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# Tracial gauge norms on finite von Neumann algebras satisfying the weak Dixmier property

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**Abstract**

In this paper we set up a representation theorem for tracial gauge norms on finite von Neumann algebras satisfying the weak Dixmier property in terms of Ky Fan norms. Examples of tracial gauge norms on finite von Neumann algebras satisfying the weak Dixmier property include unitarily invariant norms on finite factors (type $\mathrm{II}_1$ factors and $M_n(\mathbb{C})$) and symmetric gauge norms on $L^\infty[0,1]$ and $\mathbb{C}^n$. As the first application, we obtain that the class of unitarily invariant norms on a type $\mathrm{II}_1$ factor coincides with the class of symmetric gauge norms on $L^\infty[0,1]$ and von Neumann's classical result [J. von Neumann, Some matrix-inequalities and metrization of matrix-space, Tomsk. Univ. Rev. 1 (1937) 286–300] on unitarily invariant norms on $M_n(\mathbb{C})$. As the second application, Ky Fan's dominance theorem [Ky Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, Proc. Natl. Acad. Sci. USA 37 (1951) 760–766] is obtained for finite von Neumann algebras satisfying the weak Dixmier property. As the third application, some classical results in non-commutative $L^p$-theory (e.g., non-commutative Hölder's inequality, duality and reflexivity of non-commutative $L^p$-spaces) are obtained for general unitarily invariant norms on finite factors. We also investigate the extreme points of $\mathfrak{N}(\mathcal{M})$, the convex compact set (in the pointwise weak topology) of normalized unitarily invariant norms (the norm of the identity operator is 1) on a finite factor $\mathcal{M}$. We obtain all extreme points of $\mathfrak{N}(M_2(\mathbb{C}))$ and some extreme points of $\mathfrak{N}(M_n(\mathbb{C}))$ ($n \geqslant 3$). For a type $\mathrm{II}_1$ factor $\mathcal{M}$, we prove that if $t$ ($0 \leqslant t \leqslant 1$) is a rational number then the Ky Fan $t$th norm is an extreme point of $\mathfrak{N}(\mathcal{M})$.



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**Contents**



## 1. Introduction

The unitarily invariant norms were introduced by von Neumann [21] for the purpose of metrizing matrix spaces. Von Neumann, together with his associates, established that the class of unitarily invariant norms of $n \times n$ complex matrices coincides with the class of symmetric gauge functions of their $s$-numbers. These norms have now been variously generalized and utilized in several contexts. For example, Schatten [17,18] defined norms on two-sided ideals of completely continuous operators on an arbitrary Hilbert space; Ky Fan [13] studied Ky Fan norms and obtained his dominance theorem. The unitarily invariant norms play a crucial role in the study of function spaces and group representations (see e.g. [12]) and in obtaining certain bounds of importance in quantum field theory (see [20]). For historical perspectives and surveys of unitarily invariant norms, see Schatten [17,18], Hewitt and Ross [9], Gohberg and Krein [7] and Simon [20].

The theory of non-commutative $L^p$-spaces has been developed under the name "non-commutative integration" beginning with pioneer work of Segal, Dixmier, and Kunze. Since then the theory has been extensively studied, extended and applied by Nelson, Haagerup, Fack, Kosaki, Junge, Xu, and many others. The recent survey by Pisier and Xu [15] presents a rather complete picture on non-commutative integration and contains a lot of references. This theory is still a very active subject of investigation. Some tools in the study of the usual commutative $L^p$-spaces still work in the non-commutative setting. However, most of the time, new techniques must be invented. To illustrate the difficulties one may encounter in studying the non-commutative $L^p$-spaces, we mention here one basic well-known fact. Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. The basic fact states that the usual triangle inequality for the absolute values of complex numbers is no longer valid for the absolute values of operators, namely, in general, we do not have $|S+T| \leqslant |S|+|T|$ for $S, T \in \mathcal{B}(\mathcal{H})$, where $|S| = (S^*S)^{1/2}$ is the absolute value of $S$. Despite such difficulties, by now the strong parallelism between non-commutative and classical Lebesgue integration is well known.

Motivated by von Neumann's theorem and the analogies between non-commutative and classical $L^p$-spaces, in this paper, we will systematically study tracial gauge norms on finite von Neumann algebras that satisfy the weak Dixmier property. Before stating the main theorem and its consequences, we explain some of the notation and terminology that will be used throughout the paper.

In this paper, a finite von Neumann algebra $(\mathcal{M}, \tau)$ means a von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state $\tau$. A finite von Neumann algebra $(\mathcal{M}, \tau)$ is said to satisfy the *weak Dixmier property* if for every positive operator $T \in \mathcal{M}$, $\tau(T)$ is in the *operator norm* closure of the convex hull of $\{S \in \mathcal{M}\colon\ S \text{ and } T \text{ are equi-measurable, i.e., } \tau(S^n) = \tau(T^n) \text{ for all } n = 0, 1, 2, \ldots\}$. Recall that finite factors satisfy the *Dixmier property*: if $T \in \mathcal{M}$, then $\tau(T)$ is in the *operator norm* closure of the convex hull of $\{UTU^*\colon\ U \in \mathcal{M} \text{ is a unitary operator}\}$ and hence satisfy the weak Dixmier property. In Section 3.5, we prove that a finite von Neumann algebra $(\mathcal{M}, \tau)$ satisfies the weak Dixmier property if and only if either $(\mathcal{M}, \tau)$ can be identified as a von Neumann subalgebra of $(M_n(\mathbb{C}), \tau_n)$ that contains all diagonal matrices, where $\tau_n$ is the normalized trace on $M_n(\mathbb{C})$, or $\mathcal{M}$ is diffuse. Throughout the paper, we will reserve the notation $\|\cdot\|$ for the *operator norm* on von Neumann algebras.

A *tracial gauge* norm $|||\cdot|||$ on a finite von Neumann algebra $(\mathcal{M}, \tau)$ is a norm on $\mathcal{M}$ satisfying $|||T||| = |||\,|T|\,|||$ for all $T \in \mathcal{M}$ (gauge invariant) and $|||S||| = |||T|||$ if $S$ and $T$ are two equi-measurable positive operators in $\mathcal{M}$ (tracial). For a finite von Neumann algebra $(\mathcal{M}, \tau)$, let $\mathrm{Aut}(\mathcal{M}, \tau)$ be the set of $*$-automorphisms on $\mathcal{M}$ that preserve the trace. A *symmetric gauge* norm $|||\cdot|||$ on a finite von Neumann algebra $(\mathcal{M}, \tau)$ is a gauge norm on $\mathcal{M}$ satisfying $|||\theta(T)||| = |||T|||$ for all positive operators $T \in \mathcal{M}$ and $\theta \in \mathrm{Aut}(\mathcal{M}, \tau)$. A *unitarily invariant* norm $|||\cdot|||$ on a finite von Neumann algebra $(\mathcal{M}, \tau)$ is a norm on $\mathcal{M}$ satisfying $|||UTW||| = |||T|||$ for all $T \in \mathcal{M}$ and unitary operators $U, W$ in $\mathcal{M}$. On $(L^\infty[0,1], \int_0^1 dx)$ and $(\mathbb{C}^n, \tau)$, where $\tau((x_1, \ldots, x_n)) = \frac{x_1+\cdots+x_n}{n}$, a norm is a tracial gauge norm if and only if it is a symmetric gauge norm. A norm on a finite factor is a tracial gauge norm if and only if it is a unitarily invariant norm. A *normalized* norm is one that assigns the value 1 to the identity operator (which is also denoted by 1).

In [5], Fack and Kosaki defined $\mu_s(T)$, *the generalized s-numbers of an operator* $T$ in a finite von Neumann algebra $(\mathcal{M},\tau)$ by

$$\mu_s(T)=\inf\bigl\{\|TE\|\colon\ E \text{ is a projection in } \mathcal{M} \text{ with } \tau(1-E)\leqslant s\bigr\},\quad 0\leqslant s\leqslant 1.$$

For $0<t\leqslant 1$, *the Ky Fan tth norm*, $|||T|||_{(t)}$, on a finite von Neumann algebra $(\mathcal{M},\tau)$ is defined by

$$|||T|||_{(t)}=\frac{1}{t}\int_0^t \mu_s(T)\,ds.$$

Then $|||\cdot|||_{(t)}$ is a tracial gauge norm on $(\mathcal{M},\tau)$. Note that $|||T|||_{(1)}=\tau(|T|)=\|T\|_1$ is the trace norm.

Let $n\in\mathbb{N}$, $a_1\geqslant a_2\geqslant\cdots\geqslant a_n\geqslant a_{n+1}=0$ and $f(x)=a_1\chi_{[0,\frac{1}{n})}(x)+a_2\chi_{[\frac{1}{n},\frac{2}{n})}(x)+\cdots+a_n\chi_{[\frac{n-1}{n},1]}(x)$. For $T\in\mathcal{M}$, define $|||T|||_f=\int_0^1 f(s)\mu_s(T)\,ds$. Then

$$|||T|||_f=\sum_{k=1}^{n}\frac{k(a_k-a_{k+1})}{n}|||T|||_{(\frac{k}{n})}.$$

Therefore, $|||T|||_f$ is a tracial gauge norm on $(\mathcal{M},\tau)$. Note that if $f(x)$ is the constant 1 function on $[0,1]$, then $|||T|||_f=|||T|||_{(1)}=\|T\|_1=\tau(|T|)$.

Let $\mathcal{F}=\{f(x)=a_1\chi_{[0,\frac{1}{n})}(x)+a_2\chi_{[\frac{1}{n},\frac{2}{n})}(x)+\cdots+a_n\chi_{[\frac{n-1}{n},1]}(x)\colon\ a_1\geqslant a_2\geqslant\cdots\geqslant a_n\geqslant 0,\ \frac{a_1+\cdots+a_n}{n}\leqslant 1,\ n=1,2,\ldots\}$. In Section 7, we prove the following representation theorem, which is the main result of this paper.

**Theorem A.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property. If $|||\cdot|||$ is a normalized tracial gauge norm on $\mathcal{M}$, then there is a subset $\mathcal{F}'$ of $\mathcal{F}$ containing the constant 1 function on $[0,1]$ such that for every $T\in\mathcal{M}$,*

$$|||T|||=\sup\bigl\{|||T|||_f\colon\ f\in\mathcal{F}'\bigr\},$$

*where $|||T|||_f$ is defined as above.*

To prove Theorem A, we firstly prove the following technical theorem in Section 4.

**Theorem B.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. Then $\mathcal{M}_{1,|||\cdot|||}=\{T\in\mathcal{M}\colon\ |||T|||\leqslant 1\}$ is closed in the weak operator topology.*

The Russo–Dye theorem [16] and the Kadison–Peterson theorem [10] on convex hulls of unitary operators in von Neumann algebras and the idea of Dixmier's averaging theorem [2] play fundamental roles in the proof of Theorem B. An important consequence of Theorem B is the following corollary which enables us to apply the powerful techniques of normal conditional expectations from finite von Neumann algebras to abelian von Neumann subalgebras.

**Corollary 1.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. If $\mathcal{A}$ is a separable abelian von Neumann subalgebra of $\mathcal{M}$ and $\mathbf{E}_{\mathcal{A}}$ is the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{A}$ preserving $\tau$, then $|||\mathbf{E}_{\mathcal{A}}(T)||| \leqslant |||T|||$ for all $T\in\mathcal{M}$.*

The notion of *dual norms* plays a key role in the proof of Theorem A. Let $|||\cdot|||$ be a norm on a finite von Neumann algebra $(\mathcal{M},\tau)$. Then the dual norm $|||\cdot|||^{\#}$ is defined by

$$|||T|||^{\#}=\sup\bigl\{\bigl|\tau(TX)\bigr|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\bigr\},\quad T\in\mathcal{M}.$$

In Section 5, we study the dual norms systematically. By applying Corollary 1 and careful analysis, we prove the following theorem.

**Theorem C.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. Then $|||\cdot|||^{\#}$ is also a tracial gauge norm on $\mathcal{M}$ and $|||\cdot|||^{\#\#}=|||\cdot|||$.*

Combining Corollary 1, Theorem C and the following theorem on non-increasing rearrangements of functions (see [8, 10.13] for instance), we prove Theorem A in Section 7.

**Hardy–Littlewood–Pólya.** *Let $f(x), g(x)$ be non-negative Lebesgue measurable functions on $[0,1]$ and let $f^*(x), g^*(x)$ be the non-increasing rearrangements of $f(x), g(x)$, respectively, then $\int_0^1 f(x)g(x)\,dx\leqslant\int_0^1 f^*(x)g^*(x)\,dx$.*

Now we state some important consequences of Theorem A. Since there is a natural one-to-one correspondence between Ky Fan $t$th norms on finite von Neumann algebras (satisfying the weak Dixmier property) and Ky Fan $t$th norms on $(L^\infty[0,1],\int_0^1 dx)$ or $(\mathbb{C}^n,\tau)$, the first application of Theorem A is the following.

**Theorem D.** *Let $(\mathcal{M},\tau)$ be a diffuse finite von Neumann algebra (or a von Neumann subalgebra of $\mathcal{M}_n(\mathbb{C})$, $\tau=\tau_n|_{\mathcal{M}}$, such that $\mathcal{M}$ contains all diagonal matrices). Then there is a one-to-one correspondence between tracial gauge norms on $(\mathcal{M},\tau)$ and symmetric gauge norms on $(L^\infty[0,1],\int_0^1 dx)$ (or $(\mathbb{C}^n,\tau')$, $\tau'((x_1,\dots,x_n))=\frac{x_1+\cdots+x_n}{n}$, respectively). Namely:*

1. *If $|||\cdot|||$ is a tracial gauge norm on $(\mathcal{M},\tau)$ and $\theta$ is an embedding from $(L^\infty[0,1],\int_0^1 dx)$ into $(\mathcal{M},\tau)$ (or $x_1\oplus\cdots\oplus x_n$ is the diagonal matrix with diagonal elements $x_1,\dots,x_n$, respectively), then $|||f(x)|||'=|||\theta(f(x))|||$ defines a symmetric gauge norm on $(L^\infty[0,1],\int_0^1 dx)$ (or $|||(x_1,\dots,x_n)|||'=|||x_1\oplus\cdots\oplus x_n|||$ defines a symmetric gauge norm on $(\mathbb{C}^n,\tau')$, respectively).*
2. *If $|||\cdot|||'$ is a symmetric gauge norm on $(L^\infty[0,1],\int_0^1 dx)$ (or $(\mathbb{C}^n,\tau')$ respectively), then $|||T|||=|||\mu_s(T)|||'$ (or $|||T|||=|||(s_1(T),\dots,s_n(T))|||'$, respectively) defines a tracial gauge norm on $(\mathcal{M},\tau)$.*

As consequences of Theorem D, we have the following corollary and von Neumann's theorem.

**Corollary 2.** *There is a one-to-one correspondence between unitarily invariant norms on a type* $\mathrm{II}_1$ *factor* $(\mathcal{M}, \tau)$ *and symmetric gauge norms on* $(L^\infty[0,1], \int_0^1 dx)$. *Namely*:

1. *If* $|||\cdot|||$ *is a unitarily invariant norm on* $\mathcal{M}$ *and* $\theta$ *is an embedding from* $(L^\infty[0,1], \int_0^1 dx)$ *into* $(\mathcal{M}, \tau)$, *then* $|||f(x)|||' = |||\theta(f(x))|||$ *defines a symmetric gauge norm on* $(L^\infty[0,1], \int_0^1 dx)$.
2. *If* $|||\cdot|||'$ *is a symmetric gauge norm on* $(L^\infty[0,1], \int_0^1 dx)$, *then* $|||T||| = |||\mu_s(T)|||'$ *defines a unitarily invariant norm on* $\mathcal{M}$.

**Von Neumann.** *There is a one-to-one correspondence between unitarily invariant norms on* $M_n(\mathbb{C})$ *and symmetric gauge norms on* $(\mathbb{C}^n, \tau)$, $\tau((x_1, \ldots, x_n)) = \frac{x_1+\cdots+x_n}{n}$. *Namely*:

1. *If* $|||\cdot|||$ *is a unitarily invariant norm on* $M_n(\mathbb{C})$, *then* $|||(x_1, \ldots, x_n)|||' = |||x_1 \oplus \cdots \oplus x_n|||$ *defines a symmetric gauge norm on* $(\mathbb{C}^n, \tau)$.
2. *If* $|||\cdot|||'$ *is a symmetric gauge norm on* $(\mathbb{C}^n, \tau)$, *then* $|||T||| = |||(s_1(T), \ldots, s_n(T))|||'$ *defines a unitarily invariant norm on* $M_n(\mathbb{C})$.

Theorem D establishes the one-to-one correspondence between tracial gauge norms on finite von Neumann algebras satisfying the weak Dixmier property and symmetric gauge norms on abelian von Neumann algebras. The following theorem further establishes the one-to-one correspondence between the dual norms on finite von Neumann algebras satisfying the weak Dixmier property and the dual norms on abelian von Neumann algebras, which plays a key role in the studying of duality and reflexivity of the completion of type $\mathrm{II}_1$ factors with respect to unitarily invariant norms.

**Theorem E.** *Let* $(\mathcal{M}, \tau)$ *be a diffuse finite von Neumann algebra* (*or a von Neumann subalgebra of* $\mathcal{M}_n(\mathbb{C})$, $\tau = \tau_n|_{\mathcal{M}}$, *such that* $\mathcal{M}$ *contains all diagonal matrices*). *If* $|||\cdot|||$ *is a tracial gauge norm on* $(\mathcal{M}, \tau)$ *corresponding to the symmetric gauge norm* $|||\cdot|||_1$ *on* $(L^\infty[0,1], \int_0^1 dx)$ (*or* $(\mathbb{C}^n, \tau')$, *respectively*) *as in Theorem* D, *then* $|||\cdot|||^\#$ *on* $\mathcal{M}$ *is the tracial gauge norm corresponding to the symmetric gauge norm* $|||\cdot|||_1^\#$ *on* $(L^\infty[0,1], \int_0^1 dx)$ (*or* $(\mathbb{C}^n, \tau')$, *respectively*) *as in Theorem* D.

The second consequence of Theorem A is the following theorem.

