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Perfect dodecagon quadrangle systems

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ABSTRACT

A dodecagon quadrangle is the graph consisting of two cycles: a 12-cycle $(x_1, x_2, \dots, x_{12})$ and a 4-cycle (x_1, x_4, x_7, x_{10}). *A dodecagon quadrangle system* of order n and index ρ [DQS] is a pair (X, H), where X is a finite set of n vertices and H is a collection of edge disjoint dodecagon quadrangles (called *blocks*) which partitions the edge set of ρK_n, with vertex set X. *A dodecagon quadrangle system* of order n is said to be *perfect* [PDQS] if the collection of 4-cycles contained in the dodecagon quadrangles form a 4-cycle system of order n and index μ. In this paper we determine completely the spectrum of *DQSs* of index one and of PDQSs with the inside 4-cycle system of index one.

1. In memory of Lucia

All the results contained in this paper were obtained by Lucia. She completed this research during the last week of February of 2008. This is Lucia's last paper.

Her intention and wish was to give a talk on this subject at Combinatorics 2008, in June 2008, in Costermano (Italy). However, Lucia's life changed dramatically: on March 17, after medical treatment in Milan, where she usually went since 2006, her oncologist communicated to her that she had only a few months to live.

From that moment, she passed through extremely difficult periods and terrible moments. After a period of chemotherapy and radiotherapy, which she faced with great courage, after a period of unbelievable suffering, she passed away on July 21.

Lucia was only 34 years old.

I, my wife and my son Giuseppe, will never forget the last four months of her life, full of great sadness and sorrow.

Lucia's splendid smile will remain with us forever. (*Mario Gionfriddo*)

2. Introduction

A λ-fold *m-cycle system* of order n is a pair (X, C), where X is a finite set of n elements, called *vertices*, and C is a collection of edge disjoint m-cycles which partitions the edge set of λK_n (the complete graph with vertex set X, where every pair of vertices is joined by λ edges). In this case, $|C| = \lambda n(n-1)/2m$. Often the integer number λ is also called the *index* of the system. When $\lambda = 1$, we will simply say *m-cycle system*. A 3-cycle is also called a *triple* and so a λ-fold 3-*cycle system* will also be called a λ-fold 3-*triple system*. When $\lambda = 1$, we have the well-known definition of *Steiner triple system* (or simply *triple system*).

Fairly recently the spectrum (i.e., the set of all n such that a *m-cycle systems* of order n exists) has been determined to be [1–12]:

(1) $n \geq m$, if $n > 1$,

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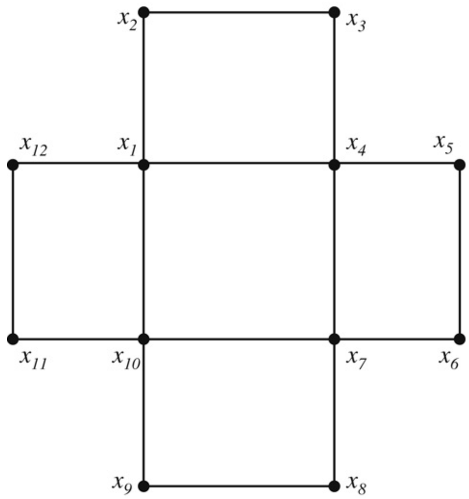

Fig. 1. Dodecagon quadrangle.

(2) n is odd, and
(3) $\frac{n(n-1)}{2m}$ is an integer.

The spectrum for λ-fold *m-cycle systems* for $\lambda \geq 2$ is still an open problem.

The graph given in Fig. 1 is called a *dodecagon quadrangle* and will be also denoted by $[(x_1), x_2, x_3, (x_4), x_5, x_6, (x_7), x_8, x_9, (x_{10}), x_{11}, x_{12}]$.

A dodecagon quadrangle system of order n and index ρ [DQS] is a pair (X, H), where X is a finite set of n vertices and H is a collection of edge disjoint dodecagon quadrangles (called *blocks*) which partitions the edge set of ρK_n, with vertex set X.

A dodecagon quadrangle system (X, H) of order n and index ρ is said to be *perfect* [*PDQS*] if the collection of all of the *inside* 4-cycles contained in the dodecagon quadrangles form a μ-fold 4-cycle system of order n. Usually, this μ-fold 4-cycle system is also said to be *nested* in the DQS (X, H).

