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On ϵ_K beyond lowest order in the operator product expansion

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ABSTRACT

We analyse the structure of long distance (LD) contributions to the CP-violating parameter ϵ_K, that generally affect both the *absorptive* (Γ_{12}) and the *dispersive* (M_{12}) parts of the K^0–$\bar{K}^0$ mixing amplitude. We point out that, in a consistent framework, in addition to LD contributions to $\mathrm{Im}\,\Gamma_{12}$, estimated recently by two of us, also LD contributions to $\mathrm{Im}\,M_{12}$ have to be taken into account. Estimating the latter contributions the impact of LD effects on ϵ_K is significantly reduced (from −6.0% to −3.6%). The overall effect of LD corrections and of the superweak phase being different from 45° is summarised by the multiplicative factor $\kappa_\epsilon = 0.94 \pm 0.02$.

1. Introduction

Some of the most important tests of the Standard Model (SM) are offered by CP-violating observables, that in this model are supposed to originate from a single CP-odd phase in the CKM matrix [1]. In particular, the crucial test is the hierarchy of CP-violating effects in B_d, B_s and K systems predicted by this model. Indeed the most prominent CP-violating observables in these three systems, $S_{\psi K_S}$, $S_{\psi\phi}$ and ϵ_K, predicted by the SM, differ by orders of magnitude

$$S_{\psi K_S} \approx 2/3, \qquad S_{\psi\phi} \approx 4 \times 10^{-2}, \qquad |\epsilon_K| \approx 2 \times 10^{-3}. \tag{1}$$

Extensive analyses of the Unitarity Triangle have shown a spectacular consistency of the data for $S_{\psi K_S}$ and ϵ_K, within the parametric and theoretical uncertainties in ϵ_K, that until recently were rather sizable. The size of $S_{\psi\phi}$ measured by CDF [2] and DØ [3] appears to be by one order of magnitude larger than predicted by the SM, but the large experimental errors preclude any definitive conclusions.

Recently the consistency of the measured values for $S_{\psi K_S}$ and ϵ_K within the SM has been challenged in [4,5] due to two facts:

- The improved value of the relevant hadronic parameter $\hat{B}_K$ from unquenched lattice QCD that enters the evaluation of ϵ_K. This parameter is now not only known with an accuracy of 4% [6,7] but turns out to be significantly lower than previously found in lattice calculations, suppressing by 10% the previous estimates of ϵ_K.
- A more careful look at ϵ_K, that identified an additional suppression of $|\epsilon_K|$, summarised by a multiplicative factor $\kappa_\epsilon = 0.92 \pm 0.02$ [5] to the previously adopted formula for ϵ_K.

In view of these two suppressions, as demonstrated in [5], the size of CP violation measured in $B_d \to \psi K_S$ might be insufficient to describe ϵ_K within the SM. Clarifying this new tension is important as the $S_{\psi K_S} - \epsilon_K$ correlation in the SM is presently the most important direct relation between CP violation in the B_d and K systems that can be tested experimentally.

The correction calculated in [5] originates from two factors: (i) the difference of the superweak phase ϕ_ϵ from 45°, and (ii) the long-distance contribution to ϵ_K arising from the imaginary part of the *absorptive* amplitude of the K^0–$\bar{K}^0$ mixing, Γ_{12}. The latter effect has been estimated with the help of the $\Delta I = 1/2$ dominance in $K \to \pi\pi$ decays and the experimental value for ϵ'/ϵ.

In the present Letter we point out that at the same level of accuracy other effects should be considered, in particular the long distance contributions to the imaginary part of the *dispersive* amplitude M_{12}. While this topic has been the subject of intensive discussions in the mid 1980's, it is important to have a fresh look at this issue in view of the decrease of the error in $\hat{B}_K$ and of the theoretical advances during the last twenty five years.

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Our Letter is organized as follows. In Section 2 we present general formulae from which the different contributions to ϵ_K can be clearly identified. In Section 3 we discuss ϵ_K using the Operator Product Expansion (OPE). This allows us to identify the most important, still missing, long-distance contributions to $\mathrm{Im}\, M_{12}$. In Section 4 we estimate the size of these contributions in the framework of Chiral Perturbation Theory (CHPT), and briefly compare our findings with previous literature. We conclude in Section 5.