**Theorem F.** *Let* $(\mathcal{M}, \tau)$ *be a finite von Neumann algebra satisfying the weak Dixmier property and* $S, T \in \mathcal{M}$. *If* $|||S|||_{(t)} \leqslant |||T|||_{(t)}$ *for all Ky Fan t-th norms,* $0 \leqslant t \leqslant 1$, *then* $|||S||| \leqslant |||T|||$ *for all tracial gauge norms* $|||\cdot|||$ *on* $\mathcal{M}$.

As a corollary, we obtain the following

**Ky Fan's dominance theorem.** *(See [13].) If* $S, T \in M_n(\mathbb{C})$ *and* $|||S|||_{(k/n)} \leqslant |||T|||_{(k/n)}$, *i.e.*, $\sum_{i=1}^k s_i(S) \leqslant \sum_{i=1}^k s_i(T)$ *for* $1 \leqslant k \leqslant n$, *then* $|||S||| \leqslant |||T|||$ *for all unitarily invariant norms* $|||\cdot|||$ *on* $M_n(\mathbb{C})$.

A unitarily invariant norm $|||\cdot|||$ on a type $\mathrm{II}_1$ factor $\mathcal{M}$ is called *singular* if $\lim_{\tau(E)\to 0+} |||E||| > 0$ and *continuous* if $\lim_{\tau(E)\to 0+} |||E||| = 0$. The following theorem is proved in Section 11.

**Theorem G.** *Let $|||\cdot|||$ be a unitarily invariant norm on $\mathcal{M}$ and let $\mathcal{T}$ be the topology induced by $|||\cdot|||$ on $\mathcal{M}_{1,\|\cdot\|} = \{T \in \mathcal{M}\colon \|T\| \leqslant 1\}$. If $|||\cdot|||$ is singular, then $\mathcal{T}$ is the operator norm topology on $\mathcal{M}_{1,\|\cdot\|}$. If $|||\cdot|||$ is continuous, then $\mathcal{T}$ is the measure topology* (*in the sense of Nelson* [14]) *on $\mathcal{M}_{1,\|\cdot\|}$.*

Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and let $|||\cdot|||$ be a unitarily invariant norm on $\mathcal{M}$. We denote by $\overline{\mathcal{M}}_{|||\cdot|||}$ the completion of $\mathcal{M}$ with respect to $|||\cdot|||$. Let $\widetilde{\mathcal{M}}$ be the completion of $\mathcal{M}$ with respect to the measure topology in the sense of Nelson [14]. In Section 12, we prove that there is an injective map from $\overline{\mathcal{M}}_{|||\cdot|||}$ into $\widetilde{\mathcal{M}}$ that extends the identity map from $\mathcal{M}$ onto $\mathcal{M}$. An element in $\widetilde{\mathcal{M}}$ can be identified with a closed, densely defined operator affiliated with $\mathcal{M}$ (see [14]). So generally speaking, an element in $\overline{\mathcal{M}}_{|||\cdot|||}$ should be treated as an unbounded operator. We will consider the following two questions in Section 13:

**Question 1.** Under what conditions is $\overline{\mathcal{M}}_{|||\cdot|||^\#}$ the dual space of $\overline{\mathcal{M}}_{|||\cdot|||}$ in the following sense: for every $\phi \in \overline{\mathcal{M}}_{|||\cdot|||}^{\#}$, there is a unique $X \in \overline{\mathcal{M}}_{|||\cdot|||^\#}$ such that

$$\phi(T) = \tau(TX), \quad \forall T \in \overline{\mathcal{M}}_{|||\cdot|||},$$

and $\|\phi\| = |||T|||$?

**Question 2.** Under what conditions is $\overline{\mathcal{M}}_{|||\cdot|||}$ a reflexive Banach space?

Let $|||\cdot|||_1$ be the symmetric gauge norm on $(L^\infty[0,1], \int_0^1 dx)$ corresponding to $|||\cdot|||$ on $\mathcal{M}$ as in Corollary 2. Then the same questions can be asked about $\overline{L^\infty[0,1]}_{|||\cdot|||_1}$, the completion of $L^\infty[0,1]$ with respect to $|||\cdot|||_1$.

As further consequences of Theorem A, we prove the following theorems that answer the Questions 1 and 2, respectively.

**Theorem H.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor, $|||\cdot|||$ be a unitarily invariant norm on $\mathcal{M}$ and $|||\cdot|||^\#$ be the dual unitarily invariant norm on $\mathcal{M}$. Let $|||\cdot|||_1$ be the symmetric gauge norm on $(L^\infty[0,1], \int_0^1 dx)$ corresponding to $|||\cdot|||$ on $\mathcal{M}$ as in Corollary* 2. *Then the following conditions are equivalent*:

1. *$\overline{\mathcal{M}}_{|||\cdot|||^\#}$ is the dual space of $\overline{\mathcal{M}}_{|||\cdot|||}$ in the sense of Question* 1;
2. *$\overline{L^\infty[0,1]}_{|||\cdot|||_1^\#}$ is the dual space of $\overline{L^\infty[0,1]}_{|||\cdot|||_1}$ in the sense of Question* 1;
3. *$|||\cdot|||$ is a continuous norm on $\mathcal{M}$*;
4. *$|||\cdot|||_1$ is a continuous norm on $L^\infty[0,1]$.*

**Theorem I.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor, $|||\cdot|||$ be a unitarily invariant norm on $\mathcal{M}$ and let $|||\cdot|||^\#$ be the dual unitarily invariant norm on $\mathcal{M}$. Let $|||\cdot|||_1$ be the symmetric gauge norm on $(L^\infty[0,1], \int_0^1 dx)$ corresponding to $|||\cdot|||$ on $\mathcal{M}$ as in Corollary* 2. *Then the following conditions are equivalent*:

1. $\overline{\mathcal{M}_{|||\cdot|||}}$ *is a reflexive space*;
2. $\overline{L^\infty[0,1]_{|||\cdot|||_1}}$ *is a reflexive space*;
3. *both* $|||\cdot|||$ *and* $|||\cdot|||^\#$ *are continuous norms on* $\mathcal{M}$;
4. *both* $|||\cdot|||_1$ *and* $|||\cdot|||_1^\#$ *are continuous norms on* $L^\infty[0,1]$.

A key step to proving Theorem H is based on the following fact: if $|||\cdot|||$ is a continuous unitarily invariant norm on $\mathcal{M}$ and $\phi \in \overline{\mathcal{M}_{|||\cdot|||}}^\#$, then the restriction of $\phi$ to $\mathcal{M}$ is an ultraweakly continuous linear functional, i.e., $\phi$ is in the predual space of $\mathcal{M}$. A significant advantage of our approach is that we develop a relatively complete theory of unitarily invariant norms on type $\mathrm{II}_1$ factors before handling unbounded operators. Indeed, unbounded operators are slightly involved only in the last two sections (Sections 12 and 13). Compared with the classical methods (e.g., [19]), which have to do a lot of subtle analysis on unbounded operators, our methods are much simpler.

Let $\mathcal{M}$ be a finite factor. Recall that a norm $|||\cdot|||$ on $\mathcal{M}$ is called a normalized norm if $|||1|||=1$. Let $\mathfrak{N}(\mathcal{M})$ be the set of normalized unitarily invariant norms on $\mathcal{M}$. Then $\mathfrak{N}(\mathcal{M})$ is a convex compact set in the pointwise weak topology. Let $\mathfrak{N}_{\mathrm{e}}(\mathcal{M})$ be the set of extreme points of $\mathfrak{N}(\mathcal{M})$. By the Krein–Milman theorem, $\mathfrak{N}(\mathcal{M})$ is the closure of the convex hull of $\mathfrak{N}_{\mathrm{e}}(\mathcal{M})$ in the pointwise weak topology. So it is an interesting question of characterizing the set $\mathfrak{N}_{\mathrm{e}}(\mathcal{M})$. In Section 10, we prove the following theorems.

**Theorem J.** $\mathfrak{N}_{\mathrm{e}}(M_2(\mathbb{C})) = \{\max\{t\|T\|, \|T\|_1\}\colon 1/2 \leqslant t \leqslant 1\}$*, where* $\|T\|_1 = \tau_2(|T|)$.

**Theorem K.** *If* $\mathcal{M}$ *is a type* $\mathrm{II}_1$ *factor and* $t$ *is a rational number such that* $0 \leqslant t \leqslant 1$*, then the Ky Fan* $t$*th norm is an extreme point of* $\mathfrak{N}(\mathcal{M})$.

This paper is almost self-contained and we do not assume any backgrounds on non-commutative $L^p$-theory.

## 2. Preliminaries

### *2.1. Nonincreasing rearrangements of functions*

Throughout this paper, we denote by $m$ the Lebesgue measure on $[0,1]$. In the following, a measurable function and a measurable set mean a Lebesgue measurable function and a Lebesgue measurable set, respectively. For two measurable sets $A$ and $B$, $A=B$ means $m((A\setminus B)\cup(B\setminus A))=0$.

Let $f(x)$ be a real measurable function on $[0,1]$. The *non-increasing rearrangement function*, $f^*(x)$, of $f(x)$ is defined by

$$f^*(x) = \begin{cases} \sup\{y\colon m(\{f>y\})>x\}, & 0\leqslant x<1; \\ \operatorname{ess\,inf} f, & x=1. \end{cases} \tag{2.1}$$

We summarize some useful properties of $f^*(x)$ in the following proposition.

**Proposition 2.1.** *Let* $f(x), g(x)$ *be real measurable functions on* $[0,1]$*. Then we have the following*:

1. *$f^*(x)$ is a non-increasing, right-continuous function on* $[0,1]$ *such that* $f^*(0) = \operatorname{ess\,sup} f$;
2. *if $f(x)$ and $g(x)$ are bounded functions and* $\int_0^1 f^n(x)\,dx = \int_0^1 g^n(x)\,dx$ *for all* $n = 0, 1, 2, \ldots,$ *then* $f^*(x) = g^*(x)$;
3. *$f(x)$ and $f^*(x)$ are equi-measurable and* $\int_0^1 f(x)\,dx = \int_0^1 f^*(x)\,dx$ *when either integral is well defined.*

### 2.2. *Invertible measure-preserving transformations on* $[0,1]$

Let $\mathfrak{G} = \{\phi\colon\ \phi(x)$ is an invertible measure-preserving transformation on $[0,1]\}$. It is well known that $\mathfrak{G}$ acts on $[0,1]$ ergodically (see [6, pp. 3, 4], for instance), i.e., for a measurable subset $A$ of $[0,1]$, $\phi(A) = A$ for all $\phi \in \mathfrak{G}$ implies that $m(A) = 0$ or $m(A) = 1$.

**Lemma 2.2.** *Let $A, B$ be two measurable subsets of* $[0,1]$ *such that $m(A) = m(B)$. Then there is a $\phi \in \mathfrak{G}$ such that $\phi(A) = B$.*

**Proof.** We can assume that $m(A) = m(B) > 0$. Since $\mathfrak{G}$ acts ergodically on $[0,1]$, there is a $\phi \in \mathfrak{G}$ such that $m(\phi(A) \cap B) > 0$. Let $B_1 = \phi(A) \cap B$ and $A_1 = \phi^{-1}(B_1)$. Then $m(A_1) = m(B_1)$ and $\phi(A_1) = B_1$. By Zorn's lemma and maximality arguments, we prove the lemma. □

**Corollary 2.3.** *Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be disjoint measurable subsets of* $[0,1]$ *such that $m(A_k) = m(B_k)$ for $1 \leqslant k \leqslant n$. Then there is a $\phi \in \mathfrak{G}$ such that $\phi(A_k) = B_k$ for $1 \leqslant k \leqslant n$.*

**Proof.** We can assume that $A_1 \cup \cdots \cup A_n = B_1 \cup \cdots \cup B_n = [0,1]$. By Lemma 2.2, there is a $\phi_k \in \mathfrak{G}$ such that $\phi_k(A_k) = B_k$, $1 \leqslant k \leqslant n$. Define $\phi(x) = \phi_k(x)$ for $x \in A_k$. Then $\phi \in \mathfrak{G}$ and $\phi(A_k) = B_k$ for $1 \leqslant k \leqslant n$. □

For $f(x) \in L^\infty[0,1]$, define $\tau(f) = \int_0^1 f(x)\,dx$. The following theorem is a version of the Dixmier's averaging theorem (see [3] or [11]) and it has a similar proof.

**Theorem 2.4.** *Let $f(x) \in L^\infty[0,1]$ be a real function. Then $\tau(f)$ is in the $L^\infty$-norm closure of the convex hull of $\{f \cdot \phi(x)\colon\ \phi \in \mathfrak{G}\}$.*

We end this subsection with the following proposition.

**Proposition 2.5.** *If $\phi(x)$ is an invertible measure-preserving transformation on* $[0,1]$, *then*

$$\theta(f) = f \circ \phi$$

*is a $*$-automorphism of $L^\infty[0,1]$ preserving $\tau$. Conversely, if $\theta$ is a $*$-automorphism of $L^\infty[0,1]$ preserving $\tau$, then there is an invertible measure-preserving transformation on* $[0,1]$ *such that*

$$\theta(f) = f \circ \phi$$

*for all $f(x) \in L^\infty[0,1]$.*

**Proof.** The first part of the proposition is easy to see. Suppose $\theta$ is a $*$-automorphism of $L^\infty[0,1]$. Let $\phi(x) = \theta(f)(x)$, where $f(x) \equiv x$. Then it is easy to see the second part of the proposition. □

*2.3. s-Numbers of operators in type* $\mathrm{II}_1$ *factors*

In [5], Fack and Kosaki give a rather complete exposition of generalized $s$-numbers of $\tau$-measurable operators affiliated with semi-finite von Neumann algebras. For the sake of reader's convenience and our purpose, we provide sufficient details on $s$-numbers of bounded operators in finite von Neumann algebras in the following. We will define $s$-numbers of bounded operators in finite von Neumann algebras from the point of view of non-increasing rearrangement of functions.

The following lemma is well known. The proof is an easy exercise.

**Lemma 2.6.** *Let* $(\mathcal{A}, \tau)$ *be a separable* (*i.e., with separable predual*) *diffuse abelian von Neumann algebra with a faithful normal trace* $\tau$ *on* $\mathcal{A}$. *Then there is a* $*$*-isomorphism* $\alpha$ *from* $(\mathcal{A}, \tau)$ *onto* $(L^\infty[0,1], \int_0^1 dx)$ *such that* $\tau = \int_0^1 dx \circ \alpha$.

Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and let $\tau$ be the unique trace on $\mathcal{M}$. For $T \in \mathcal{M}$, there is a separable diffuse abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ containing $|T|$. By Lemma 2.6, there is a $*$-isomorphism $\alpha$ from $(\mathcal{A}, \tau)$ onto $(L^\infty([0,1], \int_0^1 dx)$ such that $\tau = \int_0^1 dx \circ \alpha$. Let $f(x) = \alpha(|T|)$ and $f^*(x)$ be the non-increasing rearrangement of $f(x)$ (see (2.1)). Then *the $s$-numbers of $T$*, $\mu_s(T)$, are defined as

$$\mu_s(T) = f^*(s), \quad 0 \leqslant s \leqslant 1.$$

**Lemma 2.7.** $\mu_s(T)$ *does not depend on* $\mathcal{A}$ *and* $\alpha$.

**Proof.** Let $\mathcal{A}_1$ be another separable diffuse abelian von Neumann subalgebra of $\mathcal{M}$ containing $|T|$ and let $\beta$ be a $*$-isomorphism from $(\mathcal{A}_1, \tau)$ onto $(L^\infty[0,1], \int_0^1 dx)$ such that $\tau = \int_0^1 dx \cdot \beta$. Let $g(x) = \beta(|T|)$. For every number $n = 0, 1, 2, \ldots$, $\int_0^1 f^n(x)\,dx = \tau(|T|^n) = \int_0^1 g^n(x)\,dx$. Since both $f(x)$ and $g(x)$ are bounded positive functions, by 2 of Proposition 2.1, $f^*(x) = g^*(x)$ for all $x \in [0,1]$. $\square$

**Corollary 2.8.** *For* $T \in \mathcal{M}$ *and* $p \geqslant 0$, $\tau(|T|^p) = \int_0^1 \mu_s(T)^p\,ds$.

The following lemma says that the above definition of $s$-numbers coincides with the definition of $s$-numbers given by Fack and Kosaki. Recall that $\mathcal{P}(\mathcal{M})$ is the set of projections in $\mathcal{M}$.

