A variational technique for the computation of the vibration frequencies of mechanical systems governed by nonsymmetric matrices

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In this paper an algorithm for the solution of linear eigenvalue problems governed by ill-conditioned nonsymmetric matrices that are typical in dynamic structural analysis in the presence of nonconservative loads is proposed. The real eigenpairs (x, λ) *are formulated as the minimizers of a suitable nonnegative functional, which plays a role analogous to that of the Rayleigh quotient for positive definite matrices. The proposed method, which is similar to a Rayleigh iterative scheme, has proven itself to be substantially unaffected by extremely high dispersions of the real eigenvalues. The method is illustrated by means of examples that correspond to beams subject to nonconservative loads.*

Keywords: variational methods, projection technique, eigenvalue computation, vibration frequencies, nonconservative loads

Introduction

Matrix eigenvalue problems are encountered in many engineering computations, particularly when harmonic oscillations of elastic mechanical systems are considered. If attention is restricted to the structural analysis domain, the classical linear model, in the presence of conservative loads, leads to the generalized eigenvalue problem

$$\mathbf{Ku} - \Omega^2\mathbf{Mu} = \mathbf{0} \tag{1}$$

where both the $n \times n$ stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$ are symmetric and positive definite. Upon Cholesky factorization[1] of the matrix $\mathbf{M}$, problem (1) can be reduced to the equivalent standard form

$$(\mathbf{D} - \lambda\mathbf{I})\mathbf{y} = \mathbf{0} \tag{2}$$

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where

$$\mathbf{D} = \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T} \qquad \mathbf{y} = \mathbf{L}^T\mathbf{u}$$
$$\mathbf{L}\mathbf{L}^T = \mathbf{M} \qquad \lambda = \Omega^2 \tag{3}$$

The matrix $\mathbf{D}$ is symmetric and positive definite and turns out to be strongly ill conditioned when its dimensions (i.e., the dimension of the approximation to the related continuous problem) increase. The basic methods[1–5] of solving problem (2) are matrix transformation methods (such as Jacobi or QR and QZ algorithms) and vector iteration methods (such as the power method or the Rayleigh iteration scheme). In structural analysis, few eigenvalues and eigenvectors of large (often sparse) matrices are of practical interest. Therefore transformation methods may be too time-consuming; moreover, they tend to fail in the presence of an increasing ill-conditioning matrix unless arithmetic precision is substantially improved. This is not the case for vector iteration methods, which do not modify the matrix entries (thus preserving the matrix sparsity) and can be advantageously employed to obtain partial eigensolutions, especially when good eigenvector pre-

dictions are available. In this context the simultaneous iteration method of Jennings[6] and Clint and Jennings[7] should be remembered. This method can be seen as a generalization of the power method for obtaining a subset of dominant eigenvalues and their corresponding eigenvectors.

Alternatively, in the presence of ill conditioning, the Rayleigh iteration method can be adopted.[1-4,8] In this method the first eigenvalue of **D** (and the corresponding eigenvector) can be determined by minimizing the Rayleigh quotient, i.e.,

$$\min \frac{\mathbf{y}^T\mathbf{D}\mathbf{y}}{\|\mathbf{y}\|_2^2} \tag{4}$$

When the coordinate functions adopted to discretize the continuous problem do not satisfy all the kinematic boundary conditions, this approach turns out to be particularly useful. In this case the free minimum problem (4) can be restated as a minimum problem under a set of M linear homogeneous constraints,

$$\mathbf{B}\mathbf{y} = \mathbf{0} \tag{5}$$

which correspond to the boundary conditions not satisfied by the coordinate functions. As is well known, these constraints can be imposed by adopting a projection operator P defined as follows[9]:

$$\mathbf{P} = \mathbf{I} - \mathbf{B}^T(\mathbf{B}\mathbf{B}^T)^{-1}\mathbf{B} \tag{6}$$

which projects any given vector $\mathbf{y} \in \mathbb{R}^N$ onto the subspace defined by (5). This projection procedure can also be adopted to determine the desired dominant set of eigenpairs. In fact, when the nth eigenvalue is to be found, the matrix **B** will be augmented by $n - 1$ additional rows corresponding to the eigenvectors already determined.[10]

