Available online at www.sciencedirect.com
SciVerse ScienceDirect
Advances in Mathematics 229 (2012) 1912–1923
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Existence of maximizers for Sobolev–Strichartz inequalities

Luca Fanelli [a,*], Luis Vega [a], Nicola Visciglia [b]

[a] *Universidad del Pais Vasco, Departamento de Matemáticas, Apartado 644, 48080, Bilbao, Spain*
[b] *Università di Pisa, Dipartimento di Matematica, Largo B. Pontecorvo 5, 56100 Pisa, Italy*

Received 24 March 2011; accepted 13 December 2011

Available online 29 December 2011

Communicated by Carlos E. Kenig

Abstract

We prove the existence of maximizers of Sobolev–Strichartz estimates for a general class of propagators, involving relevant examples, as for instance the wave, Dirac and the hyperbolic Schrödinger flows.

MSC: 35J10; 35L05

Keywords: Fourier restriction theorems; Strichartz estimates

1. Introduction

In the recent years, an increasing interest has been devoted to the problem of existence of maximizers for Strichartz inequalities, and more in general Fourier restriction theorems.

Recall that, given two Banach spaces $(X, |\,.\,|_X)$, $(Y, |\,.\,|_Y)$, and a linear and continuous operator $T \in \mathcal{L}(X, Y)$, it is customary to define maximizer for T any $x_0 \in X$ such that

$$\|x_0\|_X = 1, \qquad \|Tx_0\|_Y = \sup_{\|x\|_X=1} \|Tx\|_Y. \tag{1.1}$$

* Corresponding author.
E-mail addresses: luca.fanelli@ehu.es (L. Fanelli), luis.vega@ehu.es (L. Vega), viscigli@dm.unipi.it (N. Visciglia).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.12.012

Notice that the existence (as well as the definition) of maximizers depends on the specific norm introduced on X and Y.

In this paper we consider the existence of maximizers for a large class of propagators of the form $e^{ith(D)}$, and we focus our attention to cases in which some Sobolev–Strichartz estimates of the following type hold

$$\left\|e^{ith(D)} f\right\|_{L^r_{t,x}} \leqslant C\|f\|_{\dot H^s}, \quad s>0,\ r>2, \tag{1.2}$$

where in general f is a vector valued function $f=(f_1,\dots,f_n):\mathbb{R}^d\to\mathbb{C}^n$ and

$$\|f\|^2_{\dot H^s}=\sum_{j=1}^n \|f_j\|^2_{\dot H^s}. \tag{1.3}$$

Concerning the definition of the norm $L^r_{t,x}$ in the l.h.s. of (1.2) recall that in general for any given norm $|\,.\,|_{\mathbb{C}^n}$ on $\mathbb{C}^n$ one can define the corresponding mixed Lebesgue norms $L^p_t L^q_x$ for vector valued functions $F=F(t,x):\mathbb{R}^{1+d}\to\mathbb{C}^n$ as follows

$$\left\|F(t,x)\right\|^p_{L^p_t L^q_x}=\int\left(\int\left|F(t,x)\right|^q_{\mathbb{C}^n}\,dx\right)^{\frac{p}{q}}dt,$$

where $1\leqslant p,q\leqslant\infty$. In the case $p=q=r$ we shall write shortly

$$L^r_t L^r_x = L^r_{t,x}.$$

Remark 1.1. Recall that all the norms on $\mathbb{C}^n$ are equivalent and hence also the corresponding norms induced on the spaces $L^p_t L^q_x$ are equivalent. Due to this fact in (1.2) it is not necessary to specify the norm on $\mathbb{C}^n$ with respect to which we are working with (of course provided that the corresponding constant C is suitably modified). On the other hand the existence of maximizers for the inequality (1.2) could be affected in principle by changing the corresponding norm on $\mathbb{C}^n$.