The following examples prove that for $n = 33$ there exist both *perfect* systems and *non perfect* systems. In the following examples the vertex set is always Z_{33}.

Example 1. The following system is a *DQS*(33) of index one, but it is not a PDQS.
Base block (mod 33):

[(0), 11, 21, (1), 7, 14, (18), 6, 8, (3), 17, 9].

Example 2. The following system is a *PDQS*(33) having index four. The inside 4-cycle system has index one.
Base blocks (mod 33):

[(0), 32, 7, (1), 4, 20, (17), 18, 31, (8), 13, 29],
[(0), 20, 11, (2), 13, 9, (17), 24, 31, (7), 8, 19],
[(0), 14, 29, (3), 7, 9, (17), 5, 11, (6), 28, 31],
[(0), 16, 22, (4), 9, 19, (17), 30, 15, (5), 26, 14].

In this paper we determine completely the spectrum of *DQS*(n) having index one and the spectrum of *PDQS*(n) having index four (minimum value possible) with the *inside* 4-cycle system of index one.

In what follows, when the index is not specified it is understood to be one.

3. Necessary existence conditions and existence of DQS and PQDS of minimum order

In this section we prove some necessary existence conditions for *DQSs* and for PDQSs.

Theorem 3.1. (1) *If* (X, H) *is a DQS of order* n *and index one, then* $n \equiv 1 \mod 32$, $n \geq 33$. (2) *There exist DQSs of minimum order* $n = 33$.

Proof. (1) Let (X, H) be a DQS of order n (and index one).

Since every block of H consists of 16 edges with vertices in X and all the vertices of a *dodecagon quadrangle* have degree an even integer number, it follows that:

(i) $b = |H| = \frac{\binom{n}{2}}{16}$,

(ii) n is an odd integer number.

From which:

$$n \equiv 1 \bmod 32, \quad n \geq 33.$$

(2) See Example 1. □

Theorem 3.2. *(1) If (X, H) is a PDQS of order n, index ρ, with the inside 4-cycle system of index one, then $\rho = 4$ and $n \equiv 1 (\bmod 8)$, $n \geq 17$. (2) There exist PDQSs of minimum order $n = 17$. (3) There exist PDQSs of order $n = 25$.*

Proof. (1) Let (X, H) be a PDQS of order n having an inside 4-cycle system of index one. Then:

(i) every block of H consists of 16 edges with vertices in X,

(ii) all the vertices of a *dodecagon quadrangle* have even degree which implies that n is odd, and

(iii) the number b of blocks of H is equal to the number c of blocks of the inside 4-cycle system.

Then, since:

$$b = \frac{\binom{n}{2}}{16}\rho, \qquad c = \frac{\binom{n}{2}}{4},$$

it follows that:

$$\rho = 4, \qquad b = \frac{\binom{n}{2}}{4}.$$

Hence: $n \equiv 1 \bmod 8$, $n \geq 17$.

(2) The following system, defined on Z_{17} is a PDQS having index four. The inside 4-cycle system has index one. Base blocks (mod 17):

$$[(0), 9, 4, (1), 16, 15, (8), 12, 6, (3), 7, 2], \quad \text{and}$$
$$[(0), 1, 11, (2), 3, 5, (10), 14, 7, (4), 12, 6].$$

(3) The following system, defined on Z_{25} is a PDQS having index four. The inside 4-cycle system has index one. Base blocks (mod 25):

$$[(0), 10, 2, (1), 5, 11, (13), 22, 12, (6), 9, 8],$$
$$[(0), 7, 14, (2), 9, 15, (13), 24, 4, (5), 8, 12] \quad \text{and}$$
$$[(0), 11, 19, (3), 1, 22, (13), 8, 18, (4), 7, 12]. \quad \square$$

4. Construction $n \to n + 32$ for DQSs and construction $n \to n + 16$ for PDQSs

In this section we give a construction for *DQSs* and a construction for PDQSs.