**Lemma 2.9.** *For* $0 \leqslant s \leqslant 1$,

$$\mu_s(T) = \inf\bigl\{\|TE\|:\ E \in \mathcal{P}(\mathcal{M}),\ \tau\bigl(E^\perp\bigr) = s\bigr\}.$$

**Proof.** By the polar decomposition and the definition of $\mu_s(T)$, we may assume that $T$ is positive. Let $\mathcal{A}$ be a separable diffuse abelian von Neumann subalgebra of $\mathcal{M}$ containing $T$ and let $\alpha$ be a $*$-isomorphism from $(\mathcal{A}, \tau)$ onto $(L^\infty[0,1], \int_0^1 dx)$ such that $\tau = \int_0^1 dx \cdot \alpha$. Let $f(x) = \alpha(T)$ and let $f^*(x)$ be the non-increasing rearrangement of $f(x)$. Then $\mu_s(T) = f^*(s)$. By the definition of $f^*$,

$$m\bigl(\bigl\{f^* > \mu_s(T)\bigr\}\bigr) = \lim_{n\to\infty} m\left(\left\{f^* > \mu_s(T) + \frac{1}{n}\right\}\right) \leqslant s$$

and

$$m\big(\{f^* \geqslant \mu_s(T)\}\big) \geqslant \lim_{n\to\infty} m\left(\left\{f^* > \mu_s(T) - \frac{1}{n}\right\}\right) \geqslant s.$$

Since $f^*$ and $f$ are equi-measurable, $m(\{f > \mu_s(T)\}) \leqslant s$ and $m(\{f \geqslant \mu_s(T)\}) \geqslant s$. Therefore, there is a measurable subset $A$ of $[0,1]$, $\{f > \mu_s(T)\} \subset [0,1] \setminus A \subset \{f \geqslant \mu_s(T)\}$, such that $m([0,1]\setminus A) = s$ and $\|f(x)\chi_A(x)\|_\infty = \mu_s(T)$ and $\|f(x)\chi_B(x)\|_\infty \geqslant \mu_s(T)$ for all $B \subset [0,1]\setminus A$ such that $m(B) > 0$. Let $F = \alpha^{-1}(\chi_A)$. Then $\tau(F^\perp) = s$, $\|TF\| = \|\alpha^{-1}(f\chi_A)\|_\infty = \mu_s(T)$ and $\|TF'\| \geqslant \mu_s(T)$ for all non-zero subprojections $F'$ of $F^\perp$. This proves that $\mu_s(T) \geqslant \inf\{\|TE\|\colon E \in \mathcal{P}(\mathcal{M}),\ \tau(E^\perp) = s\}$. Similarly, for every $\epsilon > 0$, there is a projection $F_\epsilon \in \mathcal{M}$ such that $\tau(F_\epsilon^\perp) = s + \epsilon$, $\|TF_\epsilon\| = \mu_{s+\epsilon}(T)$ and $\|TF'\| \geqslant \mu_{s+\epsilon}(T)$ for all non-zero subprojections $F'$ of $F_\epsilon^\perp$. Suppose $E \in \mathcal{M}$ is a projection such that $\tau(E^\perp) = s$. Then $\tau(E \wedge F_\epsilon^\perp) = \tau(E) + \tau(F_\epsilon^\perp) - \tau(E \vee F_\epsilon^\perp) = 1 + \epsilon - \tau(E \vee F^\perp) \geqslant \epsilon > 0$. Hence, $\|TE\| \geqslant \|T(E \wedge F_\epsilon^\perp)\| \geqslant \mu_{s+\epsilon}(T)$. This proves that $\inf\{\|TE\|\colon E \in \mathcal{P}(\mathcal{M}),\ \tau(E^\perp) = s\} \geqslant \mu_{s+\epsilon}(T)$. Since $\mu_s(T)$ is right-continuous, $\mu_s(T) \leqslant \inf\{\|TE\|\colon E \in \mathcal{P}(\mathcal{M}),\ \tau(E^\perp) = s\}$. □

**Corollary 2.10.** *Let $S, T \in \mathcal{M}$. Then $\mu_s(ST) \leqslant \|S\|\mu_s(T)$ for $s \in [0,1]$.*

We refer to [4,5] for other interesting properties of $s$-numbers of operators in type $\mathrm{II}_1$ factors.

### *2.4. s-Numbers of operators in finite von Neumann algebras*

Throughout this paper, a finite von Neumann algebra $(\mathcal{M}, \tau)$ means a finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state $\tau$. An *embedding* of a finite von Neumann algebra $(\mathcal{M}, \tau)$ into another finite von Neumann algebra $(\mathcal{M}_1, \tau_1)$ means a $*$-isomorphism $\alpha$ from $\mathcal{M}$ to $\mathcal{M}_1$ such that $\tau = \tau_1 \circ \alpha$. Let $(\mathcal{L}(\mathcal{F}_2), \tau')$ be the free group factor with the faithful normal trace $\tau'$. Then the reduced free product von Neumann algebra $\mathcal{M}_1 = (\mathcal{M}, \tau) * (\mathcal{L}(\mathcal{F}_2), \tau')$ is a type $\mathrm{II}_1$ factor with a (unique) faithful normal trace $\tau_1$ such that the restriction of $\tau_1$ to $\mathcal{M}$ is $\tau$. So every finite von Neumann algebra can be embedded into a type $\mathrm{II}_1$ factor.

**Definition 2.11.** Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and $T \in \mathcal{M}$. If $\alpha$ is an embedding of $(\mathcal{M}, \tau)$ into a type $\mathrm{II}_1$ factor $(\mathcal{M}_1, \tau_1)$, then *the s-numbers of $T$* are defined as

$$\mu_s(T) = \mu_s\big(\alpha(T)\big).$$

Similar to the proof of Lemma 2.7, we can see that $\mu_s(T)$ is well defined, i.e., does not depend on the choice of $\alpha$ and $\mathcal{M}_1$.

Let $T \in (M_n(\mathbb{C}), \tau_n)$, where $\tau_n$ is the normalized trace on $M_n(\mathbb{C})$. Then $|T|$ is unitarily equivalent to a diagonal matrix with diagonal elements $s_1(T) \geqslant \cdots \geqslant s_n(T) \geqslant 0$. In the classical matrices theory [1,7], $s_1(T), \ldots, s_n(T)$ are also called $s$-numbers of $T$. It is easy to see that the relation between $\mu_s(T)$ and $s_1(T), \ldots, s_n(T)$ is the following

$$\mu_s(T) = s_1(T)\chi_{[0,1/n)}(s) + s_2(T)\chi_{[1/n,2/n)}(s) + \cdots + s_n(T)\chi_{[n-1/n,1]}(s). \tag{2.2}$$

Since no confusions will arise, we will use both $s$-numbers for matrices in $M_n(\mathbb{C})$. We refer to [1,7] for other interesting properties of $s$-numbers of matrices.

We end this section by the following definition.

**Definition 2.12.** Positive operators $S$ and $T$ in a finite von Neumann algebra $(\mathcal{M}, \tau)$ are *equi-measurable* if $\mu_s(S) = \mu_s(T)$ for $0 \leqslant s \leqslant 1$.

By 2 of Proposition 2.1 and Corollary 2.8, positive operators $S$ and $T$ in a finite von Neumann algebra $(\mathcal{M}, \tau)$ are equi-measurable if and only if $\tau(S^n) = \tau(T^n)$ for all $n = 0, 1, 2, \ldots$.

## 3. Tracial gauge semi-norms on finite von Neumann algebras satisfying the weak Dixmier property

### 3.1. Gauge semi-norms

**Definition 3.1.** Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. A semi-norm $|||\cdot|||$ on $\mathcal{M}$ is called *gauge invariant* if for every $T \in \mathcal{M}$,

$$|||T||| = \big|\big|\big||T|\big|\big|\big|.$$

**Lemma 3.2.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $|||\cdot|||$ be a semi-norm on $\mathcal{M}$. Then the following conditions are equivalent*:

1. $|||\cdot|||$ *is gauge invariant*;
2. $|||\cdot|||$ *is left unitarily invariant, i.e., for every unitary operator $U \in \mathcal{M}$ and operator $T \in \mathcal{M}$, $|||UT||| = |||T|||$*;
3. *for operators $A, T \in \mathcal{M}$, $|||AT||| \leqslant \|A\| \cdot |||T|||$.*

**Proof.** "3 ⇒ 2" and "2 ⇒ 1" are easy to see. We only prove "1 ⇒ 3." We need to prove that if $\|A\| < 1$, then $|||AT||| \leqslant |||T|||$. Since $\|A\| < 1$, there are unitary operators $U_1, \ldots, U_k$ such that $A = \frac{U_1 + \cdots + U_k}{k}$ (see [10,16]). Since $|U_1T| = \cdots = |U_kT| = |T|$, $|||AT||| = |||\frac{U_1T + \cdots + U_kT}{k}||| \leqslant \frac{|||U_1T||| + \cdots + |||U_kT|||}{k} \leqslant |||T|||$. □

**Corollary 3.3.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $|||\cdot|||$ be a gauge invariant semi-norm on $\mathcal{M}$ such that $|||TV||| = |||T|||$ for every unitary operator $V \in \mathcal{M}$ and operator $T \in \mathcal{M}$. If $0 \leqslant S \leqslant T$, then $|||S||| \leqslant |||T|||$.*

**Proof.** Since $0 \leqslant S \leqslant T$, there is an operator $A \in \mathcal{M}$ such that $S = ATA^*$ and $\|A\| \leqslant 1$. Similar to the proof of Lemma 3.2, $|||S||| = |||ATA^*||| \leqslant \|A\| \cdot |||T||| \cdot \|A^*\| \leqslant |||T|||$. □

**Definition 3.4.** A *normalized* semi-norm on a finite von Neumann algebra $(\mathcal{M}, \tau)$ is a semi-norm $|||\cdot|||$ such that $|||1||| = 1$.

By Lemma 3.2, we have the following corollary.

**Corollary 3.5.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $|||\cdot|||$ be a normalized gauge semi-norm on $\mathcal{M}$. Then for every $T \in \mathcal{M}$,*

$$|||T||| \leqslant \|T\|.$$

A *simple* operator in a finite von Neumann algebra $(\mathcal{M}, \tau)$ is an operator $T = a_1E_1 + \cdots + a_nE_n$, where $E_1, \ldots, E_n$ are projections in $\mathcal{M}$ such that $E_1 + \cdots + E_n = 1$.

**Corollary 3.6.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra, $|||\cdot|||_1$ and $|||\cdot|||_2$ be two gauge invariant semi-norms on $\mathcal{M}$. Then $|||\cdot|||_1 = |||\cdot|||_2$ on $\mathcal{M}$ if $|||T|||_1 = |||T|||_2$ for all positive simple operators $T \in \mathcal{M}$.*

**Proof.** Without loss of generality, assume $|||1|||_1 = |||1|||_2 = 1$. Let $T \in \mathcal{M}$ be a positive operator. By the spectral decomposition theorem, there is a sequence of positive simple operators $T_n \in \mathcal{M}$ such that $\lim_{n\to\infty} \|T - T_n\| = 0$. By Corollary 3.5, $\lim_{n\to\infty} |||T - T_n|||_1 = \lim_{n\to\infty} |||T - T_n|||_2 = 0$. By the assumption of the corollary, $|||T_n|||_1 = |||T_n|||_2$. Hence, $|||T|||_1 = |||T|||_2$. Since both $|||\cdot|||_1$ and $|||\cdot|||_2$ are gauge invariant, $|||\cdot|||_1 = |||\cdot|||_2$. $\square$

*3.2. Tracial gauge semi-norms*

**Definition 3.7.** Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra. A semi-norm $|||\cdot|||$ on $\mathcal{M}$ is called *tracial* if $|||S||| = |||T|||$ for every two equi-measurable positive operators $S, T$ in $\mathcal{M}$. A semi-norm $|||\cdot|||$ on $\mathcal{M}$ is called a *tracial gauge semi-norm* if it is both tracial and gauge invariant.

Since for a positive operator $T$ in a finite von Neumann algebra $(\mathcal{M},\tau)$, $\|T\| = \lim_{n\to\infty}(\tau(T^n))^{\frac{1}{n}}$, the operator norm $\|\cdot\|$ is a tracial gauge norm on $(\mathcal{M},\tau)$. Another less obvious example of a tracial gauge norm on $(\mathcal{M},\tau)$ is the non-commutative $L^1$-norm: $\|T\|_1 = \tau(|T|) = \int_0^1 \mu_s(T)\,ds$. The less obvious part is to show that $\|\cdot\|_1$ satisfies the triangle inequality. The following lemma overcomes this difficulty.

**Lemma 3.8.** $\|A\|_1 = \sup\{|\tau(UA)|\colon\ U \in \mathcal{U}(\mathcal{M})\}$, *where $\mathcal{U}(\mathcal{M})$ is the set of unitary operators in $\mathcal{M}$.*

**Proof.** By the polar decomposition theorem, there is a unitary operator $V \in \mathcal{M}$ such that $A = V|A|$. By the Schwarz inequality, $|\tau(UA)| = |\tau(UV|A|)| = |\tau(UV|A|^{1/2}|A|^{1/2})| \leqslant \tau(|A|)^{1/2} \cdot \tau(|A|)^{1/2} = \tau(|A|)$. Hence $\|A\|_1 \geqslant \sup\{|\tau(UA)|\colon\ U \in \mathcal{U}(\mathcal{M})\}$. Let $U = V^*$, we obtain $\|A\|_1 \leqslant \sup\{|\tau(UA)|\colon\ U \in \mathcal{U}(\mathcal{M})\}$. $\square$

**Corollary 3.9.** $\|A + B\|_1 \leqslant \|A\|_1 + \|B\|_1$.

**Lemma 3.10.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and let $|||\cdot|||$ be a gauge invariant semi-norm on $\mathcal{M}$. Then $|||\cdot|||$ is tracial if $|||S||| = |||T|||$ for every two equi-measurable positive simple operators $S, T$ in $\mathcal{M}$.*

**Proof.** We can assume that $|||1||| = 1$. Let $A, B$ be two equi-measurable positive operators in $\mathcal{M}$. By the spectral decomposition theorem, there are two sequences of positive simple operators $A_n$, $B_n$ in $\mathcal{M}$ such that $A_n$ and $B_n$ are equi-measurable and $\lim_{n\to\infty} \|A - A_n\| = \lim_{n\to\infty} \|B - B_n\| = 0$. By Corollary 3.5, $\lim_{n\to\infty} |||A - A_n||| = \lim_{n\to\infty} |||B - B_n||| = 0$. By the assumption of the lemma, $|||A_n||| = |||B_n|||$. Hence, $|||A||| = |||B|||$. $\square$

*3.3. Symmetric gauge semi-norms*

**Definition 3.11.** Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and let $\mathrm{Aut}(\mathcal{M},\tau)$ be the set of $*$-automorphisms on $\mathcal{M}$ preserving $\tau$. A semi-norm $|||\cdot|||$ on $\mathcal{M}$ is called *symmetric* if

$$\big|\big|\big|\theta(T)\big|\big|\big| = |||T|||, \quad \forall T \in \mathcal{M},\ \theta \in \mathrm{Aut}(\mathcal{M},\tau).$$

A semi-norm $|||\cdot|||$ on $\mathcal{M}$ is called a *symmetric gauge* semi-norm if it is both symmetric and gauge invariant.

**Example 3.12.** Let $\mathcal{M}=\mathbb{C}^n$ and $\tau(T)=\frac{x_1+\cdots+x_n}{n}$, where $T=(x_1,\ldots,x_n)\in\mathbb{C}^n$. Then $\mathrm{Aut}(\mathcal{M},\tau)$ is the set of permutations on $\{1,\ldots,n\}$. So a semi-norm $|||\cdot|||$ on $\mathcal{M}$ is a symmetric gauge semi-norm if and only if for every $(x_1,\ldots,x_n)\in\mathbb{C}^n$ and a permutation $\pi$ on $\{1,\ldots,n\}$,

$$\big|\big|\big|(x_1,\ldots,x_n)\big|\big|\big|=\big|\big|\big|\big(|x_1|,\ldots,|x_n|\big)\big|\big|\big|,$$

and

$$\big|\big|\big|(x_1,\ldots,x_n)\big|\big|\big|=\big|\big|\big|(x_{\pi(1)},\ldots,x_{\pi(n)})\big|\big|\big|.$$

**Lemma 3.13.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and let $|||\cdot|||$ be a semi-norm on $\mathcal{M}$. If $|||\cdot|||$ is tracial gauge invariant, then $|||\cdot|||$ is symmetric gauge invariant.*

**Proof.** Let $\theta\in\mathrm{Aut}(\mathcal{M},\tau)$ and $T\in\mathcal{M}$. We need to prove that $|||\theta(T)|||=|||T|||$. Since $|\theta(T)|=\theta(|T|)$ and $|||\cdot|||$ is gauge invariant, we can assume that $T$ is positive. Since $\theta\in\mathrm{Aut}(\mathcal{M},\tau)$, $T$ and $\theta(T)$ are equi-measurable. Hence, $|||T|||=|||\theta(T)|||$. □

**Example 3.14.** Let $\mathcal{M}=\mathbb{C}\oplus M_2(\mathbb{C})$ and $\tau(a\oplus B)=\frac{a}{2}+\frac{\tau_2(B)}{2}$, where $\tau_2$ is the normalized trace on $M_2(\mathbb{C})$. Define $|||a\oplus B|||=|a|/2+\tau_2(|B|)$. Then $|||\cdot|||$ is a symmetric gauge norm but not a tracial gauge norm. Note that $1\oplus 0$ and $0\oplus 1$ are equi-measurable, but $1/2=|||1\oplus 0|||\neq|||0\oplus 1|||=1$.

$\mathrm{Aut}(\mathcal{M},\tau)$ acts on $\mathcal{M}$ *ergodically* if $\theta(T)=T$ for all $\theta\in\mathrm{Aut}(\mathcal{M},\tau)$ implies $T=\lambda 1$.

**Lemma 3.15.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and let $|||\cdot|||$ be a semi-norm on $\mathcal{M}$. If* $\mathrm{Aut}(\mathcal{M},\tau)$ *acts on $\mathcal{M}$ ergodically, then the following are equivalent*:

1. $|||\cdot|||$ *is a tracial gauge semi-norm*;
2. $|||\cdot|||$ *is a symmetric gauge semi-norm.*

**Proof.** "1 ⇒ 2" by Lemma 3.13. We need to prove "2 ⇒ 1." By Corollary 3.6, we need to prove $|||S|||=|||T|||$ for two equi-measurable simple operators $S,T$ in $\mathcal{M}$. Similar to the proof of Corollary 2.3, there is a $\theta\in\mathrm{Aut}(\mathcal{M},\tau)$ such that $S=\theta(T)$. Hence $|||S|||=|||T|||$. □

**Corollary 3.16.** *A semi-norm on $(L^\infty[0,1],\int_0^1 dx)$ or $(\mathbb{C}^n,\tau)$ is a tracial gauge norm if and only if it is a symmetric gauge norm, where $\tau((x_1,\ldots,x_n))=\frac{x_1+\cdots+x_n}{n}$.*

### *3.4. Unitarily invariant semi-norms*

**Definition 3.17.** Let $(\mathcal{M},\tau)$ be a von Neumann algebra. A semi-norm $|||\cdot|||$ on $\mathcal{M}$ is *unitarily invariant* if $|||UTV|||=|||T|||$ for all $T\in\mathcal{M}$ and unitary operators $U,V\in\mathcal{M}$.

**Proposition 3.18.** *Let $|||\cdot|||$ be a semi-norm on $\mathcal{M}$. Then the following statements are equivalent*:

1. $|||\cdot|||$ *is unitarily invariant*;
2. $|||\cdot|||$ *is gauge invariant and unitarily conjugate invariant, i.e.,* $|||UTU^*||| = |||T|||$ *for all* $T \in \mathcal{M}$ *and unitary operators* $U \in \mathcal{M}$;
3. $|||\cdot|||$ *is left-unitarily invariant and* $|||T||| = |||T^*|||$ *for every* $T \in \mathcal{M}$;
4. *for all operators* $T, A, B \in \mathcal{M}$, $|||ATB||| \leqslant \|A\| \cdot |||T||| \cdot \|B\|$.