Estimates of the kind (1.2) with a loss of derivatives with respect to the datum f, are natural in some relevant cases, as for instance the wave and Dirac equations. Let us point our attention to the Cauchy problem for the following system

$$\begin{cases} i\partial_t u + h(D)u = 0,\\ u(0,x)=f(x),\end{cases} \tag{1.4}$$

where $u(t,x)=(u_1(t,x),\dots,u_n(t,x)):\mathbb{R}^{1+d}\to\mathbb{C}^n$, $f(x)=(f_1(x),\dots,f_n(x)):\mathbb{R}^d\to\mathbb{C}^n$. Here we denote by $h(D)=\mathcal{F}^{-1}(h(\xi)\mathcal{F})$, where $\mathcal{F}$ is the standard Fourier transform; in addition, the symbol $h(\xi)\in\mathcal{M}_{n\times n}(\mathbb{C})$, $n\geqslant 1$, is assumed to be a matrix-valued function $h(\xi)=(h_{ij}(\xi))_{ij=1,\dots,n}$. In the sequel, we always make the following abstract assumptions:

(H1) there exists $0<s<\frac{d}{2}$ such that the problem (1.4) is globally well-posed in $\dot H^s$, and the unique solution is given via propagator $u(t,x)=e^{ith(D)}f(x)$;

(H2) the flow $e^{ith(D)}$ is unitary onto $\dot{H}^s$, i.e.

$$\left\|e^{ith(D)} f\right\|_{\dot{H}^s} = \|f\|_{\dot{H}^s} \quad \forall t \in \mathbb{R},$$

where s is the same as in (H1), and $\|\cdot\|_{\dot{H}^s}$ is defined in (1.3).

In order to introduce our problem, assume for the moment that $h_{i,j}(\xi)$ is homogeneous of some degree $k > 0$ for every $i, j \in \{1, \dots, n\}$, namely

$$h_{ij}(\xi) : \mathbb{R}^d \to \mathbb{C}, \qquad h(\lambda\xi) = \lambda^k h(\xi) \quad \forall \lambda > 0. \tag{1.5}$$

Then, Eq. (1.4) is invariant under the scaling $v_\lambda(t,x) = u(\lambda^k t, \lambda x)$; as a consequence, if a Strichartz estimate of the following type holds

$$\left\|e^{ith(D)} f\right\|_{L_t^p L_x^q} \leqslant C\|f\|_{\dot{H}^s}, \tag{1.6}$$

for some $s > 0$ and some constant $C > 0$, then p and q have to satisfy the following scaling condition

$$\frac{k}{p} + \frac{d}{q} = \frac{d}{2} - s. \tag{1.7}$$

Moreover, by the Sobolev embedding $\dot{H}^s \subset L^{\frac{2d}{d-2s}}$, for $0 < s < d/2$, and the $\dot{H}^s$-preservation

$$\left\|e^{ith(D)} f\right\|_{L_t^\infty \dot{H}_x^s} = \|f\|_{\dot{H}^s}$$

(assumption (H2) above), we get

$$\left\|e^{ith(D)} f\right\|_{L_t^\infty L_x^{\frac{2d}{d-2s}}} \leqslant C\|f\|_{\dot{H}^s}, \quad 0 < s < \frac{d}{2}, \tag{1.8}$$

for some constant $C > 0$. Hence, if an estimate of the type (1.6) holds with $p < q$, then by interpolation with (1.8) we obtain

$$\left\|e^{ith(D)} f\right\|_{L_{t,x}^r} \leqslant C\|f\|_{\dot{H}^s}, \quad 0 < s < \frac{d}{2}, \ r = \frac{2(k+d)}{d-2s}. \tag{1.9}$$

The aim of this paper is to prove that, as soon as an estimate as (1.9) holds, with a strictly positive $s > 0$, then the best constant of the estimate is achieved by some maximizing functions, independently on the norm which is fixed on the target $\mathbb{C}^n$ (and consequently on the corresponding definition of the L_{tx}^r-norm).

The answer to analogous question about the existence of maximizers for Sobolev inequalities is addressed to the well-known results by Aubin, Talenti and Lions [1,21,14,15]. More recently, several authors treated separately different propagators. We firstly mention Kunze [12], who proved the existence of maximizers of the $L_{t,x}^6$-Strichartz inequality for the 1D Schrödinger propagator. Later, Foschi [8] succeeded in characterizing the best constant of the inequality and also the shape of the maximizers, for the 1D and 2D-Schrödinger propagators. In the same paper, the author treats the $L^6_{t,x}$ (in 2D) and the $L^4_{t,x}$ (in 3D) Strichartz estimates for the wave equation, which hold to the scale of $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$-initial data. Recently, Bez and Rogers [3] computed the best constant and described the shape of maximizers of the $L^4_{t,x}$-Strichartz estimate for the wave equation, with initial data in the energy space $\dot{H}^1 \times L^2$, in space dimension $d = 5$. We also mention the papers by Shao [19] for the Schrödinger equation and by Bulut [5] for the wave equation, in which the existence of maximizers for anisotropic Strichartz estimates in spaces $L^p_t L^q_x$, for general couples (p, q), is also proved. Finally, Ramos [17] proves that there exist maximizers for the isotropic Strichartz estimates for the wave equation, at the scale $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$. We mention also the papers by Bégout and Vargas [2], and Merle and Vega [16], for the analogous about the Strichartz estimates for the Schrödinger propagator at the L^2-scale.

