Asymptotic series and Stieltjes continued fractions for a gamma function ratio

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ABSTRACT

An analysis is given for the expansion (60 terms) of a gamma function ratio discussed by Stieltjes and others. A Stieltjes continued fraction is derived, affording lower and upper bound (but lacking a rigorous proof), along with continued fraction for the odd and even series.

1. INTRODUCTION

We study the ratio of gamma functions

$$y(n) = \frac{(n-1)\,\Gamma(^1/_2 n)}{\sqrt{2n}\,\Gamma(\frac{1}{2}n+\frac{1}{2})}, \quad (n = 2, 3, \ldots) \tag{1}$$

and the related function

$$f(n) = y(n)/(1-1/n), \tag{2}$$

by developing series in powers of n^{-1} and converting these (in several ways) to continued fractions (c.f.'s). Our interest was initiated from a study of divergent series occurring in statistics [2], [8], [9]. In particular, it was expedient to find a well-known statistic whose expectation was known in closed form and for which a nontrivial series expansion in a parameter could be developed. The second sample moment in sampling from a standard normal density provided an answer. Specifically, for the random sample $(x_1, x_2, \ldots, x_n)$ of independent and identically distributed variates from N(0, 1), we consider the mean value of $\sqrt{m_2}$ (i.e., $E\sqrt{m_2}$) where $m_2 = \Sigma(x_j - \bar{x})^2/n$. Note that in many statistical applications it is more appropriate to consider $s_x^2 = nm_2/(n-1)$, but for our immediate purpose (which concerns mathematical properties), we prefer the second central moment rather than the variance. The series expansion of $E\sqrt{m_2}$ in descending powers of n, the sample size, turns out to be a divergent series whose sum is y(n) given in (1). There is little point in studying Em_2^s, $s = 1, 3/2, 2, \ldots$, since the expression has a terminating series for integer s, and for half-integers involves no new development over the case $s = {}^1/_2$. Again, our main interest is in properties of y(n), deduced from series and c.f.'s, for n = 2, 3,..., although we may from time to time extend this domain.

The ratio y(n), or one closely related to it, has attracted the attention of mathematicians from time to time over the last century or longer (see, for example, Perron [7], pp. 31-6; also Mitrinovič [6], pp. 286-8). Recently, inequalities for $y^2(n)$ have been derived from purely statistical concepts (see, for example, Gurland [5], Gokhale [4], and Uppuluri [12]; also Gautshi [3] has derived results for n not restricted to integers and from a mathematical viewpoint). Of considerable interest from the asymptotic analysis approach and the use of c.f.'s are the remarks to be found in the works of Stieltjes written towards the end of the last century. In letter number 153 to Hermite (written around November, 1888; see [10]) Stieltjes uses the expressions

$$\left\{\begin{aligned} \frac{\Gamma(a)\Gamma(n)}{\Gamma(a+n)} &= \int_0^\infty \left[\frac{1-e^{-y}}{y}\right]^{a-1} y^{a-1} e^{-ny}\,dy \\ \frac{\Gamma(a)\Gamma(n-a+1)}{\Gamma(n+1)} &= \int_0^\infty \left[\frac{e^{y}-1}{y}\right]^{a-1} y^{a-1} e^{-ny}\,dy \end{aligned}\right. \tag{3}$$

along with power series for the first factors in the integrands, to obtain the series

$$\frac{\Gamma(n-a+1)}{\Gamma(n+1)} = \frac{1}{n^a}\left[1 + \frac{a}{n}c_1 + \frac{a(a+1)}{n^2}c_2 + \frac{a(a+1)(a+2)}{n^3}c_3 + \ldots\right] \tag{4a}$$

$$\frac{\Gamma(n)}{\Gamma(n+a)} = \frac{1}{n^a}\left[1 - \frac{a}{n}c_1 + \frac{a(a+1)}{n^2}c_2 - \frac{a(a+1)(a+2)}{n^3}c_3 - \ldots\right] \tag{4b}$$

where $c_1, c_2, \ldots$ are polynomials in a. He remarks that (4b) reduces to $[n(n+1)\ldots(n+m-1)]^{-1}$ when a is a

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positive integer m, and converges for $|n| > m-1$; in all other cases, it diverges. One cannot help but notice the tantalizing fact that the series parts of (4a) and (4b) are the same when n is replaced by -n. Using, at least formally,

$$\frac{\Gamma(-n)}{\Gamma(-n+a)} = \frac{\sin\pi(n-a)}{\sin\pi n} \times \frac{\Gamma(n-a+1)}{\Gamma(n+1)}$$