2. Notation and general formulae

Indirect CP violation originates in the weak phase difference between the (off-diagonal elements of the) Hermitian matrices M and Γ which control the time evolution of a neutral meson system. For the K^0–$\bar{K}^0$ system one has

$$i\frac{d}{dt}\begin{pmatrix}|K^0(t)\rangle\\ |\bar{K}^0(t)\rangle\end{pmatrix} = \left(M - i\frac{\Gamma}{2}\right)\begin{pmatrix}|K^0(t)\rangle\\ |\bar{K}^0(t)\rangle\end{pmatrix}. \tag{2}$$

Defining the eigenvectors

$$|K_{S(L)}\rangle = \frac{1}{\sqrt{2(1+|\bar{\epsilon}|^2)}}\left[(1+\bar{\epsilon})\left|K^0\right\rangle \mp (1-\bar{\epsilon})\left|\bar{K}^0\right\rangle\right], \tag{3}$$

the following phase-convention-independent relation holds:

$$\frac{\mathrm{Re}(\bar{\epsilon})}{1+|\bar{\epsilon}|^2} = \frac{\mathrm{Im}(\Gamma_{12}M_{12}^*)}{4|M_{12}|^2+|\Gamma_{12}|^2}\left[1+\mathcal{O}\left(\mathrm{Im}\left(\frac{\Gamma_{12}}{M_{12}}\right)\right)\right]. \tag{4}$$

This represents indeed the indirect CP-violating parameter measured from the semileptonic charge asymmetries [8] or the Bell–Steinberger relation [9]. The experimental smallness of $\mathrm{Re}(\bar{\epsilon})$ makes the expansion to first non-trivial order in the weak phases an excellent approximation. At this level of accuracy we can identify $\mathrm{Re}(\bar{\epsilon})$ with the real part of the complex quantity ϵ_K, defined in terms of the $K \to 2\pi$ amplitudes,

$$\epsilon_K = \frac{2\eta_{+-}+\eta_{00}}{3}, \qquad \eta_{ij} = \frac{\mathcal{A}(K_L \to \pi^i\pi^j)}{\mathcal{A}(K_S \to \pi^i\pi^j)}. \tag{5}$$

The two parameters are indeed related by $\epsilon_K = \bar{\epsilon} + i\xi$, where ξ is the weak phase of the $K^0 \to (2\pi)_{I=0}$ amplitude, namely

$$\xi = \frac{\mathrm{Im}\, A_0}{\mathrm{Re}\, A_0}, \quad A_0 \equiv \mathcal{A}\big(K^0 \to (2\pi)_{I=0}\big). \tag{6}$$

Expanding to first non-trivial order in the weak phases we have

$$\Delta m_K = m_L - m_S = 2\,\mathrm{Re}(M_{12}),$$
$$\Delta\Gamma = \Gamma_S - \Gamma_L = -2\,\mathrm{Re}(\Gamma_{12}). \tag{7}$$

Introducing also the so-called superweak phase, $\phi_\epsilon = \arctan(2\Delta m_K/\Delta\Gamma)$, the expression for $\mathrm{Re}(\bar{\epsilon})$ becomes

$$\mathrm{Re}(\epsilon_K) = \mathrm{Re}(\bar{\epsilon}) = \cos\phi_\epsilon \sin\phi_\epsilon\left[\frac{\mathrm{Im}\, M_{12}}{2\,\mathrm{Re}\, M_{12}} - \frac{\mathrm{Im}\,\Gamma_{12}}{2\,\mathrm{Re}\,\Gamma_{12}}\right]. \tag{8}$$

A further simplification arises by the observation that the $|(2\pi)_{I=0}\rangle$ final state largely saturates the neutral kaon decay widths. Since

$$\Gamma_{21} = \Gamma_{12}^* = \sum_f \mathcal{A}\big(K^0 \to f\big)\mathcal{A}\big(\bar{K}^0 \to f\big)^*, \tag{9}$$

the $|(2\pi)_{I=0}\rangle$ dominance in the sum over final states implies

$$\frac{\mathrm{Im}\,\Gamma_{12}}{\mathrm{Re}\,\Gamma_{12}} \approx -2\frac{\mathrm{Im}\, A_0}{\mathrm{Re}\, A_0} = -2\xi. \tag{10}$$

Expressing $\mathrm{Re}\, M_{12}$ in terms of Δm_K and using Eq. (10) we arrive at

$$\mathrm{Re}(\epsilon_K) = \cos\phi_\epsilon \sin\phi_\epsilon\left[\frac{\mathrm{Im}\, M_{12}}{\Delta m_K} + \xi\right], \tag{11}$$

which is consistent with

$$\epsilon_K = e^{i\phi_\epsilon}\sin\phi_\epsilon\left[\frac{\mathrm{Im}\, M_{12}}{\Delta m_K} + \xi\right]. \tag{12}$$

The equation above allows us to calculate ϵ_K by taking ϕ_ϵ and Δm_K from experiment and calculating $\mathrm{Im}\, M_{12}$ and ξ in a given model, in particular the SM. In Ref. [5] only *short distance* contributions to $\mathrm{Im}\, M_{12}$, represented by the well known box diagrams, have been included, while ξ has been calculated by relating it to the ratio ϵ'/ϵ and taking the latter from experiment. As we will discuss in the following, this approach is not fully consistent: in this way $\mathrm{Im}\,\Gamma_{12}$ and $\mathrm{Im}\, M_{12}$ are evaluated at a different order in the OPE. In particular, *long distance* contributions to $\mathrm{Im}\, M_{12}$, which are of the same order of $\mathrm{Im}\,\Gamma_{12}$ (the latter giving rise to the ξ term in Eq. (12)), are missing.