**Proof.** "$1 \Rightarrow 4$" is similar to the proof of Lemma 3.2. "$4 \Rightarrow 3$," "$3 \Rightarrow 2$," and "$2 \Rightarrow 1$" are routine. □

For a unitary operator $U \in \mathcal{M}$, let $\theta(T) = UTU^*$. Then $\theta \in \operatorname{Aut}(\mathcal{M}, \tau)$. By Proposition 3.18, we have the following.

**Corollary 3.19.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $|||\cdot|||$ be a symmetric, gauge invariant semi-norm on $\mathcal{M}$. Then $|||\cdot|||$ is a unitarily invariant semi-norm on $\mathcal{M}$.*

**Example 3.20.** Let $\mathcal{M} = \mathbb{C}^n$, $n \geqslant 2$ and $\tau((x_1, \ldots, x_n)) = \frac{x_1+\cdots+x_n}{n}$. Define $|||(x_1, \ldots, x_n)||| = |x_1|$. Then $|||\cdot|||$ is a unitarily invariant semi-norm but not a symmetric gauge semi-norm on $\mathcal{M}$.

**Lemma 3.21.** *Let $(\mathcal{M}, \tau)$ be a finite factor and $|||\cdot|||$ be a semi-norm on $\mathcal{M}$. Then the following conditions are equivalent*:

1. $|||\cdot|||$ *is a tracial gauge semi-norm*;
2. $|||\cdot|||$ *is a symmetric gauge semi-norm*;
3. $|||\cdot|||$ *is a unitarily invariant semi-norm.*

**Proof.** "$1 \Rightarrow 2$" by Lemma 3.13 and "$2 \Rightarrow 3$" by Corollary 3.19. We need to prove "$3 \Rightarrow 1$." By Corollary 3.6, we need to prove $|||S||| = |||T|||$ for two equi-measurable positive simple operators $S, T \in \mathcal{M}$. Suppose $S = a_1E_1 + \cdots + a_nE_n$ and $T = a_1F_1 + \cdots + a_nF_n$, where $E_1 + \cdots + E_n = 1$ and $F_1 + \cdots + F_n = 1$ and $\tau(E_k) = \tau(F_k)$ for $1 \leqslant k \leqslant n$. Since $\mathcal{M}$ is a factor, there is a unitary operator $U \in \mathcal{M}$ such that $E_k = UF_kU^*$ for $1 \leqslant k \leqslant n$. Therefore, $S = UTU^*$ and $|||S||| = |||T|||$. □

*3.5. Weak Dixmier property*

**Definition 3.22.** A finite von Neumann algebra $(\mathcal{M}, \tau)$ satisfies *the weak Dixmier property* if for every positive operator $T \in \mathcal{M}$, $\tau(T)$ is in the operator norm closure of the convex hull of $\{S \in \mathcal{M}\colon\ S \text{ and } T \text{ are equi-measurable}\}$.

A finite factor $(\mathcal{M}, \tau)$ satisfies the *Dixmier property* (see [2,11]): for every operator $T \in \mathcal{M}$, $\tau(T)$ is in the operator norm closure of the convex hull of $\{UTU^*\colon\ U \in \mathcal{U}(\mathcal{M})\}$. Hence finite factors satisfy the weak Dixmier property. In the following, we will characterize finite von Neumann algebras satisfying the weak Dixmier property.

There is a central projection $P$ in a finite von Neumann algebra $(\mathcal{M}, \tau)$ such that $P\mathcal{M}$ is of type I and $(1 - P)\mathcal{M}$ is of type II. A type II von Neumann algebra is diffuse, i.e, there are no non-zero minimal projections in the von Neumann algebra. Furthermore, there are central projections $P_1, \dots, P_n, \dots$ in $\mathcal{M}$, such that $P_1 + \dots + P_n + \dots = P$ and $P_n\mathcal{M} = \mathcal{A}_n \otimes M_n(\mathbb{C})$, $\mathcal{A}_n$ is abelian. We can decompose $\mathcal{A}_n$ into an atomic part $\mathcal{A}_n^{\mathrm{a}}$ and a diffuse part $\mathcal{A}_n^{\mathrm{c}}$, i.e., there is a projection $Q_n$ in $\mathcal{A}_n$, $\mathcal{A}_n^{\mathrm{a}} = Q_n\mathcal{A}_n$, such that $Q_n = E_{n1} + E_{n2} + \cdots$, where $E_{nk}$ is a minimal projection in $\mathcal{A}_n^{\mathrm{a}}$ and $\tau(E_{nk}) > 0$, and $\mathcal{A}_n^{\mathrm{c}} = (1 - Q_n)\mathcal{A}_n$ is diffuse. Let $\mathcal{M}_{\mathrm{a}} = \sum_{\oplus} \mathcal{A}_n^{\mathrm{a}} \otimes M_n(\mathbb{C})$ and $\mathcal{M}_{\mathrm{c}} = \sum_{\oplus} \mathcal{A}_n^{\mathrm{c}} \otimes M_n(\mathbb{C}) \oplus (1 - P)\mathcal{M}$. Then $\mathcal{M} = \mathcal{M}_{\mathrm{a}} \oplus \mathcal{M}_{\mathrm{c}}$. We call $\mathcal{M}_{\mathrm{a}}$ the *atomic part* of $\mathcal{M}$ and $\mathcal{M}_{\mathrm{c}}$ the *diffuse part* of $\mathcal{M}$. A finite von Neumann algebra $(\mathcal{M}, \tau)$ is *atomic* if $\mathcal{M} = \mathcal{M}_{\mathrm{a}}$ and is *diffuse* if $\mathcal{M} = \mathcal{M}_{\mathrm{c}}$.

**Lemma 3.23.** *Let $(\mathcal{M}, \tau)$ be a finite-dimensional von Neumann algebra such that for every two non-zero minimal projections $E, F \in \mathcal{M}$, $\tau(E) = \tau(F)$. Then $(\mathcal{M}, \tau)$ satisfies the weak Dixmier property.*

**Proof.** Since $\mathcal{M}$ is finite-dimensional, $\mathcal{M} \cong M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_r}(\mathbb{C})$. Since $\tau(E) = \tau(F)$ for every two non-zero minimal projections $E, F \in \mathcal{M}$, $(\mathcal{M}, \tau)$ can be embedded into $(M_n(\mathbb{C}), \tau_n)$, where $n = k_1 + \cdots + k_r$. So we can assume that $(\mathcal{M}, \tau)$ is a von Neumann subalgebra of $(M_n(\mathbb{C}), \tau_n)$ such that $\mathcal{M}$ contains all diagonal matrices $a_1E_1 + \cdots + a_nE_n$. Now for every positive operator $T \in \mathcal{M}$, there is a unitary operator $U \in \mathcal{M}$ such that $UTU^* = a_1E_1 + \cdots + a_nE_n$, $a_1, \dots, a_n \geqslant 0$ and $\tau(T) = \frac{a_1 + \cdots + a_n}{n}$. Then $\tau(T) = \frac{\sum_\pi (a_{\pi(1)}E_1 + \cdots + a_{\pi(n)}E_n)}{n!}$. $\square$

**Lemma 3.24.** *Let $(\mathcal{M}, \tau)$ be a diffuse finite von Neumann algebra. Then $(\mathcal{M}, \tau)$ satisfies the weak Dixmier property.*

**Proof.** Let $\mathcal{A}$ be a separable diffuse abelian von Neumann subalgebra of $\mathcal{M}$. By Lemma 2.6, there is a $*$-isomorphism $\alpha$ from $(\mathcal{A}, \tau)$ onto $(L^\infty[0,1], \int_0^1 dx)$ such that $\int_0^1 dx \cdot \alpha = \tau$. For a positive operator $T \in \mathcal{M}$, there is an operator $S \in \mathcal{A}$ such that $\alpha(S) = \mu_s(T)$. Hence $\tau(T) = \tau(S) = \int_0^1 \mu_s(T)\,ds$. By Theorem 2.4, for any $\epsilon > 0$, there are $S_1, \dots, S_n$ in $\mathcal{A}$ such that $S, S_1, \dots, S_n$ are equi-measurable and $\|\tau(S) - \frac{S_1 + \cdots + S_n}{n}\| < \epsilon$. Hence $(\mathcal{M}, \tau)$ satisfies the weak Dixmier property. $\square$

**Lemma 3.25.** *Let $(\mathcal{M}, \tau)$ be an atomic finite von Neumann algebra with two minimal projections $E$ and $F$ in $\mathcal{M}$ such that $\tau(E) \neq \tau(F)$. Then $(\mathcal{M}, \tau)$ does not satisfy the weak Dixmier property.*

**Proof.** Since $(\mathcal{M}, \tau)$ is an atomic finite von Neumann algebra, $\mathcal{M} \cong M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C}) \oplus \cdots$. Let $E_{ij}$ be minimal projections in $M_{k_i}$ such that $\sum E_{ij} = 1$. Without loss of generality, assume that $\tau(E_{11}) > \tau(E_{21}) \geqslant \tau(E_{31}) \geqslant \cdots$. Let

$$T = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{k_1} \oplus A,$$

where

$$A = \begin{pmatrix} \frac{1}{2} & & \\ & \ddots & \\ & & (\frac{1}{2})^{k_2} \end{pmatrix} \oplus \begin{pmatrix} (\frac{1}{2})^{k_2+1} & & \\ & \ddots & \\ & & (\frac{1}{2})^{k_2+k_3} \end{pmatrix} \oplus \cdots.$$

If $T_1 \in \mathcal{M}$ and $T$ are equi-measurable, then

$$T_1 = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{k_1} \oplus A_1,$$

where $A$ and $A_1$ are equi-measurable. Hence, if $\tau(T)$ is in the operator norm closure of the convex hull of $\{S \in \mathcal{M}\colon\ S \text{ and } T \text{ are equi-measurable}\}$, then $\tau(T) = 1$. This is a contradiction. □

Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and let $E \in \mathcal{M}$ be a non-zero projection. The *induced finite von Neumann algebra* $(\mathcal{M}_E, \tau_E)$ is the von Neumann algebra $\mathcal{M}_E = E\mathcal{M}E$ with a faithful normal trace $\tau_E(ETE) = \frac{\tau(ETE)}{\tau(E)}$. The proof of the following lemma is similar to the proof of Lemma 3.25.

**Lemma 3.26.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra such that $\mathcal{M}_{\mathrm{a}} \neq 0$ and $\mathcal{M}_{\mathrm{c}} \neq 0$. Then $\mathcal{M}$ does not satisfy the weak Dixmier property.*

**Proof.** Let $P$ be the central projection such that $\mathcal{M}_{\mathrm{a}} = P\mathcal{M}$ and $\mathcal{M}_{\mathrm{c}} = (1 - P)\mathcal{M}$. Let $\mathcal{A}$ be a separable diffuse abelian von Neumann subalgebra of $(\mathcal{M}_{\mathrm{c}}, \tau_{1-P})$. By Lemma 2.6, there is a positive operator $A$ in $\mathcal{M}_{\mathrm{c}}$ such that $\mu_s(A) = \frac{1-s}{2}$ with respect to $(\mathcal{M}_{\mathrm{c}}, \tau_{1-P})$. Consider $T = P + A(1 - P)$. Then

$$\mu_s(T) = \begin{cases} 1, & 0 \leqslant s < \tau(P); \\ \frac{1-s}{2\tau(1-P)} \leqslant \frac{1}{2}, & \tau(P) \leqslant s \leqslant 1 \end{cases}$$

with respect to $(\mathcal{M}, \tau)$. If $T_1 \in \mathcal{M}$ and $T$ are equi-measurable, then $T_1 = P + A_1$ such that $A_1$ and $A$ are equi-measurable. Hence, if $\tau(T)$ is in the operator norm closure of the convex hull of $\{S \in \mathcal{M}\colon\ S \text{ and } T \text{ are equi-measurable}\}$, then $\tau(T) = 1$. This is a contradiction. □

Summarizing Lemmas 3.23–3.26, we can characterize finite von Neumann algebras satisfying the weak Dixmier property as the following theorem.

**Theorem 3.27.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra. Then $\mathcal{M}$ satisfies the weak Dixmier property if and only if $\mathcal{M}$ satisfies one of the following conditions*:

1. *$\mathcal{M}$ is finite-dimensional (hence atomic) and for every two non-zero minimal projections $E, F \in \mathcal{M}$, $\tau(E) = \tau(F)$, or equivalently, $(\mathcal{M}, \tau)$ can be identified as a von Neumann subalgebra of $(M_n(\mathbb{C}), \tau_n)$ that contains all diagonal matrices*;
2. *$\mathcal{M}$ is diffuse.*

**Corollary 3.28.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and $E \in \mathcal{M}$ be a non-zero projection. Then $(\mathcal{M}_E, \tau_E)$ also satisfies the weak Dixmier property.*

The following example shows that we cannot replace the weak Dixmier property by the following condition: $\tau(T)$ is in the operator norm closure of the convex hull of $\{\theta(T)\colon \theta \in \mathrm{Aut}(\mathcal{M}, \tau)\}$.

**Example 3.29.** $(\mathbb{C} \oplus M_2(\mathbb{C}), \tau)$, $\tau(a \oplus B) = \frac{1}{3}a + \frac{2}{3}\tau_2(B)$, satisfies the weak Dixmier property. On the other hand, let $T = 1 \oplus 2 \in \mathbb{C} \oplus M_2(\mathbb{C})$. Then for every $\theta \in \mathrm{Aut}(\mathcal{M}, \tau)$, $\theta(T) = T$. Hence, $\tau(T)$ is not in the operator norm closure of the convex hull of $\{\theta(T)\colon \theta \in \mathrm{Aut}(\mathcal{M}, \tau)\}$.

### 3.6. A comparison theorem

The following theorem is the main result of this section.

**Theorem 3.30.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property. If $|||\cdot|||$ is a normalized tracial gauge semi-norm on $\mathcal{M}$, then for all $T \in \mathcal{M}$,*

$$\|T\|_1 \leqslant |||T||| \leqslant \|T\|.$$

*In particular, every tracial gauge semi-norm on $\mathcal{M}$ is a norm.*

**Proof.** By Corollary 3.5, $|||T||| \leqslant \|T\|$ for every $T \in \mathcal{M}$. To prove $\|T\|_1 \leqslant |||T|||$, we can assume $T \geqslant 0$. Let $\epsilon > 0$. Since $(\mathcal{M}, \tau)$ satisfies the weak Dixmier property, there are $S_1, \ldots, S_k$ in $\mathcal{M}$ such that $T, S_1, \ldots, S_k$ are equi-measurable and $\|\tau(T) - \frac{S_1+\cdots+S_k}{k}\| < \epsilon$. By Corollary 3.5, $|||\tau(T) - \frac{S_1+\cdots+S_k}{k}||| \leqslant \|\tau(T) - \frac{S_1+\cdots+S_k}{k}\| < \epsilon$. Hence $\|T\|_1 = |\tau(T)| \leqslant |||\frac{S_1+\cdots+S_k}{k}||| + \epsilon \leqslant \frac{|||S_1|||+\cdots+|||S_k|||}{k} + \epsilon = |||T||| + \epsilon$. $\square$

By Theorem 3.30 and Lemma 3.21, we have the following corollary.

**Corollary 3.31.** *Let $(\mathcal{M}, \tau)$ be a finite factor and let $|||\cdot|||$ be a normalized unitarily invariant norm on $\mathcal{M}$. Then*

$$\|T\|_1 \leqslant |||T||| \leqslant \|T\|, \quad \forall T \in \mathcal{M}.$$

*In particular, every unitarily invariant semi-norm on a finite factor is a norm.*

By Theorem 3.30 and Lemma 3.15, we have the following corollary.

**Corollary 3.32.** *Let $|||\cdot|||$ be a normalized symmetric gauge semi-norm on $(L^\infty[0,1], \int_0^1 dx)$ (or $(\mathbb{C}^n, \tau)$, where $\tau((x_1, \ldots, x_n)) = \frac{x_1+\cdots+x_n}{n}$). Then*

$$\|T\|_1 \leqslant |||T||| \leqslant \|T\|, \quad \forall T \in L^\infty[0,1] \ \big(or\ \mathbb{C}^n\big).$$

*In particular, every symmetric gauge semi-norm on $(L^\infty[0,1], \int_0^1 dx)$ (or $(\mathbb{C}^n, \tau)$, respectively) is a norm.*

## 4. Proof of Theorem B

To prove Theorem B, we need the following lemmas.

**Lemma 4.1.** *Let $E_1, \ldots, E_n$ be projections in $\mathcal{M}$ such that $E_1 + \cdots + E_n = 1$ and $T \in \mathcal{M}$. Then $S = E_1TE_1 + \cdots + E_nTE_n$ is in the convex hull of $\{UTU^*\colon U \in \mathcal{U}(\mathcal{M})\}$.*

**Proof.** Let $T=(T_{ij})$ be the matrix with respect to the decomposition $1=E_1+\cdots+E_n$. Let $U=-E_1+E_2+\cdots+E_n$. Then simple computation shows that

$$\frac{1}{2}\big(UTU^*+T\big)=\begin{pmatrix} T_{11} & 0 & \cdots & 0 \\ 0 & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & T_{n2} & \cdots & T_{nn} \end{pmatrix}=E_1TE_1+(1-E_1)T(1-E_1).$$

By induction, $S=E_1TE_1+\cdots+E_nTE_n$ is in the convex hull of $\{UTU^*\colon U\in\mathcal{U}(\mathcal{M})\}$. $\square$

**Corollary 4.2.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and let $|||\cdot|||$ be a unitarily invariant norm on $\mathcal{M}$. Let $E_1,\ldots,E_n$ be projections in $\mathcal{M}$ such that $E_1+\cdots+E_n=1$ and $T\in\mathcal{M}$ and $S=E_1TE_1+\cdots+E_nTE_n$. Then $|||S|||\leqslant|||T|||$.*

Recall that for a (non-zero) finite projection $E$ in $\mathcal{M}$, $\tau_E(ETE)=\frac{\tau(ETE)}{\tau(E)}$ is the induced trace on $\mathcal{M}_E=E\mathcal{M}E$.