Stieltjes produces the expression

$$\frac{\Gamma(-n)}{\Gamma(-n+a)} = \frac{1}{n^a}\left[1+\frac{a}{n}c_1+\frac{a(a+1)}{n^2}c_2+\dots\right]\frac{\sin\pi(n-a)}{\sin\pi n}, \quad (5)$$

and points out that if n is replaced by -n in (4b), then $(-n)^a$ has to be replaced by $n^a(\sin\pi n)/\sin\pi(n-a)$. He remarks "Existe-t-il une formule plus générale qui embrasse les formules 5 (our (4b)) et 5' (our (5)) en même temps ?" Only a few lines earlier in this letter he had remarked "J'ai vainement cherché (il y a quelques années) à établir une théorie satisfaisante de ces séries divergentes", surely a surprising comment considering his considerable contributions to the subject including the Stieltjes c.f.. Aside from this, Stieltjes was putting his finger on the sensitive relation in these gamma function ratios, namely the continuation of the functions through the origin $n = 0$. The problem of discovering a single expression for (4a) and (4b) seems to be still unresolved, and a solution might be a source of illumination for the c.f.'s we derive for y(n) and f(n) in the sequel. Another strange aspect of the Stieltjes comments is that (i) he did not comment on the evident reciprocity in (4) when $a = 1/2$ (which is clearly closely related to y(n)), and (ii) he returned to the subject, although not explicity saying so, in letter 299 ([10], March, 1891) showing among other things that

$$x\left[\frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})}\right]^2 = 1+\frac{2}{8x-1+}\;\frac{1.3}{8x+\dots}\;.$$

In our notation this is equivalent, in terms of (1), to

$$y(n) = \left[1-\frac{1}{n}\right]\sqrt{\left\{1+\frac{2}{4n-1+}\;\frac{1.3}{4n+}\;\frac{3.5}{4n+}\;\frac{5.7}{4n+\dots}\right\}} \quad (6)$$

which incidentally provides monotonic sequences of bounding approximants to y(n) when $4n > 1$. By elementary manipulation, we deduce

$$\frac{n^2y^2(n)-(n-1)^2}{n^2y^2(n)+(n-1)^2} = \frac{n}{4n^2+}\;\frac{1}{4/3+}\;\frac{1}{4n^2/5+}\;\frac{1}{4/7+}\;\frac{1}{4n^2/9+\dots} \quad (7)$$

so that the left side is an odd function in n, leading to

$$y(-n) = (1-1/n^2)/y(n) \quad (8)$$

which defines y(n) for negative n (complex values not being of especial interest at this stage). Note that $ny(n)/(n-1)$ or f(n) is (4b) when n is replaced by n/2.

Having given a brief historical background, we now describe in this note (i) an assessment of the rate of divergence of the series for y(n), (ii) show that the coefficients are (apart from denominators consisting of powers of 2) integers, (iii) describe c.f.'s for the even and odd parts of y(n) and f(n), and (iv) give a Stieltjes type c.f. for f(n) providing new tentative bounds.

2. INTEGRAL REPRESENTATIONS AND SERIES

2.1. y(n)

From the beta-function

$$y(n) = \frac{(n-1)}{\sqrt{2n\pi}}\int_0^1 t^{\frac{1}{2}n-1}(1-t)^{-\frac{1}{2}}\,dt \quad (n>0) \quad (9a)$$

we have from the transformation $t = \exp(-x)$,

$$y(n) = \frac{(2m-1)}{2\sqrt{m\pi}}\int_0^\infty e^{-mx}(1-e^{-x})^{-\frac{1}{2}}\,dn. \qquad (m=\tfrac{1}{2}n) \quad (9b)$$

Hence, if $y(n) \sim \sum_0^\infty e_s/n^s$, setting up a recursion for the coefficients in the power series expansion of $(1-e^{-x})^{-\frac{1}{2}}$, we find

$$2se_s+(2s-3)e_{s-1}/2!+(2s-5)e_{s-2}/3!+\dots+e_1/s! = e_0/(s+1)! \quad (10)$$

for $s = 2, 3, \dots$; $e_0 = 1$, $e_1 = 1/4$.