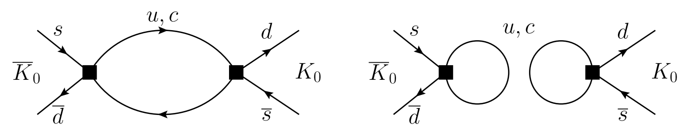

Fig. 1. Contractions of the leading $|\Delta S| = 1$ four-quark effective operators contributing to M_{12} at $\mathcal{O}(G_F^2)$.

3. Decomposition of $\mathrm{Re}(\epsilon_K)$ using the OPE

As shown in Eq. (8) the evaluation of ϵ_K requires the knowledge of the weak phases of both M_{12} and Γ_{12}. In this respect, we should emphasize that $\mathrm{Im}\, M_{12}$ and $\mathrm{Im}\,\Gamma_{12}$ are both generated at $\mathcal{O}(G_F^2)$. Since $\mathrm{Re}\, M_{12}$ and $\mathrm{Re}\,\Gamma_{12}$ are very similar in size ($\phi_\epsilon \approx 43.5°$), we should consistently evaluate $\mathrm{Im}\, M_{12}$ and $\mathrm{Im}\,\Gamma_{12}$ at the same order in the OPE.

The relevant effective Hamiltonians are $\mathcal{H}_{\Delta S=2}$ (contributing to $\mathrm{Im}\, M_{12}$ only) and $\mathcal{H}_{\Delta S=1}$ (contributing to both $\mathrm{Im}\, M_{12}$ and $\mathrm{Im}\,\Gamma_{12}$). The leading term in the OPE is the short-distance contribution to $\mathrm{Im}\, M_{12}$,

$$\mathrm{Im}\, M_{12}^{(6)} \equiv \mathrm{Im}\, M_{12}^{\mathrm{SD}} = \frac{1}{2m_K}\mathrm{Im}\big(\big\langle\bar{K}^0\big|\mathcal{H}_{\Delta S=2}^{(6)}\big|K^0\big\rangle\big)^* \tag{13}$$

where

$$\mathcal{H}_{\Delta S=2}^{(6)} = \frac{G_F^2 m_W^2}{16\pi^2} \times F_0 \times Q^{(6)},$$
$$Q^{(6)} = (\bar{s}d)_{V-A}(\bar{s}d)_{V-A}, \tag{14}$$

is the dimension-six $\Delta S = 2$ effective Hamiltonian. The operator $Q^{(6)}$ does not mix with other operators and the imaginary part of its Wilson coefficient is dominated by terms proportional to the top-quark Yukawa coupling.[1] At this order in the OPE one is neglecting terms generated by two insertions of $\Delta S = 1$ operators (see Fig. 1) which cannot be absorbed into the coefficient of $Q^{(6)}$. For consistency, this implies one should set $\mathrm{Im}\,\Gamma_{12}$ to zero, since $\mathrm{Im}\,\Gamma_{12}$ is the absorptive part of the diagrams in Fig. 1. In other words, the leading order result is obtained with the following substitutions in Eq. (11):

$$\mathrm{Im}\, M_{12} \to \mathrm{Im}\, M_{12}^{(6)} = \mathrm{Im}\, M_{12}^{\mathrm{SD}} \quad \text{and} \quad \xi \to 0. \tag{15}$$

Going one step forward requires taking into account:

[1] The explicit expression of the coefficient function F_0, depending on quark masses and CKM elements, can be found in [10].

1. non-local contributions to both $\mathrm{Im}\, M_{12}$ and $\mathrm{Im}\, \Gamma_{12}$ generated by the $\mathcal{O}(G_F)$ dimension-six $\Delta S = 1$ operators,
2. local contributions to $\mathrm{Im}\, M_{12}$ generated by dimension-eight $\Delta S = 2$ operators of $\mathcal{O}(G_F^2)$.