Here we give a unified (and simple) proof of the existence of maximizers of (1.9), when $s > 0$, which involves a large class of examples of propagators. Our main result is the following.

Theorem 1.1. *Let assumptions* (H1), (H2) *be satisfied for some* $0 < s < \frac{d}{2}$, *and let* $h(\xi)$ *satisfy* (1.5) *for some* $k > 0$. *Moreover, assume that, for some* $2 \leqslant p < q \leqslant \infty$

$$\left\| e^{ith(D)} f \right\|_{L^p_t L^q_x} \leqslant C \|f\|_{\dot{H}^s}, \tag{1.10}$$

so that, for $r = \frac{2(k+d)}{d-2s}$, *we also have*

$$\left\| e^{ith(D)} f \right\|_{L^r_{t,x}} \leqslant M \|f\|_{\dot{H}^s}, \tag{1.11}$$

with

$$M := \sup_{\|f\|_{\dot{H}^s}=1} \left\| e^{ith(D)} f \right\|_{L^r_{t,x}}. \tag{1.12}$$

Then there exists $f_0 \in \dot{H}^s$ *such that*

$$\|f_0\|_{\dot{H}^s} = 1, \qquad \left\| e^{ith(D)} f_0 \right\|_{L^r_{t,x}} = M. \tag{1.13}$$

Remark 1.2. We underline that in Theorem 1.1 no specific norm is fixed on the target $\mathbb{C}^n$ (and hence also on the corresponding $L^r_{t,x}$ norm). Nevertheless we prove in general the existence of maximizers for the Strichartz estimate (1.11). We also underline that in Theorem 1.1 we assume that the norm on $\dot{H}^s$ is the one defined in (1.3). Indeed the Hilbert structure of the norm $\|.\|_{\dot{H}^s}$ is crucial in our argument.

Remark 1.3. Notice that the conditions $2 \leqslant p < q \leqslant \infty$ imply that $r > 2$, which will be crucial in the proof of Theorem 1.1.

Remark 1.4. As we see in the sequel, the proof of Theorem 1.1 is quite simple, and it is based on a suitable variant of a classic result by Brézis and Lieb [4,13], see Section 2 below. On the other hand, the technique we use does not allow us to consider the case $s = 0$ for which, in fact, some extra ingredients (being typically suitable improvements of Strichartz estimates) are needed, as substitutes of (1.10).

We shall now give some examples of applications of the previous theorem.

Example 1.1 *(Wave propagator).* The Strichartz estimates for the wave propagator $e^{it|D|}$ (see [10,11]), in dimension $d \geqslant 2$, are the following:

$$\left\| e^{it|D|} f \right\|_{L_t^p L_x^q} \leqslant C \| f \|_{\dot{H}^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}, \tag{1.14}$$

under the admissibility condition

$$\frac{2}{p} + \frac{d-1}{q} = \frac{d-1}{2}, \quad p \geqslant 2, \qquad (p,q) \neq (2,\infty). \tag{1.15}$$

In this case, the gap of derivatives $\frac{1}{p} - \frac{1}{q} + \frac{1}{2} \geqslant 0$ is null only in the case of the energy estimate $(p,q) = (\infty, 2)$. In particular, we have

$$\left\| e^{it|D|} f \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}} \leqslant C \| f \|_{\dot{H}^{\frac{1}{2}}}, \quad d \geqslant 2, \tag{1.16}$$

which is in fact the original estimate proved by Strichartz in [20]. More generally, by Sobolev embedding one also obtains that

$$\left\| e^{it|D|} f \right\|_{L_{t,x}^{\frac{2(d+1)}{d-1-2\sigma}}} \leqslant C \| f \|_{\dot{H}^{\frac{1}{2}+\sigma}}, \quad 0 \leqslant \sigma < \frac{d-1}{2}, \ d \geqslant 2. \tag{1.17}$$

Theorem 1.1 gives a short and simple proof of the fact that, for any dimension $d \geqslant 2$, and $0 < \sigma < \frac{d-1}{2}$, the best constant in (1.17) is achieved on some maximizing function f.