Theorem 4.1. *A DQS of order $n + 32$ can be constructed from a DQS of order n.*

Proof. Let $Z_{9,i} = Z_9 \times \{i\}$, for $i = 1, 2, \ldots, 4k, a, b, c, d$ (all distinct elements), where $(0, i) = 0$ for every $i = 1, 2, \ldots, 4k, a, b, c, d$. Further, let $(x, i) = x_i$. Let (A, H_1) be a $DQS(n)$ of order $n = 32k + 1$, $n \geq 33$ and let (B, H_2) be a $DQS(33)$ of order 33, where:

$$A = Z_{9,1} \cup Z_{9,2} \cup \cdots \cup Z_{9,4k}, \quad \text{and}$$
$$B = Z_{9,a} \cup Z_{9,b} \cup Z_{9,c} \cup Z_{9,d}.$$

Define on $A \cup B$ the family H^* of dodecagon quadrangles such that: $H_1 \subseteq H^*$, $H_2 \subseteq H^*$.

Further, for every $i = 1, 2, \ldots, 4k$ and for every $j = a, b, c, d$, if:

$$\Phi(1)_{i,j} = \{[(1_j), 5_i, 7_j, (3_i), 8_j, 7_i, (2_j), 6_i, 4_j, (1_i), 3_j, 2_i]\}$$
$$\Phi(2)_{i,j} = \{[(3_j), 6_i, 1_j, (4_i), 2_j, 2_i, (4_j), 8_i, 6_j, (3_i), 5_j, 5_i]\}$$
$$\Phi(3)_{i,j} = \{[(3_j), 4_i, 5_j, (6_i), 6_j, 2_i, (4_j), 5_i, 2_j, (8_i), 1_j, 7_i]\}$$
$$\Phi(4)_{i,j} = \{[(5_j), 8_i, 3_j, (7_i), 4_j, 5_i, (6_j), 4_i, 8_j, (1_i), 7_j, 2_i]\}$$

then:

$$\Phi(1)_{i,j} \subseteq H^*, \qquad \Phi(2)_{i,j} \subseteq H^*,$$
$$\Phi(3)_{i,j} \subseteq H^*, \qquad \Phi(4)_{i,j} \subseteq H^*.$$

If $X = A \cup B$ and $H^* = H_1 \cup H_2 \cup \Phi(1)_{i,j} \cup \Phi(2)_{i,j} \cup \Phi(3)_{i,j} \cup \Phi(4)_{i,j}$ then, examining by difference methods that every pair of distinct elements of X is contained in exactly one *dodecagon quadrangle* of H^*, it is straightforward to verify that (X, H^*) is a DQS of order $n + 32$.

We observe that the number of blocks of H^*, counting $\Phi(u)_{i,j}$ for every $i = 1, 2, \dots, 4k$ and for every $j = a, b, c, d$, is:

$$|H^*| = |H_1| + |H_2| + |\Phi(1)_{i,j}| + |\Phi(2)_{i,j}| + |\Phi(3)_{i,j}| + |\Phi(4)_{i,j}|$$
$$= \frac{\binom{32k+1}{2}}{16} + \frac{\binom{33}{2}}{16} + 64k = 32k^2 + 65k + 33,$$

which is exactly the number of blocks of a $DQS(32k + 33)$:

$$\frac{\binom{32k+33}{2}}{16} = 32k^2 + 65k + 33.$$

So, the proof is complete. □

Theorem 4.2. *A PDQS of order $n + 16$ can be constructed from a PDQS of order n.*

Proof. Let $Z_{4k,i} = Z_{4k} \times \{i\}$, $i = 1, 2$, and let $Z_{8,j} = Z_8 \times \{j\}$, $j = a, b$. Further, let $(x, i) = x_i$.

Let (A, H_1) be a $PDQS(n)$ of order $n = 8k + 1$, $n \geq 17$, and let (B, H_2) be a $PDQS(17)$ of order 17, where:

$$A = Z_{4k,1} \cup Z_{4k,2} \cup \{\infty\} \quad \text{and}$$
$$B = Z_{8,a} \cup Z_{8,b} \cup \{\infty\}.$$