The structure of the subleading terms in $\mathrm{Im}\, M_{12}$ is very similar to the $\mathcal{O}(G_F^2)$ long-distance contributions to $K \to \pi \nu \bar{\nu}$, discussed in Ref. [11]. The relevant effective Hamiltonian changes substantially if we choose a renormalization scale above or below the charm mass. Keeping the charm as explicit degree of freedom, dimension-eight operators are safely negligible and the key quantity to evaluate is

$$\mathcal{T}_{12} = -i \int d^4x \langle K^0 | T\big[\mathcal{H}^{(u,c)}_{|\Delta S|=1}(x) \mathcal{H}^{(u,c)}_{|\Delta S|=1}(0)\big] | \bar{K}^0 \rangle, \tag{16}$$

where the superscript in $\mathcal{H}^{(u,c)}_{\Delta S=1}$ denotes that the we have two dynamical up-type quarks. The absorptive part of $\mathcal{T}_{12}$ contributes to Γ_{12}, while the dispersive part contributes to M_{12}. In the latter case the leading term in the expansion in local operators should be subtracted, being already included in $\mathrm{Im}\, M_{12}^{(6)}$. In principle, extracting the subleading contribution to $\mathrm{Im}\, M_{12}$ directly from Eq. (16) is the best strategy: the result would be automatically scale independent. However, in practice this is far from being trivial also on the lattice, given the disconnected diagrams in Fig. 1.

Following a purely analytical approach, we can integrate out the charm and renormalize $\mathcal{H}_{\Delta S=1}$ below the charm mass. This allows to identify ξ with the weak phase of the A_0 amplitude, that, as mentioned, has already been estimated in Ref. [5] (see also [12]). On the other hand, $\mathrm{Im}\, M_{12}$ assumes the form

$$\begin{aligned} \mathrm{Im}\, M_{12} &= \mathrm{Im}\, M_{12}^{\mathrm{SD}} + \mathrm{Im}\, M_{12}^{\mathrm{LD}}, \\ \mathrm{Im}\, M_{12}^{\mathrm{LD}} &= \mathrm{Im}\, M_{12}^{\text{non-local}} + \mathrm{Im}\, M_{12}^{(8)}, \end{aligned} \tag{17}$$

where $\mathrm{Im}\, M_{12}^{\text{non-local}}$ and $\mathrm{Im}\, M_{12}^{(8)}$ are not separately scale independent. The structure of the dimension-eight operators obtained integrating out the charm, and an estimate of their impact on ϵ_K, has been presented in Ref. [13]. According to this estimate, $\mathrm{Im}\, M_{12}^{(8)}$ is less than 1% of the leading term.

The smallness of $\mathrm{Im}\, M_{12}^{(8)}$ can be understood by the following dimensional argument. First, it should be noted that the CKM suppression of the dimension-eight operators is $(V_{cs}^* V_{cd})^2$, namely the same CKM factor of the genuine charm contribution in $\mathcal{H}^{(6)}_{\Delta S=2}$. Second, even if we are not able to precisely evaluate the hadronic matrix elements of the dimension-eight operators, we expect

$$\langle \bar{K}^0 | Q_i^{(8)} | K^0 \rangle = \mathcal{O}(1) \times m_K^2 \times \langle \bar{K}^0 | Q^{(6)} | K^0 \rangle. \tag{18}$$

According to this scaling, the contribution of $\mathrm{Im}\, M_{12}^{(8)}$ is an $\mathcal{O}(m_K^2/m_c^2 \approx 15\%)$ correction of the charm contribution (charm–charm loops) to $\mathrm{Im}\, M_{12}^{(6)}$, which itself is an $\mathcal{O}(15\%)$ correction of the total dimension-six contribution. We are thus left with an overall $\mathcal{O}(2\%)$ naive suppression of $\mathrm{Im}\, M_{12}^{(8)}$ with respect to $\mathrm{Im}\, M_{12}^{(6)}$. According to the explicit evaluation in Ref. [13], the actual numerical impact is even smaller.

The only potentially large long-distance contribution to $\mathrm{Im}\, M_{12}$ is the contribution of the non-local terms enhanced by the $\Delta I = 1/2$ rule. For this purpose, we observe that if we had a single weak operator in $\mathcal{H}_{\Delta S=1}$, this would generate the same weak phase to both $\mathrm{Im}\, M_{12}^{\mathrm{LD}}$ and $\mathrm{Im}\, \Gamma_{12}$. As we discuss in more detail in the next section, this is what happens to lowest order in CHPT, where the $\Delta I = 1/2$ part $\mathcal{H}_{\Delta S=1}$ has only one operator, with effective coupling G_8. Decomposing $\mathrm{Im}\, M_{12}^{\mathrm{LD}}$ as a leading term proportional to G_8^2, and a subleading term with different effective coupling

