(2, \mathbb{R})$ & $1 \times \mathbb{R}$ & $\mathbb{Z}$ & \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{List of admissible non-compact simply connected simple Lie groups $G$ with finite center, their maximal compact subgroups $K$ and sphere transitive representations $\rho$ of $K$}
\begin{tabular}{|c|c|c|c|}
\hline
$G$ & $K$ & $\rho$ & Condition \\
\hline
1 & $SL(n, \mathbb{R})$ & $Spin(n)$ & $\rho_n$ & $n \geq 3$ \\
1a & $n = 3$ & $SU(2)$ & $\mu_2$ & \\
1b & $n = 4$ & $SU(2) \times SU(2)$ & $\mu_2, \rho_3$ & \\
1c & $n = 5$ & $Sp(2)$ & $v_2$ & \\
1d & $n = 6$ & $SU(4)$ & $\mu_4$ & \\
1e & $n = 7, 9$ & $Spin(n)$ & $\Delta_n$ & \\
2 & $SO_\rho(n, m)$ & $Spin(n) \times Spin(m)$ & $\rho_n, \rho_m$ & $n, m \neq 2$ \\
3 & $Spin(n)$ & $Sp(n)$ & $v_n, v_m$ & $n \geq 2$ \\
3a & $n = 2$ & $Spin(5)$ & $\rho_5$ & \\
4 & $Sp(n, m)$ & $Sp(n) \times Sp(m)$ & $v_n, v_m$ & \\
4a & $Sp(2, m)$ & $Spin(5) \times Sp(m)$ & $\rho_5$ & \\
4b & $Sp(1, m)$ & $Spin(1) \times Sp(m)$ & $\rho_3, v_1 \otimes v_m$ & \\
5 & $SO(n, \mathbb{C})$ & $Spin(n)$ & $\rho_n$ & $n \geq 7$ \\
6 & $SU(n)$ & $SU(6) \times SU(2)$ & $\mu_n$ & $n \geq 2$ \\
7 & $E_6^\times$ & $Sp(4)$ & $v_4$ & \\
8 & $E_7^\times$ & $SU(6) \times SU(2)$ & $\mu_6, \mu_2, \rho_3$ & \\
9 & $E_7^5$ & $SU(8)$ & $\mu_8$ & \\
10 & $E_7^{-5}$ & $Spin(12) \times SU(2)$ & $\rho_12, \mu_2, \rho_3$ & \\
11 & $E_7^8$ & $Spin(16)$ & $\rho_16$ & \\
12 & $E_7^{-24}$ & $E_7 \times SU(2)$ & $\mu_2, \rho_3$ & \\
13 & $F_4^4$ & $Sp(1) \times Sp(3)$ & $v_1, \rho_3, v_3, v_1 \otimes v_3$ & \\
14 & $F_4^{-20}$ & $Spin(9)$ & $\rho_9, \Delta_9$ & \\
15 & $G_2^\mathbb{C}$ & $SU(2) \times SU(2)$ & $\mu_2, \rho_3, \rho_4$ & \\
16 & $G_2^C$ & $G_2$ & $\phi_1$ & \\
\hline
\end{tabular}
\end{table}

Table 3

List of admissible simple simply connected Lie groups $G$ with finite center whose maximal compact subgroup $K$ has representation $\rho(K) = SU(n)$ or $Sp(n)$

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\rho(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$SL(2, \mathbb{C}), Sp(1, m), \tilde{SL}(3, \mathbb{R})$</td>
</tr>
<tr>
<td></td>
<td>$SO_o(n, n), n \geq 4$</td>
</tr>
<tr>
<td></td>
<td>$E^6, E^7, E^{3,5}, E_6^{24}, F_4 \times G_2$</td>
</tr>
<tr>
<td>$2$</td>
<td>$SL(n, \mathbb{C})$</td>
</tr>
<tr>
<td></td>
<td>$E^6$</td>
</tr>
<tr>
<td></td>
<td>$E^7$</td>
</tr>
<tr>
<td></td>
<td>$SL(6, \mathbb{R}), \tilde{SO}_0(6, m)$</td>
</tr>
<tr>
<td>$3$</td>
<td>$\tilde{SL}(5, \mathbb{R}), \tilde{SO}_0(5, m)$</td>
</tr>
<tr>
<td>$4$</td>
<td>$Sp(n, \mathbb{C}), SU^*(2n), Sp(n, m)$</td>
</tr>
<tr>
<td></td>
<td>$F_4^6$</td>
</tr>
<tr>
<td></td>
<td>$F_4^{10}$</td>
</tr>
</tbody>
</table>


Spherical spinor representation $\Delta_n$ of $Spin(m)$ for $m = 3, 5, 6$ and its projection on one of the two $SU(2)$ factors for $n = 4$ are also sphere transitive representations. \(\square\)

Assume now that the universal covering $\tilde{G}$ of an admissible group $G$ with finite center has infinite center. If $G$ is simple, then $\tilde{G}$ is one of the groups of Table 1, where the center $Z(\tilde{G})$ and a maximal compact subgroup $K'$ are given. This information can be extracted from [13] or [10]. It is known also that the normalizer $N_{\tilde{G}}(K')$ has the form $N_{\tilde{G}}(K') = \mathbb{R} \times K'$ and the center $Z(\tilde{G})$ is given by

$$Z(\tilde{G}) = \mathbb{Z} \times \mathbb{Z}_r$$

where $\mathbb{Z}$ is an infinite cyclic subgroup of $N_{\tilde{G}}(K') = \mathbb{R} \times K'$ with a generator $z = (1, a)$, $a \in Z(K')$ is an element of a finite order and $\mathbb{Z}_r \subset Z(K)$ is a cyclic subgroup of a finite order $r$, see [13]. Denote by $p\mathbb{Z}$ the cyclic subgroup of $Z(\tilde{G})$ with generator $pz$. Then $\tilde{G}(p) := \tilde{G}/p\mathbb{Z}$ is a simple Lie group with finite center and any simple Lie group with finite center and the universal covering $\tilde{G}$ is finitely covered by $\tilde{G}(p)$ for some $p$. Due to this, the description of admissible $G$-manifolds of type I where $G$ is a simple Lie group with finite center and universal covering $\tilde{G}$ reduces to description of sphere transitive representations of a maximal compact subgroup $K(p) = T^1 \times K'$ of the group $G = \tilde{G}(p) = \tilde{G}/p\mathbb{Z}$. The result is stated in the following theorem.