The coefficients $\{e_s\}$ can be generated from (10), and to proceed as far as 60 terms extended precision arithmetic is needed (Table 1). An inspection shows that there is a distinct sign pattern (two negatives followed by two positives) with a marked increase in magnitude starting at e_{11}. The first four terms would scarcely suggest that e_{60} is of order 10^{49}. Note that

$$y(2) \sim 1-0.375-5.46875E-02-8.7890625E-03 +1.800537106E-03+1.842498778E-03 -\underline{5.538463578E-03}-1.275569200E-03,$$

the underscored term being the smallest numerically, so that summing to the term before the smallest (following the concept that in certain classical asymptotic series the error is less than the magnitude of the first term omitted) yields $y(2) = 0.5652$ approx. as against the correct value 0.5641896. Similarly, summing to the numerically smallest term yields $y(2) = 0.5646$, a slightly better approximation. The corresponding results for $n = 5$ are 0.8407481, 0.8407484 (summing to the smallest) against 0.8407487 (true). The assessments (knowing, *a priori*, the correct values) are acceptable.

It is clearly advantageous to study the even and odd parts of the series from which (Table 2) we have asymptotically

$$\sqrt{|e_{2s}/e_{2s-2}|} \sim (2s-5/2)/\pi, \quad (11a)$$

$$\sqrt{|e_{2s+1}/e_{2s-1}|} \sim (2s-1/2)/\pi . \quad (11b)$$

Improvements on these can be found by setting for example

$$\sqrt{|e_{2s}/e_{2s-2}|} \sim \frac{(2s-5/2)}{\pi}\left[a_0^{(s)} + \frac{a_1^{(s)}}{s} + \dots + \frac{a_r^{(s)}}{s^r}\right]$$

as suggested by Bender and Wu [1]. For our purpose we only need the dominant terms, and find

$$|e_{2s}| \sim 0.2116\,(2/\pi)^{2s}\,\Gamma^2(s-1/4)$$
$$|e_{2s+1}| \sim 0.1143\,(2/\pi)^{2s}\,\Gamma^2(s+3/4) \quad (12)$$

where e_s has the same sign as $(-1)^{[\frac{1}{2}s]}$. As for the accuracy of (12) we have the approximants 3.91E-02, 2.94E-02, 7.76E00, 5.82E00, 1.96E09, 1.46E11, 3.44E36 and 2.58E36 for $|e_s|$, $s = 3, 4, 11, 12, 24, 25, 49, 50$, respectively, with reasonable agreement (Table 1). It should, of course, be noted that there is no guarantee that (12) would still be acceptable if we analyzed more terms (the first million, for example), and unsuccessful attempts using (1) or its integral representations have run up against the problem of the definition of the function in the neighborhood of the origin, as foreseen by Stieltjes.

Since we shall refer to the odd and even parts of y(n), we note the following definitions and representations.

Even part

$$y_e(n) = [y(n) + y(-n) - 2]/2$$
$$= \frac{1}{\sqrt{2n\pi}} \int_0^\infty e^{-2nx}\{n\sqrt{\coth x} - \sqrt{\tanh x} - n/\sqrt{x}\}dx \quad (13a)$$

Odd part

$$y_0(n) = [y(n) - y(-n) - 2e_1/n]/2$$
$$= \frac{-1}{\sqrt{2n\pi}} \int_0^\infty e^{-2nx}\{\sqrt{x} - \sqrt{\tanh x}\}\{n + \sqrt{\coth x/x}\}dx. \quad (13b)$$

Since $x > \tanh x$ for $x > 0$, it is evident that the odd part, as defined in (13b), is negative for $n > 0$.

2.2. f(n)

As for the integral representation for y(n) in (9b), we have

$$f(n) = \sqrt{m/\pi} \int^\infty e^{-mx}(1-e^{-x})^{-\frac{1}{2}}\,dx, \quad (14)$$

so that corresponding to (10) we find for $f(n) \sim \Sigma b_s/n^s$,

$$2sb_s + \frac{(2s-1)(2s-3)}{2!}b_{s-1} + \frac{(2s-1)(2s-3)(2s-5)}{3!}b_{s-2}$$
$$+ \dots + \frac{(2s-1)(2s-3)\dots .1}{s!}b_1 = \frac{(2s-1)(2s-3)\dots .1}{(s+1)!}.$$
$$(s=1,2,\dots;\ b_0=1). \quad (15)$$