In this paper the vibrations of mechanical systems under nonconservative loading[11,12] will be considered. Consequently, the matrix **D** of (2) becomes nonsymmetric, and yet the corresponding eigenvalue problem has all real eigenvalues as long as the external load stays below the critical (flutter) load. As a matter of fact, the flutter load is defined as the load level at which two eigenvalues and the corresponding eigenvectors tend to coincide. The mechanical counterpart of this phenomenon consists of a vibrational motion of increasing amplitude. Therefore the calculation of the flutter load is reduced to the repeated search for a limited number of (real) dominant eigenpairs for an increasing value of the load parameter until the coalescence of two eigenpairs is reached. In this context, matrix transformation methods[13] may still provide very poor results. An alternative approach consists of the derivation, from the continuous mechanical problem, of an extended variational principle in order to obtain an eigenvalue problem governed by a symmetric matrix.[14] Unfortunately, these matrices may not be positive definite, and therefore it is necessary to resort to semiheuristic relaxation procedures.[15] A very powerful method to determine a subset of real or complex eigenpairs of real nonsymmetric matrices without resorting to complex variables is that of Jennings and Stewart.[16] Nevertheless, when additional constraints such as (5) are imposed, a variational formulation of problem (2) incorporating these constraints by means of a projection technique may still result in a very direct and efficient method. In fact, once the projection operator of (6) is defined, problem (2), restricted to the quotient space $\mathbb{R}^N/\mathbf{P}$, may be restated as

$$\mathbf{PDPy} - \lambda\mathbf{Py} = \mathbf{0} \tag{7}$$

Equation (7) may be rewritten as follows:

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0} \tag{8}$$

where $\mathbf{A} = \mathbf{PDP}$ (with rank $N - M$), $\mathbf{x} = \mathbf{Py} \in \mathbb{R}^N/\mathbf{P}$ as required by the original problem and the property $\mathbf{P}^2 = \mathbf{P}$ has been accounted for. By exploiting this approach, a variational formulation of problem (8) playing a role similar to that of the Rayleigh quotient is proposed. In fact, the following non-negative quotient will be considered:

$$J(\mathbf{x}, \lambda) = \frac{\|\mathbf{Ax} - \lambda\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} \qquad \mathbf{x} \in \mathbb{R}^N/\mathbf{P} \tag{9}$$

Since the proposed procedure is aimed at determining real eigenvalues, the minima of $J(\mathbf{x}, \lambda)$ will vanish for these eigenvalues and their corresponding eigenvectors only. Hence without resorting to a complex variable formulation, the iterative procedure will be directed to find the (null) minima of J. As will be made clear in the following, this method will present the same good behavior, with respect to ill conditioning, as the classical Rayleigh iteration scheme.

Analytical considerations

The quotient $J(\mathbf{x}, \lambda)$ of (9) to be minimized can be given the explicit form,

$$J(\mathbf{x}, \lambda) = \lambda^2 - \frac{\mathbf{x}^T(\mathbf{A}^T + \mathbf{A})\mathbf{x}}{\|\mathbf{x}\|_2^2}\lambda + \frac{\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}}{\|\mathbf{x}\|_2^2} \tag{10}$$

In order to reduce the proposed method to a Rayleigh iteration scheme the first step will consist of finding the relationship between λ and $\mathbf{x}$ at the minima of J. Setting $J(\mathbf{x}, \lambda) = 0$, (7) yields

$$\lambda = \frac{1}{2}\frac{\mathbf{x}^T(\mathbf{A}^T + \mathbf{A})\mathbf{x}}{\|\mathbf{x}\|_2^2} \pm i\sqrt{R(\mathbf{x})} \tag{11}$$

where

$$R(\mathbf{x}) = \bar{\mu}(\mathbf{x}) - \bar{\lambda}^2(\mathbf{x}) \tag{12}$$

with

$$\bar{\mu}(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x}}{\|\mathbf{x}\|_2^2} \qquad \bar{\lambda}(\mathbf{x}) = \frac{1}{2}\frac{\mathbf{x}^T(\mathbf{A}^T + \mathbf{A})\mathbf{x}}{\|\mathbf{x}\|_2^2}$$