**Theorem 3.6.** (1) All admissible $\tilde{G}$-manifolds $M = \tilde{G} \times K V_\rho$ of a simple simply connected Lie groups $\tilde{G}$ with an infinite center such that the corresponding effective group $G = \tilde{G}/\Gamma'$ has finite center are described as follows:

- $\tilde{G}$ is one of the group from Table 1, $K = K' \times \mathbb{R}$ is the corresponding normalizer of a maximal compact subgroup $K'$ and $\rho = 1 \times \rho_2^{m} : K \rightarrow SO(2)$ is the 2-dimensional representation of $K = K' \times \mathbb{R}$ or one of the following representations:
  - If $G$ is in the first row, then $\rho = \rho_{n+1}$ or
    - for $n = 2$ a representation $\rho : K = \mathbb{R} \times \mathbb{R} \rightarrow \rho(K) = SO(2)$,
    - for $n = 3$ the representations $\rho : K = SU(2) \times \mathbb{R} \rightarrow SO(4)$ with the image $SU(2)$ or $U(2)$,
    - for $n = 4$ the representations $\rho : K = SU(2) \times SU(2) \times \mathbb{R} \rightarrow SO(4)$ with the image $SU(2)$ or $U(2)$ or $\rho : K \rightarrow SO(3)$,
    - for $n = 5$ the representations $\rho : K = Sp(2) \times \mathbb{R} \rightarrow SO(8)$ with the image $Sp(2)$ or $T^1 \cdot Sp(2)$,
    - for $n = 6$ the representations $\rho : K = SU(4) \times \mathbb{R} \rightarrow SO(8)$ with the image $SU(4)$ or $U(4)$,
    - for $n = 7$ or $9$ the representation $\rho = \Delta_n \times 1$.
  - If $G$ is in the second row, then $\rho$ is one of the representations of $K = SU(p) \times SU(q) \times \mathbb{R}$ with the image $\rho(K) = SU(p), U(p), SU(q), U(q)$.
  - If $G$ is in the rows 3,4, then $K = SU(n) \times \mathbb{R}$ and $\rho(K) = SU(n)$ or $U(n)$.
  - If $G$ is in the row 5, then $\rho = \rho_{10} \times 1 : Spin(10) \times \mathbb{R} \rightarrow SO(10)$.
(2) All admissible $\tilde{G}$-manifolds $M = \tilde{G} \times_K V_\rho$ of type II where $\tilde{G} = G_1 \times G_2$ is a simply connected Lie group with infinite center such that the effective group $\tilde{G}/\Gamma$ has finite center are described as follows:

If $\rho(K)$ is not a semisimple group, that is $\rho(K) = T^1 \cdot SU(m)$ or $T^1 \cdot Sp(m)$, then $G_1$ is a group from Table 1 with $K_1 = K'/\Gamma \subset \mathbb{R}$ and $G_2$ is either a group from Table 3 or a group from the first row with $n = 3, 4, 5, 6$ or rows $2, 3, 4$ of Table 1 and $\rho(K_1 \times id) = T^1$, $\rho(id \times K_2) = SU(m)$ or $Sp(m)$. If $\rho(K)$ is a semisimple group that is $Sp(1) \cdot Sp(m)$ then $G_1$ is one of the groups from the first row for $n = 3, 4$, second row for $p = 2$ or $q = 2$, rows 3, 4 for $n = 2$ of Table 1 or a group from the first row of Table 3 and $G_2$ is a group from row 1 or 4 of Table 3 or a group from the first row for $n = 3, 4, 5$ or the second row for $p = 2$ or $q = 2$ of Table 1 and $\rho(K_1 \times id) = Sp(1)$, $\rho(K_2) = Sp(m)$.

Proof. (1) Let $M = \tilde{G} \times_K V_\rho$ be an admissible $\tilde{G}$-manifold as in theorem. Then $\Gamma$ contains the cyclic subgroup $p\mathbb{Z}$ for some $p$ and the description of the standard $\tilde{G}$-manifolds reduces to enumeration of sphere transitive representations of a maximal compact subgroup $K(p) = T^1 \times K'$ of the group $G = \tilde{G}(p) = \tilde{G}/p\mathbb{Z}$. It can be easily done using Table 1.

The proof of (2) when the group $\tilde{G} = G_1 \times G_2$ is not simple is similar. In this case we may assume that $G_1$ has infinite center, hence is one of the group of Table 1. If the group $G_2$ has finite center it is one of the groups of Table 2 whose maximal compact subgroup $K_2$ has a factor isomorphic to $SU(m)$ or $Sp(m)$. All these groups are enumerated in Table 3. If $G_2$ has an infinite center, it is a group from Table 1 with the same property.

Remark. Let $M = G \times_K V_\rho$ be an admissible $G$-manifold of a simple Lie group $G$ and the universal covering $\tilde{G}$ of $G$ has infinite center and $K'$ is a maximal compact subgroup of $\tilde{G}$. Since $K_1/\Gamma$ is a (connected) maximally compact subgroup of the group $G/\Gamma$ where $\Gamma \subset Z(G)$ is the kernel of effectivity of the action of $G$ on $M$ we may assume that $K' \subset K_1 \subset N_G(K') = \mathbb{R} \times K'$. Moreover, if $\Gamma$ is a finite group (i.e. $\Gamma \subset \mathbb{Z}_p$), then $K_1 = K$. If $\Gamma$ is infinite subgroup, i.e. it contains a cyclic subgroup $m\mathbb{Z}$ of $\mathbb{Z} \subset Z(G)$ with a generator $m\mathbb{Z}$, then $K_1/\Gamma = (\mathbb{R} \times K)/\Gamma = T^1 \cdot Z_p/(\Gamma \cap Z_p)$ and $K_1 = \mathbb{R} \times K$. Hence, a description of admissible $G$-manifolds reduces to description of sphere transitive representations of the groups $K$ and $\mathbb{R} \times K$. We will not give details since such manifolds $M = G \times_K V_\rho$ does not admit an invariant metric of non-positive curvature.

4. Invariant metrics on an admissible $G$-manifold

4.1. Invariant metrics of non-positive curvature on admissible manifolds of a compact Lie group $G$

Assume that $G$ is a compact group, then an admissible $G$-manifold is a vector space $M = V$ with the linear sphere transitive action of a group $G$ from the Borel list. Invariant metrics on such a $G$-manifold $V$ were described by L. Verdiani [18], who calculate also their curvature tensor. The description of invariant metrics is especially simple in the case of the groups $G = SO(n), G_2 \leq SO(7)$ and $Spin(7) \leq SO(8)$. Using his results, we get the following necessary and sufficient conditions that an invariant metric has negative (or non-positive) curvature.

We denote by $g_0$ a Euclidean $G$-invariant metric on $V = \mathbb{R}^n$, and by $S^{n-1}$ the unit sphere in $(V, g_0)$ and we identify $V \setminus \{0\}$ with $\mathbb{R}^+ \times S^{n-1}$ via the map $V \ni v = te \mapsto (t, e) \in \mathbb{R}^+ \times S^{n-1}$ where $t = |v|$ is the radial coordinate. We denote also by $W : e \mapsto W(e) = T_eS^{n-1} \subset V$ the tangent distribution of $S^{n-1}$ and we identify $W(e)$ with $(te)^1 = T_e(tS^{n-1})$ for $t > 0$.

Proposition 4.1. Let $G \leq SO(V), V = \mathbb{R}^n$ be one of the following sphere transitive linear groups: $G = SO(n), G_2(n = 7)$ or $Spin(7)(n = 8)$. Then

(a) [18,20] any complete $G$-invariant metric on $V$ at a point $v = te \in V \setminus \{0\}$ where $e \in S^{n-1}$ has the form

$$g = \sigma^2(t)dt^2 + \eta^2(t)g_0|_{W(e)}$$

where $t$ is the radial coordinate, $g_0$ is the standard Euclidean metric in $V$, $W(e) = T_eS^{n-1}$ and $\sigma$ and $\eta$ are even positive smooth functions on $\mathbb{R}$ and $\sigma(0) = \eta(0)$.