For example,

$$f(n) \sim 1 + 1/(4n) + 1/(32n^2) - 5/(128n^3) - 21/(2048n^4) + ..$$

and an extended tabulation (Table 3) brings out the possibility that the general coefficient is an integer divided by a power of 2. A heuristic approach to this property is to consider

$$f(n) \sim \int_0^\infty \frac{\exp[\psi(x) - x]}{\sqrt{\pi x}}\,dx, \quad (16)$$

where

$$\psi(x) = (x - x^2/2)/m + (x^2/2 - x^3/3)/m^2 + \dots .$$

Let

$$\exp\psi(x) = 1 + k_1 x + k_2 x^2/2! + \dots$$

so that after differentiation and simplification,

$$mk_{s+1} = (s+1)k_s - sk_{s-1} \qquad (s=1,2,\dots) \quad (17)$$

with $k_0 = 1$, $k_1 = 1/m$. Clearly k_s is an integer-valued polynomial of degree s in 1/m. But, using

$$\int_0^\infty \frac{e^{-x}x^r\,dx}{\sqrt{\pi x}} = \frac{(2r-1)(2r-3)\dots 1}{2^r}, \quad (r=1,2,\dots)$$

a typical term in (16) is $(2r)!\,k_r/(2^r r!\,r!)$, where the factorial is always an integer for r a positive integer. Hence, the asymptotic expansion of f(n) (and also that of y(n) since $e_s = b_s - b_{s-1}$) in descending powers of n has coefficients which are odd integers divided by powers of 2.

Note also from (1), (2) and (8) that

$$f(n) = \sqrt{(\tfrac{1}{2}n)}\,\Gamma(\tfrac{1}{2}n)/\Gamma(\tfrac{1}{2}n + \tfrac{1}{2}) \quad (n>0) \quad (18a)$$

with the reciprocal relation

$$f(n)\,f(-n) = 1. \quad (18b)$$

The odd and even parts are of interest as integrals.

Odd part

$$f_0(n) = [f(n) - f(-n)]/2$$

$$= \sqrt{2n/\pi}\int_0^\infty e^{-2nx}\sqrt{\tanh x}\,dx \quad (19a)$$

Even part

$$f_e(n) = [f(n) + f(-n)]/2$$
$$= \sqrt{2n/\pi} \int_0^\infty e^{-2nx} \sqrt{\coth x}\, dx \qquad (19b)$$

so that for $n > 0$, $f_0(n)$ and $f_e(n)$ are positive.

3. CONTINUED FRACTIONS RELATED TO y(n)

3.1. Complete series

The Stieltjes c.f. for $y(n)$ in (6) would scarcely be discovered from a direct attack on the series for $y(n)$; however, it would appear if we searched the Padé table derived from the series for $f^2(n)$, but it would elude us if we studied the Padé table for $y^2(n)$.

Defining the c.f. for $y(n)$ by

$$y(n) = \frac{na_0}{n+b_1 -} \frac{a_1}{n+b_2 -} \cdots \quad (a_0 = 1) \qquad (20)$$

and evaluating a_s, b_s ($s = 1$ to 30), it turns out that the a's are negative (excepting a_0), $b_1 > 0$, b_2 to b_9 are negative and alternate in sign thereafter. It will be seen (Table 4) that the sequence $\{-a_s\}$ is steadily increasing, the more so for larger s, whereas $\{b_{2s}\}$ for $s > 5$ also increases at a much slower rate. The odd sequence $\{-b_{2s+1}\}$ also increases slowly and may well be bounded by 2. If this is the case, then for $n > 2$ all the *computed* partial numerators and denominators are positive and successive convergents will provide increasing (even) and decreasing (odd) sequences bounding the true value (assuming the pattern exhibited in the table holds for additional terms). As a numerical illustration, when $n = 2$, the 26th to 30th convergents are : $0.56418\underline{796}$, $0.5641\underline{9528}$, $0.56418\underline{831}$, $0.5641\underline{9454}$, and $0.56418\underline{856}$, underscored digits showing discrepancies from the true value 0.56418958. The rate of "convergence" is anything but spectacular and in fact the error of the 30th approximant is 1.0E-06 as compared to 2.7E-08 for the corresponding term of Stieltjes square root form given in (6).

Is the situation improved by using the reciprocal relation $y(n)y(-n) = (1 - 1/n^2)$? By an equivalence transformation in (20), we can see that for $n > 2$, under the assumptions made for the c.f. for $y(n)$, that upper and lower bounds will become available for $y(2)$, for example, using $y(-2)$. The 29th and 30th approximants to

$y(-2) = (3/4)\sqrt{\pi} = 1.329340388179$ are $1.32934039\underline{414}$ and $1.32934038\underline{672}$ leading to approximants to $y(2)$ with error in the region of at most 2. 0E-09, showing an improvement on the previous two assessments.