Example 1.2 *(Dirac equation).* Consider the (massless) Dirac operator

$$\mathcal{D} := \frac{1}{i} \sum_{j=1}^{3} \alpha_j \partial_j,$$

which is defined in dimension $d = 3$. Here $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}_{4\times 4}(\mathbb{C})$ are the so-called *Dirac matrices*, which are 4×4-hermitian matrices, $\alpha_j^t = \overline{\alpha_j}$, $j = 1, 2, 3$. They are defined as

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

or equivalently $\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$, where σ_j is the j-th 2×2-Pauli matrix, $j = 1, 2, 3$.

Since $\mathcal{D}^2 = -\Delta I_{4\times 4}$, the Strichartz estimates for the massless Dirac operator are the same as for the 3D wave equation (see [6]):

$$\left\| e^{it\mathcal{D}} f \right\|_{L_t^p L_x^q} \leqslant C \| f \|_{\dot{H}^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}, \tag{1.18}$$

with the admissibility condition

$$\frac{2}{p} + \frac{2}{q} = 1, \quad p > 2. \tag{1.19}$$

In particular we have

$$\left\| e^{it\mathcal{D}} f \right\|_{L_{t,x}^{\frac{8}{2-2\sigma}}} \leqslant C \| f \|_{\dot{H}^{\frac{1}{2}+\sigma}}, \quad 0 \leqslant \sigma < 1. \tag{1.20}$$

Hence also in this case a loss of derivatives with respect to the initial datum is natural in the estimate. Also in this case Theorem 1.1 applies, and it proves that there exist maximizers for (1.20), in the range $0 < \sigma < 1$, which at our knowledge is not a known fact.

Example 1.3 *(Hyperbolic Schrödinger equation).* Let us now consider the hyperbolic Schrödinger operator $L := \sum_{j=1}^{m} \partial_j^2 - \sum_{j=m+1}^{d} \partial_j^2$, for $d \geqslant 2$ and $1 \leqslant m < d$. In this case, Strichartz estimates are the same as for the Schrödinger propagator, namely

$$\left\| e^{itL} f \right\|_{L_t^p L_x^q} \leqslant C \| f \|_{\dot{H}^s}, \tag{1.21}$$

with the admissibility condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} - s, \quad p \geqslant 2, \qquad (p,q) \neq (2,\infty). \tag{1.22}$$

In particular, one has

$$\left\| e^{itL} f \right\|_{L_{t,x}^{\frac{2(d+2)}{d-2s}}} \leqslant C \| f \|_{\dot{H}^s}, \tag{1.23}$$

for any $0 \leqslant s < \frac{d}{2}$. The only case in which the existence of maximizers for (1.23) is known is $d = 2$, $s = 0$. Indeed, the result by Rogers and Vargas in [18] contains all the ingredients which are necessary to prove the profile decomposition for bounded sequences in L^2, with respect to the propagator e^{itL}, in dimension $d = 2$; the existence of maximizers follows from this fact by a general argument in the spirit of [19]. It is a matter of fact that Theorem 1.1 applies for any $d \geqslant 2$ and $0 < s < \frac{d}{2}$, but we remark that it cannot include the case $s = 0$. We finally remark, our argument is rather simple and does not involve any profile decomposition-theorems associated to the propagator.

In the statement of Theorem 1.1, the homogeneity assumption (1.5) is required; in fact, this is just put in order to write the explicit dependence of r on k, s in (1.11). Motivated by the case of the wave equation, in which the homogeneity property does not hold (see Example 1.4 below), we now state a more general version of the previous theorem.