It is worth noting that $R(x) \geq 0$ and vanishes for the eigenvectors corresponding to real eigenvalues only. Therefore for real eigenvalues, (11) reduces to

$$\lambda = \frac{1}{2}\frac{\mathbf{x}^T(\mathbf{A}^T + \mathbf{A})\mathbf{x}}{\|\mathbf{x}\|_2^2} \tag{13}$$

which can be alternatively deduced by making $J(\mathbf{x}, \lambda)$ stationary with respect to λ. Hence if a vector $\mathbf{x} \in \mathbb{R}^N/\mathbf{P}$ exists for which $R(\mathbf{x})$ vanishes, then the pair $(\mathbf{x}, \lambda)$, with λ given by (13), makes $J(\mathbf{x}, \lambda)$ vanish too, and consequently, the problem of minimizing $J(\mathbf{x}, \lambda)$ can be restated as the minimization of $R(\mathbf{x})$.

Finally, an alternative expression for $R(\mathbf{x})$ is reported which can be useful in order to gain some insight into the general behavior of the function to be minimized. Since $\mathbf{x}$ is assumed to be a real vector,

$$[\mathbf{x}^T(\mathbf{A}^T + \mathbf{A})\mathbf{x}]^2 = 4(\mathbf{x}^T\mathbf{A}\mathbf{x})^2 = 4\mathbf{x}^T\mathbf{A}^T\mathbf{x} \cdot \mathbf{x}^T\mathbf{A}\mathbf{x} \quad (14)$$

Substitution of (14) into (12) yields the following expression for $R(\mathbf{x})$:

$$R(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{A}^T\mathbf{Q}(\mathbf{x})\mathbf{A}\mathbf{x}}{\|\mathbf{x}\|_2^2} \quad (15)$$

where $\mathbf{Q}(\mathbf{x})$ is a Householder reflector[3,4];

$$\mathbf{Q}(\mathbf{x}) = \mathbf{I} - \hat{\mathbf{x}}\hat{\mathbf{x}}^T \quad (16)$$

and the circumflex indicates normalization of the vector acted upon. It should be noted that if $\mathbf{Q}$ were independent of $\mathbf{x}$, (15) would correspond to the Rayleigh quotient for $\mathbf{A}^T\mathbf{Q}\mathbf{A}$. This circumstance strongly suggests that the local behavior of $R(\mathbf{x})$ will not be very different from that of the Rayleigh quotient, provided that the search for a local minimum does not imply relevant corrections to the current vector $\mathbf{x}$.

Numerical procedure

The problem of minimizing $R(\mathbf{x})$ can be solved by means of a relaxation procedure similar to the one adopted for the Rayleigh iteration scheme. If $\mathbf{x}_n$ represents the current solution at step n, the solution at step $n + 1$ will be obtained as

$$\mathbf{x}_{n+1} = \hat{\mathbf{x}}_n - \alpha_n\hat{\mathbf{r}}_n \qquad (\alpha_n > 0) \quad (17)$$

where $\mathbf{r}_n$ represents the gradient of $R(\mathbf{x})$ evaluated at $\mathbf{x} = \hat{\mathbf{x}}_n$ and

$$\mathbf{r}(\mathbf{x}) = \nabla_x R(\mathbf{x}) = \frac{2}{\|\mathbf{x}\|_2^4}\left[(\mathbf{A}^T\mathbf{A} - \bar{\mu}(\mathbf{x})\mathbf{I}) - 2\bar{\lambda}(\mathbf{x})\left(\frac{\mathbf{A}^T + \mathbf{A}}{2} - \bar{\lambda}(\mathbf{x})\mathbf{I}\right)\right]\mathbf{x} \quad (18)$$

Hence by making use of (17), a normalized new solution $\hat{\mathbf{x}}_{n+1}$ is obtained which is legitimate, since $R(\hat{\mathbf{x}}_{n+1}) = R(\mathbf{x}_{n+1})$.