Define on $A \cup B$ the family H^* of dodecagon quadrangles such that: $H_1 \subseteq H^*$, $H_2 \subseteq H^*$. Further, for every $i \in Z_{4k}$, let:

$$\Delta(1)_a = \{[((i+1)_1), 4_a, (i+2)_1, (2_a), (i+2)_2, 8_a, ((i+1)_2), 7_a, (i+3)_2, (1_a), (i+3)_1, 3_a]\},$$
$$\Delta(2)_a = \{[((i+1)_1), 6_a, (i+2)_1, (4_a), (i+2)_2, 2_a, ((i+1)_2), 1_a, (i+3)_2, (3_a), (i+3)_1, 5_a]\},$$
$$\Delta(3)_a = \{[((i+1)_1), 8_a, (i+2)_1, (6_a), (i+2)_2, 4_a, ((i+1)_2), 3_a, (i+3)_2, (5_a), (i+3)_1, 7_a]\},$$
$$\Delta(4)_a = \{[((i+1)_1), 2_a, (i+2)_1, (8_a), (i+2)_2, 6_a, ((i+1)_2), 5_a, (i+3)_2, (7_a), (i+3)_1, 5_a]\},$$
$$\Delta(1)_b = \{[((i+1)_1), 4_b, (i+2)_1, (2_b), (i+2)_2, 8_b, ((i+1)_2), 7_b, (i+3)_2, (1_b), (i+3)_1, 3_b]\},$$
$$\Delta(2)_b = \{[((i+1)_1), 6_b, (i+2)_1, (4_b), (i+2)_2, 2_b, ((i+1)_2), 1_b, (i+3)_2, (3_b), (i+3)_1, 5_b]\},$$
$$\Delta(3)_b = \{[((i+1)_1), 8_b, (i+2)_1, (6_b), (i+2)_2, 4_b, ((i+1)_2), 3_b, (i+3)_2, (5_b), (i+3)_1, 7_b]\},$$
$$\Delta(4)_b = \{[((i+1)_1), 2_b, (i+2)_1, (8_b), (i+2)_2, 6_b, ((i+1)_2), 5_b, (i+3)_2, (7_b), (i+3)_1, 5_b]\}.$$

Then:

$$\Delta(1)_a \subseteq H^*, \qquad \Delta(2)_a \subseteq H^*, \qquad \Delta(3)_a \subseteq H^*, \qquad \Delta(4)_a \subseteq H^*, \quad \text{and}$$
$$\Delta(1)_b \subseteq H^*, \qquad \Delta(2)_b \subseteq H^*, \qquad \Delta(3)_b \subseteq H^*, \qquad \Delta(4)_b \subseteq H^*.$$

If $X = A \cup B$ and

$$H^* = H_1 \cup H_2 \cup \Delta(1)_a \cup \Delta(2)_a \cup \Delta(3)_a \cup \Delta(4)_a \cup \Delta(1)_b \cup \Delta(2)_b \cup \Delta(3)_b \cup \Delta(4)_b$$

then, examining by difference methods that every pair of distinct elements of X is contained in exactly four *dodecagon quadrangles* of H^* and that the *inside* 4-cycles form a 4-cycle system of the same order and index one, it is straightforward to verify that (X, H^*) is a PDQS of order $n + 16$ and index four (minimum possible).

We observe that the number of blocks of H^* is:

$$|H^*| = |H_1| + |H_2| + |\Delta(1)_a| + |\Delta(2)_a| + |\Delta(3)_a| + |\Delta(4)_a| + |\Delta(1)_b| + |\Delta(2)_b| + |\Delta(3)_b| + |\Delta(4)_b|$$
$$= \frac{\binom{8k+1}{2}}{16}4 + \frac{\binom{17}{2}}{16}4 + 32k = 8k^2 + 33k + 34,$$

which is exactly the number of blocks of a $PDQS(8k + 17)$ of index four:

$$\frac{\binom{8k+17}{2}}{16}4 = 8k^2 + 33k + 34.$$

So, the proof is complete. □

5. Conclusions

Collecting together the results of the previous sections, we have the following conclusive theorems:

Theorem 5.1. *There exists a DQS of order* n *and index one if and only if* $n \equiv 1 \bmod 32$, $n \geq 33$.

Proof. It is sufficient to see the statements of Theorems 3.1 and 4.1. □

Theorem 5.2. *There exists a PDQS of order* n *and minimum index if and only if* $n \equiv 1 \bmod 8$, $n \geq 17$.

Proof. It is sufficient to see the statements of Theorems 3.2 and 4.2. □

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