**Lemma 4.3.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. Suppose $T,E_1,\ldots,E_n\in\mathcal{M}$, $T\geqslant 0$, $E_1+\cdots+E_n=1$. Then $|||T|||\geqslant|||\tau_{E_1}(E_1TE_1)E_1+\cdots+\tau_{E_n}(E_nTE_n)E_n|||$.*

**Proof.** We may assume that $|||1|||=1$. Since $\mathcal{M}$ satisfies the weak Dixmier property, by Corollary 3.28, $(\mathcal{M}_{E_i},\tau_{E_i})$ also satisfies the weak Dixmier property, $1\leqslant i\leqslant n$. Let $\epsilon>0$. There are operators $S_{i1},\ldots,S_{ik}$ in $\mathcal{M}_{E_i}$ such that $E_iTE_i,S_{i1},\ldots,S_{ik}$ are equi-measurable and

$$\left\|\frac{S_{i1}+\cdots+S_{ik}}{k}-\tau_{E_i}(E_iTE_i)E_i\right\|<\epsilon.$$

Let $S_j=S_{1j}E_1+\cdots+S_{nj}E_n$, $1\leqslant j\leqslant k$. Then $T,S_1,\ldots,S_n$ are equi-measurable and

$$\left\|\frac{S_1+\cdots+S_k}{k}-\big(\tau_{E_1}(E_1TE_1)E_1+\cdots+\tau_{E_n}(E_nTE_n)E_n\big)\right\|<\epsilon.$$

By Corollary 3.5,

$$\left|\left|\left|\frac{S_1+\cdots+S_k}{k}-\big(\tau_{E_1}(E_1TE_1)E_1+\cdots+\tau_{E_n}(E_nTE_n)E_n\big)\right|\right|\right|<\epsilon.$$

Hence, $|||\tau_{E_1}(E_1TE_1)E_1+\cdots+\tau_{E_n}(E_nTE_n)E_n|||\leqslant|||T|||+\epsilon$. Since $\epsilon>0$ is arbitrary, we obtain the lemma. $\square$

**Corollary 4.4.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. If $\mathcal{A}$ is a finite-dimensional abelian von Neumann subalgebra of $\mathcal{M}$ and $\mathbf{E}_{\mathcal{A}}$ is the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{A}$ preserving $\tau$, then for every $T\in\mathcal{M}$, $|||\mathbf{E}_{\mathcal{A}}(T)|||\leqslant|||T|||$.*

**Proof.** Let $\mathcal{A} = \{E_1, \dots, E_n\}''$ such that $E_1 + \cdots + E_n = 1$. Then for every $T \in \mathcal{M}$,

$$\mathbf{E}_{\mathcal{A}}(T) = \tau_{E_1}(E_1 T E_1)E_1 + \cdots + \tau_{E_n}(E_n T E_n)E_n.$$

By Corollary 4.2 and Lemma 4.3, $|||\mathbf{E}_{\mathcal{A}}(T)||| \leqslant |||T|||$. $\square$

**Proof of Theorem B.** By Lemma 3.13 and Corollary 3.19, $|||\cdot|||$ is unitarily invariant. Suppose $T_\alpha$ is a net in $\mathcal{M}_{1,|||\cdot|||}$ such that $\lim_\alpha T_\alpha = T$ in the weak operator topology. Let $T = U|T|$ be the polar decomposition of $T$. Then $\lim_\alpha U^*T_\alpha = |T|$ in the weak operator topology. Since $|||\cdot|||$ is unitarily invariant, $|||UT_\alpha||| \leqslant 1$ and $|||\,|T|\,||| = |||T|||$. So we may assume that $T \geqslant 0$ and $T_\alpha = T_\alpha^*$. By the spectral decomposition theorem and Corollary 3.5, to prove $|||T||| \leqslant 1$, we need to prove $|||S||| \leqslant 1$ for every positive simple operator $S$ such that $S \leqslant T$. Let $S = a_1E_1 + \cdots + a_nE_n$ and $\epsilon > 0$. Since $\lim_\alpha T_\alpha = T \geqslant S$, $\lim_\alpha E_iT_\alpha E_i = E_iTE_i \geqslant a_iE_i$ for $1 \leqslant i \leqslant n$. Hence, $\lim_\alpha \tau_{E_i}(E_i(T_\alpha + \epsilon)E_i) \geqslant a_i + \epsilon > a_i$. So there is a $\beta$ such that $\tau_{E_1}(E_1(T_\beta + \epsilon)E_1)E_1 + \cdots + \tau_{E_n}(E_n(T_\beta + \epsilon)E_n)E_n \geqslant S$. By Lemma 4.3 and Corollary 3.3, $1 + \epsilon \geqslant |||T_\beta + \epsilon||| \geqslant |||\tau(E_1(T_\beta + \epsilon)E_1)E_1 + \cdots + \tau(E_n(T_\beta + \epsilon)E_n)E_n||| \geqslant |||S|||$. Since $\epsilon > 0$ is arbitrary, $|||S||| \leqslant 1$. $\square$

**Proof of Corollary 1.** Since $\mathcal{A}$ is a separable abelian von Neumann algebra, there is a sequence of finite-dimensional abelian von Neumann subalgebras $\mathcal{A}_n$ such that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}$ and $\mathcal{A}$ is the closure of $\bigcup_n \mathcal{A}_n$ in the strong operator topology. Let $\mathbf{E}_{\mathcal{A}_n}$ be the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{A}_n$ preserving $\tau$. Then for every $T \in \mathcal{M}$, $\mathbf{E}_{\mathcal{A}}(T) = \lim_{n\to\infty} \mathbf{E}_{\mathcal{A}_n}(T)$ in the strong operator topology. By Theorem B and Corollary 4.4, $|||\mathbf{E}_{\mathcal{A}}(T)||| \leqslant |||T|||$. $\square$

In the following we give some other useful corollaries of Theorem B.

**Corollary 4.5.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. Suppose $0 \leqslant T_1 \leqslant T_2 \leqslant \cdots \leqslant T$ in $\mathcal{M}$ such that $\lim_{n\to\infty} T_n = T$ in the weak operator topology. Then $\lim_{n\to\infty} |||T_n||| = |||T|||$.*

**Proof.** By Corollary 3.3, $|||T_1||| \leqslant |||T_2||| \leqslant \cdots \leqslant |||T|||$. Hence, $\lim_{n\to\infty} |||T_n||| \leqslant |||T|||$. By Theorem B, $\lim_{n\to\infty} |||T_n||| \geqslant |||T|||$. $\square$

**Corollary 4.6.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and $|||\cdot|||_1$ and $|||\cdot|||_2$ be two tracial gauge norms on $\mathcal{M}$. Then $|||\cdot|||_1 = |||\cdot|||_2$ on $\mathcal{M}$ if $|||T|||_1 = |||T|||_2$ for every operator $T = a_1E_1 + \cdots + a_nE_n$ in $\mathcal{M}$ such that $a_1, \dots, a_n \geqslant 0$ and $\tau(E_1) = \cdots = \tau(E_n) = \frac{1}{n}$, $n = 1, 2, \dots$.*

**Proof.** We need only to prove $|||T|||_1 = |||T|||_2$ for every positive operator $T$ in $\mathcal{M}$. By Theorem 3.27, $\mathcal{M}$ is either a finite-dimensional von Neumann algebra such that $\tau(E) = \tau(F)$ for arbitrary two non-zero minimal projections in $\mathcal{M}$ or $\mathcal{M}$ is diffuse. If $\mathcal{M}$ is a finite-dimensional von Neumann algebra such that $\tau(E) = \tau(F)$ for arbitrary two non-zero minimal projections in $\mathcal{M}$, then the corollary is obvious. If $\mathcal{M}$ is diffuse, by the spectral decomposition theorem, there is a sequence of operators $T_n \in \mathcal{M}$ satisfying the following conditions:

1. $0 \leqslant T_1 \leqslant T_2 \leqslant \cdots \leqslant T$,

2. $T_n = a_{n1}E_{n1} + \cdots + a_{nn}E_{nn}$, $a_{n1}, \ldots, a_{nn} \geqslant 0$ and $\tau(E_{n1}) = \cdots = \tau(E_{nn}) = \frac{1}{n}$,
3. $\lim_{n\to\infty} T_n = T$ in the weak operator topology.

By the assumption of the corollary, $|||T_n|||_1 = |||T_n|||_2$. By Corollary 4.5, $|||T|||_1 = |||T|||_2$. $\square$

**Corollary 4.7.** *Let $\mathcal{M}$ be a type* $\mathrm{II}_1$ *factor and $|||\cdot|||_1$ and $|||\cdot|||_2$ be two unitarily invariant norms on $\mathcal{M}$. Then $|||\cdot|||_1 = |||\cdot|||_2$ on $\mathcal{M}$ if $|||\cdot|||_1 = |||\cdot|||_2$ on all type* $\mathrm{I}_n$ *subfactors of $\mathcal{M}$, $n = 1, 2, \ldots$.*

## 5. Ky Fan norms on finite von Neumann algebras

Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and $0 \leqslant t \leqslant 1$. For $T \in \mathcal{M}$, define the *Ky Fan tth norm* by

$$|||T|||_{(t)} \begin{cases} \|T\|, & t = 0; \\ \frac{1}{t}\int_0^t \mu_s(T)\,ds, & 0 < t \leqslant 1. \end{cases}$$

Let $\mathcal{M}_1 = (\mathcal{M}, \tau) * (\mathcal{L}_{\mathcal{F}_2}, \tau')$ be the reduced free product von Neumann algebra of $\mathcal{M}$ and the free group factor $\mathcal{L}_{\mathcal{F}_2}$. Then $\mathcal{M}_1$ is a type $\mathrm{II}_1$ factor with a faithful normal trace $\tau_1$ such that the restriction of $\tau_1$ to $\mathcal{M}$ is $\tau$. Recall that $\mathcal{U}(\mathcal{M}_1)$ is the set of unitary operators in $\mathcal{M}_1$ and $\mathcal{P}(\mathcal{M}_1)$ is the set of projections in $\mathcal{M}_1$.

**Lemma 5.1.** *For $0 < t \leqslant 1$, $t|||T|||_{(t)} = \sup\{|\tau_1(UTE)|\colon U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\}$.*

**Proof.** We may assume that $T$ is a positive operator. Let $\mathcal{A}$ be a separable diffuse abelian von Neumann subalgebra of $\mathcal{M}_1$ containing $T$ and let $\alpha$ be a $*$-isomorphism from $(\mathcal{A}, \tau_1)$ onto $(L^\infty[0,1], \int_0^1 dx)$ such that $\tau_1 = \int_0^1 dx \cdot \alpha$. Let $f(x) = \alpha(T)$ and let $f^*(x)$ be the non-increasing rearrangement of $f(x)$. Then $\mu_s(T) = f^*(s)$. By the definition of $f^*$(see (2.1)),

$$m\big(\{f^* > f^*(t)\}\big) = \lim_{n\to\infty} m\left(\left\{f^* > f^*(t) + \frac{1}{n}\right\}\right) \leqslant t$$

and

$$m\big(\{f^* \geqslant f^*(t)\}\big) \geqslant \lim_{n\to\infty} m\left(\left\{f^* > f^*(t) - \frac{1}{n}\right\}\right) \geqslant t.$$

Since $f^*$ and $f$ are equi-measurable, $m(\{f > f^*(t)\}) \leqslant t$ and $m(\{f \geqslant f^*(t)\}) \geqslant t$. Therefore, there is a measurable subset $A$ of $[0,1]$, $\{f > f^*(t)\} \subset A \subset \{f \geqslant f^*(t)\}$, such that $m(A) = t$. Since $f(x)$ and $f^*(x)$ are equimeasurable, $\int_A f(s)\,ds = \int_0^t f^*(s)\,ds$. Let $E' = \alpha^{-1}(\chi_A)$. Then $\tau_1(E') = t$ and $\tau_1(TE') = \int_A f(s)\,ds = \int_0^t f^*(s)\,ds = t|||T|||_{(t)}$. Hence, $t|||T|||_{(t)} \leqslant \sup\{|\tau_1(UTE)|\colon U \in \mathcal{U}(\mathcal{M}),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\}$.

We need to prove that if $E$ is a projection in $\mathcal{M}_1$, $\tau_1(E) = t$, and $U \in \mathcal{U}(\mathcal{M}_1)$, then $t|||T|||_{(t)} \geqslant |\tau_1(UTE)|$. By the Schwarz inequality, $|\tau_1(UTE)| = \tau_1(EUT^{1/2}T^{1/2}E) \leqslant \tau_1(U^*EUT)^{1/2} \times \tau_1(ET)^{1/2}$. By Corollary 2.8, $\tau_1(ET) = \int_0^1 \mu_s(ET)\,ds$. By Corollary 2.10, $\mu_s(ET) \leqslant \min\{\mu_s(T), \mu_s(E)\|T\|\}$. Note that $\mu_s(E) = 0$ for $s \geqslant \tau_1(E) = t$. Hence, $\tau_1(ET) \leqslant \int_0^t \mu_s(T)\,ds = t|||T|||_t$. Similarly, $\tau_1(U^*EUT) \leqslant t|||T|||_t$. So $|\tau_1(UTE)| \leqslant t|||T|||_t$. This proves that $t|||T|||_{(t)} \geqslant \sup\{|\tau_1(UTE)|\colon U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\}$. $\square$

**Theorem 5.2.** *For $0 \leqslant t \leqslant 1$, $|||\cdot|||_{(t)}$ is a normalized tracial gauge norm on $(\mathcal{M}, \tau)$.*

**Proof.** We only prove the triangle inequality, since the other parts are obvious. We may assume that $0 < t \leqslant 1$. Let $S, T \in \mathcal{M}$. By Lemma 5.1, $t|||S+T|||_{(t)} = \sup\{|\tau_1(U(S+T)E)|:\ U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\} \leqslant \sup\{|\tau_1(USE)|:\ U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\} + \sup\{|\tau_1(UTE)|:\ U \in \mathcal{U}(\mathcal{M}_1),\ E \in \mathcal{P}(\mathcal{M}_1),\ \tau_1(E) = t\} = t|||S|||_{(t)} + t|||T|||_{(t)}$. $\square$

**Proposition 5.3.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and $T \in (\mathcal{M}, \tau)$. Then $|||T|||_{(t)}$ is a non-increasing continuous function on* $[0, 1]$.

**Proof.** Let $0 < t_1 < t_2 \leqslant 1$.

$$|||T|||_{(t_1)} - |||T|||_{(t_2)} = \frac{1}{t_1}\int_0^{t_1} \mu_s(T)\,ds - \frac{1}{t_2}\int_0^{t_2} \mu_s(T)\,ds$$

$$= \frac{\frac{1}{t_1}\int_0^{t_1} \mu_s(T)\,ds - \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \mu_s(T)\,ds}{t_2(t_2 - t_1)} \geqslant 0.$$

Since $\mu_s(T)$ is right-continuous, $|||T|||_{(t)}$ is a non-increasing continuous function on $[0, 1]$. $\square$

**Example 5.4.** The Ky Fan $\frac{k}{n}$th norm of a matrix $T \in (\mathcal{M}_n(\mathbb{C}), \tau_n)$ is

$$|||T|||_{\left(\frac{k}{n}\right)} = \frac{s_1(T) + \cdots + s_k(T)}{k}, \quad 1 \leqslant k \leqslant n.$$

## 6. Dual norms of tracial gauge norms on finite von Neumann algebras satisfying the weak Dixmier property

### 6.1. Dual norms

Let $|||\cdot|||$ be a norm on a finite von Neumann algebra $(\mathcal{M}, \tau)$. For $T \in \mathcal{M}$, define

$$|||T|||^{\#}_{\mathcal{M}} = \sup\bigl\{\bigl|\tau(TX)\bigr|:\ X \in \mathcal{M},\ |||X||| \leqslant 1\bigr\}.$$

When no confusion arises, we simply write $|||\cdot|||^{\#}$ instead of $|||\cdot|||^{\#}_{\mathcal{M}}$.

**Lemma 6.1.** *$|||\cdot|||^{\#}$ is a norm on $\mathcal{M}$.*

**Proof.** If $T \neq 0$, $|||T|||^{\#} \geqslant \tau(TT^*)/|||T^*||| > 0$. It is easy to see that $|||\lambda T|||^{\#} = |\lambda| \cdot |||T|||^{\#}$ and $|||T_1 + T_2|||^{\#} \leqslant |||T_1|||^{\#} + |||T_2|||^{\#}$. $\square$

**Definition 6.2.** $|||\cdot|||^{\#}$ is called the *dual norm* of $|||\cdot|||$ on $\mathcal{M}$ with respect to $\tau$.

The next lemma follows directly from the definition of dual norm.

**Lemma 6.3.** *Let $|||\cdot|||$ be a norm on a finite von Neumann algebra $(\mathcal{M},\tau)$ and let $|||\cdot|||^{\#}$ be the dual norm on $\mathcal{M}$. Then for $S,T\in\mathcal{M}$, $|\tau(ST)|\leqslant |||S|||\cdot|||T|||^{\#}$.*

The following corollary is a generalization of Hölder's inequality for bounded operators in finite von Neumann algebras.