(b) The Riemannian $G$-manifold $(V, g)$ has negative curvature if and only if the following conditions hold:

1) the even function $c(t) = \frac{\eta^{1+n}/\sigma}{\sigma}$ satisfies $c(0) = 1$, $c'(t) > 0$ for $t > 0$ and
2) $\sigma''(0) < 3\eta''(0)$.
Proof. (b) If \( N, Y_i, i = 1, \ldots, n - 1 \), form a \( g \)-orthonormal basis at \( v = te \in V \), \( N \) is normal to orbits, then the sectional curvatures are given by (cf. [18, §6])

\[
K(Y_i, Y_j) = g\left(R(Y_i, Y_j)Y_j, Y_i\right) = -\frac{\left(\eta t + 2\eta'\right)\sigma + \sigma'(\eta t + \eta)}{t\eta\sigma^3},
\]

\[
K(Y_i, N) = g\left(R(Y_i, N)N, Y_i\right) = \frac{\sigma''(0) - 3\eta''(0)}{\sigma(0)^3}.
\]

In terms of \( c(t) \) we have

\[
K(Y_i, Y_j) = \frac{1}{t^2}\frac{\eta^2}{\sigma^2}(1 - c^2), \quad K(Y_i, N) = \frac{-c'}{t\eta\sigma}.
\]

Note that \( c(0) = \eta(0)/\sigma(0) = 1 \), so for \( t > 0 \) the conditions 1), 2) are necessary and sufficient conditions that the sectional curvatures are negative. \( \square \)

4.2. Invariant metrics on an admissible \( G \)-manifold of non-compact semisimple Lie group \( G \)

In this subsection we describe the curvature tensor \( R \) of an invariant metric \( g \) on an admissible \( G \)-manifold \( M = G \times_K V_\rho \) where \( G \) is a non-compact connected semisimple Lie group.

Let \( M = G \times_K V_\rho \) be such a manifold and \( g \) a \( G \)-invariant metric on \( M \). Recall that the natural projection \( \pi : M \to G/K \) has the structure of a homogeneous vector bundle over the homogeneous manifold \( P = G/K \) and \( P \) is imbedded in \( M \) as the zero section. The induced metric on the fiber \( \pi^{-1}(o) = [(e, V_\rho)] \approx V_\rho \) is a \( \rho(K) \)-invariant metric. We can also identify \( V_\rho \) with the normal space \( T_o \perp P \) of the singular orbit \( P = Go \) at \( o \). We fix a reductive decomposition

\[
g = \mathfrak{k} + m
\]

and identify \( m \) with the tangent space \( T_o P \) of the orbit \( P \). Then \( T_o M = m \oplus V_\rho = T_o P \oplus T_o \perp P \) is an orthogonal decomposition of the tangent space \( T_o M \).

Let \( v \in V_\rho \) be a unit vector (with respect to the Euclidean metric \( g_o = g|_o \)) and \( H = G_{[e, v]} = K_v \) the stabilizer of the point \( [e, v] \). We denote by \( c(t) \) the naturally parametrized normal (to orbits) geodesic starting from the point \( c(0) = o \) in the direction \( v \) which is \( H \)-invariant. We identify the open submanifold of regular points \( M_{\text{reg}} = M \setminus P \) with a direct product

\[
\mathbb{R}^+ \times G/H = M_{\text{reg}},
\]

\[
\mathbb{R}^+ \times G/H \ni (t, aH) \mapsto ac(t) \in M_{\text{reg}}.
\]

Then \( c(t) = (t, eH) \) and the restriction to \( M_{\text{reg}} \) of the \( G \)-invariant Riemannian metric \( g \) on \( M \) can be written as

\[
g = dt^2 + g_t
\]

where \( g_t \) is a one parameter family of \( G \)-invariant metrics of \( G/H \).

We fix a reductive decomposition

\[
\mathfrak{k} = \mathfrak{h} + n, \quad \mathfrak{h} = \text{Lie } H
\]

and will use the identification

\[
n = T_{t_0} \rho(K)(tv) = T_{eH} K/H, \quad T_e(t)M = \mathbb{R} T_t + T_{c(t)}(Gc(t)) = \mathbb{R} T_t + n + m = V_\rho + m
\]

where \( T_t = c(t)' \) is the tangent vector of the geodesic \( c(t) \). In particular, we have

\[
T_{c(t)}(Gc(t)) = p := n + m.
\]
We denote by $\tilde{K}$ the normalizer of $K$ in $G$ and by $\tilde{\mathfrak{m}}$ the orthogonal (with respect to the Killing form $B$) complement to the Lie algebra $\tilde{\mathfrak{k}}$ in $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{k} + \tilde{\mathfrak{m}}$$

is the symmetric decomposition associated with the symmetric space $G/\tilde{K}$. We denote by $\theta$ the corresponding involutive automorphism of $\mathfrak{g}$ and define an $\text{Ad}(\tilde{K})$-invariant Euclidean metric $Q$ on $\mathfrak{g}$ by

$$Q(X, Y) = -B(X, \theta Y), \quad X, Y \in \mathfrak{g}.$$  

Note that the symmetric decomposition is $Q$-orthogonal and the adjoint operator $\text{ad}_X$ is $Q$-symmetric (resp., $Q$-skew symmetric) for $X \in \mathfrak{m}$ (resp., $X \in \tilde{\mathfrak{k}}$). We have $Q$-orthogonal decompositions

$$\tilde{\mathfrak{k}} = \mathfrak{k} \oplus \mathbb{R}^d$$

where $\mathbb{R}^d \leq \mathfrak{z} = Z(\tilde{\mathfrak{k}})$ is a commutative ideal of $\tilde{\mathfrak{k}}$ and

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{n} + \mathfrak{m} = \mathfrak{h} + \tilde{\mathfrak{n}} + \tilde{\mathfrak{m}}$$

(4.1)

where $\mathfrak{n} = \mathfrak{n} + \mathbb{R}^d$ and $\mathfrak{m} = \mathbb{R}^d + \tilde{\mathfrak{m}}$. Note that $\tilde{\mathfrak{k}} = \mathfrak{k}$ if and only if the center of $G$ is finite.

For $X \in \mathfrak{g}$, we denote by $X^*$ the corresponding Killing vector field in $M$. Note that

$$X^*_{c(t)}(t) = X_p$$

where $X_p$ is the projection of $X$ onto $p = n + m = T_{c(t)}(Gc(t))$.

For $t > 0$, we can write

$$g(X^*, Y^*)_{c(t)} = g_t(X, Y) = Q(P_tX, Y) = Q(X, P_tY), \quad \text{for } X, Y \in p$$

where $P_t$ is an $\text{Ad}(H)$-invariant $Q$-symmetric positive definite endomorphism of $p$. Conversely, any 1-parametric family $P_t \in \text{End}(p)$, $t > 0$ of positively defined $\text{Ad}(H)$-invariant endomorphisms defines a $G$-invariant Riemannian metric $g = dt^2 + g_t$ on $M_{\text{reg}}$.

The following formula describes the Levi-Civita connection $\nabla$ of a $G$-invariant metric $g = dt^2 + g_t$ on $M_{\text{reg}}$ and its curvature tensor $R$ in terms of the operator $P_t$, the Levi-Civita connection $\nabla'$ of the invariant metric $g_t$, $t > 0$ on $G/H$ and its curvature tensor $R'$. Here $T = \frac{\partial}{\partial t}$ is the unit normal vector field on $M_{\text{reg}}$, which commutes with Killing fields $X^*$, $X \in \mathfrak{g}$.