It is worth mentioning that $\nabla_x R(\mathbf{x}) = \mathbf{0}$ is just the stationarity condition for $R(\mathbf{x})$. Furthermore, it can be easily verified that the gradient $\mathbf{r}_n$ is orthogonal to the current vector $\mathbf{x}_n$ exactly as for the Rayleigh quotient method.

The further task is to define the relaxation parameter α_n so as to determine the convergence of the iterative procedure; obviously, a global convergence condition cannot be formulated since the objective function is assumed to be locally convex only. However, if the current solution $\mathbf{x}_n$ is assumed to lie in one of the convexity domains, it will be sufficient to require that α_n satisfy the standard convergence condition,[9,17]

$$\Delta R(\alpha_n) = R(\hat{\mathbf{x}}_n - \alpha_n\hat{\mathbf{r}}_n) - R(\hat{\mathbf{x}}_n) < 0 \quad (19)$$

By making use of (12) and (18), $\Delta R(\alpha_n)$ can be given the explicit form,

$$\Delta R(\alpha_n) = \left[\frac{-\alpha_n}{(1 + \alpha_n^2)^2}\right](c_{1_n} + c_{2_n}\alpha_n + c_{3_n}\alpha_n^2 + c_{4_n}\alpha_n^3) \quad (20)$$

with

$$c_{1_n} = 2\Omega_n + 4\omega_n\lambda_n \quad (21a)$$

$$c_{2_n} = -P_n + \mu_n + 4\omega_n^2 - 2\rho_n\lambda_n - 2\lambda_n \quad (21b)$$

$$c_{3_n} = 2\Omega_n - 4\omega_n\rho_n \quad (21c)$$

$$c_{4_n} = -P_n + \rho_n^2 + \mu_n - \lambda_n^2 \quad (21d)$$

where

$$\Omega_n = \hat{\mathbf{x}}_n^T\mathbf{A}^T\mathbf{A}\hat{\mathbf{r}}_n \quad (22a)$$

$$\omega_n = \tfrac{1}{2}\hat{\mathbf{x}}_n^T(\mathbf{A}^T + \mathbf{A})\hat{\mathbf{r}}_n \quad (22b)$$

$$P_n = \hat{\mathbf{r}}_n^T\mathbf{A}^T\mathbf{A}\hat{\mathbf{r}}_n \quad (22c)$$

$$\rho_n = \tfrac{1}{2}\hat{\mathbf{r}}_n^T(\mathbf{A}^T + \mathbf{A})\hat{\mathbf{r}}_n \quad (22d)$$

$$\mu_n = \bar{\mu}(\hat{\mathbf{x}}_n) \quad (22e)$$

$$\lambda_n = \bar{\lambda}(\hat{\mathbf{x}}_n) \quad (22f)$$

It is worth noting that in a certain neighborhood of $\alpha_n = 0$, $\Delta R(\alpha_n)$ itself is a convex function of its argument, since it is the restriction of $R(x)$ (locally convex) to the straight line $\hat{\mathbf{x}}_n - \alpha_n\hat{\mathbf{r}}_n$. Furthermore, having $\hat{\mathbf{r}}_n$ the same direction as $\nabla_x R(\hat{\mathbf{x}}_n)$, for $\alpha_n > 0$ sufficiently small, it turns out that $\Delta R(\alpha_n) < 0$. This is sufficient to infer that

$$\left.\frac{d\,\Delta R(\alpha_n)}{d\alpha_n}\right|_{\alpha_n = 0} = -c_{1_n} < 0 \quad (23)$$

$$\left.\frac{d^2\,\Delta R(\alpha_n)}{d\alpha_n^2}\right|_{\alpha_n = 0} = -2c_{2_n} > 0 \quad (24)$$

or equivalently,

$$c_{1_n} > 0 \qquad c_{2_n} < 0 \quad (25)$$

Finally, by making use of equations (12), (21), and (22), c_{4_n} can be restated as follows:

$$c_{4_n} = R(\hat{\mathbf{x}}_n) - R(\hat{\mathbf{r}}_n) \quad (26)$$

Consequently, since a minimum of $R(\mathbf{x})$ is desired, if $\hat{\mathbf{x}}_n$ represents a "good" approximation to the solution, the orthogonality between $\hat{\mathbf{x}}_n$ and $\hat{\mathbf{r}}_n$ will imply $R(\hat{\mathbf{r}}_n) > R(\hat{\mathbf{x}}_n)$ or, equivalently, $c_{4_n} < 0$ and

$$\lim_{\alpha_n \to \infty} \Delta R(\alpha_n) = -c_{4_n} > 0 \quad (27)$$

Hence a first conclusion can be derived.

Note that the condition $\Delta R(\alpha_n) < 0$ is satisfied if α_n is chosen in the interval $[0, \alpha_n^*]$ where α_n^* is the smallest positive root of the cubic polynomial $P_n(\alpha_n)$ contained in the right-hand side of (20). Furthermore, taking into

account that the term within brackets in the definition (20) of $\Delta R(\alpha_n)$ is negative for any $\alpha_n > 0$, the condition $P_n(0) = c_{1_n} > 0$ and (27) agree with a general result of the theory of algebraic equations, i.e., that the polynomial $P_n(\alpha_n)$ has at least one real root. In case of multiple real roots, α_n^* will be given by the smallest positive one.

A further problem concerns how to choose α_n within the interval $[0,\alpha_n^*]$ so as to minimize $\Delta R(\alpha_n)$. In principle, taking into account the local convexity of $\Delta R(\alpha_n)$, a local minimum of $\Delta R(\alpha_n)$ can be determined by computing the smallest positive root of the equation,

$$\frac{d\,\Delta R(\alpha_n)}{d\alpha_n} = 0 \tag{28}$$

which is made explicit in the form

$$-(c_{1_n} + 2c_{2_n}\alpha_n + 3c_{3_n}\alpha_n^2 + 4c_{4_n}\alpha_n^3)(1 + \alpha_n^2) + 4\alpha_n(c_{1_n}\alpha_n + c_{2_n}\alpha_n^2 + c_{3_n}\alpha_n^3 + c_{4_n}\alpha_n^4) = 0 \tag{28'}$$

In order to avoid the iterative solution of (28'), in the first steps of the procedure outlined a suboptimal value of the relaxation parameter is proposed, i.e., $\alpha_n \cong 0.4\alpha_n^*$. Nevertheless, as numerical experiments have shown, after a few steps the value of α_n becomes very small, and a good estimate of the optimal value can be obtained by eliminating third- and higher-order powers of α_n in (28'), i.e.,

$$3(c_{1_n} - c_{3_n})\tilde{\alpha}_n^2 - 2c_{2_n}\tilde{\alpha}_n - c_{1_n} = 0 \tag{29}$$

resulting in

$$\tilde{\alpha}_n = \frac{c_{2_n} + \sqrt{c_{2_n}^2 + 3c_{1_n}(c_{1_n} - c_{3_n})}}{3(c_{1_n} - c_{3_n})} \tag{30}$$

The ambiguity of the sign of the square root is easily solved by observing that for $c_{1_n} > 0$ and $c_{2_n} < 0$, $\tilde{\alpha}_n > 0$ if the "plus" alternative is chosen, whatever the sign of $c_{1_n} - c_{3_n}$. When a good guess of the first step vector is available (as in the examples reported in the next section), $\alpha_n = \tilde{\alpha}_n$ can be assumed from the first step.