$$\mathrm{Im}\, M_{12}^{\mathrm{LD}} = \mathrm{Im}\, M_{12}^{\mathrm{LD}}|_{G_8^2} + \mathrm{Im}\, M_{12}^{\mathrm{LD}}|_{\text{non-}G_8^2}, \tag{19}$$

we can write

$$\mathrm{Im}\, M_{12}^{\mathrm{LD}}|_{G_8^2} = \mathrm{Re}\, M_{12}^{\mathrm{LD}}|_{G_8^2} \times \frac{\mathrm{Im}[(G_8^*)^2]}{\mathrm{Re}[(G_8^*)^2]}, \tag{20}$$

and identify the weak phase of G_8 with ξ. As a result,

$$\mathrm{Im}\, M_{12}^{\mathrm{LD}}|_{G_8^2} \approx \mathrm{Re}\, M_{12}^{\mathrm{LD}}|_{G_8^2} \times (-2\xi) \approx -\xi \times \big(\Delta m_K^{\mathrm{LD}}|_{G_8^2}\big). \tag{21}$$

This allow us to re-write Eq. (11) as follows

$$\mathrm{Re}(\epsilon_K) = \cos\phi_\epsilon \sin\phi_\epsilon \left[\frac{\mathrm{Im}\, M_{12}^{(6)}}{\Delta m_K} + \xi\left(1 - \frac{\Delta m_K^{\mathrm{LD}}|_{G_8^2}}{\Delta m_K}\right) + \delta_{\mathrm{Im}\, M_{12}}\right], \tag{22}$$

where $\delta_{\mathrm{Im}\, M_{12}}$ encodes the subleading terms in $\mathrm{Im}\, M_{12}^{\mathrm{LD}}|_{\text{non-}G_8^2}$ (including also $\mathrm{Im}\, M_{12}^{(8)}$). Note that, in the limit where the contribution of G_8 saturates Δm_K, the contribution of ξ would be absent. This is exactly what we should expect, since in this limit M_{12} and Γ_{12} would have the same weak phase but for the short-distance contribution to $\mathrm{Im}\, M_{12}$.

4. Estimate of long-distance effects in CHPT

A convenient framework for estimating the long-distance contribution to M_{12} is provided by Chiral Perturbation Theory (CHPT). In this framework π, K and η fields are identified with the would-be Goldstone bosons arising from the $SU(3)_L \times SU(3)_R \to SU(3)_{L+R}$ symmetry breaking of the QCD action in the limit of vanishing light quark masses. Low-energy amplitudes involving these mesons, expanded in powers of their masses and momenta, are evaluated by means of an effective Lagrangian written in terms of the pseudo-Goldstone boson fields.

The lowest-order effective Lagrangian describing non-leptonic $\Delta S = 1$ decays has only two operators, transforming as $(8_L, 1_R)$ and $(27_L, 1_R)$ under the $SU(3)_L \times SU(3)_R$ chiral group. Moreover, only the $(8_L, 1_R)$ operator has a phenomenologically large coefficient, being responsible for the enhancement of $\Delta I = 1/2$ amplitudes. As a result, the only term in the effective Lagrangian relevant to our calculation is

$$\mathcal{L}^{(2)}_{|\Delta S|=1} = F^4 G_8 \big(\partial^\mu U^\dagger \partial_\mu U\big)_{23} + \text{h.c.}, \tag{23}$$

where, as usual, we define

$$U = \exp(i\sqrt{2}\Phi/F),$$

$$\Phi = \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{bmatrix}, \tag{24}$$

and F can be identified with the pion decay constant ($F \approx 92$ MeV). The effective coupling G_8 can be determined by $K \to 2\pi$ amplitudes. Neglecting the $(27_L, 1_R)$ operator and evaluating the $K \to 2\pi$ amplitudes at tree level leads to

$$A_0 = \mathcal{A}\big(K^0 \to (2\pi)_{I=0}\big) = \sqrt{2} F G_8 \big(m_K^2 - m_\pi^2\big), \tag{25}$$

which implies $|G_8| \approx 9 \times 10^{-6}$ $(\mathrm{GeV})^{-2}$. As far as the weak phase of G_8 is concerned, at this level of accuracy we have $\mathrm{Im}(G_8)/\mathrm{Re}(G_8) = \xi$.

In principle $\mathcal{L}^{(2)}_{|\Delta S|=1}$ could contribute to M_{12} already at $\mathcal{O}(p^2)$, via the tree-level diagram in Fig. 2 (left). However, considering the

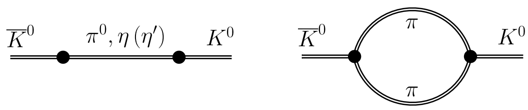

Fig. 2. Tree-level and one-loop diagrams contributing to $\bar{K}^0$–K^0 mixing in CHPT.