**Lemma 4.2.** For $X, Y \in p$ we have

$$\nabla_{X^*} Y^*|_{c(t)} = \nabla_{X^*} Y^* + g_t(S_tX, Y)T,$$

where $S_tX = -\nabla_T X^* = -\nabla_X T$ is the shape operator of a regular orbit $Gc(t)$ whose value at $c(t)$ is given by

$$S_t = -\frac{1}{2} P_t^{-1} P_t' X.$$  

To calculate the curvature $R$, we use the standard formula (see [4, p. 183]) for the curvature tensor $R'$ of an invariant metric $g_t$ on the homogeneous manifold $Gc(t) = G/H$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + p$. We define a bilinear map

$$U_t : p \times p \to p, \quad 2g_t(U_t(X, Y), Z) = g_t([Z, X]_p, Y) + g_t([Z, Y]_p, X), \quad \forall X, Y, Z \in p.$$  

Then the covariant derivative of a Killing vector field $Y^*$ and the curvature tensor $R'$ at a point $c(t)$ can be written as

$$\nabla_{X^*} Y^*|_{c(t)} = -\frac{1}{2} [X, Y]_p + U_t(X, Y), \quad X, Y \in p.$$  

(4.2)

$$g_t(R'(X, Y)Y, X) = -\frac{3}{4}||[X, Y]_p||^2 - \frac{1}{2} g_t([X, Y]_p, X) + \frac{1}{2} g_t([X, Y]_p, Y)$$

$$+ ||U_t(X, Y)||^2 - g_t(U_t(X, X), U_t(Y, Y)) \quad X, Y \in p.$$  

(4.3)

The following lemma expresses $U_t(X, Y)$ in terms of the operator $P_t$. 
Lemma 4.3. The following formulas hold

\[ U_t(X, Y) = P_t^{-1}(\theta \text{ad}_X \theta P_t Y + \theta \text{ad}_Y \theta P_t X), \quad \text{for } X, Y, Z \in \mathfrak{p}, \]

\[ U_t(X, Y) = \frac{1}{2} P_t^{-1}([P_t X, Y]_\mathfrak{p} - [X, P_t Y]_\mathfrak{p}) \quad X, Y \in \tilde{\mathfrak{m}}, \]

\[ U_t(X, Y) = \frac{1}{2} P_t^{-1}([P_t X, Y]_\mathfrak{p} + [X, P_t Y]_\mathfrak{p}) \quad X \in \tilde{\mathfrak{n}}, Y \in \tilde{\mathfrak{m}}, \]

\[ U_t(X, Y) = \frac{1}{2} P_t^{-1}(-[P_t X, Y]_\mathfrak{p} + [X, P_t Y]_\mathfrak{p}) \quad X, Y \in \tilde{\mathfrak{n}}. \]

The following proposition describes the curvature tensor \( R \) of the manifold \( (M_{\text{reg}} = \mathbb{R}^+ \times G/H, g = dt^2 + g_t) \) at the point \( c(t), t > 0 \).

Proposition 4.4. For any \( X, Y \in \mathfrak{p} \), the following formulas hold:

(a) \( g_{c(t)}(R(X, Y)Y, X) = -\frac{3}{4} Q(P_t[X, Y]_\mathfrak{p}, [X, Y]_\mathfrak{p}) - \frac{1}{2} Q([X, Y]_\mathfrak{p}, P_t X) + \frac{1}{2} Q([X, Y]_\mathfrak{p}, P_t Y) + Q(P_t U(X, Y), U(X, Y)) - Q(P_t U(X, Y), U(Y, X)) + \frac{1}{4} Q(P_t' X, Y)^2 - \frac{1}{4} Q(P_t' X, X)Q(P_t' Y, Y). \)

(b) \( g_{c(t)}(R(X, Y)Y, T) = \frac{3}{4} Q([X, Y], P_t Y) - \frac{1}{2} Q(P_t' X, U(Y, X)) + \frac{1}{2} Q(P_t' Y, U(Y, X)). \)

(c) \( g_{c(t)}(R(X, T)T, X) = Q\left(-\frac{1}{2} P_t'' + \frac{1}{4} P_t' P_t^{-1} P_t'\right)X, X. \)

Proof of this proposition is straightforward and it is similar to the proof of Proposition 1.9 of [8].

4.3. The curvature of an invariant diagonal metric on \( M_{\text{reg}} \)

Now we specify the endomorphism \( P_t, t > 0 \) which determines the metric \( g = dt^2 + g_t \) on the manifold \( M_{\text{reg}} = \mathbb{R}^+ \times G/H \subseteq M = G \times_K V_p \). Recall that we have a \( Q \)-orthogonal decomposition (4.1) of the Lie algebra \( g \) into \( \text{Ad}(H) \)-submodules. We decompose \( \text{Ad}(H) \)-modules \( \tilde{\mathfrak{n}} \) and \( \tilde{\mathfrak{m}} \) into a direct sum of \( Q \)-orthogonal submodules

\[ \tilde{\mathfrak{n}} = \mathfrak{n}_0 \oplus \cdots \oplus \mathfrak{n}_p, \quad \tilde{\mathfrak{m}} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q \]

such that \( \mathfrak{p} = \tilde{\mathfrak{n}} + \tilde{\mathfrak{m}} = \sum_{i=0}^p \mathfrak{n}_i + \sum_{i=1}^q \mathfrak{m}_i \) where \( \mathfrak{n}_0 := \mathbb{R}^d \subseteq Z(\tilde{\mathfrak{g}}) \) is a trivial module. With respect to this decomposition we define \( P_t \in \text{End}(\mathfrak{p}) \) as a diagonal endomorphism of the form \( P_t = \text{diag}(f_1^2(t), \ldots, f_p^2(t), h_1^2(t), \ldots, h_q^2(t)) \) where \( f_i \) and \( h_j \) are positive functions on \( \mathbb{R}^+ \). The \( G \)-invariant metric \( g \) on \( M_{\text{reg}} \) associated to \( P_t \), is called a diagonal metric. The restriction of the metric \( g \) to \( \mathfrak{p} = T_{c(t)}(Gc(t)) \) is given by

\[ g_t = \sum_i f_i^2(t)Q|_{\mathfrak{n}_i} + \sum_j h_j^2(t)Q|_{\mathfrak{m}_j}. \]

Note that if all \( \text{Ad}(H) \)-modules \( \mathfrak{n}_i, \mathfrak{m}_j \) are irreducible and mutually inequivalent then any invariant metric \( g \) on \( M_{\text{reg}} \) is a diagonal metric. One can easily calculate the bilinear form \( U_t(X, Y) \) for the diagonal metric \( g \) as follows:

\[ U_t(X, Y) = \sum_r \frac{h_r^2 - h_r^2}{2f_r^2}[X, Y]_{\mathfrak{n}_r}, \quad X \in \mathfrak{m}_j, \quad Y \in \mathfrak{m}_j, \]

\[ U_t(X, Y) = \sum_r \frac{f_r^2 + h_r^2}{2h_r^2}[X, Y]_{\mathfrak{n}_r}, \quad X \in \mathfrak{n}_i, \quad Y \in \mathfrak{m}_j, \]

\[ U_t(X, Y) = \sum_r \frac{f_r^2 - f_r^2}{2f_r^2}[X, Y]_{\mathfrak{n}_r}, \quad X \in \mathfrak{n}_i, \quad Y \in \mathfrak{n}_j. \]
Using these formulas, we can specify the formulas for the curvature $R$ from Proposition 4.4 as follows (we write $|X|^2_Q$ for $Q(X, X)$):