3.2. Odd and even parts

The c.f.'s for $y_0(n)$ and $y_e(n)$ defined in (13) are of Stieltjes type. In fact

$$ny_0(n) = \frac{-p_0^*}{n^2} \, \frac{p_1^*}{+1} \, \frac{q_1^*}{+n^2} \, \frac{p_2^*}{+1} \, \frac{q_2^*}{+n^2} + \cdots,$$
$$(p_0^* = 0.0703125) \qquad (22a)$$

$$y_e(n) = \frac{-p_0}{n^2 +} \, \frac{p_1}{1 +} \, \frac{q_1}{n^2 +} \, \frac{p_2}{1 +} \, \frac{q_2}{n^2 +} \cdots$$
$$(p_0 = 0.21875) \qquad (22b)$$

where the partial numerators (as far as the computations go) are positive (Table 5). Hence, since

$$y(n) = e_0 + e_1/n + y_0(n) + y_e(n)$$
$$(e_0 = 1,\ e_1 = -3/4) \qquad (23)$$

it follows that using the Stieltjes odd (even) approximants from (22a) and (22b) in (23) will lead to approximants to $y(n)$ less (greater) than the true value. For example, the 28th and 29th approximants to $y(2) = 0.5641895835$ are

$$y_{28}(2) = 1 - 0.75/2 - 0.053234993 - 0.007575335$$
$$= 0.56418967,$$

$$y_{29}(2) = 1 - 0.75/2 - 0.053235032 - 0.007575461$$
$$= 0.56418951.$$

It will be seen from Table 5 that there are the asymptotes

$$\begin{cases} p_s^* \sim s^2, & q_s^* \sim s(s-1), \\ p_s \sim s(s-1), & q_s \sim s^2. \end{cases} \qquad (24)$$

Can it be proved from the integral forms in (13) that the c.f.'s in (22) are convergent for $n > 0$, for example, and have the asymptotes (24) ?

4. CONTINUED FRACTIONS FOR f(n)

4.1. Complete series

Stieltjes c.f. has been computed and takes the form

$$n^{-1} f(n) = \frac{1}{n-1} \, \frac{1/4}{+n -} \, \frac{1/8}{1+} \, \frac{p_2}{n} \, \frac{p_2}{1+} \, \frac{p_3}{n-} \, \frac{p_3}{1} \, \frac{p_4}{+n} \, \frac{p_4}{\cdot} \qquad (25)$$

where

p_2 = 1.37500000 00	p_3 = 7.32438017 -01
p_4 = 2.93279411 00	p_5 = 1.37362135 00
p_6 = 4.46116414 00	p_7 = 2.02809281 00
p_8 = 5.97095265 00	p_9 = 2.69058071 00
p_{10} = 7.46772663 00	p_{11} = 3.35872173 00
p_{12} = 8.95466981 00	p_{13} = 4.03118846 00
p_{14} = 1.04338010 01	p_{15} = 4.70713644 00

$p_{16} = 1.19064989\ 01$	$p_{17} = 5.38598397\ 00$
$p_{18} = 1.33737577\ 01$	$p_{19} = 6.06730750\ 00$
$p_{20} = 1.48363244\ 01$	$p_{21} = 6.75078574\ 00$
$p_{22} = 1.62947783\ 01$	$p_{23} = 7.43616714\ 00$
$p_{24} = 1.77495811\ 01$	$p_{25} = 8.12324957\ 00$
$p_{26} = 1.92011091\ 01$	$p_{27} = 8.81186645\ 00$
$p_{28} = 2.06496781\ 01$	$p_{29} = 9.50187496\ 00$
$p_{30} = 2.20955700\ 01$	

As might be expected, the even and odd sequences of partial numerators belong to different classes but both increase. The fact that the partial numerators in (25) are in pairs is noteworthy. Numerical studies [n = 1 (1) 5] bring out a periodicity of four with respect to increasing and decreasing sequences; for example, the 3 (4) 57 approximants exceed the true value whereas the 5 (4) 55 approximants are deficient. To understand this property, a contraction process applied to (25) gives

$$f(n) = 1 + \frac{1}{4n - \frac{1}{2} +} \; \frac{4^2 p_1 p_2}{4n +} \; \frac{4^2 p_2 p_3}{4n +} \cdots \qquad (p_1 = 1/8) \qquad (26)$$

which clearly leads to bounding sequences since the p's are positive. The resemblance to the Stieltjes form (6), namely

$$f^2(n) = 1 + \frac{2}{4n-1 +} \; \frac{1.3}{4n +} \cdots$$

is remarkable. Can one conclude that

$$f(n) = 1 + \frac{1}{4n - \frac{1}{2} + n\Psi_1(n^2)} \qquad (27a)$$

$$f^2(n) = 1 + \frac{2}{4n - 1 + n\Psi_2(n^2)}, \qquad (27b)$$

where

$$\Psi_i(n^2) = \int_0^\infty \frac{d\sigma_i(t)}{t + n^2} \qquad (i = 1, 2)$$

are Stieltjes transforms, with a consequent relation between Ψ_1 and Ψ_2 ?