$\mathcal{O}(p^2)$ relation among π^0, η and kaon masses (i.e. the Gell-Mann–Okubo mass formula), this contribution vanishes [14]. As a result, the first non-vanishing contribution to M_{12} generated by $\mathcal{L}^{(2)}_{|\Delta S|=1}$ arises only at $\mathcal{O}(p^4)$.

At $\mathcal{O}(p^4)$ we should evaluate loop amplitudes with two insertions of $\mathcal{L}^{(2)}_{|\Delta S|=1}$ and tree-level diagrams with the insertion of appropriate $\mathcal{O}(p^4)$ counterterms. Among all these $\mathcal{O}(p^4)$ contributions, the only model-independent, and presumably dominant, contribution to M_{12} is the non-analytic one generated by the pion-loop amplitude in Fig. 2 (right),

$$\begin{aligned} T_{12}^{(\pi\pi)} &= \mathcal{A}^{(\pi\pi)}\left(\bar{K}^0 \to K^0\right) \\ &= -\frac{3}{16\pi^2} F^2 \left(G_8^*\right)^2 \left(m_K^2 - m_\pi^2\right)^2 \\ &\quad \times \left[\sqrt{1-4r_\pi^2}\left(\log\frac{1+\sqrt{1-4r_\pi^2}}{1-\sqrt{1-4r_\pi^2}} - i\pi\right) + \log\left(\frac{m_\pi^2}{\mu^2}\right)\right], \end{aligned} \tag{26}$$

with $r_\pi^2 = m_\pi^2/m_K^2$ and where we have absorbed all finite (mass-independent) terms in the definition of the renormalization scale μ. This is the only contribution which has an absorptive part. As a consequence, its weak phase can be unambiguously related to the weak phase of the $K^0 \to (2\pi)_{I=0}$ amplitude to all orders in the chiral expansion. In addition, it is the only contribution that survives in the limit of $SU(2)_L \times SU(2)_R$ CHPT, which is known to represent a good approximation of the full $\mathcal{O}(p^4)$ amplitude in several K-decay observables where contributions from counterterms are fully under control (see e.g. Ref. [15]).

A CHPT calculation of M_{12} complete to $\mathcal{O}(p^4)$ would require consideration of loops involving kaons and η's, as well as $\mathcal{O}(p^4)$ local counterterms. However, all these additional pieces are not associated with any physical cut. As such, they can effectively be treated as a local term whose overall weak phase cannot be related to the phase of the $K^0 \to (2\pi)_{I=0}$ amplitude.[2] On account of the above considerations,[3] we refrain from a full $\mathcal{O}(p^4)$ CHPT calculation, and we focus on the pion-loop non-analytic contribution only. Using the relation $T_{12}^{(\pi\pi)} = 2m_K M_{12}^{(\pi\pi)}(\mu)$, the result in Eq. (26) implies

$$M_{12}^{(\pi\pi)}(\mu) = -\frac{3}{64\pi^2 m_K}\left(A_0^*\right)^2\left[\log\left(\frac{m_K^2}{\mu^2}\right) + \mathcal{O}\left(\frac{m_\pi^2}{m_K^2}\right)\right]. \tag{27}$$

The absorptive part in Eq. (26) is nothing but the leading $|(2\pi)_{I=0}\rangle$ contribution to Γ_{12}, which gives rise to the relation (10). The dispersive part is the dominant contribution to M_{12} in the leading-log approximation. The close link of these two terms is a further confirmation that we cannot neglect the long-distance contribution to $\operatorname{Im} M_{12}$ if we want to keep track of all the $\mathcal{O}(\xi)$ terms in ϵ_K.

Using the result in Eq. (27) we can estimate the contribution to $\operatorname{Im} M_{12}$ proportional to G_8, which enters in the phenomenological formula for $\operatorname{Re}(\epsilon_K)$ in Eq. (22). Setting $\mu = 800$ MeV and varying it in the interval 0.6–1 GeV leads to

$$\frac{\Delta m_K^{\mathrm{LD}}|_{G_8^2}}{\Delta m_K^{\exp}} = \frac{2\operatorname{Re} M_{12}^{(\pi\pi)}}{\Delta m_K^{\exp}} = 0.4 \pm 0.2. \tag{28}$$

Note that the result has a well-defined sign since G_8 (or A_0) appears squared in $M_{12}^{(\pi\pi)}$. Using this result in Eq. (22) we find a suppression of the ξ term relative to the estimate in [5], where only the LD contribution to $\operatorname{Im}\Gamma_{12}$ has been taken into account.

