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LARGE SCALE NETWORK ANALYSIS WITH APPLICATIONS TO TRANSPORTATION, COMMUNICATION AND INFERENCE NETWORKS

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1. Introduction

The study of large scale networks has been mainly motivated by practical problems, like transportation problems, reliability problems. The problems usually involve finding the optimal paths in the networks and they are rather similar in nature. These different networks can be unified into a more general form of network, the semiring network. Attempts to describe such networks in a general setting is not new. In [13], Shier has described an algebraic structure to study reliability problem. Carré [6] has given an excellent description of semiring networks and their properties using matrices. In this paper, we shall describe semiring networks and show how to deal with large matrices. The latter is particularly important because in large-scale networks, even computation on the computer presents some difficulties as the amount of random access memory in every computer is limited and the computation time may be long.

2. Semirings

Let S be a non-empty set with two binary operations, $\oplus$ and $\otimes$. The system $\langle S, \oplus, \otimes\rangle$ is called a semiring if
(1) $\langle S, \oplus\rangle$ is a commutative semigroup,
(2) $\langle S, \otimes\rangle$ is a semigroup,
(3) for any $a, b, c \in S$,

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$
$$(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a).$$

For any pair of integers, n and m, let $M_{n\times m}$ be the class of all $n \times m$ matrices over the semiring $\langle S, \oplus, \otimes\rangle$. Define the operation $\oplus$ on $M_{n\times m}$ as usual: if

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$A=[a_{ij}]$ and $B=[b_{ij}]$ are matrices in $\boldsymbol{M}_{n\times m}$, let

$$A \oplus B = [a_{ij} \oplus b_{ij}], \quad \begin{array}{l} i=1,\ldots,n, \\ j=1,\ldots,m. \end{array}$$

Then $\langle \boldsymbol{M}_{n\times m}, \oplus \rangle$ is a commutative semigroup.

Let n, m, k be three given positive integers. Let $A=[a_{ij}]$ be a matrix in $\boldsymbol{M}_{n\times m}$, $B=[b_{ij}]$ be a matrix in $\boldsymbol{M}_{m\times k}$. Define the matrix $C=[c_{ij}]$ as follows:

$$c_{ij} = \bigoplus_{r=1,\ldots,m} (a_{ir} \otimes b_{rj}).$$

Let this operation be denoted by $C = A \otimes B$. In the case where $m=k=n$, $\langle \boldsymbol{M}_{m\times n}, \otimes \rangle$ is a semigroup. Hence the following theorem holds.

Theorem 2.1. *Let $\langle S, \oplus, \otimes \rangle$ be a semiring. For a given positive integer n, $\langle \boldsymbol{M}_{n\times n}, \oplus, \otimes \rangle$ is a semiring.*

Suppose n is a factor of two numbers, p and m, which are greater than 1. For each matrix, $A=[a_{ij}]$, in $\boldsymbol{M}_{n\times n}$, define a collection of $p\times p$ matrices A_{ij}, where $i, j = 1, \ldots, m$, as follows.

$$A_{ij} = [a_{(i-1)p+h,(j-1)p+k}], \qquad h, k = 1, \ldots, p.$$

Then the matrix $A \otimes B$ can also be obtained as specified in the following theorem.

Theorem 2.2. *Let A, B be matrices in $\boldsymbol{M}_{n\times n}$, where $n = p\times m$. Then $C = A\otimes B$ iff*

$$C_{ij} = \bigoplus_{r=1,\ldots,m} (A_{ir} \otimes B_{rj}).$$

This theorem is very useful for very large matrices because it gives us a way of decomposing the matrices.

3. Semiring networks

Let Ω denote a given finite directed graph with vertex set $V(\Omega)$ and edge set $E(\Omega)$. In this paper, we assume that $V(\Omega) = \{1, \ldots, n\}$. Suppose S_1 and S_2 are two semirings whose additive semigroups have zero elements (i.e. additive identity elements) and whose multiplicative semigroups have identity elements. Let ω be a mapping such that

$$\omega : V(\Omega) \to S_1$$
$$\omega : E(\Omega) \to S_2.$$

The function ω is called the weight function and the ordered pair $\langle \Omega, \omega \rangle$ is called a semiring network.

For a semiring network $\langle \Omega, \omega \rangle$ over the semirings $\langle S_1, \oplus, \otimes \rangle$ and $\langle S_2, \oplus, \otimes \rangle$, let $\boldsymbol{P}(\Omega)$ denote the class of all paths of the directed graphs Ω. Extend the weight function, ω to paths of Ω as follows: if $P = (v_0, v_1, \ldots, v_k)$ is a path in $\boldsymbol{P}(\Omega)$, define

$$\omega(P) = \omega(v_0, v_1) \otimes \omega(v_1, v_2) \otimes \cdots \otimes \omega(v_{k-1}, \omega_k).$$

Observe that ω maps P into an element of S_2.