**Corollary 6.4.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and let $|||\cdot|||$ be a gauge norm on $\mathcal{M}$. Then for $S,T\in\mathcal{M}$, $\|ST\|_1\leqslant |||S|||\cdot|||T|||^{\#}$.*

**Proof.** By Lemma 3.8, $\|ST\|_1=\sup\{|\tau(UST)|:\ U\in\mathcal{U}(\mathcal{M})\}$. By Lemmas 6.3 and 3.2, $|\tau(UST)|\leqslant |||US|||\cdot|||T|||^{\#}=|||S|||\cdot|||T|||^{\#}$. $\square$

**Proposition 6.5.** *If $|||\cdot|||$ is a unitarily invariant norm on a finite von Neumann algebra $(\mathcal{M},\tau)$, then $|||\cdot|||^{\#}$ is also a unitarily invariant norm on $\mathcal{M}$.*

**Proof.** Let $U$ be a unitary operator. Then $|||UT|||^{\#}=\sup\{|\tau(UTX)|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\}=\sup\{|\tau(TXU)|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\}=\sup\{|\tau(TX)|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\}=|||T|||$ and $|||TU|||^{\#}=\sup\{|\tau(TUX)|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\}=\sup\{|\tau(TX)|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\}=|||T|||$. $\square$

**Proposition 6.6.** *If $|||\cdot|||$ is a symmetric gauge norm on a finite von Neumann algebra $(\mathcal{M},\tau)$, then $|||\cdot|||^{\#}$ is also a symmetric gauge norm on $(\mathcal{M},\tau)$.*

**Proof.** Let $\theta\in\mathrm{Aut}(\mathcal{M},\tau)$. Then $|||\theta(T)|||^{\#}=\sup\{|\tau(\theta(T)X)|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\}=\sup\{|\tau(\theta(T\theta^{-1}(X)))|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\}=\sup\{|\tau(T\theta^{-1}(X))|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\}=\sup\{|\tau(TX)|:\ X\in\mathcal{M},\ |||X|||\leqslant 1\}=|||T|||$. $\square$

**Lemma 6.7.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. If $T\in\mathcal{M}$ is a positive operator, then*

$$|||T|||^{\#}=\sup\bigl\{\tau(TX):\ X\in\mathcal{M},\ X\geqslant 0,\ XT=TX,\ |||X|||\leqslant 1\bigr\}.$$

**Proof.** Let $\mathcal{A}$ be a separable abelian von Neumann subalgebra of $\mathcal{M}$ containing $T$ and let $\mathbf{E}_{\mathcal{A}}$ be the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{A}$ preserving $\tau$. For every $Y\in\mathcal{M}$ such that $|||Y|||\leqslant 1$, let $X=\mathbf{E}_{\mathcal{A}}(Y)$. By Corollary 1, $|||\,|X|\,|||=|||X|||\leqslant|||Y|||\leqslant 1$. Furthermore, $|\tau(TY)|=|\tau(\mathbf{E}_{\mathcal{A}}(TY))|=|\tau(T\mathbf{E}_{\mathcal{A}}(Y))|=|\tau(TX)|\leqslant\tau(T|X|)$. Hence,

$$|||T|||^{\#}=\sup\bigl\{\tau(TX):\ X\in\mathcal{M},\ X\geqslant 0,\ XT=TX,\ |||X|||\leqslant 1\bigr\}.\qquad\square$$

**Lemma 6.8.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. Suppose $T=a_1E_1+\cdots+a_nE_n$ is a positive simple operator in $\mathcal{M}$. Then*

$$\begin{aligned}|||T|||&=\sup\bigl\{\tau(TX):\ X=b_1E_1+\cdots+b_nE_n\geqslant 0 \textit{ and } |||X|||^{\#}\leqslant 1\bigr\}\\&=\sup\left\{\sum_{k=1}^{n}a_kb_k\tau(E_k):\ X=b_1E_1+\cdots+b_nE_n\geqslant 0 \textit{ and } |||X|||^{\#}\leqslant 1\right\}.\end{aligned}$$

**Proof.** By Lemma 6.7, $|||T|||^{\#} = \sup\{|\tau(TX)|\colon X \in \mathcal{M},\ X \geqslant 0,\ XT = TX,\ |||X||| \leqslant 1\}$. Let $\mathcal{A} = \{E_1, \ldots, E_n\}''$ and let $\mathbf{E}_{\mathcal{A}}$ be the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{A}$ preserving $\tau$. Then $S = \mathbf{E}_{\mathcal{A}}(X) = \tau_{E_1}(E_1XE_1)E_1 + \cdots + \tau_{E_n}(E_nXE_n)E_n$ is a positive operator, $\tau(TX) = \tau(\mathbf{E}_{\mathcal{A}}(TX)) = \tau(T\mathbf{E}_{\mathcal{A}}(X)) = \tau(TS)$, and $|||S||| \leqslant |||X|||$ by Corollary 4.4. Combining the definition of dual norm, this proves the lemma. □

**Corollary 6.9.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. Suppose $S, T$ are equi-measurable, positive simple operators in $\mathcal{M}$. Then $|||S|||^{\#} = |||T|||^{\#}$.*

**Theorem 6.10.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. Then $|||\cdot|||^{\#}$ is also a tracial gauge norm on $\mathcal{M}$. Furthermore, if $|||1||| = 1$, then $|||1|||^{\#} = 1$.*

**Proof.** By Lemma 3.13, $|||\cdot|||$ is a symmetric gauge norm on $\mathcal{M}$. By Proposition 6.6, Corollary 6.9 and Lemma 3.10, $|||\cdot|||^{\#}$ is a tracial gauge norm on $\mathcal{M}$. Note that $|||1||| = 1$, hence, $|||1|||^{\#} \geqslant \tau(1 \cdot 1) = 1$. On the other hand, by Theorem 3.30, $|||1|||^{\#} = \sup\{|\tau(X)|\colon X \in \mathcal{M}, |||X||| \leqslant 1\} \leqslant \sup\{|||X|||\colon X \in \mathcal{M}, |||X||| \leqslant 1\} \leqslant 1$. □

**Corollary 6.11.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. If $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ satisfying the weak Dixmier property, then $|||\cdot|||^{\#}_{\mathcal{N}}$ is the restriction of $|||\cdot|||^{\#}_{\mathcal{M}}$ to $\mathcal{N}$.*

**Proof.** Let $|||\cdot|||_1 = |||\cdot|||^{\#}_{\mathcal{N}}$ and let $|||\cdot|||_2$ be the restriction of $|||\cdot|||^{\#}_{\mathcal{M}}$ to $\mathcal{N}$. By Theorem 6.10, both $|||\cdot|||_1$ and $|||\cdot|||_2$ are tracial gauge norms on $\mathcal{N}$. By Lemma 3.6, to prove $|||\cdot|||_1 = |||\cdot|||_2$, we need to prove $|||T|||_1 = |||T|||_2$ for every positive simple operator $T \in \mathcal{N}$. Let $\mathcal{A}$ be a finite-dimensional abelian von Neumann subalgebra of $\mathcal{N}$ containing $T$. By Lemma 6.8, $|||T|||^{\#}_{\mathcal{M}} = |||T|||^{\#}_{\mathcal{N}} = |||T|||^{\#}_{\mathcal{A}}$. So $|||T|||_1 = |||T|||_2$. □

### 6.2. Dual norms of Ky Fan norms

For $(x_1, \ldots, x_n) \in \mathbb{C}^n$, $\tau(x) = \frac{x_1 + \cdots + x_n}{n}$ defines a trace on $\mathbb{C}^n$. For $1 \leqslant k \leqslant n$, the Ky Fan $\frac{k}{n}$th norm on $(\mathbb{C}^n, \tau)$ is $|||(x_1, \ldots, x_n)|||_{(\frac{k}{n})} = \frac{x_1^* + \cdots + x_k^*}{k}$, where $(x_1^*, \ldots, x_n^*)$ is the decreasing rearrangement of $(|x_1|, \ldots, |x_n|)$. Let $\mathbf{\Gamma} = \{(x_1, \ldots, x_n) \in \mathbb{C}^n\colon x_1 \geqslant x_2 \geqslant x_k = x_{k+1} = \cdots = x_n \geqslant 0, \frac{x_1 + \cdots + x_k}{k} \leqslant 1\}$ and $\mathcal{E}$ be the set of extreme points of $\mathbf{\Gamma}$.

The proof of the following lemma is an easy exercise.

**Lemma 6.12.** *$\mathcal{E}$ consists of $k+1$ points*: $(k, 0, \ldots), (\frac{k}{2}, \frac{k}{2}, 0, \ldots), \ldots, (\frac{k}{k-1}, \ldots, \frac{k}{k-1}, 0, \ldots)$, $(1, 1, \ldots, 1)$ *and* $(0, 0, \ldots, 0)$.

The following lemma is well known. For a proof we refer to [8, 10.2].

**Lemma 6.13.** *Let $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n \geqslant 0$ and $t_1, \ldots, t_n \geqslant 0$. If $t_1^* \geqslant t_2^* \geqslant \cdots \geqslant t_n^*$ is the decreasing rearrangement of $t_1, \ldots, t_n$, then $s_1t_1^* + \cdots + s_nt_n^* \geqslant s_1t_1 + \cdots + s_nt_n$.*

**Lemma 6.14.** *For $T \in (M_n(\mathbb{C}), \tau_n)$,*

$$|||T|||^{\#}_{(\frac{k}{n})} = \max\left\{\frac{k}{n}\|T\|, \|T\|_1\right\}.$$

**Proof.** Let $|||T|||_1 = |||T|||^{\#}_{(\frac{k}{n})}$ and $|||T|||_2 = \max\{\frac{k}{n}\|T\|, \|T\|_1\}$. Then both $|||\cdot|||_1$ and $|||\cdot|||_2$ are unitarily invariant norms on $M_n(\mathbb{C})$. To prove $|||\cdot|||_1 = |||\cdot|||_2$, we need only to prove $|||T|||_1 = |||T|||_2$ for every positive matrix $T$ in $M_n(\mathbb{C})$. We can assume that

$$T = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix},$$

where $s_1, \ldots, s_n$ are $s$-numbers of $T$ such that $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$. By Lemmas 6.8 and 6.13,

$$|||T|||_1 = \sup\left\{\frac{\sum_{i=1}^n s_i t_i}{n}\colon (t_1, \ldots, t_n) \in \mathbf{\Gamma}\right\} = \sup\left\{\frac{\sum_{i=1}^n s_i t_i}{n}\colon (t_1, \ldots, t_n) \in \mathcal{E}\right\}.$$

Note that $\|T\| = s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n \geqslant 0$. By Lemma 6.12 and simple computations, $|||T|||_1 = \max\{\frac{k}{n}\|T\|, \|T\|_1\} = |||T|||_2$. $\square$

The next lemma simply follows from the definition of dual norms.

**Lemma 6.15.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and $|||\cdot|||$, $|||\cdot|||_1$, $|||\cdot|||_2$ be norms on $\mathcal{M}$ such that*

$$|||T|||_1 \leqslant |||T||| \leqslant |||T|||_2, \quad \forall T \in \mathcal{M}.$$

*Then*

$$|||T|||_2^{\#} \leqslant |||T|||^{\#} \leqslant |||T|||_1^{\#}, \quad \forall T \in \mathcal{M}.$$

**Corollary 6.16.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and $|||\cdot|||_1$, $|||\cdot|||_2$ be equivalent norms on $\mathcal{M}$. Then $|||\cdot|||_1^{\#}$ and $|||\cdot|||_2^{\#}$ are equivalent norms on $\mathcal{M}$.*

**Theorem 6.17.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and $0 \leqslant t \leqslant 1$. Then*

$$|||T|||^{\#}_{(t)} = \max\{t\|T\|, \|T\|_1\}, \quad \forall T \in \mathcal{M}.$$

**Proof.** Firstly, we assume $t = \frac{k}{n}$ is a rational number. Let $\mathcal{N}_r$ be a type $I_{rn}$ subfactor of $\mathcal{M}$. Then the restriction of $|||\cdot|||_{(t)}$ to $\mathcal{N}_r$ is $|||\cdot|||_{(\frac{rk}{rn})}$. By Lemma 6.14 and Corollary 6.11, $|||T|||^{\#}_{(t)} = \max\{t\|T\|, \|T\|_1\}$ for $T \in \mathcal{N}_r$. By Corollary 4.7, $|||T|||^{\#}_{(t)} = \max\{t\|T\|, \|T\|_1\}$ for all $T \in \mathcal{M}$. Now assume $t$ is an irrational number. Let $t_1$, $t_2$ be two rational numbers such that $t_1 < t < t_2$. By Lemma 6.15, for every $T \in \mathcal{M}$,

$$\max\{t_2\|T\|, \|T\|_1\} \leqslant |||T|||^{\#}_{(t)} \leqslant \max\{t_1\|T\|, \|T\|_1\}.$$

Since $t_1 \leqslant t \leqslant t_2$ are arbitrary, $|||T|||^{\#}_{(t)} = \max\{t\|T\|, \|T\|_1\}$. $\square$

*6.3. Proof of Theorem C*

**Lemma 6.18.** *Let $n \in \mathbb{N}$ and $\tau$ be an arbitrary faithful state on $\mathbb{C}^n$. If $|||\cdot|||$ is a norm on $(\mathbb{C}^n, \tau)$ and $|||\cdot|||^\#$ is the dual norm with respect to $\tau$, then $|||\cdot|||^{\#\#} = |||\cdot|||$.*

**Proof.** By Lemma 6.3, $|||T|||^{\#\#} = \sup\{|\tau(TX)|\colon X \in \mathbb{C}^n, |||X|||^\# \leqslant 1\} \leqslant |||T|||$. We need to prove $|||T||| \leqslant |||T|||^{\#\#}$. By the Hahn–Banach theorem, there is a continuous linear functional $\phi$ on $\mathbb{C}^n$ with respect to the topology induced by $|||\cdot|||$ on $\mathbb{C}^n$ such that $|||T||| = \phi(T)$ and $\|\phi\| = 1$. Since all norms on $\mathbb{C}^n$ induce the same topology, there is an element $Y \in \mathbb{C}^n$ such that $\phi(S) = \tau(SY)$ for all $S \in \mathbb{C}^n$. By the definition of dual norm, $|||Y|||^\# = \|\phi\| = 1$. By Lemma 6.3, $|||T||| = \phi(T) = \tau(TY) \leqslant |||T|||^{\#\#}$. $\square$

**Proof of Theorem C.** By Theorem 6.10, both $|||\cdot|||^{\#\#}$ and $|||\cdot|||$ are tracial gauge norms on $\mathcal{M}$. By Corollary 3.6, to prove $|||\cdot|||^{\#\#} = |||\cdot|||$, we need to prove that $|||T||| = |||T|||^{\#\#}$ for every positive simple operator $T \in \mathcal{M}$. Let $\mathcal{A}$ be the abelian von Neumann subalgebra generated by $T$. By Corollary 6.11 and Lemma 6.18, $|||T|||^{\#\#}_{\mathcal{M}} = |||T|||^{\#\#}_{\mathcal{A}} = |||T|||$. $\square$

## 7. Proof of Theorem A

Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra.

**Lemma 7.1.** *Let $n \in \mathbb{N}$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant a_{n+1} = 0$ and $f(x) = a_1\chi_{[0,\frac{1}{n})}(x) + a_2\chi_{[\frac{1}{n},\frac{2}{n})}(x) + \cdots + a_n\chi_{[\frac{n-1}{n},1]}(x)$. For $T \in \mathcal{M}$, define*

$$|||T|||_f = \int_0^1 f(s)\mu_s(T)\,ds. \tag{7.1}$$

*Then*

$$|||T|||_f = \sum_{k=1}^{n} \frac{k(a_k - a_{k+1})}{n} |||T|||_{\left(\frac{k}{n}\right)}. \tag{7.2}$$

**Proof.** Since $t|||T|||_{(t)} = \int_0^t \mu_s(T)\,ds$, summation by parts shows that

$$\begin{aligned} |||T|||_f &= \int_0^1 f(s)\mu_s(T)\,dt = a_1 \int_0^{\frac{1}{n}} \mu_s(T)\,ds + a_2 \int_{\frac{1}{n}}^{\frac{2}{n}} \mu_s(T)\,ds + \cdots + a_n \int_{\frac{n-1}{n}}^{1} \mu_s(T)\,ds \\ &= \sum_{k=1}^{n} \frac{k(a_k - a_{k+1})}{n} |||T|||_{\left(\frac{k}{n}\right)}. \qquad \square \end{aligned}$$

**Corollary 7.2.** *The norm $|||\cdot|||_f$ defined as above is a tracial gauge norm on $\mathcal{M}$ and $|||1|||_f = \int_0^1 f(x)\,dx = \frac{a_1+\cdots+a_n}{n}$.*

**Lemma 7.3.** *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $\{|||\cdot|||_\alpha\}$ be a set of tracial gauge norms on $(\mathcal{M}, \tau)$ such that $|||1|||_\alpha \leqslant 1$ for all $\alpha$. For* *every $T \in \mathcal{M}$, define*

$$|||T||| = \sup_{\alpha} |||T|||_{\alpha}.$$

*Then $|||\cdot||| \triangleq \bigvee_{\alpha} |||\cdot|||_{\alpha}$ is also a tracial gauge norm on $(\mathcal{M}, \tau)$.*

**Proof.** By Corollary 3.5, $|||T||| \leqslant \|T\|$ is well defined. It is easy to check that $|||\cdot|||$ is a tracial gauge norm on $(\mathcal{M}, \tau)$. $\square$

**Proof of Theorem A.** Let

$$\mathcal{F}' = \Big\{\mu_s(X)\colon\ X \in \mathcal{M},\ |||X|||^{\#} \leqslant 1,\ X = b_1F_1 + \cdots + b_kF_k \geqslant 0,$$
$$\text{where } F_1 + \cdots + F_k = 1 \text{ and } \tau(F_1) = \cdots = \tau(F_k) = \frac{1}{k},\ k = 1, 2, \ldots\Big\}.$$

For every positive operator $X \in \mathcal{M}$ such that $|||X|||^{\#} \leqslant 1$, $\int_0^1 \mu_s(X)\,ds = \tau(X) = \|X\|_1 \leqslant |||X|||^{\#} \leqslant 1$ by Theorem 3.30. Hence $\mathcal{F}' \subset \mathcal{F}$ and $\mu_s(1) = \chi_{[0,1]}(s) \in \mathcal{F}'$ by Theorem 6.10. For $T \in \mathcal{M}$, define

$$|||T|||' = \sup\big\{|||T|||_f\colon\ f \in \mathcal{F}'\big\}.$$

By Corollary 7.2, $|||\cdot|||'$ is a tracial gauge norm on $\mathcal{M}$. To prove that $|||\cdot|||' = |||\cdot|||$, by Corollary 4.6, we need prove that $|||T|||' = |||T|||$ for every positive operator $T \in \mathcal{M}$ such that $T = a_1E_1 + \cdots + a_nE_n$ and $\tau(E_1) = \cdots = \tau(E_n) = \frac{1}{n}$.

By Lemma 6.8 and Theorem C,

$$|||T||| = \sup\left\{\frac{1}{n}\sum_{k=1}^{n} a_kb_k\colon\ X = b_1E_1 + \cdots + b_nE_n \geqslant 0 \text{ and } |||X|||^{\#} \leqslant 1\right\}.$$

Note that if $X = b_1E_1 + \cdots + b_nE_n$ is a positive simple operator in $\mathcal{M}$ and $|||X|||^{\#} \leqslant 1$, then $\mu_s(X) \in \mathcal{F}'$ and $|||T|||_{\mu_s(X)} = \int_0^1 \mu_s(X)\mu_s(T)\,ds = \frac{1}{n}\sum_{k=1}^{n} a_k^*b_k^*$, where $\{a_k^*\}$ and $\{b_k^*\}$ are non-increasing rearrangements of $\{a_k\}$ and $\{b_k\}$, respectively. By Lemma 6.13, $|||T||| \leqslant \sup\{|||T|||_f\colon\ f \in \mathcal{F}'\} = |||T|||'$.