Since our estimate of $\Delta m_K^{\mathrm{LD}}|_{G_8^2}$ is not the result of a complete calculation at fixed order in the chiral expansion, it is worthwhile to cross-check it using a different argument. For this purpose, we note that the only relevant contribution to M_{12}, beside the two-pion intermediate state, is expected to arise from the tree-level η' exchange (Fig. 2 left) [17]. We can thus decompose M_{12} as follows:

$$M_{12} \approx M_{12}^{\mathrm{SD}} + M_{12}^{\mathrm{LD}}|_{\pi\pi} + M_{12}^{\mathrm{LD}}|_{\eta'}. \tag{29}$$

According to this decomposition it is clear that, as far as long-distance contributions are concerned, we can trade the evaluation of $M_{12}^{\mathrm{LD}}|_{\pi\pi}$ for that of $M_{12}^{\mathrm{LD}}|_{\eta'}$. An estimate of the η' contribution to M_{12} goes beyond pure CHPT, where it can be considered as a free parameter (the leading contribution to the $\mathcal{O}(p^4)$ local terms). However, its impact can be estimated in the large N_c limit, extending the underlying symmetry from $SU(3)_L \times SU(3)_R$ to $U(3)_L \times U(3)_R$. Within this framework the operator basis must be extended and we cannot directly relate the phase of the η' exchange amplitude to the phase of G_8. According to the recent analysis in Ref. [17], the η' amplitude gives a negative contribution to Δm_K:

$$2\operatorname{Re} M_{12}^{\mathrm{LD}}\big|_{\eta'} = \Delta m_K^{\mathrm{LD}}\big|_{\eta'} \approx -0.3\,\Delta m_K^{\exp}. \tag{30}$$

Most important for our analysis, this contribution is found to be induced at the quark level by the operator $(\bar{s}d)_{V-A} \times (\bar{u}u)_{V-A}$ only [17]. This implies that the η' exchange has a vanishing weak phase in the standard CKM phase convention:

$$\operatorname{Im} M_{12}^{\mathrm{LD}}\big|_{\eta'} = 0. \tag{31}$$

Using this result in Eq. (11), and using the relation (21) for the $\pi\pi$ contribution, we get

$$\operatorname{Re}(\epsilon_K) = \cos\phi_\epsilon \sin\phi_\epsilon\left[\frac{\operatorname{Im} M_{12}^{(6)}}{\Delta m_K^{\exp}} + \xi\,\frac{\Delta m_K^{\mathrm{SD}} + \Delta m_K^{\mathrm{LD}}|_{\eta'}}{\Delta m_K^{\exp}}\right], \tag{32}$$

where the G_8 term (i.e. the $\pi\pi$ contribution), is manifestly absent. Denoting as ρ the coefficient of the ξ term in Eq. (32), and combining Eq. (30) with the NLO short-distance estimate of $\operatorname{Re} M_{12}$, namely $\Delta m_K^{\mathrm{SD}} = (0.7 \pm 0.1)\Delta m_K^{\exp}$ [10,18], we get $\rho = 0.4 \pm 0.1$. This result is well consistent with the value $\rho = 0.6 \pm 0.2$ obtained from Eq. (22) with the direct evaluation of the $\pi\pi$ contribution in Eq. (28).

We rate the direct evaluation of the $\pi\pi$ loop as the most reliable estimate of ρ. As a consequence, our final phenomenological expression for ϵ_K is

$$\epsilon_K = \sin\phi_\epsilon e^{i\phi_\epsilon}\left[\frac{\operatorname{Im} M_{12}^{(6)}}{\Delta m_K} + \rho\xi\right] \quad \text{with } \rho = 0.6 \pm 0.3, \tag{33}$$

where we have conservatively increased by 50% the error in Eq. (28) to take into account the sub-leading contributions of $\operatorname{Im} M_{12}^{\mathrm{LD}}|_{\text{non-}G_8^2}$. For $\rho = 1$ our result reduces to the one in [5]. The contribution calculated in this Letter, resulting in $\rho < 1$, completes the estimate of the terms of $\mathcal{O}(\xi)$ in ϵ_K.

Following the notation of Ref. [5], we summarise the corrections to ϵ_K due to LD effects and $\phi_\epsilon \neq 45°$, via the introduction of the phenomenological factor κ_ϵ, defined by

[2] For a recent, elucidating discussion about the role of kaon loops in CHPT, see [16].