In some applications, the zero elements may not exist, but this can be easily remedied. Consider an element λ that satisfies the following properties with the operations in $\langle S_2, \oplus, \otimes \rangle$:

$$\begin{aligned}
&\lambda \oplus \lambda = \lambda \\
&\lambda \otimes \lambda = \lambda \\
&a \oplus \lambda = \lambda \oplus a = a \quad \text{for all } a \text{ in } S_2, \\
&a \otimes \lambda = \lambda \otimes a = \lambda \quad \text{for all } a \text{ in } S_2.
\end{aligned}$$

Then $\langle S_2 \cup \{\lambda\}, \oplus, \otimes \rangle$ is also a semiring. Hence we may include λ in the set S_2 as the zero element in the semiring. Now, we are ready to define matrices for representing networks.

Let $\langle \Omega, \omega \rangle$ be a given semiring network. Suppose Ω has n vertices, $V(\Omega) = \{1, 2, \ldots, n\}$. Define its associated matrix $A = [a_{ij}]$, $i, j = 1, \ldots, n$, as follows:

$$\text{for } i = j, \quad a_{ii} = \text{identity of } \langle S_2, \otimes \rangle,$$

$$\text{for } i \neq j, \quad a_{ij} = \begin{cases} \omega(i, j) & \text{if } (i, j) \in W(\Omega), \\ \lambda & \text{otherwise.} \end{cases}$$

Given an associated matrix of a semiring network, define a sequence of $n \times n$ matrices, $A_0, A_1, \ldots, A_t, \ldots$ as follows:

$$\begin{aligned}
&A_0 = A \\
&A_1 = A_0 \otimes A_0 \\
&\vdots \\
&A_{t+1} = A_t \otimes A_t \\
&\vdots
\end{aligned}$$

We say that the sequence $A_0, A_1, A_2, \ldots$ converges to A^* if for some non-negative integer t,

$$A^* = A_t = A_{t+1} = A_{t+2} = \cdots$$

In such a case, we call A^* the induced matrix of $\langle \Omega, \omega \rangle$.

4. Basic problems of semiring networks and their applications

Consider some basic problems of the semiring network. For any given semiring network, its induced matrix does not always exist. So the problem is, under what condition does it exist? If it exists, what are the uses of the induced matrix? Finally, how can we compute A^*?

Existence of the induced matrix A^*

It is a well known fact that the induced matrix does not necessarily exist, as testified by the shortest path problem for the transportation network that contains negative cycles. For what types of semiring networks does the induced matrix exist? In [6], Carré has studied this problem. We shall give a brief outline of the ideas presented in his paper.

In most of the applications (path problems, reliability problems, network flow problems), the additive operation $\oplus$ is usually either the minimum or maximum operation. Hence, to generalize this idea into semiring networks, we may assume that $a \oplus a = a$ and introduce an ordering (or partial ordering) to the semiring. This ordering is determined by the additive operation $\oplus$.

For a semiring, define an order relation as follows:

$$a \leqslant b \text{ if } a \oplus b = a,$$

and define the strict order relation as

$$a < b \text{ if } a \leqslant b \text{ and } a \neq b.$$

Then the following theorem gives a condition for the existence of the induced matrix.

Theorem 4.1. *If a semiring network does not contain any cycle 0 with $\omega(C) < e$ (where e is the multiplicative identity), then the induced matrix A^* exists.*

Uses of the induced matrix

Each (i, j) entry of the induced matrix A^* gives the value of an 'optimal' path from vertex i to vertex j. The reason why we call it the 'optimal' path may not be very obvious until we look at the applications of the induced matrix. For example, in the shortest path problem in which the edge-lengths are non-negative real numbers, the induced matrix exists and is actually the distance matrix (matrix whose entries show the distance from one vertex to another) [1, 4, 5]. Similarly, for the reliability problem described in [13], the induced matrix is the reliability matrix.

From the induced matrix, many other properties of the networks can be obtained mainly because many properties of the networks can be defined in terms of the entries of the induced matrix. Let us illustrate this by the transportation problem with the semiring structure $\langle \boldsymbol{R} \cup \{\infty\}, \boldsymbol{L}, +\rangle$ for its edges, where : denotes the minimum operation.

The radius of a vertex, centres, centroids, total distance of a network are defined in terms of the distance of the shortest path between two vertices. the radius of a vertex k is the maximum of $a_{ik}^* + a_{kj}^*$, where i, j runs from 1 to n and it can be found as follows:

(1) from the kth row of A^*, $a_{k1}^*, a_{k2}^*, \ldots, a_{kn}^*$ find the maximum, a_{kj}^*, of all elements in the kth row.

(2) find the maximum, a_{ik}^*, of all the elements $a_{1k}^*, a_{2k}^*, \ldots, a_{nk}^*$, in the kth column of A^*.