As numerical illustrations, the 55th and 57th convergents of (25) yield 1.253198 < f(1) < 1.253417 for $\sqrt{\pi/2} = 1.253314$, whereas the same convergents for $f(2) = 2/\sqrt{\pi} = 1.128379167$ are 1.12837910 and 1.12837922.

Lastly, we remark that there is no clear conclusion possible with respect to the even convergents of (25), for the contracted form now becomes

$$f(n) = \frac{n}{n - 1/4 +} \; \frac{1/32}{n - p_2 + p_1 +} \; \frac{p_2^2}{n - p_3 + p_2 +} \; \frac{p_3^2}{n - p_4 + p_3 +} \cdots \qquad (28)$$

in which it seems likely that the pattern of signs of the partial denominators for any n > 0 will sooner or later change, since $p_{2s} > p_{2s-1}$.

4.2. Odd and even parts

There are Stieltjes type c.f.'s for the odd and even parts of f(n) [see (19a), (19b) and (Table 6)]

$$f_0(n) = \frac{0.25\,n}{n^2 +} \; \frac{p_1^*}{1 +} \; \frac{q_1^*}{n^2 +} \cdots \qquad (29a)$$

$$f_e(n) = \frac{n^2}{n^2 -} \; \frac{p_1}{1 +} \; \frac{q_1}{n^2 +} \cdots \qquad (29b)$$

where, apart from p_1, the p's and q's are positive and increase with s. The first c.f. (29a) corresponds to an infinite integral involving $\sqrt{\tanh x}$; Stieltjes ([11], p. 383, (15)) gives a corresponding c.f. for one involving tanh x. The second (29b) corresponds to an infinite integral involving $\sqrt{\coth x}$.

5. CONCLUDING REMARKS

A number of outstanding problems are posed in this note.

(i) Can analytical expressions be found for the coefficients in y(n), similar to (11a), (11b) ? Will these expressions hold, failing analytic methods, if the numerical analysis is carried further ?

(ii) Related to (1) is the problem of discovering a single expression for y(n) valid in a neighborhood of the origin (n = 0).

(iii) Can convergence regions be established for the c.f.'s corresponding to y(n), f(n), and their odd and even parts ?

(iv) Recalling the findings of the studies by Gurland [5], Gokhale [4], Uppuluri [12], and Gautshi [3], similar tentative inequalities can be set up from (20), (22), (25), and (26). For example, for $m > \frac{1}{2}$, from (26)

$$\ell_1 < \ell_2 < \frac{\sqrt{m}\,\Gamma(m)}{\Gamma(m + \frac{1}{2})} < u_2 < u_1$$

where

$$u_1 = (16m + 1) / (16m - 1),$$

$$\ell_1 = [(16m)^2 + 16m + 11] / [(16m)^2 - 16m + 11],$$

$$u_2 = \frac{11(16m)^3 + 11(16m)^2 + 830(16m) + 709}{11(16m)^3 - 11(16m)^2 + 830(16m) - 709},$$

$$\ell_2 = \frac{709(16m)^4 + 709(16m)^3 + 150969(16m)^2 + 143170(16m) + 10721}{709(16m)^4 - 709(16m)^3 + 150969(16m)^2 - 143170(16m) + 107218}$$

Are these valid ?

In any event, whether these problems can be solved or not, the series for y(n) and f(n) provide proving grounds for new and old techniques of summation.

TABLE 1.

Coefficients of Terms of Powers of n^{-1} in $E\sqrt{m_2}$.

Population Sampled, N(0,1)