Let us first introduce the notations

$$\dot{H}^s = \dot{H}^{s_1} \times \cdots \times \dot{H}^{s_n}, \qquad s := (s_1, \ldots, s_n),$$
$$\| f \|_{\dot{H}^s}^2 = \sum_{i=1}^{n} \| f_i \|_{\dot{H}^{s_i}}^2 \tag{1.24}$$

where $f = (f_1, \dots, f_n) \in \dot{H}^s$. Consider Eq. (1.4); in addition to (H1) and (H2) (where the norm $\dot{H}^s$ in this case is the more general one defined in (1.24)) assume:

(H3) there exist $j \in \{1, \dots, n\}$ and $2 \leqslant p < q \leqslant \infty$ such that, if $(u_1, \dots, u_n)$ solves (1.4) then

$$\|u_j\|_{L_t^p L_x^q} \leqslant C_j \|f\|_{\dot{H}^s};$$

(H4) if $u = (u_1, \dots, u_n)$ solves (1.4), then

$$u_\lambda = \left(\lambda^{\frac{d}{2}-s_1} u_1(\lambda t, \lambda x), \dots, \lambda^{\frac{d}{2}-s_n} u_n(\lambda t, \lambda x)\right)$$

solves (1.4), for any $\lambda > 0$.

Notice that, by a scaling argument, we have

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - s_j.$$

Moreover, interpolating with the energy estimate one also obtains

$$\|u_j\|_{L^r_{t,x}} \leqslant M \|f\|_{\dot{H}^s}, \tag{1.25}$$

where $r = \frac{2(1+d)}{d-2s_j}$. We have the following result.

Theorem 1.2. *Assume the operator $h(D)$ is such that* (H1)–(H4) *are satisfied for some $s_1, \dots, s_n \in \mathbb{R}$, $2 \leqslant p < q \leqslant \infty$, and $0 < s_j < \frac{d}{2}$, where j is the one in* (H3). *Consider estimate* (1.25) *and let*

$$M := \sup_{\|f\|_{\dot{H}^s}=1} \|u_j\|_{L^r_{t,x}}. \tag{1.26}$$

Then there exists $f_0 \in \dot{H}^s$ such that

$$\|f_0\|_{\dot{H}^s} = 1, \qquad \|v_j\|_{L^r_{t,x}} = M \tag{1.27}$$

where $(v_1, \dots, v_n) = e^{ith(D)} f_0$.

Remark 1.5. In other words, Theorem 1.2 states that the problem of existence of maximizers can be solved analogously for Sobolev–Strichartz estimates involving one single component of the solution of equation (1.4). In a completely analogous way, one could treat the case of estimates for a selected group of components of the solution. The proof of Theorem 1.2 is completely identical to the one of Theorem 1.1: indeed it is sufficient to repeat the same argument for the composition operator $T_j(t) = \pi_j \circ e^{ith(D)}$, where $\pi_j : \mathbb{C}^N \to \mathbb{C}$ is the projection onto the j-th component. We will omit the details of the proof.

We pass now to show the main application of Theorem 1.2, on solutions of the wave equation.

Example 1.4 *(Wave equation).* Consider now the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \mathbb{R}^{1+d}, \\ u(0,x) = u_0(x), & \partial_t u(0,x) = u_1(x). \end{cases} \tag{1.28}$$

The vector-variable $V := (u, \partial_t u)^t$ is uniquely given by $V(t,x) = e^{ith(D)}V_0(x)$, where $V_0 = (u_0, u_1)^t$, and $h(D) = \begin{pmatrix} 0 & -i \\ -i\Delta & 0 \end{pmatrix}$. The Strichartz estimates for (1.28) can be immediately deduced by (1.17), writing the solution u as

$$u(t,\cdot) = \cos\bigl(t|D|\bigr)u_0(\cdot) + \frac{\sin(t|D|)}{|D|}u_1(\cdot);$$

in particular, in terms of the vector variable V, denoting by $f = (u_0, u_1)^t$, one has

$$\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-1-2\sigma}}} \leqslant C\|f\|_{\dot{H}^{\frac{1}{2}+\sigma}\times\dot{H}^{-\frac{1}{2}+\sigma}}, \quad 0 \leqslant \sigma < \frac{d-1}{2},\ d \geqslant 2. \tag{1.29}$$

Foschi [8] proved that the best constant in (1.29) is achieved, in the cases $d = 2, 3$, $\sigma = 0$; moreover, he can characterize the shape of the maximizers. Later, Bulut [5] proved the same (also the anisotropic version of (1.29), with $p \neq q$); she can treat the range $\frac{1}{2} \leqslant \sigma < \frac{d-1}{2}$, with $d \geqslant 3$. More recently, Bez and Rogers [3] can prove the existence of maximizers for (1.29), and characterize their shape, in the case $d = 5$, $\sigma = \frac{1}{2}$.