**Proposition 4.5.** The curvature of the diagonal metric (4.5) on the manifold $M_{reg} = \mathbb{R}^+ \times G/H \subseteq M = G \times K V_\rho$ at the point $c(t)$ is given by

$$g(R(X,Y),X,X) = \sum_r \left( \frac{f_r^4 + f_r^4 + 2f_r^2 f_r' + 2f_r^2 f_r'^2 - 2f_r^2 f_r'}{4f_r^2} |X,Y|_{n_r}^2 \right)$$

$$+ \frac{f_r^2 + f_r^2}{2} |X,Y|_h^2 - f_r f_r' f_r' |X|_Q^2 |Y|_Q^2 \quad \text{for} \quad X \in n_i, \quad Y \in n_j,$$

$$g(R(X,Y),X,X) = \sum_r \left( \frac{h_r^4 + h_r^4 - 2h_r^2 f_r^2 - 2h_r^2 f_r'^2 - 2h_r^2 h_r'}{4f_r^2} |X,Y|_{n_r}^2 \right)$$

$$- \frac{h_r^2 + h_r^2}{2} |X,Y|_h^2 - h_r h_r' h_r' |X|_Q^2 |Y|_Q^2 \quad \text{for} \quad X \in m_i, \quad Y \in m_j,$$

$$g(R(X,Y),X,X) = \sum_r \left( \frac{f_r^4 + f_r^4 + 2f_r^2 h_r^2 + 2h_r^2 h_r'^2 - 2f_r^2 h_r'}{4h_r^2} |X,Y|_{m_r}^2 \right)$$

$$- f_r h_r h_r' |X|_Q^2 |Y|_Q^2 \quad \text{for} \quad X \in n_i, \quad Y \in m_j,$$

$$g(R(X,Y),T,T) = 0,$$

$$g(R(T,T),X,Y) = -f_r f_r'' Q(X,Y) \quad \text{for} \quad X \in n_i,$$

$$= -h_r h_r'' Q(X,Y) \quad \text{for} \quad X \in m_j.$$

**4.4. A construction of invariant metric of non-positive curvature on an admissible $G$-manifold of a semisimple Lie group $G$ with finite center**

Now we construct an invariant metric of non-positive curvature on an admissible $G$-manifold $M$ where $G$ is a semisimple group with finite center. The construction is based on the following two propositions. The first proposition gives sufficient conditions that a fiber of the canonical fibration $\pi : M = G \times K V_\rho \rightarrow G/K$ of a cohomogeneity one Riemannian $G$-manifold $(M, g)$ is totally geodesic. As above, we identify the tangent space $T_{c(t)} M$ at any point $c(t)$ of the geodesic $c(t)$ with $T_{c(t)}(M) = \mathbb{R}^+ T + n + m = V + m$.

**Proposition 4.6.** Let $(M = G \times K V, g)$ be a cohomogeneity one Riemannian $G$-manifold and $c(t)$ the normal geodesic as above. If for any $t \neq 0$ the decomposition $T_{c(t)} M = V + m$ is $g$-orthogonal, then the fibers of the projection $\pi : G \times K V_\rho \rightarrow G/K$ are totally geodesic submanifolds.

**Proof.** The proof follows from the next two lemmas.

**Lemma 4.7.** Let $(N = G/H, g)$ be a homogeneous Riemannian manifold and $g = h + p$ a reductive decomposition of $g$. The orbit $S = K o, o = eH$ of a closed connected subgroup $K$ of $G$ which contains $H$ is a totally geodesic submanifold if the orthogonal complement $m$ to $n := f \cap p$ in $p$ is $\text{ad}(f)$-invariant.

**Lemma 4.8.** Let $M = \mathbb{R} \times N$ be a manifold with a metric of the form $g = dt^2 + g_t$, where $g_t$ is a one-parameter family of metrics on the manifold $N$. If a submanifold $S \subseteq \mathbb{R} \times N$ is totally geodesic with respect to any metric $g_t$, then $\mathbb{R} \times S$ is a totally geodesic submanifold of the Riemannian manifold $(M, g)$.

The first lemma shows that $Kc(t) = K/H$ is a totally geodesic submanifold of the orbit $N = Gc(t) = G/H$ and then the second lemma implies that the fiber $V \setminus 0 = \pi^{-1}(o) \cap M_{reg}$ is a totally geodesic submanifold of $M_{reg} = \mathbb{R}^+ \times G/H$. □
Proposition 4.9. Let $M = G \times_K V_\rho$ be an admissible $G$-manifold of a semisimple Lie group $G$ and $H$ the stabilizer of a regular point $c = [e, v] \in M$. The orbit $S = Kc \approx \rho(K)v = K/H$ is a sphere and we identify the tangent space $T_cS$ with $n$ where $\mathfrak{t} = \mathfrak{h} + n$ is a reductive decomposition. The standard $K$-invariant metric of $S$ with curvature 1 is defined by $\text{Ad}(H)$-invariant metric $g_{\text{can}}$ on $n$ which is diagonal with respect to a decomposition

\[ n = n_1 + \cdots + n_p \]

of $\text{Ad}(H)$-module $n$ into $\text{Ad}(H)$-submodules $n_i$, that is

\[ g_{\text{can}} = \sum c_i Q|_{n_i} \]

where $c_i, i = 1, \ldots, p$, are positive constants. We define a positive constant $C$ by

\[ C = \max\{c_i, i = 1, \ldots, p\}. \quad (4.6) \]

In the following proposition, we identify the invariant metric on homogeneous manifold $G/H$ having a reductive decomposition $g = \mathfrak{h} + \mathfrak{p}$ with the corresponding $\text{Ad}(H)$-invariant Euclidean metric on $\mathfrak{p}$.

**Proposition 4.9.** Let $M = G \times_K V_\rho$ be an admissible $G$-manifold of a semisimple Lie group $G$ and $M_{\text{reg}}$ the submanifold of regular points which is identified with $\mathbb{R}^+ \times G/H$. Then a diagonal metric $g$ on $M_{\text{reg}}$ of the form

\[ g = dt^2 + f^2(t)g_{\text{can}} + f_0^2(t)Q|_{n_0} + h^2(t)Q|_{\mathfrak{m}} \quad (4.7) \]

extends to a smooth metric on $M$ if $f$ is a smooth odd function on $\mathbb{R}$ with $|f'(0)| = 1$, $f > 0$ on $\mathbb{R}^+$ and $f_0$ and $h$ are smooth positive even functions on $\mathbb{R}$. (Here we use notations $(4.4)$.)

**Proof.** It follows from Theorem 1 and Corollary 2 of [18]. If $h$ and $f_0$ are even functions, the conditions of Corollary 2 of [18] are satisfied for the part $f_0^2(t)Q|_{n_0} + h^2(t)Q|_{\mathfrak{m}}$ of the metric. The part $dt^2 + f^2(t)g_{\text{can}}$ is a $K$-invariant metric on $V_\rho$. Note that $g_{\text{can}|c(t)} = \frac{1}{t^2}g_0|_{n}$ where $g_0$ is the standard (Euclidean) metric of $V_\rho$ and by Theorem 1 of [18] the metric $g$ has a smooth extension if the function $\eta(t) = f(t)/t$ is even and $\eta^2(0) = 1$. \[ \square \]

Note that if the group $G$ has finite center, there is no submodule $n_0$ and therefore $\mathfrak{n} = n$, $\mathfrak{m} = \mathfrak{m}$. Now we are able to state the main theorem. In its proof we use the inequality

\[ |[X, Y]|_Q^2 \leq 4|X|_Q^2|Y|_Q^2 \]

which can be checked using the following remarks.