The full iteration procedure can be summarized as follows:

Solve the problem

$$\min \frac{\mathbf{y}^T\mathbf{D}\mathbf{y}}{\|\mathbf{y}\|_2^2} \mid \mathbf{B}\mathbf{y} = \mathbf{0}$$

or equivalently, solve the problem

$$\mathbf{PDP}\mathbf{y} - \lambda\mathbf{P}\mathbf{y} = \mathbf{0}$$

where $\mathbf{P} = \mathbf{I} - \mathbf{B}^T(\mathbf{B}\mathbf{B}^T)^{-1}\mathbf{B}$

1. Calculate the projection matrix **P** according to (5).
2. Calculate $\mathbf{A} = \mathbf{PDP}$
3. Consider the nth current approximation $\mathbf{y}_n$ of the searched eigenvector (or select the first step vector).
4. Calculate $\hat{\mathbf{x}}_n = (\mathbf{P}\mathbf{y}_n)\hat{}$ if $n = 1$.
5. Calculate $\hat{\mathbf{r}}_n = \hat{\mathbf{r}}(\hat{\mathbf{x}}_n)$ according to (18).
6. Determine the parameter $\tilde{\alpha}_n$ according to (30).
7. Calculate the updated approximation $\mathbf{x}_{n+1} = \hat{\mathbf{x}}_n - \tilde{\alpha}_n\hat{\mathbf{r}}_n$.
8. Test tolerance ϵ for $J(\mathbf{x}_{n+1}, \lambda_{n+1})$, where λ is given by (13), and go back to step 3 if required.

Numerical examples

The first example is a purely algebraic example developed to check the convergence properties of the proposed algorithm. The eigenvalue problem was governed by the matrix A, reported in *Figure 1*, which is nonsymmetric and shows a low condition number. In the first iteration the trial vector $\mathbf{x}^i$ was defined by assuming

$$\begin{aligned} x_j^i &= A_{ji}/(A_{ii} - A_{jj}) \qquad i \neq j \\ x_j^i &= 1 \qquad\qquad\qquad\quad i = j \end{aligned} \tag{31}$$

This choice ensured convergence to the ith eigenvalue. The choice $\alpha_n = \tilde{\alpha}_n$ was made in each iteration. In order to have some results for comparison the same problem was also solved by using a secant implicit polynomial iteration[18] and QZ algorithm,[13] which gave the same results up to six significant figures. The results given by the proposed procedure, adopting three different convergence tolerances ϵ for $J(\mathbf{x}, \lambda)$, are summarized in *Table 1*, where the numbers in parentheses are the numbers of iterations required for convergence.

The second and third examples were obtained from a classical mechanical problem, i.e., a vibrating cantilever beam subjected to a follower axial force P (Beck's problem).[11,12] The lateral displacement was discretized by means of a trigonometric series expansion in order to work with a matrix which explicitly showed an increasing condition number. Assuming the displacements to be functions of time and space independently, the lateral displacement can be written as

$$\eta(z) = \sum_{n=1}^{N} u_n\left(\cos\frac{2n-1}{2}\frac{\pi z}{l} - 1\right)e^{i\Omega t} \tag{32}$$

Consequently, the coefficients of the stiffness matrix **K** and the mass matrix **M** take the form

$$K_{nm} = \frac{1}{2}\delta_{nm}\left(\frac{2n-1}{2}\pi\right)^4 - \frac{1}{2}\delta_{nm}p\left(\frac{2n-1}{2}\pi\right)^2 - p\left(\frac{2n-1}{2}\pi\right)(-1)^n \tag{33}$$

$$\mathbf{A} \equiv \begin{vmatrix} 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 3 & -1 & 2 & 1 & -2 & -1 & 0 & 1 & 2 \\ 2 & 1 & 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 9 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -2 & 11 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 15 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 17 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 & 0 & -3 & 20 & -4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 26 \end{vmatrix}$$

Figure 1. Nonsymmetric matrix adopted in example 1

Table 1. Eigenvalues of the nonsymmetrical matrix shown in *Figure 1* and number of iterations required for convergence