Now notice that Theorem 1.2 applies (with the choice $j = 1$ in assumption (H3)). It implies that there exist maximizers for (1.29), for any dimension $d \geqslant 2$ and any $0 < \sigma < \frac{d-1}{2}$. We remark that Ramos [17] proves a refinement of (1.29) in the case $\sigma = 0$, which in particular implies the existence of maximizers, in this specific case of the wave equation.

The rest of the paper is devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

Before starting with the proof of our main theorem, we need to recall two fundamental results which will play a role in the sequel. The first one is a variant of a well-known result obtained by Brézis and Lieb in [4,13].

Proposition 2.1. *(See [7].) Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H}, L^p(\mathbb{R}^d))$ for a suitable $p \in (2,\infty)$. Let $\{h_n\}_{n\in\mathbb{N}} \in \mathcal{H}$ such that:*

$$\|h_n\|_{\mathcal{H}} = 1; \tag{2.1}$$

$$\lim_{n\to\infty} \|Th_n\|_{L^p(\mathbb{R}^d)} = \|T\|_{\mathcal{L}(\mathcal{H},L^p(\mathbb{R}^d))}; \tag{2.2}$$

$$h_n \rightharpoonup \bar{h} \neq 0; \tag{2.3}$$

$$T(h_n) \to T(\bar{h}) \quad \text{a.e. in } \mathbb{R}^d. \tag{2.4}$$

Then $h_n \to \bar{h}$ in $\mathcal{H}$, in particular $\|\bar{h}\|_{\mathcal{H}} = 1$ and $\|T(\bar{h})\|_{L^p(\mathbb{R}^d)} = \|T\|_{\mathcal{L}(\mathcal{H},L^p(\mathbb{R}^d))}$.

Remark 2.1. The main difference between Proposition 2.1 and Lemma 2.7 in [13] is that we only need to assume weak convergence in the Hilbert space $\mathcal{H}$ for the maximizing sequence h_n. On the other hand the argument in [13] works for operators defined between general Lebesgue spaces and not necessarily in the Hilbert spaces framework.

Proposition 2.1 has been proved in [7], in the scalar case. The proof in the vector-case is completely analogous and will be omitted. The second tool we need is a byproduct of a well-known result by Gérard in [9], in which the lack of compactness of the Sobolev embedding $\dot{H}^s \subset L^{\frac{2d}{d-2s}}$ is classified.

Proposition 2.2. *(See [9].) Let $0 < s < \frac{d}{2}$, and let $w_n \in \dot{H}^s$ be a sequence such that*

$$\|w_n\|_{\dot{H}^s} = 1, \quad n = 1, 2, \dots, \tag{2.5}$$

$$\inf_{n \geqslant 1} \|w_n\|_{L_x^{\frac{2d}{d-2s}}} \geqslant \epsilon > 0. \tag{2.6}$$

Then there exist a sequence of parameters $\lambda_n > 0$, a sequence of centers $x_n \in \mathbb{R}^d$, and a non-zero function $0 \neq v \in \dot{H}^s$ such that

$$v_n(x) := \lambda_n^{\frac{d}{2}-s} w_n\big(\lambda_n(x - x_n)\big) \rightharpoonup v, \tag{2.7}$$

weakly in $\dot{H}^s$, as $n \to \infty$.

Remark 2.2. We remark that the condition $s > 0$ is crucial in the previous proposition, which is in fact false in the case $s = 0$.

We are now ready to perform the proof of our main theorem.

2.1. Proof of Theorem 1.1

Let M be as in (1.12) and let $u_n \in \dot{H}^s$ be a maximizing sequence, i.e.

$$\|u_n\|_{\dot{H}^s} = 1, \qquad \lim_{n \to \infty} \big\|e^{ith(D)} u_n\big\|_{L^r_{t,x}} = \big\|e^{ith(D)}\big\|_{\mathcal{L}(\dot{H}^s, L^r_{t,x})} = M, \tag{2.8}$$

with $r = \frac{2(k+d)}{d-2s}$. Our aim is to prove that, by a suitable remodulation of u_n, we can obtain a new maximizing sequence for which Proposition 2.1 applies. In fact, since u_n is uniformly bounded in $\dot{H}^s$, it admits a weak limit, which in principle could be zero.