s	e_s		s	e_s	
0	0.1000000000000000	01	30	-0.5561563144197839	15
1	-0.7500000000000000	00	31	-0.6561036165408340	17
2	-0.2187500000000000	00	32	0.4903294437166505	17
3	-0.7031250000000000	-01	33	0.6592523246557520	19
4	0.2880859375000000	-01	34	-0.4928991304269193	19
5	0.5895996093750000	-01	35	-0.7492645185553590	21
6	-0.3544616699218750	-01	36	0.5604012017898723	21
7	-0.1632728576660156	00	37	0.9563445097595795	23
8	0.1101402044296265	00	38	-0.7155002382831404	23
9	0.8957180678844452	00	39	-0.1362142738068739	26
10	-0.6334559656679630	00	40	0.1019363302629313	26
11	-0.8045845820568602	01	41	0.2152690534739988	28
12	0.5819407935196068	01	42	-0.1611322438038073	28
13	0.1067670059217926	03	43	-0.3755419264931757	30
14	-0.7814824188229068	02	44	0.2811514913009373	30
15	-0.1959839821832750	04	45	0.7198315371443462	32
16	0.1444461086177711	04	46	-0.5389928207672613	32
17	0.4751954396992378	05	47	-0.1509589401875378	35
18	-0.3517534368957456	05	48	0.1130503806626766	35
19	-0.1470431744280302	07	49	0.3450331550664175	37
20	0.1091602568267512	07	50	-0.2584208385470577	37
21	0.5653650491703796	08	51	-0.8564346854071110	39
22	-0.4205587430749474	08	52	0.6415168872472400	39
23	-0.2643818606156279	10	53	0.2301117860638066	42
24	0.1969559019470220	10	54	-0.1723830088467729	42
25	0.1477539850916502	12	55	-0.6672411131174865	44
26	-0.1101938612757657	12	56	0.4998912531874082	44
27	-0.9725147345571507	13	57	0.2082143106603326	47
28	0.7259141246039853	13	58	-0.1560042793970636	47
29	0.7445873515644643	15	59	-0.6974175373234660	49

NOTE: Reduced to 16 digits from the original 70-digit output.

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TABLE 3. Coefficients in Series for y(n) and f(n)

	y(n)		f(n)
s	N_s	$\phi(s)$	N_s^*
0	1	0	1
1	-3	2	1
2	-7	5	1
3	-9	7	-5
4	59	11	-21
5	483	13	399
6	-2323	16	869
7	-42801	18	-39325
8	923923	23	-334477
9	30055511	25	28717403
10	-170042041	28	59697183
11	-8639161167	30	-8400372435
12	99976667055	34	-34429291905
13	7336972779615	36	7199255611995
14	-42962450319915	39	14631594576045
15	-4309733345367105	41	-4251206967062925

$(e_s = N_s/2^{\phi(s)},\ b_s = N_s^*/2^{\phi(s)})$

TABLE 2. Analysis of Coefficients in y(n)

s	$-e_{2s}/e_{2s-2}$	$\sqrt{\left\|\frac{e_{2s}}{e_{2s-2}}\right\|}$	$r_s = (2s - \frac{5}{2})/\pi$	s	e_{2s+1}/e_{2s-1}	$\sqrt{\left\|\frac{e_{2s+1}}{e_{2s-1}}\right\|}$	$r_s' = (2s-\frac{1}{2})/\pi$
1	0.2188	0.4677	-	1	0.0938	0.3062	0.4775
2	0.1317	0.3629	0.4775	2	0.8385	0.9157	0.1141
3	1.2304	1.1092	1.1141	3	2.7692	1.6641	1.7507
4	3.1073	1.7627	1.7507	4	5.4860	2.3422	2.3873
5	5.7514	2.3982	2.3873	5	8.9826	2.9971	3.0239
6	9.1869	3.0310	3.0239	6	13.2698	3.6428	3.6606
7	13.4289	3.6645	3.6606	7	18.3562	4.2844	4.2972
8	18.4836	4.2993	4.2972	8	24.2466	4.9241	4.9338
9	24.3519	4.9348	4.9338	9	30.9437	5.5627	5.5704
10	31.0332	5.5707	5.5704	10	38.4489	6.2007	6.2070
11	38.5267	6.2070	6.2070	11	46.7630	6.8384	6.8437
12	46.8320	6.8434	6.8437	12	55.8866	7.4757	7.4803
13	55.9485	7.4799	7.4803	13	65.8199	8.1129	8.1169
14	65.8761	8.1164	8.1169	14	76.5631	8.7500	8.7535
15	76.6146	8.7530	8.7535	15	88.1164	9.3870	9.3901
16	88.1640	9.3896	9.3901	16	100.4799	10.0240	10.0268
17	100.5241	10.0262	10.0268	17	113.6537	10.6608	10.6634
18	113.6949	10.6628	10.6634	18	127.6378	11.2977	11.3000
19	127.6764	11.2994	11.3000	19	142.4322	11.9345	11.9366
20	142.4686	11.9360	11.9366	20	158.0371	12.5713	12.5732
21	158.0715	12.5726	12.5732	21	174.4524	13.2080	13.2099
22	174.4849	13.2093	13.2099	22	191.6781	13.8448	13.8465
23	191.7090	13.8459	13.8465	23	209.7143	14.4815	14.4831
24	209.7438	14.4825	14.4831	24	228.5609	15.1182	15.1197
25	228.5891	15.1192	15.1197	25	248.2181	15.7549	15.7563
26	248.2450	15.7558	15.7563	26	268.6857	16.3916	16.3930
27	268.7116	16.3924	16.3930	27	289.9639	17.0283	17.0296
28	289.9887	17.0291	17.0296	28	312.0526	17.6650	17.6662
29	312.0764	17.6657	17.6662	29	334.9518	18.3017	18.3028