Eigenvalue number	Tolerance ϵ on $J(x,\lambda)$			Polynomial iteration and QZ algorithm
	1×10^{-3}	1×10^{-4}	1×10^{-5}	
1	1.45802 (112)	1.45721 (135)	1.45711 (193)	1.45710
2	3.34424 (157)	3.33684 (458)	3.33669 (581)	3.33666
3	3.90030 (295)	3.89866 (387)	3.89801 (698)	3.89799
4	6.49073 (76)	6.48978 (115)	6.48976 (124)	6.48976
5	9.44530 (55)	9.44546 (65)	9.44553 (76)	9.44553
6	11.7137 (31)	11.7147 (37)	11.7147 (44)	11.7147
7	14.7489 (47)	14.7491 (58)	14.7491 (65)	14.7491
8	16.9238 (46)	16.9232 (66)	16.9231 (70)	16.9231
9	22.0708 (64)	22.0720 (75)	22.0722 (96)	22.0722
10	23.9155 (65)	23.9138 (99)	23.9137 (124)	23.9137

Numbers in parentheses are the numbers of iterations required for convergence.

$$M_{nm} = 1 + \frac{1}{2}\delta_{nm} + \frac{2}{\pi(2n-1)}(-1)^n + \frac{2}{\pi(2m-1)}(-1)^m \quad (34)$$

Moreover, the circular frequency Ω and the axial force P are made nondimensional according to the following relations (*Figure 3*):

$$\omega^2 = \frac{\Omega^2 l^4 m}{EI} \qquad p = \frac{Pl^2}{EI} \quad (35)$$

where l is the beam length, m is the mass per unit length, E is the Young modulus, and I is the moment of inertia of the cross section. The second example ($p = 0$) refers to the natural vibrations of the unloaded beam, which is characterized by a symmetric stiffness matrix. The first two frequencies have been determined for a number N of coordinate functions increasing from 2 to 64. Correspondingly, the condition number increased up to 7 orders of magnitude.

Figure 2 shows the comparison of the values of the first circular frequency obtained through the proposed method with those given by the QZ algorithm and by subroutines EVCCG (single precision complex) and DEVCCG (double precision complex) of the IMSL library[13] adopting an error tolerance 1×10^{-3}. In order to define the first step vectors the criterion reported in (31) was adopted. *Table 2* shows that the frequencies obtained with the proposed method, adopting single precision, converge toward the exact values in spite of the strongly increasing condition number. It is worth noting that routine DEVCCG required a storage memory for the matrix A four times greater than that of the proposed method.

The third example refers to the evolution of the first two vibration frequencies of a cantilever beam subjected to a follower axial force. The initial guessed vectors at each load level were profitably assumed to coincide with the eigenvectors of the previous load level. *Table 3* shows the evolution of the nondimensional frequencies and points out how the first two

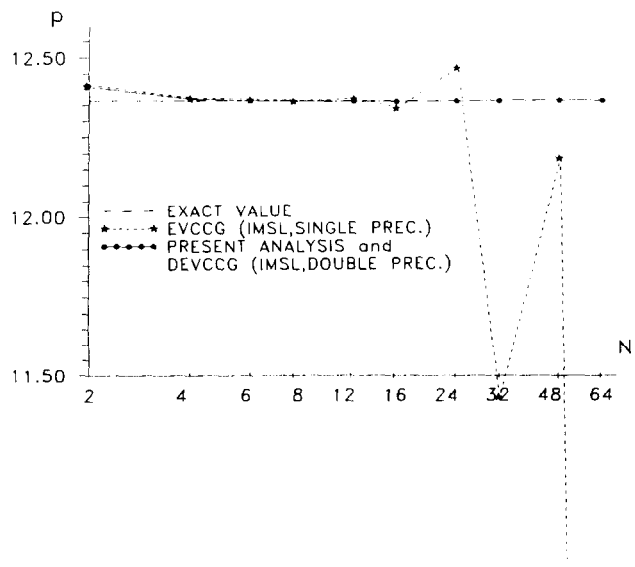

Figure 2. First circular frequency of an unloaded cantilever beam versus the number N of coordinate functions

Table 2. First two nondimensional circular frequencies ω_1, ω_2 of an unloaded cantilever beam for an increasing number N of coordinate functions

N	ω_1^2	ω_2^2	$\frac{K_{nn}}{K_{11}} = (2n-1)^4$
2	12.413051	575.41707	81
4	12.371062	491.63844	2401
6	12.365141	487.11571	14641
8	12.363569	486.15829	50625
12	12.362729	485.70077	279841
16	12.362520	485.59450	923521
24	12.362413	485.54120	4879681
32	12.362387	485.52846	15752961
48	12.362374	485.52201	81450625
64	12.362370	485.52046	260144641
Exact values	12.362363	485.51882	