(3) then the radius is $a_{ik}^* + a_{kj}^*$.

The centre of a network is the vertex with the smallest radius and it can be found from the induced matrix A^* as follows:

(1) Let $R = (r_1, \ldots, r_n)$ be the vector such that

$$r_1 = \text{minimum}\{a_{i1}^*, \ldots, a_{in}^*\}$$

and let $C = (c_1, \ldots, c_n)$ be the vector such that

$$c_i = \text{maximum}\{a_{1i}^*, \ldots, a_{ni}^*\}:$$

(2) let $S = R + C$; then vertex i is the centre if

$$s_i = \text{minimum}\{s_1, \ldots, s_n\}.$$

The total distance at a vertex i is defined as

$$\sum_{j=1}^{n} a_{ij}^* + \sum_{j=1}^{n} a_{ji}^*,$$

and it can be obtained by adding the sum of all entries in the ith row and the sum of all entries in the ith column. The total distance of a network defined as

$$\sum_{(i,j)\in V\times V} a_{ij}^*$$

and it is equal to twice the sum of all the entries of A^*. To compare two different network designs, the total distance may be used as a form of measurement. Another measure that may be considered is the cost-effective ratio for two networks, which is defined as

$$\frac{\text{decrease in total distance}}{\text{increase in cost}},$$

and this can be obtained by computing the total distances of the networks and their respective costs.

Another application is to find a shortest path from a vertex i to another vertex j. Such a path can be obtained by the following algorithm:

(1) If $a_{ij}^* = \lambda$, then there is no path from vertex i to vertex j; stop. Otherwise, go to Step (2).

(2) Take the ith row, R_i and jth column, C_j of the induced matrix A^*.
(3) Let $S = R_i + C_j$. If there exists $k \neq i, j$ such that $s_k = a^*_{ij}$, then k is an internal vertex of a path from i to j; go to Step (4). Otherwise, (i, j) is such a path; stop.
(4) Use this algorithm to find a shortest path from i to k and a shortest path from k to j. Then these two paths constitute a shortest path from i to j.

An example in which the weight of vertices is defined is the inference network. Consider the network with vertex weight and edge weight both defined on $\langle [0, 1], \Gamma, x \rangle$, where Γ denotes the maximum operation. The inferenced value at a vertex v is

$$\max\{\omega(u) \times \omega(P)\},$$

where u runs through every vertex in $V(\Omega)$, and p runs through every path from u to v. the inferenced value of all the vertices can be found from the induced matrix A^* as follows:

(1) Let $W = (w_1, \ldots, w_n)$, where $w_i = \omega(i)$.
(2) Let $W' = W \otimes A^*$, where $\otimes$ is the matrix multiplication for matrices over the semiring $\langle [0, 1], \Gamma, x \rangle$.

Then the entries of the vector W' gives the inferenced value of each vertex in $\langle \Omega, \omega \rangle$.

The above examples do not exhaust all the applications of the induced matrix. Other types of applications occur in network flow problems, assignment problems, critical path analysis, reliability problems [2, 3, 13].

Various algorithms computing A^*

Many algorithms have evolved for computing the induced matrix. Two such algorithms are the cascade algorithm and Floyd's algorithm [5–9]. These two algorithms operate on whole matrices. Variations of these two algorithms that decompose large matrices have also been developed [10, 11]. We shall briefly describe the cascade algorithm and Floyd's algorithm below.

The Cascade Algorithm

For $i = 1$ to n
for $j = 1$ to n

$$a_{ij} = \bigoplus_{k=1,\ldots,n} (a_{ik} \otimes a_{kj})$$

Floyd's Algorithm

For $k = 1$ to n
for each pair $i, j \in \{1, \ldots, n\}$, let $c_{ij} = a_{ik} \otimes a_{kj}$ let $A = C \oplus A$ (where C denotes the matrix $[c_{ij}]$)

An algorithm using decomposition of matrices is given blow. This makes use of Theorem 2.2.

Algorithm using decomposition to compute A^*

Let $A^{[0]} = A$ and for each non-negative integer t,

$$A_{ij}^{[t+j]} = \bigoplus_{k=1,\ldots,m} (A_{ik}^{[t]} \otimes A_{kj}^{[t]}), \quad i, j \in \{1, \ldots, m\},$$

where the matrices $A_{ij}^{[t]}$, $i, j \in \{1, \ldots, m\}$, are submatrices of $A[t]$ as defined in Section 2.

5. Conclusion

In this paper, we see that the concept of semiring networks can be used to unify various network problems. The idea of imposing algebraic structure is by no means new. Other types of structures have also been discussed [13]. These trends may lead to the study of general algebraic networks.

For large networks, various methods have been devised to simplify and improve efficiency of algorithms for finding properties of networks using matrices. The availability of software that provides whole matrix operations allows us to consider algorithms which have been formerly deemed as inefficient.

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