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TABLE 4. J Fraction for y(n)

s	$-a_s$	b_s
1	0.7813	0.7500
2	0.0656	-1.0500
3	1.7536	-0.6900
4	2.2140	-0.2439
5	5.7605	-0.8169
6	6.1982	-0.1366
7	11.9196	-0.9090
8	12.0411	-0.0507
9	20.2186	-0.9895
10	19.7538	0.0266
11	30.6492	-1.0637
12	29.3435	0.0986
13	43.2053	-1.1338
14	40.8156	0.1671
15	57.8821	-1.2008
16	54.1742	0.2329
17	74.6758	-1.2654
18	69.4226	0.2966
19	93.5832	-1.3281
20	86.5636	0.3584
21	114.6015	-1.3891
22	105.5997	0.4187
23	137.7283	-1.4487
24	126.5330	0.4777
25	162.9616	-1.5071
26	149.3652	0.5356
27	190.2994	-1.5644
28	174.0982	0.5924
29	219.7401	-1.6208
		0.6483

NOTE: Reduced to 4 decimal places from the original 70-digit out.

TABLE 5. Stieltjes c.f.'s for the Odd and Even Parts of y(n)

s	$y_o(n)$ p_s^*	q_s^*	$y_e(n)$ p_s	q_s
1	8.38541667 -01	1.93067417 00	1.31696429 -01	1.09870611 00
2	3.89678156 00	5.88056825 00	2.10181904 00	4.15252976 00
3	8.94478582 00	1.18352963 01	6.05008488 00	9.20293048 00
4	1.59890141 01	1.97925819 01	1.20013848 01	1.62506931 01
5	2.50310506 01	2.97515640 01	1.99548759 01	2.52965089 01
6	3.60715727 01	4.17117990 01	2.99100214 01	3.63408246 01
7	4.91109503 01	5.56730188 01	4.18664726 01	4.93839383 01
8	6.41494145 01	7.16350449 01	5.58239900 01	6.44260590 01
9	8.11871230 01	8.95977503 01	7.17824019 01	8.14673394 01
10	1.00224190 02	1.09561041 02	8.97415795 01	1.00507895 02
11	1.21260700 02	1.31524843 02	1.09701424 02	1.21547817 02
12	1.44296720 02	1.55489103 02	1.31661856 02	1.44587177 02
13	1.69332298 02	1.81453783 02	1.55622813 02	1.69626033 02
14	1.96367456 02	1.81369003 02	1.81584244 02	1.96134692 02

(see (22a), (22b))

TABLE 6. Stieltjes Type c.f.'s for the Odd and Even Parts of f(n)

s	$f_o(n)$ p_s^*	q_s^*	$f_e(n)$ p_s	q_s
1	1.56250000 -01	1.09062500 00	0.03125000 00	0.35937500 00
2	2.09571096 00	4.14821091 00	0.88111413 00	2.38052330 00
3	6.04144385 00	9.20087291 00	3.78874594 00	6.46798952 00
4	1.19908144 01	1.62504700 01	8.70586198 00	1.25497392 01
5	1.99426176 01	2.52979272 01	1.56265510 01	2.06285522 01
6	2.98962052 01	3.63437732 01	2.45494201 01	3.07053781 01
7	4.18511813 01	4.93883431 01	3.54738826 01	4.27806954 01
8	5.58072827 01	6.44318659 01	4.83996075 01	5.68547973 01
9	7.17643232 01	8.14745066 01	6.33263784 01	7.29278835 01
10	8.97221654 01	1.00516389 02	8.02540408 01	9.10000993 01
11	1.09680704 02	1.21557608 02	9.91824789 01	1.11071555 02
12	1.31639857 02	1.44598238 02	1.20111602 02	1.33142340 02
13	1.55599565 02	1.69638327 02	1.43041333 02	1.57212534 02
14	1.81559800 02	1.96677866 02	1.67971588 02	1.83282247 02
15	2.09520638 02		1.94902211 02	

(see (29a), (29b))