K_{11} and K_{nn} are the first and the last diagonal coefficients of the stiffness matrix

Table 3. Evolution of the first two nondimensional circular frequencies ω_1, ω_2 of a cantilever beam subjected to a follower axial force $P(N = 24)$

p	ω_1^2	ω_2^2	$\langle \hat{x}_1, \hat{x}_2 \rangle$
0	12.362	485.54	4.124×10^{-3}
2.5	14.763	451.98	0.2795
5.0	17.789	417.64	0.4104
7.5	21.673	382.28	0.5232
10.0	26.789	345.50	0.6278
12.5	33.778	306.64	0.7282
15.0	43.992	264.41	0.8248
17.5	60.564	215.44	0.9151
20.0	110.77	132.61	0.9972
20.05	119.26	123.47	0.9989
20.0519	121.06	121.64	0.9994

eigenvectors tend to coincide as the nondimensional applied force approaches the flutter load $p_{cr} = 20.0519$.

The fourth example derived from the Leipholz problem, i.e., a cantilever beam subjected to a distributed (constant) follower axial load q (case A). For this problem the same coordinate functions (32) as in the case of Beck's problem were adopted. In order to show how the proposed projection technique easily takes additional constraints into account, keeping the same coordinate functions, a beam built in at one end and supported at the other also was considered, alternatively subjected to a follower (case B) and a dead (case C) axial load q. *Figure 3* shows that in contrast with the classical Leipholz problem, buckling occurs (vanishing of the first frequency) for the statically redundant beam whether or not a conservative load is present.

Final remarks

The numerical method presented proved its effectiveness in a typical class of engineering applications through some quite severe test problems. From a formal point of view it must be noted that the simplest technique was employed to generate the successive approximations of the searched eigenvectors, since the aim of the paper was to propose the basic algorithm only. Nevertheless, it should be remembered that the basic gradient method can be substantially accelerated by the "parallel tangent" technique with a negligible increase in the computational effort.[9] Furthermore, a whole collection of very fast algorithms[20] (conjugate gradient, Newton-Raphson, Davidon's, or variable metric methods) can be considered, in principle, as possible alternatives. However, it should be noted that the latter methods require the explicit evaluation of the Hessian matrix of the objective function at each step, which reduces the appeal of the expected improvements.

As far as the convergence rate of the proposed method is concerned, a standard analysis[9] shows that the order of convergence is equal to unity, as generally happens for gradient algorithms. Consequently, a relevant ill conditioning of A may reduce the convergence speed of the process. However, it should be mentioned that the numerical experiments, reported in the previous section, did not show this phenomenon, even when the eigenvalues of A differed by a factor of 10^7 (see *Table 2*).

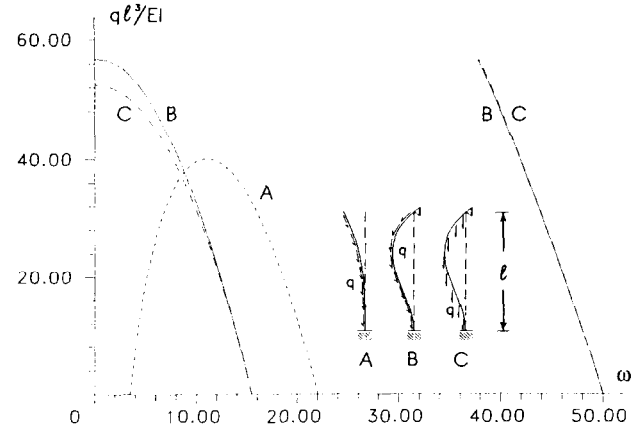

Figure 3. Cantilever beam subjected to a follower distributed axial load (A). Beam built in at one end and supported at the other subjected to a follower (B) and a dead (C) distributed axial load (number of coordinate functions $N = 24$)

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