[3] The authors warmly acknowledge Jean-Marc Gérard for triggering a discussion on this point.

$$\epsilon_K = \kappa_\epsilon \frac{e^{i\phi_\epsilon}}{\sqrt{2}} \left[\frac{\mathrm{Im}\, M_{12}^{(6)}}{\Delta m_K} \right]. \tag{34}$$

According to our result in Eq. (28), and taking into account the estimate of ξ obtained in [5], namely $\xi = -(6.0 \pm 1.5) \times 10^{-2} \times \sqrt{2}|\epsilon_K|$, the new numerical value of κ_ϵ is

$$\kappa_\epsilon = \frac{\sin\phi_\epsilon}{1/\sqrt{2}} \times \left(1 + \rho \frac{\xi}{\sqrt{2}|\epsilon_K|} \right) = 0.94 \pm 0.02. \tag{35}$$

This should be compared with 0.92 ± 0.02 in [5] and 0.92 ± 0.01 in [19], where only the long-distance contributions to $\mathrm{Im}\,\Gamma_{12}$ (not those to $\mathrm{Im}\, M_{12}$) have been included.

4.1. Comparison with previous literature

As anticipated in the introduction, the relative role of short- and long-distance contributions to ϵ_K has been widely discussed in the literature in the mid 1980's [20–28]. It is therefore useful to compare our findings to those in these earlier works.

First of all, we agree on the main conclusion of all these papers, namely that $\epsilon_K^{\mathrm{LD}}/\epsilon_K^{\mathrm{exp}}$ is small as long as ϵ'/ϵ_K is small. This is certainly correct, but it is not the point of our analysis: the issue we are addressing in this work is the size of the subleading (long-distance) contributions to ϵ_K, that vanish in the limit of vanishing ϵ'.

Second, we agree that single-particle intermediate states (π^0, η, η') do not generate a significant long-distance contribution to $\mathrm{Im}\, M_{12}$. The cancellation of π^0 and η contributions at the lowest order in the chiral expansion was noted first in [24]. The role of the η' was more debated [24–27]. The issue was clarified in [28], where it was shown that the full nonet contribution (π^0, η, η') vanishes in the large N_c limit. This is consistent with our findings, which are based on the updated and detailed analysis of the η' exchange amplitude in Ref. [17].

Having clarified that single-particle intermediate states do not generate a significant contribution to $\mathrm{Im}\, M_{12}^{\mathrm{LD}}$, we are left with the two-pion intermediate state as the potentially leading contribution to $\mathrm{Im}\, M_{12}^{\mathrm{LD}}$. A naive estimate of this contribution at the partonic level seems to indicate that it is totally negligible; however, as we have shown, this is not the case because of the $\Delta I = 1/2$ enhancement of $K \to 2\pi$ amplitudes. Our key observation is that, thanks to chiral symmetry and to the $\Delta I = 1/2$ dominance, the weak phase of this contribution can be related to ξ, and the problem is shifted to the evaluation of the two-pion contribution to Δm_K, as summarised in Eq. (21). The numerical impact of this contribution is then estimated in two ways: (i) a direct computation of the $\pi\pi$ loop in the leading-log approximation, Eq. (28), which provides a *definite sign* for this term; (ii) the difference between the experimental value of Δm_K and the sum of its short-distance contribution and the other large long-distance contribution provided by the η' exchange, which allows us to perform the useful cross-check:

$$\Delta m_K^{\mathrm{LD}}\big|_{G_8^2} \approx \Delta m_K^{\mathrm{exp}} - \left[\Delta m_K^{\mathrm{SD}} + \Delta m_K^{\mathrm{LD}}\big|_{\eta'} \right]. \tag{36}$$

We finally note that our estimate of the $\mathcal{O}(\xi)$ corrections to ϵ_K is based on the dominance of the $\Delta I = 1/2$ amplitude in $K \to 2\pi$ decays. Given the experimental smallness of $\Delta I = 3/2$ transitions, and the overall size of the effect we have evaluated (a few % correction to ϵ_K), this is certainly a very safe approximation.

5. Conclusions

In this Letter we have presented a complete analysis of ϵ_K beyond the lowest order in the OPE. In particular, we have analysed the structure of long distance (LD) contributions that affect both the *absorptive* (Γ_{12}) and *dispersive* (M_{12}) parts of the K^0–$\bar{K}^0$ mixing amplitude. We have pointed out that, in a consistent framework, in addition to LD contributions to $\mathrm{Im}\,\Gamma_{12}$, estimated recently in [5], also LD contributions to $\mathrm{Im}\, M_{12}$ have to be taken into account. Estimating the latter contributions in chiral perturbation theory, we found that they reduce by 40% the total impact of LD corrections on ϵ_K.

The overall multiplicative factor κ_ϵ in ϵ_K, summarising the effect of LD corrections and of the superweak phase being different from 45°, is increased to $\kappa_\epsilon = 0.94 \pm 0.02$, to be compared with 0.92 ± 0.02 obtained without LD contributions to $\mathrm{Im}\, M_{12}$.

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