1) We can consider an exact representation $\phi$ of a real semisimple Lie algebra $\mathfrak{g}$ with a Cartan involution $\theta$ such that $\theta$ corresponds to the minus transposition of $\phi(\mathfrak{g})$ [11, Prop. 6.28].

2) If, moreover, $g$ is simple, then the Killing form $B$ is a scalar multiple of $\text{Tr}(\phi(X)\phi(Y))$. Due to this, it is sufficient to check the formula

\[ |[X, Y]|^2 \leq 4|X|^2|Y|^2 \]

for matrices $X, Y$ where $|X|^2 = \text{Tr}XX^t$, which is straightforward.

**Theorem 4.10.** Let $M = G \times_K V_\rho$ be an admissible $G$-manifold, where $G$ is a semisimple Lie group with finite center. Then an invariant diagonal metric $g$ on $M_{\text{reg}}$ of the form

\[ g = dt^2 + f^2(t)g_{\text{can}} + h^2(t)Q|_{\mathfrak{m}} \quad (4.8) \]

is extended to a smooth invariant metric of non-positive curvature on $M$ if $f$ is a smooth odd function on $\mathbb{R}$ with $f'(0) = 1$ which is positive and convex on $\mathbb{R}^+$ and $h$ is an even convex positive function on $\mathbb{R}$ such that

\[ h^3(t)h'(t)f'(t) \geq C \cdot f^3(t) \quad \text{for } t > 0 \quad (4.9) \]
where \( C \) is the positive constant defined by (4.6).

If, moreover, the functions \( f(t), h(t) \) are strictly convex for \( t > 0 \) and the inequality (4.9) is strict, then the sectional curvature at any regular point \( x \in M_{\text{reg}} \) is negative and the sectional curvature \( K(X, Y) = 0 \) for \( X, Y \in \mathfrak{m} = T_o(Go) \) if and only if \([X, Y] = 0\). In particular, if the symmetric space \( Go = G/K \) has rank one, the metric \( g \) has negative curvature.

Such functions \( f, h \) exist, for example we can take \( a = \max\{C, 1\}, h(t) = e^{at^2} \) and \( f(t) = th(t) \).

**Proof of Theorem 4.10.** By Proposition 4.9, the metric \( g \) on \( M_{\text{reg}} \) given by (4.8) extends to a smooth metric \( g \) on \( M \). Using Proposition 4.5, we show that \((M, g)\) has non-positive curvature. It is sufficient to check that the sectional curvatures \( K(X, Y) \) and \( K(X, aT + bY) \) for \( a^2 + b^2 = 1 \) are not positive at a regular point \( c(t), t > 0 \) of a normal geodesic \( c(t) = (t, o) \subset M_{\text{reg}} \) for any orthonormal vectors \( X, Y \in T_{c(t)}(Gc(t)) = p \).

Proposition 4.5 shows that for any orthonormal vectors \( X, Y \in p \simeq T_{c(t)}Gc(t) \) and \( T = \frac{\partial}{\partial t} \), we have

\[
g_{c(t)}(R(X, Y)Y, T) = 0 \quad \text{and} \quad g_{c(t)}(R(X, T)T, X) \leq 0
\]

since \( f \) and \( h \) are convex. Now we estimate the sectional curvature \( g_{c(t)}(R(X, Y)Y, X) \) for \( X, Y \in \mathfrak{m} \) as follows (note that \( f_i = f \sqrt{c_i} \) where \( c_i \) is coefficient in \( g_{\text{can}} = \sum c_i \mathfrak{Q}_{\alpha_i} \)).

\[
g_{c(t)}(R(X, Y)Y, X) = -h^2 ||X, Y||_{\mathfrak{h}}^2 - \sum_i \frac{4h^2 f_i^2 + 3f_i^4}{4f_i^2} ||X, Y||_{\mathfrak{h}}^2 Q - h^2 h' ||X||_{Q}^2 ||Y||_{Q}^2 \leq 0,
\]

If \( X \in \mathfrak{n}, Y \in \mathfrak{m} \) then using the inequality

\[
||X, Y||_{Q}^2 \leq 4||X||_{Q}^2 ||Y||_{Q}^2
\]

we get

\[
g_{c(t)}(R(X, Y)Y, X) = \frac{f_i c_i}{4h^2} \frac{4}{||X, Y||_{\mathfrak{h}}^2} - f_i h f_i h' ||X||_{Q}^2 ||Y||_{Q}^2 \\
\leq \frac{f_i c_i}{h^2} \left[f^3 c_i - f'h'^3 \right] ||X||_{Q}^2 ||Y||_{Q}^2 \leq \frac{f_i c_i}{h^2} \left[f^3 c - f'h'^3 \right] ||X||_{Q}^2 ||Y||_{Q}^2 \leq 0.
\]

To prove that the sectional curvature \( K(X, Y) \) for \( X, Y \in \mathfrak{n} \) is not positive, we use Proposition 4.6. It shows that the submanifold

\[
\mathbb{R}^+ \times K/H = \left[ (e, V_{\rho} \setminus \{0\}) \right]
\]

is a totally geodesic submanifold of \( M_{\text{reg}} \). One can easily check that the curvature of the manifold \( \mathbb{R}^+ \times K/H \) with the metric \( dt^2 + f^2(t) \mathfrak{g}_{\text{can}} \) is non-positive if \( f'' \geq 0 \) and \( f'(0) = 1 \), and is negative if \( f'' > 0 \) and \( f'(0) = 1 \).

It is clear now that the sectional curvatures at a regular point \( x \in M_{\text{reg}} \) are negative if the inequalities are strict. To finish the proof, we calculate the sectional curvatures \( K(X, Y) = g_{c(t)}(R(X, Y)Y, X) \) at the singular point \( c(0) \) using the limit procedure as follows:

\[
\lim_{t \to 0} g_{c(t)}(R(X, Y)Y, X) = \begin{cases} 
-\frac{1}{h'(0)} ||X, Y||_{Q}^2 & \text{if } X, Y \in \mathfrak{m} \\
-\frac{f'(0)}{h''(0)} & \text{if } X, Y \in V \\
-\frac{h'(0)}{h''(0)} & \text{if } X \in \mathfrak{n}, Y \in \mathfrak{m}
\end{cases}
\]

This implies the last statement of the theorem. \( \square \)

As another application of Proposition 4.5, we construct an invariant metric of negative curvature on the regular \( G \)-manifold \( M = \mathbb{R} \times G/K \) where \( K \) is a compact subgroup of a connected, non-compact semisimple Lie group \( G \) under the assumption \([\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{g} \subseteq \mathfrak{t} \supseteq \mathfrak{h} \) where \( \mathfrak{h} \) is a maximal compact subalgebra of the Lie algebra \( \mathfrak{g} \). Such a subgroup \( K \) exists if and only if \( \dim Z(\mathfrak{h}) > 0 \), in particular when \( G \) has infinite center. Since \( \mathfrak{h} \) is the Lie algebra of a maximal
compact subgroup of the adjoint group $\text{Ad}(G)$, the Lie algebra $\mathfrak{g}$ has a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. As before, we have

$$\mathfrak{k} = \mathfrak{k} \oplus \mathfrak{z}, \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{k} + (\mathfrak{z} + \mathfrak{m})$$

where $\mathfrak{z} \neq 0$ is a commutative ideal of $\mathfrak{k}$. We identify $\mathfrak{p}$ with the tangent space $T_o(G/K)$ and an $\text{Ad}(K)$-invariant metric on $\mathfrak{p}$ with associated $G$-invariant metric on $G/K$. In particular, any two positive functions $f(t), h(t)$ on $\mathbb{R}$ define $G$-invariant metric

$$g = dt^2 + f^2(t)Q|_{\mathfrak{k}} + h^2(t)Q|_{\mathfrak{m}}.$$  

(4.10)