Notice that, by Sobolev embedding,

$$\big\|e^{ith(D)} u_n\big\|_{L_t^\infty L_x^{\frac{2d}{d-2s}}} \leqslant C \big\|e^{ith(D)} u_n\big\|_{L_t^\infty \dot{H}_x^s} = C \|u_n\|_{\dot{H}_x^s} = C, \tag{2.9}$$

for some constant $C > 0$. Moreover, by assumption (1.10), there exist $2 \leqslant p < q \leqslant \infty$ such that

$$\big\|e^{ith(D)} u_n\big\|_{L_t^p L_x^q} \leqslant C \|u_n\|_{\dot{H}_x^s} = C, \tag{2.10}$$

for another constant $C > 0$. Hence, by (2.8), (2.10) and the Hölder inequality we can estimate, for n sufficiently large,

$$\frac{M}{2} \leqslant \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x}} \leqslant \left\| e^{ith(D)} u_n \right\|^{1-\frac{p}{r}}_{L^\infty_t L_x^{\frac{2d}{d-2s}}} \left\| e^{ith(D)} u_n \right\|^{\frac{p}{r}}_{L^p_t L^q_x}$$
$$\leqslant C^{\frac{p}{r}} \left\| e^{ith(D)} u_n \right\|^{1-\frac{p}{r}}_{L^\infty_t L_x^{\frac{2d}{d-2s}}}$$

(where we have used (1.7)). The last estimate implies that

$$\left\| e^{ith(D)} u_n \right\|_{L^\infty_t L_x^{\frac{2d}{d-2s}}} \geqslant \left(\frac{M}{2C^{\frac{p}{r}}} \right)^{\frac{r}{r-p}} =: \epsilon > 0, \tag{2.11}$$

for n sufficiently large. As a consequence, there exists a sequence of times $t_n \in \mathbb{R}$ such that

$$\left\| e^{it_n h(D)} u_n \right\|_{L_x^{\frac{2d}{d-2s}}} \geqslant \frac{\epsilon}{2} > 0, \tag{2.12}$$

for any n sufficiently large. Now denote by

$$w_n(x) = e^{it_n h(D)} u_n(x),$$

and notice that w_n is still a maximizing sequence, i.e. $\| e^{ith(D)} w_n \|_{L^r_{t,x}} \to M$, as $n \to \infty$. Moreover, we have

$$\| w_n \|_{\dot{H}^s} = \| u_n \|_{\dot{H}^s} = 1, \tag{2.13}$$
$$\| w_n \|_{L_x^{\frac{2d}{d-2s}}} \geqslant \frac{\epsilon}{2} > 0; \tag{2.14}$$

hence, by Proposition 2.2, there exist two sequences $\lambda_n > 0$, $x_n \in \mathbb{R}^d$, and a non-zero function $0 \neq v \in \dot{H}^s$, such that

$$v_n(x) := \lambda_n^{\frac{d}{2}-s} w_n\big(\lambda_n(x - x_n)\big) = \lambda_n^{\frac{d}{2}-s} e^{it_n h(D)} u_n\big(\lambda_n(x - x_n)\big) \rightharpoonup v \neq 0 \tag{2.15}$$

weakly in $\dot{H}^s$, as $n \to \infty$. By scaling, it is easy to see that v_n is still a maximizing sequence, since $\| e^{ith(D)} v_n \|_{L^r_{t,x}} = \| e^{ith(D)} w_n \|_{L^r_{t,x}}$.

By Proposition 2.1, in order to conclude the proof it is sufficient to prove that (2.4) holds on v_n, up to subsequences, with $T = e^{ith(D)}$. Let us fix $t \in \mathbb{R}$; by the continuity of $e^{ith(D)}$ in $\dot{H}^s$, we have

$$e^{ith(D)} v_n \rightharpoonup e^{ith(D)} v$$

weakly in $\dot{H}^s$, as $n \to \infty$. Then, by the Rellich theorem, for any $R > 0$

$$e^{ith(D)} v_n \to e^{ith(D)} v \tag{2.16}$$

strongly in $L^2(B(R))$ where $B(R) = \{x \in \mathbb{R}^d\colon |x| < R\}$, as $n \to \infty$. Denote by

$$F_n(t) := \int\limits_{|x|<R} \left|e^{ith(D)}(v_n - v)\right|^2 dx = \left\|e^{ith(D)}(v_n - v)\right\|^2_{L^2(B(R))}.$$