We need to prove $|||T||| \geqslant |||T|||'$. Let $X = b_1F_1 + \cdots + b_kF_k$ be a positive operator in $\mathcal{M}$ such that $F_1 + \cdots + F_k = 1$, $\tau(F_1) = \cdots = \tau(F_k) = \frac{1}{k}$ and $|||X|||^{\#} \leqslant 1$. We need only prove that $|||T||| \geqslant |||T|||_{\mu_s(X)}$. Since $(\mathcal{M}, \tau)$ satisfies the weak Dixmier property, by Theorem 3.27, $(\mathcal{M}, \tau)$ is either a von Neumann subalgebra of $(M_n(\mathbb{C}), \tau_n)$ that contains all diagonal matrices or $\mathcal{M}$ is a diffuse von Neumann algebra. In either case, we may assume that $T = \tilde{a}_1\tilde{E}_1 + \cdots + \tilde{a}_r\tilde{E}_r$ and $X = \tilde{b}_1\tilde{F}_1 + \cdots + \tilde{b}_r\tilde{F}_r$, where $\tilde{E}_1 + \cdots + \tilde{E}_r = \tilde{F}_1 + \cdots + \tilde{F}_r = 1$ and $\tau(\tilde{E}_i) = \tau(\tilde{F}_i) = \frac{1}{r}$ for $1 \leqslant i \leqslant r$, $\tilde{a}_1 \geqslant \cdots \geqslant \tilde{a}_r \geqslant 0$ and $\tilde{b}_1 \geqslant \cdots \geqslant \tilde{b}_r \geqslant 0$. Let $Y = \tilde{b}_1\tilde{E}_1 + \cdots + \tilde{b}_r\tilde{E}_r$. Then $X$ and $Y$ are two equimeasurable operators in $\mathcal{M}$ and $\mu_s(X) = \mu_s(Y)$. By Theorem 6.10, $|||Y|||^{\#} \leqslant 1$. By Lemma 6.3,

$$|||T||| \geqslant \tau(TY) = \frac{1}{r}\sum_{i=1}^{r} \tilde{a}_i\tilde{b}_i = \int_0^1 \mu_s(Y)\mu_s(T)\,ds = \int_0^1 \mu_s(X)\mu_s(T)\,ds = |||T|||_{\mu_s(X)}. \qquad \square$$

Combining Theorem A and Lemma 3.21, we obtain the following corollary.

**Corollary 7.4.** *Let $(\mathcal{M},\tau)$ be a finite factor and let $|||\cdot|||$ be a normalized unitarily invariant norm on $\mathcal{M}$. Then there is a subset $\mathcal{F}'$ of $\mathcal{F}$ containing the constant 1 function on $[0,1]$ such that for all $T\in\mathcal{M}$, $|||T|||=\sup\{|||T|||_f\colon f\in\mathcal{F}'\}$.*

Combining Theorem A and Lemma 3.15 we obtain the following corollary.

**Corollary 7.5.** *Let $|||\cdot|||$ be a normalized symmetric gauge norm on $(L^\infty[0,1],\int_0^1 dx)$. Then there is a subset $\mathcal{F}'$ of $\mathcal{F}$ containing the constant 1 function on $[0,1]$ such that for all $T\in L^\infty[0,1]$, $|||T|||=\sup\{|||T|||_f\colon f\in\mathcal{F}'\}$.*

## 8. Proof of Theorems D and E

**Lemma 8.1.** *Let $\theta_1,\theta_2$ be two embeddings from $(L^\infty[0,1],\int_0^1 dx)$ into a finite von Neumann algebra $(\mathcal{M},\tau)$. If $|||\cdot|||$ is a tracial gauge norm on $\mathcal{M}$, then $|||\theta_1(f)|||=|||\theta_2(f)|||$ for every $f\in L^\infty[0,1]$.*

**Proof.** If $f\in L^\infty[0,1]$ is a positive function, then $\theta_1(f)$ and $\theta_2(f)$ are equi-measurable operators in $\mathcal{M}$. Hence $|||\theta_1(f)|||=|||\theta_2(f)|||$. □

**Proof of Theorem D.** We prove Theorem D for diffuse finite von Neumann algebras. The proof of the atomic case is similar. We may assume that the norms on $\mathcal{M}$ or $L^\infty[0,1]$ are normalized. By the definition of Ky Fan norms, there is a one-to-one correspondence between Ky Fan $t$th norms on $(\mathcal{M},\tau)$ and Ky Fan $t$th norms on $(L^\infty[0,1],\int_0^1 dx)$ as in Theorem D. By Lemma 7.1, Theorems 3.27 and A, there is a one-to-one correspondence between normalized tracial norms on $(\mathcal{M},\tau)$ and normalized symmetric gauge norms on $(L^\infty[0,1],\int_0^1 dx)$ as in Theorem D. □

**Example 8.2.** For $1\leqslant p\leqslant\infty$, the $L^p$-norm on $L^\infty[0,1]$ defined by

$$\|f(x)\|_p=\begin{cases}(\int_0^1|f(x)|^p\,dx)^{1/p}, & 1\leqslant p<\infty;\\ \operatorname{ess\,sup}|f|, & p=\infty\end{cases}$$

is a normalized symmetric gauge norm on $(L^\infty[0,1],\int_0^1 dx)$. By Corollaries 2 and 2.8, the induced norm

$$\|T\|_p=\begin{cases}(\tau(|T|^p))^{1/p}=(\int_0^1|\mu_s(T)|^p\,ds)^{1/p}, & 1\leqslant p<\infty;\\ \|T\|, & p=\infty\end{cases}$$

is a normalized unitarily invariant norm on a type $\mathrm{II}_1$ factor $\mathcal{M}$. The norms $\{\|\cdot\|_p\colon 1\leqslant p\leqslant\infty\}$ are called $L^p$-norms on $\mathcal{M}$.

**Corollary 8.3.** *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra satisfying the weak Dixmier property and let $|||\cdot|||$ be a tracial gauge norm on $(\mathcal{M},\tau)$. If $(\mathcal{M},\tau)$ can be embedded into a finite factor $(\mathcal{M}_1,\tau_1)$, then there is a unitarily invariant norm $|||\cdot|||_1$ on $(\mathcal{M}_1,\tau_1)$ such that $|||\cdot|||$ is the restriction of $|||\cdot|||_1$ to $(\mathcal{M},\tau)$.*

The following example shows that without the weak Dixmier property, Corollary 8.3 may fail.

**Example 8.4.** On $(\mathbb{C}^2, \tau)$, $\tau((x, y)) = \frac{1}{3}x + \frac{2}{3}y$, let $|||(x, y)||| = \frac{2}{3}|x| + \frac{1}{3}|y|$. It is easy to see that $|||\cdot|||$ is a tracial gauge norm on $(\mathbb{C}^2, \tau)$. Let $\mathcal{M}_1$ be the reduced free product of $(\mathbb{C}^2, \tau)$ with the free group factor $\mathcal{L}(\mathcal{F}_2)$. Then $\mathcal{M}_1$ is a type $\mathrm{II}_1$ factor with a faithful normal trace $\tau_1$ such that the restriction of $\tau_1$ to $\mathbb{C}^2$ is $\tau$. Suppose $|||\cdot|||_1$ is a unitarily invariant norm on $\mathcal{M}_1$ such that the restriction of $|||\cdot|||_1$ to $\mathbb{C}^2$ is $|||\cdot|||$. Let $E = (1, 0)$ and $F = (0, 1)$ in $\mathbb{C}^2$. Then $\tau_1(E) = \tau(E) < \tau(F) = \tau_1(F)$. So there is a unitary operator $U$ in $\mathcal{M}_1$ such that $UEU^* \leqslant F$. By Corollary 3.3, $\frac{2}{3} = |||E||| = |||E|||_1 = |||UEU^*|||_1 \leqslant |||F|||_1 = |||F||| = \frac{1}{3}$. This is a contradiction.

**Proof of Theorem E.** Let $|||\cdot|||_2$ be the tracial gauge norm on $\mathcal{M}$ corresponding to the symmetric gauge norm $|||\cdot|||_1^{\#}$ on $(L^\infty[0,1], \int_0^1 dx)$ as in Theorem D. By Lemma 4.6, to prove $|||\cdot|||_2 = |||\cdot|||^{\#}$ on $\mathcal{M}$, we need to prove $|||T|||_2 = |||T|||^{\#}$ for every positive simple operator $T = a_1E_1 + \cdots + a_nE_n$ in $\mathcal{M}$ such that $\tau(E_1) = \cdots = \tau(E_n) = \frac{1}{n}$. We may assume that $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$. Then $\mu_s(T) = a_1\chi_{[0,\frac{1}{n})}(s) + \cdots + a_n\chi_{[\frac{n-1}{n},1]}(s)$. By Lemma 6.8,

$$|||T|||^{\#} = \sup\left\{\frac{1}{n}\sum_{k=1}^{n} a_k b_k\colon\ X = b_1E_1 + \cdots + b_nE_n \geqslant 0 \text{ and } |||X|||^{\#} \leqslant 1\right\}.$$

By Lemma 6.13,

$$|||T|||^{\#} = \sup\left\{\frac{1}{n}\sum_{k=1}^{n} a_k b_k\colon\ X = b_1E_1 + \cdots + b_nE_n \geqslant 0,\ b_1 \geqslant \cdots \geqslant b_n \geqslant 0,\ |||X||| \leqslant 1\right\}.$$

By Theorem D and Lemma 6.8,

$$|||T|||_2 = \left|\left|\left|\mu_s(T)\right|\right|\right|^{\#} = \sup\left\{\frac{1}{n}\sum_{k=1}^{n} a_k b_k\colon\ g(s) = b_1\chi_{[0,\frac{1}{n})}(s) + \cdots + b_n\chi_{[\frac{n-1}{n},1]}(s) \geqslant 0,\ \left|\left|\left|g(s)\right|\right|\right| \leqslant 1\right\}.$$

By Lemma 6.13,

$$|||T|||_2 = \left|\left|\left|\mu_s(T)\right|\right|\right|^{\#} = \sup\left\{\frac{1}{n}\sum_{k=1}^{n} a_k b_k\colon\ g(s) = b_1\chi_{[0,\frac{1}{n})}(s) + \cdots + b_n\chi_{[\frac{n-1}{n},1]}(s) \geqslant 0,\ b_1 \geqslant \cdots \geqslant b_n \geqslant 0,\ \left|\left|\left|g(s)\right|\right|\right| \leqslant 1\right\}.$$

Note that if $b_1 \geqslant \cdots \geqslant b_n \geqslant 0$, then $\mu_s(b_1E_1 + \cdots + b_nE_n) = b_1\chi_{[0,\frac{1}{n})}(s) + \cdots + b_n\chi_{[\frac{n-1}{n},1]}(s)$. Since $|||\cdot|||$ is the tracial gauge norm on $(\mathcal{M}, \tau)$ corresponding to the symmetric gauge norm $|||\cdot|||_1$

on $(L^\infty[0,1], \int_0^1 dx)$ as in Theorem D, $|||b_1E_1 + \cdots + b_nE_n||| \leqslant 1$ if and only if $|||b_1\chi_{[0,\frac{1}{n})}(s) + \cdots + b_n\chi_{[\frac{n-1}{n},1]}(s)|||_1 \leqslant 1$. Therefore, $|||T|||_2 = |||T|||^\#$. $\square$

**Example 8.5.** If $p=1$, let $q=\infty$. If $1<p<\infty$, let $q=\frac{p}{p-1}$. Then the $L^q$-norm on $L^\infty[0,1]$ is the dual norm of the $L^p$-norm on $L^\infty[0,1]$. By Theorem E, the $L^q$-norm on a type $\mathrm{II}_1$ factor $\mathcal{M}$ is the dual norm of the $L^p$-norm on $\mathcal{M}$.

## 9. Proof of Theorem F

**Proof of Theorem F.** Let $|||\cdot|||$ be a tracial gauge norm on $\mathcal{M}$. By Lemma 7.1, $|||S|||_f \leqslant |||T|||_f$ for every $f\in\mathcal{F}$. By Theorem A, $|||S|||\leqslant|||T|||$. $\square$

**Corollary 9.1.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and $S,T\in\mathcal{M}$. If $|||S|||_{(t)}\leqslant|||T|||_{(t)}$ for all Ky Fan $t$th norms, $0\leqslant t\leqslant 1$, then $|||S|||\leqslant|||T|||$ for all unitarily invariant norms $|||\cdot|||$ on $\mathcal{M}$.*

By Corollary 9.1, we obtain Ky Fan's dominance theorem [13].

**Ky Fan's dominance theorem.** *If $S,T\in M_n(\mathbb{C})$ and $|||S|||_{(k/n)}\leqslant|||T|||_{(k/n)}$, i.e., $\sum_{i=1}^k s_i(S)\leqslant\sum_{i=1}^k s_i(T)$ for $1\leqslant k\leqslant n$, then $|||S|||\leqslant|||T|||$ for all unitarily invariant norms $|||\cdot|||$ on $M_n(\mathbb{C})$.*

## 10. Extreme points of normalized unitarily invariant norms on finite factors

In this section, we assume that $\mathcal{M}$ is a finite factor with the unique tracial state $\tau$.

### *10.1.* $\mathfrak{N}(\mathcal{M})$

Let $\mathfrak{N}(\mathcal{M})$ be the set of normalized unitarily invariant norms on $\mathcal{M}$. It is easy to see that $\mathfrak{N}(\mathcal{M})$ is a convex set. Let $\mathfrak{F}(\mathcal{M})$ be the set of complex functions defined on $\mathcal{M}$. Then $\mathfrak{F}(\mathcal{M})$ is a locally convex space such that a neighborhood of $f\in\mathfrak{F}(\mathcal{M})$ is

$$N(f,T_1,\ldots,T_n,\epsilon)=\big\{g\in\mathfrak{F}(\mathcal{M}):\ \big|g(T_i)-f(T_i)\big|<\epsilon\big\}.$$

In this topology, $f_\alpha\to f$ means $\lim_\alpha f_\alpha(T)=f(T)$ for every $T\in\mathcal{M}$. We call this topology the *pointwise weak* topology.

**Lemma 10.1.** $\mathfrak{N}(\mathcal{M})\subseteq\mathfrak{F}(\mathcal{M})$ *is a compact convex subset in the pointwise weak topology.*

**Proof.** It is clear that $\mathfrak{N}(\mathcal{M})$ is a convex subset of $\mathfrak{F}(\mathcal{M})$. Suppose $|||\cdot|||_\alpha\in\mathfrak{F}(\mathcal{M})$ and $f(T)=\lim_\alpha|||T|||_\alpha$ for every $T\in\mathcal{M}$. It is easy to check that $f(T)$ defines a unitarily invariant semi-norm on $\mathcal{M}$ such that $f(1)=1$. By Corollary 3.31, $f(T)$ is a norm and $f\in\mathfrak{N}(\mathcal{M})$. $\square$

Let $\mathfrak{N}_\mathrm{e}(\mathcal{M})$ be the subset of extreme points of $\mathfrak{N}(\mathcal{M})$. By the Krein–Milman theorem, the closure of the convex hull of $\mathfrak{N}_\mathrm{e}(\mathcal{M})$ is $\mathfrak{N}(\mathcal{M})$ in the pointwise weak topology. It is an interesting question of characterizing $\mathfrak{N}_\mathrm{e}(\mathcal{M})$. In the following, we will provide some results on $\mathfrak{N}_\mathrm{e}(\mathcal{M})$.

*10.2.* $\mathfrak{N}_{\mathrm{e}}(M_n(\mathbb{C}))$

For $n \geqslant 2$, let $1 \oplus s_2 \oplus \cdots \oplus s_n$ be the matrix

$$\begin{pmatrix} 1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{pmatrix} \in M_n(\mathbb{C}).$$

Let $|||\cdot|||$ be a normalized unitarily invariant norm on $M_n(\mathbb{C})$. For $0 \leqslant s_n \leqslant \cdots \leqslant s_2 \leqslant 1$, define

$$f(s_2, \ldots, s_n) = f_{|||\cdot|||}(s_2, \ldots, s_n) = |||1 \oplus s_2 \oplus \cdots \oplus s_n|||. \tag{10.1}$$

In the following, let $\Omega_{n-1} = \{(s_2, \ldots, s_n)\colon\ 0 \leqslant s_n \leqslant \cdots \leqslant s_2 \leqslant 1\}$.

By [7, Lemma 3.2] and Corollary 3.31, we have the following lemma.

**Lemma 10.2.** *Let $f(s_2, \ldots, s_n)$ be a function defined on $\Omega_{n-1}$. In order that $f(s_2, \ldots, s_n) = f_{|||\cdot|||}(s_2, \ldots, s_n)$ for some $|||\cdot||| \in \mathfrak{N}(M_n(\mathbb{C}))$, it is necessary and sufficient that the following conditions are satisfied*:

1. $f(s_2, \ldots, s_n) > 0$ *for all* $(s_2, \ldots, s_n) \in \Omega_{n-1}$ *and* $f(1, \ldots, 1) = 1$;
2. $f(s_2, \ldots, s_n)$ *is a convex function on* $\Omega_{n-1}$;
3. *for* $0 \leqslant s_n \leqslant s_{n-1} \leqslant \cdots \leqslant s_1,\ 0 \leqslant t_n \leqslant t_{n-1} \leqslant \cdots \leqslant t_1$, *if* $\sum_{i=1}^k s_i \leqslant \sum_{i=1}^k t_i$ *for* $1 \leqslant k \leqslant n$, *then* $s_1 \cdot f(\frac{s_2}{s_1}, \ldots, \frac{s_n}{s_1}) \leqslant t_1 \cdot f(\frac{t_2}{t_1}, \ldots, \frac{t_n}{t_1})$.

*If $f(s_2, \ldots, s_n)$ satisfies the above conditions, then $f$ satisfies*

$$\frac{1 + s_2 + \cdots + s_n}{n} \leqslant f(s_2, \ldots, s_n) \leqslant 1$$

*for all $(s_2, \ldots, s_n) \in \Omega_{n-1}$.*

Let $|||\cdot|||_1,\ |||\cdot|||_2 \in \mathfrak{N}(M_n(\mathbb{C}))$. If $|||S|||_1 = |||S|||_2$ for all $S = 1 \oplus s_2 \oplus \cdots \oplus s_n$, $(s_2, \ldots, s_n) \in \Omega_{n-1}$, then $|||T|||_1 = |||T|||_2$ for every matrix $T \in M_n(\mathbb{C})$. This implies the following lemma.