**Proposition 4.11.** Let $K$ be a compact subgroup of a connected, non-compact semisimple Lie group $G$. Assume that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \subsetneq \mathfrak{z}$. Then the regular $G$-manifold $M = \mathbb{R} \times G/K$ admits an invariant metric of negative curvature. More precisely, a metric of the form (4.10) has negative curvature if $f(t)$ and $h(t)$ are strictly convex positive functions on $\mathbb{R}$ which satisfy $h^3(t)f'(t)h'(t) > f^3(t)$ for all $t \in \mathbb{R}$. If the inequality is not strict then the curvature is non-positive.

**Proof.** The proof is similar to the proof of Theorem 4.10. □

4.5. Non-existence of invariant metric of negative curvature on some admissible $G$-manifolds

We prove now that some admissible $G$-manifolds has no invariant metric of negative curvature. To do this, we give a simple sufficient condition that the singular orbit of an admissible $G$-manifold is totally geodesic.

**Proposition 4.12.** Let $M = G \times_K \mathbb{V}_\rho$ be an admissible $G$-manifold. Assume that there exists an element $\sigma \in Z(G) \cap K$ such that $\rho(\sigma) = -\text{id}$. Then the singular orbit $P = Go$ is a totally geodesic submanifold of $M$ with respect to any $G$-invariant metric.

**Proof.** The singular orbit $P = Go$ is totally geodesic as a connected component of the set $M^\sigma$ of the fixed points of the isometry $\sigma$. □

**Corollary 4.13.** Let $M = G \times_K \mathbb{V}_\rho$ be an admissible $G$-manifold of a non-compact simply connected simple Lie group $G$ different from $\tilde{\text{SU}}(n, m)$ ($\gcd(n, m)$ is odd), $E_6^{-14}$, $E_8^8$ and $F_4^{-20}$. If $\rho(K) = \text{SO}(2n + 2)$, $\text{SU}(2n)$, $\text{Sp}(n)$, $U(2n)$, $\text{Spin}(7)$ or $\text{Spin}(9)$ then the singular orbit $P = Go$ is totally geodesic with respect to any $G$-invariant metric and therefore if rank $G/K > 1$, there is no invariant metric of negative curvature on $M$.

**Proof.** Using the explicit description of the center $Z(G)$ and a maximal compact subgroup $K$ of a simple Lie group $G$ given in [13], see also Table 2, one can check that in these cases there is an element $\sigma \in Z(G)$ with $\rho(\sigma) = -\text{id}$. Then the result follows from Proposition 4.12. □

Let $M = G \times_K \mathbb{V}_\rho$ be an admissible $G$-manifold of a semisimple Lie group $G$. Assume now that the decomposition (4.4) of the space $\mathfrak{p}$ consists of irreducible and mutually non-equivalent $\text{Ad}(H)$-modules. Then any invariant metric $g$ on $M$ is a diagonal metric and we can use formulas of Proposition 4.5 for the curvature. Note that if the associated non-compact symmetric space $G/K$ has rank greater than one, that is different from

$$\text{SO}(1, n)/\text{SO}(n), \text{SU}(1, n)/U(n), \text{Sp}(1, n)/\text{Sp}(1) \cdot \text{Sp}(n), F_4^{-20}/\text{Spin}(9)$$

then there are two commuting elements $X, Y$ in $\mathfrak{m}$. Using this, we prove the following proposition.

**Proposition 4.14.** Let $M = G \times_K \mathbb{V}_\rho$ be an admissible $G$-manifold of a semisimple Lie group $G$ such that the decomposition (4.4) of the space $\mathfrak{p}$ consists of irreducible and mutually non-equivalent $\text{Ad}(H)$-modules. If the manifold $G/K$ does not belong to the list (4.11) then $M$ does not admit an invariant metric of negative curvature.

**Proof.** By assumption, the metric of $M$ restricted to $M_{\text{reg}}$ can be written as $g = dt^2 + g_t = dt^2 + \sum f_i^2(t)Q|_{\mathfrak{n}_i} + \sum h_j^2(t)Q|_{\mathfrak{m}_j}$. Using formulas from Proposition 4.5 we see that for two orthonormal commuting vectors $X \in \mathfrak{m}_i$, ...
$Y \in m_j$ the sectional curvature

$$K(X, Y) = \frac{1}{4} \frac{d}{dt} (h_i^j) \frac{d}{dt} (h_j^i) |X|^2 |Y|^2.$$

It is sufficient to prove that $h^2(t)$ are even functions. Then $K(X, Y)|_{t=0} = 0$.

To prove this, we remark that since $K$ acts transitively on the unit sphere in the normal space $T^o_o(Go)$, there exists an element $\sigma \in K$ such that $\sigma \dot{c}(0) = -\dot{c}(0)$ where $c(t)$ is a given normal geodesic. The element $\sigma$ belongs to the normalizer $N_K(H)$. This implies that $\text{Ad}_\sigma$ preserves the subspaces $m_i$ and hence the metric $g_t|_{m_i}$. Using this, we have

$$g_{-t}(X, Y) = \sigma^* g_{-t}(X, Y) = g_t(\text{Ad}_{\sigma^{-1}} X, \text{Ad}_{\sigma^{-1}} Y) = g_t(X, Y).$$

This shows that the functions $h^2(t)$ are even. \[ \square \]

**Corollary 4.15.** The admissible $G$-manifolds $M = G \times_K V_\rho$ defined by the following data does not admit an invariant metric of negative curvature:

(a) $G = SO(p, q), K = SO(p) \times SO(q), \rho(K) = SO(p)$ for $p > 2, q \geq 2$ and $p - 1 \neq q$;

(b) $G = SL(2, \mathbb{R}), K = \text{Spin}(2), \rho = \Delta_7$ and $H = G_2$;

(c) $G = SU(p, q), K = SU(p) \times SU(q), \rho(K) = U(p)$ for $p > 2, q \geq 2$ and $p - 1 \neq q$.

**Proof.** One can check that these spaces satisfy the assumption of Proposition 4.14. \[ \square \]

### 5. Non-simply connected cohomogeneity one Riemannian manifolds of non-positive curvature

In this section we assume that $(M, g)$ is a non-simply connected complete cohomogeneity one Riemannian $G$-manifold of non-positive curvature with effective action of a connected Lie group $G$. We denote by

$$\pi: \tilde{M} \longrightarrow M$$

the universal covering of $M$ and by $\tilde{g}$ the induced metric on $\tilde{M}$. Then $\pi: \tilde{M}$ is a cohomogeneity one Riemannian $\tilde{G}$-manifold where $\tilde{G}$ is a (connected) covering group of $G$ which acts effectively on $\tilde{M}$. The kernel $Z$ of the homomorphism

$$\tilde{\pi}: \tilde{G} \longrightarrow G$$

is a discrete central subgroup of $\tilde{G}$.