By the Hölder inequality and Sobolev embedding we obtain

$$F_n(t) \leqslant CR^{\frac{2s}{d}} \left\|e^{ith(D)}(v_n - v)\right\|^2_{\dot{H}^s} \leqslant 2CR^{\frac{2s}{d}}; \tag{2.17}$$

consequently, by (2.16), the Fubini and the Lebesgue theorems we have that

$$\int\limits_{-R}^{R} F_n(t)\, dt = \int\limits_{-R}^{R} \int\limits_{|x|<R} \left|e^{ith(D)}(v_n - v)\right|^2 dx\, dt \to 0$$

as $n \to \infty$; this implies that, up to a subsequence,

$$e^{ith(D)}(v_n - v) \to 0 \quad \text{a.e. in } B(R) \times (-R, R).$$

The extraction of the subsequence depends on R; now repeat the argument on a discrete sequence of radii R_n such that $R_n \to \infty$, as $n \to \infty$ and conclude, by a diagonal argument, that there exists a subsequence of v_{n_k} of v_n such that

$$e^{ith(D)}(v_n - v) \to 0 \quad \text{a.e. in } \mathbb{R} \times \mathbb{R}^d. \tag{2.18}$$

This, together with (2.15) and Proposition 2.1, concludes the proof.

References

[1] T. Aubin, Probléme isopérimétrique et espaces de Sobolev, J. Differential Geom. 11 (1976) 573–598.

[2] P. Bégout, A. Vargas, Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation, Trans. Amer. Math. Soc. 359 (2007) 5257–5282.

[3] N. Bez, K. Rogers, A sharp Strichartz estimate for the wave equation with data in the energy space, J. Eur. Math. Soc. (JEMS), in press.

[4] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (3) (1983) 486–490.

[5] A. Bulut, Maximizers for the Strichartz inequalities for the wave equation, Differential Integral Equations 23 (2010) 1035–1072.

[6] P. D'Ancona, L. Fanelli, Strichartz and smoothing estimates for dispersive equations with magnetic potentials, Comm. Partial Differential Equations 33 (6) (2008) 1082–1112.

[7] L. Fanelli, L. Vega, N. Visciglia, On the existence of maximizers for a family of restriction theorems, Bull. London Math. Soc. 43 (4) (2011) 811–817.

[8] D. Foschi, Maximizers for the Strichartz inequality, J. Eur. Math. Soc. (JEMS) 9 (4) (2007) 739–774.

[9] P. Gérard, Description du défaut de compacité de l'injection de Sobolev, ESAIM Control Optim. Calc. Var. 3 (1998) 213–233.

[10] J. Ginibre, G. Velo, Generalized Strichartz inequalities for the wave equation, J. Funct. Anal. 133 (1) (1995) 50–68.

[11] M. Keel, T. Tao, Endpoint Strichartz estimates, Amer. J. Math. 120 (5) (1998) 955–980.

[12] M. Kunze, On the existence of a maximizer for the Strichartz inequality, Comm. Math. Phys. 243 (2003) 137–162.

[13] E. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, Ann. of Math. 118 (2) (1983) 349–374.

[14] P.L. Lions, The concentration-compactness principle in the calculus of variations. The limit case, I, Rev. Mat. Iberoam. 1 (1) (1985) 145–201.

[15] P.L. Lions, The concentration-compactness principle in the calculus of variations. The limit case, II, Rev. Mat. Iberoam. 1 (2) (1985) 45–121.

[16] F. Merle, L. Vega, Compactness at blow-up time for L^2-solutions of the critical nonlinear Schrödinger equation in $2D$, Int. Math. Res. Not. IMRN 8 (1998) 399–425.

[17] J. Ramos, Refinement of the Strichartz inequality for the wave equation with applications, 2011, submitted for publication.

[18] K. Rogers, A. Vargas, A refinement of the Strichartz inequality on the saddle and applications, J. Funct. Anal. 241 (1) (2006) 212–231.

[19] S. Shao, Maximizers for the Strichartz and the Sobolev–Strichartz inequalities for the Schrödinger equation, Electron. J. Differential Equations 2009 (3) (2009) 1–13.

[20] R. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, Duke Math. J. 44 (1977) 705–714.

[21] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 4 (1976) 353–372.