**Lemma 5.1.** The homomorphism $\pi$ induces an isomorphism of an isotropy subgroup $\tilde{G}_p \subset \tilde{G}$ of any point $p \in \tilde{M}$ onto the $\tilde{\pi}((\tilde{G}_p)) \subset G$. If $G_{\pi(p)}$ is connected then $G_{\pi(p)} = \tilde{\pi}(\tilde{G}_p)$ and if $\tilde{G}$ has finite center then $\tilde{G} = G$.

**Proof.** It follows immediately from the fact that $Z = \text{Ker} \tilde{\pi}$ has no torsion, since it acts freely on a simply connected complete manifold $M$ of non-positive curvature. Hence, $Z$ has a trivial intersection with the compact subgroup $\tilde{G}_p$ and $\tilde{\pi}(\tilde{G}_p) = \tilde{G}_p/(Z \cap \tilde{G}_p) = \tilde{G}_p$. \[ \square \]

**Theorem 5.2.** Let $M$ be a non-simply connected cohomogeneity one Riemannian $G$-manifold of non-positive curvature and $\tilde{M}$ the universal covering manifold of $M$ considered as a cohomogeneity one Riemannian $\tilde{G}$-manifold. Assume that $N_{\tilde{G}}(\tilde{K}) = \tilde{K}$ where $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$. Then $\tilde{G} = G$ and $\tilde{M}$ and $M$ have the forms

$$\left( \tilde{M} = \mathbb{R} \times G/K, \tilde{g} = dt^2 + \tilde{g}_t \right) \quad \left( M = T^1 \times G/K, g = d\varphi^2 + g_\varphi \right)$$

where $\tilde{g}_t$ (resp., $g_\varphi$) is a 1-parameter family of invariant metrics on $G/K$ which depends on the coordinate $t \in \mathbb{R}$ (resp., $\varphi \in T^1$).

For the proof we need the following elementary lemma.
Lemma 5.3. Let $M = G/H$ be a homogeneous manifold. Then any diffeomorphism $\varphi$ of $M$ which commutes with the action of $G$ has the form $\varphi = R_a : gH \mapsto gaH$ for some $a \in N_G(H)$. In particular, if $N_G(H) = H$, then $\varphi = \text{id}$.

Proof of Theorem 5.2. Since $N_{\tilde{G}}(\tilde{K}) = \tilde{K}$, $\tilde{G} = G$ and $\tilde{K} = K$. Assume that the universal covering $\tilde{M}$ of $M$ is a non-regular $G$-manifold, that is $\tilde{M} = G \times_K V_\rho$. Any deck transformation $\delta$ commutes with $G$ and induces an isometry of the orbit space $\tilde{M}/G = \mathbb{R}^+$ which is trivial. Hence $\delta$ preserves orbits. In particular, it induces an isometry of the singular orbit $\tilde{P} = G/K$, which commutes with $G$. It is trivial by Lemma 5.3. The contradiction shows that $\tilde{M}$ is a regular $G$-manifold, i.e. $(M = \mathbb{R} \times G/K, \tilde{g} = dt^2 + \tilde{g}_t)$ where $\tilde{g}_t$ is a 1-parameter family of invariant metrics on $G/K$. Since $N_{\tilde{G}}(\tilde{K}) = \tilde{K}$, any deck transformation of $\tilde{M}$ has the form $\delta : (t, x) \mapsto (t + t_0, x)$. This implies that $M = T^1 \times G/K$. □

Theorem 5.4. Let $(M = G \times_K V_\rho, g)$ be a non-simply connected cohomogeneity one complete Riemannian $G$-manifold of non-positive curvature with a singular orbit $P = G/K$. Assume that $P = G/K$ is not a Riemannian product of a torus $T^k$ and a homogeneous Riemannian manifold $P'$. (This is the case if $G$ is semisimple.) Then the universal covering $\tilde{M}$ of $M$ is a regular cohomogeneity one Riemannian $\tilde{G}$ manifold of the form

$$\tilde{M} = \mathbb{R} \times \tilde{G}/\tilde{K}, \tilde{g} = dt^2 + g_t).$$

If the group $G$ is semisimple with finite center, then $M = T^1 \times G/K$ is a Riemannian direct product of the circle $T^1$ and a symmetric space $G/K$ of non-positive curvature.

Proof. Note that $P$ is not simply connected. Assume that $\tilde{M} = \tilde{G} \times \tilde{K}$ is a non-regular cohomogeneity one $\tilde{G}$-manifold and let $\tilde{P} = \tilde{G}o, o = [e, 0]$ be the singular orbit. Let $\delta$ be a nontrivial deck transformation of $\tilde{M}$. The function $d_{\delta}^2(x) = d^2(x, \delta(x))$ is constant along orbits and $d^2(x, \delta(x))$ is constant in some neighborhood of 0 and therefore $d^2_{\delta}$ is constant in some neighborhood of 0 and therefore $d_{\delta}^2$ is constant on some open $\tilde{G}$-invariant neighborhood $M' \subset \tilde{M}$ of the singular orbit $\tilde{P}$. By 1.3(c), the Riemannian cohomogeneity one $\tilde{G}$-manifold $M'$ has a De Rham decomposition $M' = W \times \mathbb{R}$ where $W$ is a non-trivial flat factor of $M'$ and $\delta$ acts as the parallel translation on $\mathbb{R}$ i.e. $\delta(w, s) = (w, s + a)$. This again implies that the singular orbit $P$ is a torus factor in the De Rham decomposition. The contradiction shows that $\tilde{M}$ is a regular $\tilde{G}$-manifold of the form $(\mathbb{R} \times \tilde{G}/\tilde{K}, dt^2 + g_t)$ where $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$ by 1.2. If $G$ is a semisimple group with finite center, then $G/K$ is a symmetric space of non-positive curvature. The effectivity of the action of $\tilde{G}$ implies that the group $\tilde{G} = G$ and it has no center and $N_{\tilde{G}}(K) = K$. Hence we can apply Theorem 5.2. Now the last statement follows from Corollary 2.3 since the functions $f_t(t)$ are convex and periodic, hence are constant. □

Corollary 5.5. Let $G$ be a semisimple Lie group whose maximal compact subgroup is non-trivial. If $G$ has infinite center then an admissible $G$-manifold $M = G \times_K V_\rho$ with effective action of $G$ does not admit an invariant metric of non-positive curvature.

Proof. Since $K'$ is a non-trivial connected compact Lie group, the codimension of the singular orbit is $\text{dim} V_\rho \geq 2$. This implies that the universal covering of $\tilde{M}$ is not a regular $\tilde{G}$-manifold. Let $Z$ be a cyclic central subgroup of $G$ which has the trivial intersection with the maximal compact subgroup $K'$ of $G$. Then $Z$ acts freely on $M$ and $\tilde{M} = M/Z$ is a non-simply connected cohomogeneity one $G = G/Z$-manifold. Assume that $M$ admits an invariant metric $g$ of non-positive curvature. Then by Lemma 5.1 $(M, g)$ is an effective non-simply connected cohomogeneity one Riemannian $\tilde{G}$-manifold of non-positive curvature of the form $\tilde{M} = \tilde{G} \times_K V_\rho$. By Theorem 5.4, the universal covering $\tilde{M}$ of $\tilde{M}$ is a regular $G$-manifold which contradicts the assumption. □
References