**Lemma 10.3.** *Let $|||\cdot|||_1,\ |||\cdot|||_2 \in \mathfrak{N}(M_n(\mathbb{C}))$. Then $|||\cdot|||_1 = |||\cdot|||_2$ if and only if $f_{|||\cdot|||_1}(s_2, \ldots, s_n) = f_{|||\cdot|||_2}(s_2, \ldots, s_n)$ for all $(s_2, \ldots, s_n) \in \Omega_{n-1}$.*

Let $1 \leqslant m \leqslant n$. Suppose $|||\cdot|||$ is a normalized unitarily invariant norm on $M_m(\mathbb{C})$ and $g(s_2, \ldots, s_m) = g_{|||\cdot|||}(s_2, \ldots, s_m)$ is the function on $\Omega_{m-1}$ induced by $|||\cdot|||$ (see (10.1)). Define $f(s_2, \ldots, s_n)$ on $\Omega_{n-1}$ by

$$f(s_2, \ldots, s_n) = g(s_2, \ldots, s_m), \quad (s_2, \ldots, s_n) \in \Omega_{n-1}.$$

It is easy to check that $f(s_2, \ldots, s_n)$ is a function on $\Omega_{n-1}$ satisfying Lemma 10.2. By Lemmas 10.2 and 10.3, there is a unique normalized unitarily invariant norm $|||\cdot|||_1 \in \mathfrak{N}(\mathcal{M}_n(\mathbb{C}))$ such that $f(s_2, \ldots, s_n) = f_{|||\cdot|||_1}(s_2, \ldots, s_n) = g(s_2, \ldots, s_m)$ for all $(s_1, \ldots, s_n) \in \Omega_{n-1}$. (This fact can also be obtained by Corollary 7.4 and Lemma 10.3.) $|||\cdot|||_1$ is called the *induced norm* of $|||\cdot|||$.

Conversely, suppose $|||\cdot|||_1$ is a normalized unitarily invariant norm on $M_n(\mathbb{C})$ and $f(s_2,\dots,s_n)=f_{|||\cdot|||_1}(s_2,\dots,s_n)$ is the function on $\Omega_{n-1}$ induced by $|||\cdot|||_1$. If $f(s_2,\dots,s_n)=g(s_2,\dots,s_m)$ for all $(s_2,\dots,s_n)\in\Omega_{n-1}$, then $g(s_2,\dots,s_m)$ satisfies Lemma 10.2. Hence, there is a unique normalized unitarily invariant norm $|||\cdot|||$ on $M_m(\mathbb{C})$ such that $g(s_2,\dots,s_m)=g_{|||\cdot|||}(s_2,\dots,s_m)$ for all $(s_2,\dots,s_m)\in\Omega_{m-1}$. $|||\cdot|||$ is called the *reduced norm* of $|||\cdot|||_1$.

**Proposition 10.4.** *For $1\leqslant k\leqslant n$, the Ky Fan $\frac{k}{n}$th norm (see Example 5.4) on $M_n(\mathbb{C})$ is an extreme point of $\mathfrak{N}(M_n(\mathbb{C}))$.*

**Proof.** Suppose $0<\alpha<1$ and $|||\cdot|||_1,\ |||\cdot|||_2\in\mathfrak{N}(M_n(\mathbb{C}))$ satisfy $|||\cdot|||_{(\frac{k}{n})}=\alpha|||\cdot|||_1+(1-\alpha)|||\cdot|||_2$. Let $f(s_2,\dots,s_n)=f_{|||\cdot|||_{(\frac{k}{n})}}(s_2,\dots,s_n)$, $f_1(s_2,\dots,s_n)=f_{|||\cdot|||_1}(s_2,\dots,s_n)$ and $f_2(s_2,\dots,s_n)=f_{|||\cdot|||_2}(s_2,\dots,s_n)$ for $(s_2,\dots,s_n)\in\Omega_{n-1}$. Then $f(s_2,\dots,s_{n-1})=\alpha f_1(s_2,\dots,s_{n-1})+(1-\alpha)f_2(s_2,\dots,s_{n-1})$.

Since $f(s_2,\dots,s_n)=\frac{1+s_2+\cdots+s_k}{k}$, $\frac{\partial f}{\partial s_{k+1}}=\cdots=\frac{\partial f}{\partial s_n}=0$. Since $f_1(s_2,\dots,s_n)$, $f_2(s_2,\dots,s_n)$ are convex functions on $\Omega_{n-1}$, $\frac{\partial f_i}{\partial s_j}\geqslant 0$ for $i=1,2$ and $k+1\leqslant j\leqslant n$. Since $f=\alpha f_1+(1-\alpha)f_2$, $\frac{\partial f_i}{\partial s_j}=0$ for $i=1,2$ and $k+1\leqslant j\leqslant n$. This implies that $f_i(s_2,\dots,s_n)=g_i(s_2,\dots,s_k)$ for all $(s_2,\dots,s_n)\in\Omega_{n-1}$ and $i=1,2$.

By the discussions above the proposition, there are normalized unitarily invariant norms $|||\cdot|||_1$, $|||\cdot|||_2$ on $M_k(\mathbb{C})$ such that $g_i(s_2,\dots,s_k)=(g_i)_{|||\cdot|||_i}(s_2,\dots,s_k)$ for all $(s_2,\dots,s_k)\in\Omega_{k-1}$ and $i=1,2$. By Lemma 10.2,

$$g_i(s_2,\dots,s_k)\geqslant\frac{1+s_2+\cdots+s_k}{k}$$

for all $(s_2,\dots,s_k)\in\Omega_{k-1}$ and $i=1,2$. Since $f=\alpha f_1+(1-\alpha)f_2$,

$$\frac{1+s_2+\cdots+s_k}{k}=\alpha g_1(s_2,\dots,s_k)+(1-\alpha)g_2(s_2,\dots,s_k).$$

This implies that $g_1(s_2,\dots,s_k)=g_2(s_2,\dots,s_k)=\frac{1+s_2+\cdots+s_k}{k}$. So $f=f_1=f_2$. □

The proof of the following proposition is similar to that of Proposition 10.4.

**Proposition 10.5.** *Let $1\leqslant m\leqslant n$ and $|||\cdot|||$ be a normalized unitarily invariant norm on $M_m(\mathbb{C})$. If $|||\cdot|||$ is an extreme point of $\mathfrak{N}(M_m(\mathbb{C}))$, then the induced norm $|||\cdot|||_1$ on $M_n(\mathbb{C})$ is also an extreme point of $\mathfrak{N}(M_n(\mathbb{C}))$.*

**Question.** *For $n\geqslant 3$, find all extreme points of $\mathfrak{N}(\mathcal{M}_n(\mathbb{C}))$.*

### *10.3.* $\mathfrak{N}_{\mathrm{e}}(M_2(\mathbb{C}))$

In this subsection, we will prove Theorem J. We need the following auxiliary results. The following lemma is a corollary of Lemma 10.2 in the case $n=2$.

**Lemma 10.6.** *Let $f(s)$ be a function on $[0,1]$. If there is a normalized unitarily invariant norm $|||\cdot|||$ on $M_2(\mathbb{C})$ such that $f(s)=f_{|||\cdot|||}(s)=|||1\oplus s|||$, then $f(s)$ is an increasing convex function on $[0,1]$ satisfying*

$$\frac{1+s}{2}\leqslant f(s)\leqslant 1, \quad \forall s\in[0,1].$$

**Corollary 10.7.** *For $0\leqslant a\leqslant b\leqslant 1$, we have*

$$0\leqslant f'(a-)\leqslant f'(a+)\leqslant f'(b-)\leqslant f'(b+)\leqslant f'(1-)\leqslant \frac{1}{2}.$$

**Proof.** Since $f(s)$ is an increasing convex function, $0\leqslant f'(a-)\leqslant f'(a+)\leqslant f'(b-)\leqslant f'(b+)\leqslant f'(1-)$. By Lemma 10.6,

$$f'(1-)=\lim_{h\to 0+}\frac{f(1)-f(1-h)}{h}\leqslant \lim_{h\to 0+}\frac{1-(2-h)/2}{h}=\frac{1}{2}. \qquad \square$$

For $\frac{1}{2}\leqslant t\leqslant 1$, define $|||\cdot|||_{\langle t\rangle}=\max\{t\|T\|,\|T\|_1\}$.

**Lemma 10.8.** *For $1/2\leqslant t\leqslant 1$, $|||\cdot|||_{\langle t\rangle}$ is an extreme point of $\mathfrak{N}(M_2(\mathbb{C}))$.*

**Proof.** Suppose $0<\alpha<1$ and $|||\cdot|||_1$, $|||\cdot|||_2\in\mathfrak{N}(M_2(\mathbb{C}))$ such that $|||\cdot|||_{\langle t\rangle}=\alpha|||\cdot|||_1+(1-\alpha)|||\cdot|||_2$. Let $f(s)=f_{|||\cdot|||_{\langle t\rangle}}(s)$, $f_1(s)=f_{|||\cdot|||_1}(s)$ and $f_2(s)=f_{|||\cdot|||_2}(s)$. Then $f(s)=\alpha f_1(s)+(1-\alpha)f_2(s)$. Note that

$$f(s)=\begin{cases} t & 0\leqslant s\leqslant \frac{2t-1}{2};\\ \frac{s+1}{2} & \frac{2t-1}{2}\leqslant s\leqslant 1.\end{cases}$$

Hence, $f'(s)=0$ if $0\leqslant s<\frac{2t-1}{2}$ and $f'(s)=\frac{1}{2}$ if $\frac{2t-1}{2}<s\leqslant 1$. By Corollary 10.7, $f_1'(s)=f_2'(s)=0$ if $0\leqslant s<\frac{2t-1}{2}$ and $f_1'(s)=f_2'(s)=\frac{1}{2}$ if $\frac{2t-1}{2}<s\leqslant 1$. Since $f(s)$, $f_1(s)$, $f_2(s)$ are convex functions and hence continuous and $f(1)=f_1(1)=f_2(1)=1$, $f(s)=f_1(s)=f_2(s)$ for all $0\leqslant s\leqslant 1$. This implies that $|||\cdot|||_{\langle t\rangle}=|||\cdot|||_1=|||\cdot|||_2$. $\square$

**Lemma 10.9.** *The mapping: $t\to|||\cdot|||_{\langle t\rangle}$ is continuous with respect to the usual topology on $[1/2,1]$ and the pointwise weak topology on $\mathfrak{N}(M_2(\mathbb{C}))$. In particular, $\{|||\cdot|||_{\langle t\rangle}\colon 1/2\leqslant t\leqslant 1\}$ is compact in the pointwise weak topology.*

**Proof.** For every $0\leqslant s\leqslant 1$, $|||1\oplus s|||_{\langle t\rangle}=\max\{t,\frac{1+s}{2}\}$ is a continuous function on $[0,1]$. Hence, the mapping: $t\to|||\cdot|||_{\langle t\rangle}$ is continuous with respect to the usual topology on $[1/2,1]$ and the pointwise weak topology on $\mathfrak{N}(M_2(\mathbb{C}))$. $\square$

**Lemma 10.10.** *The set*

$$\mathcal{S}=\left\{|||\cdot|||\colon\ |||\cdot|||=\int_{1/2}^{1}|||\cdot|||_{\langle t\rangle}\,d\mu(t),\ \mu \text{ is a regular Borel probability measure on } [1/2,1]\right\}$$

*is a convex compact subset of $\mathfrak{N}(M_2(\mathbb{C}))$ in the pointwise weak topology.*

**Proof.** Suppose $\{|||\cdot|||_\alpha\}$ is a net in $\mathcal{S}$ such that $|||\cdot|||_\alpha \to |||\cdot||| \in \mathfrak{N}(M_2(\mathbb{C}))$ in the pointwise weak topology. Let $\mu_\alpha$ be the regular Borel probability measure on $[1/2, 1]$ corresponding to $|||\cdot|||_\alpha$. Then there is a subnet of $\mu_\alpha$ that converges weakly to a regular Borel probability measure $\mu$ on $[1/2, 1]$, i.e., for every continuous function $\phi(t)$ on $[1/2, 1]$,

$$\lim_\alpha \int_{1/2}^{1} \phi(t)\, d\mu_{\alpha_\beta}(t) = \int_{1/2}^{1} \phi(t)\, d\mu(t).$$

In particular, for every $T \in M_2(\mathbb{C})$, we have

$$|||T||| = \lim_{\alpha_\beta} |||T|||_{\alpha_\beta} = \lim_{\alpha_\beta} \int_{1/2}^{1} |||T|||_{\langle t\rangle}\, d\mu_{\alpha_\beta}(t) = \int_{1/2}^{1} |||T|||_{\langle t\rangle}\, d\mu(t).$$

Hence $|||\cdot||| \in \mathcal{S}$. □

**Lemma 10.11.** *Let $f(s)$ be a convex, increasing function on* $[0, 1]$ *such that*

$$\frac{1+s}{2} \leqslant f(s) \leqslant 1, \quad \forall s \in [0, 1].$$

*Then there is an element $|||\cdot||| \in \mathcal{S}$ such that $f(s) = |||1 \oplus s|||$.*

**Proof.** We can approximate $f$ uniformly by piecewise linear functions satisfying the conditions of the lemma. By Lemma 10.10, we may assume that $f(s)$ is a piecewise linear function. Furthermore, we may assume that $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ and $f(s)$ is linear on $[a_i, a_{i+1}]$ for $0 \leqslant i \leqslant n-1$. Let $f'(s) = \alpha_i/2$ on $[a_i, a_{i+1}]$. By Corollary 10.7, $0 = \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_{n-1} \leqslant 1$. Let $g(s) = (1-\alpha_{n-1})\|1 \oplus s\| + (\alpha_{n-1} - \alpha_{n-2})|||1 \oplus s|||_{\langle \alpha_{n-1}\rangle} + \cdots + (\alpha_1 - \alpha_0)|||1 \oplus s|||_{\langle \alpha_1\rangle} + \alpha_0\|1 \oplus s\|_1$. Then $g(1) = f(1) = 1$ and $g'(s) = \alpha_i/2$ on $[a_i, a_{i+1}]$. So $g'(s) = f'(s)$ except $s = \alpha_i$ for $1 \leqslant i \leqslant n$. Hence $f(s) = g(s)$ for all $0 \leqslant s \leqslant 1$. □

**Proof of Theorem J.** By Lemma 10.8, $\{|||\cdot|||_{\langle t\rangle}\colon\ 1/2 \leqslant t \leqslant 1\}$ are extreme points of $\mathfrak{N}(\mathcal{M})$. By Lemmas 10.10, 10.11 and 10.3, the closure of the convex hull of $\{|||\cdot|||_{\langle t\rangle}\colon\ 1/2 \leqslant t \leqslant 1\}$ in the pointwise weak topology is $\mathfrak{N}(\mathcal{M}_2(\mathbb{C}))$. By Lemmas 10.8, 10.9 and by [11, Theorem 1.4.5], $\mathfrak{N}_e(M_2(\mathbb{C})) = \{|||\cdot|||_{\langle t\rangle}\colon\ 1/2 \leqslant t \leqslant 1\}$. □

**Corollary 10.12.** *Let $f(s)$ be a function on* $[0, 1]$. *Then the following conditions are equivalent*:

1. $f(s) = f_{|||\cdot|||}(s) = |||1 \oplus s|||$ *for some normalized unitarily invariant norm* $|||\cdot|||$ *on* $M_2(\mathbb{C})$;
2. $f(s)$ *is an increasing convex function on* $[0, 1]$ *such that* $\frac{1+s}{2} \leqslant f(s) \leqslant 1$ *for all* $s \in [0, 1]$;
3. $f(s)$ *is an increasing convex function on* $[0, 1]$ *such that* $f(1) = 1$ *and* $f'(1-) \leqslant \frac{1}{2}$.

In the following, we will show how to write the $L^p$-norms on $M_2(\mathbb{C})$ in terms of extreme points of $\mathfrak{N}(M_2(\mathbb{C}))$. Recall that for $1 \leqslant p < \infty$, the $L^p$-norm of $1 \oplus s$ is

$$\|1 \oplus s\|_p = \left(\frac{1+s^p}{2}\right)^{1/p}.$$

Let $f_p(s) = f_{\|\cdot\|_p}(s) = (\frac{1+s^p}{2})^{1/p}$, $0 \leqslant s \leqslant 1$. Then $f_p(1) = 1$ and

$$f_p'(s) = \frac{s^{p-1}}{2}\left(\frac{1+s^p}{2}\right)^{1/p-1},$$

$f_p'(0) = 0$, $f_p'(1) = \frac{1}{2}$.

**Lemma 10.13.** *For $1 < p < \infty$ and $0 \leqslant s \leqslant 1$,*

$$f_p(s) = \int_{1/2}^{1} |||1 \oplus s|||_{\langle t\rangle} 4 f_p''(2t-1)\,dt.$$

**Proof.**

$$\begin{aligned}
&\int_{1/2}^{1} |||1 \oplus s|||_{\langle t\rangle} 4 f_p''(2t-1)\,dt \\
&= \int_{0}^{1} |||1 \oplus s|||_{\langle \frac{x+1}{2}\rangle} 2 f_p''(x)\,dx \\
&= \int_{0}^{s} \frac{1+s}{2} 2 f_p''(x)\,dx + \int_{s}^{1} \frac{1+x}{2} 2 f_p''(x)\,dx \\
&= (1+s)f'(s) - (1+s)f'(0) + 2f'(1) - (1+s)f'(s) - \int_{s}^{1} f_p'(x)\,dx \\
&= 1 - f_p(1) + f_p(s) = f_p(s). \qquad \square
\end{aligned}$$

**Corollary 10.14.** *For $1 < p < \infty$ and $T \in M_2(\mathbb{C})$,*

$$\|T\|_p = \int_{1/2}^{1} |||T|||_{\langle t\rangle} 4 f_p''(2t-1)\,dt.$$

*10.4. Proof of Theorem K*

**Lemma 10.15.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and let $|||\cdot|||$ be a normalized unitarily invariant norm on $\mathcal{M}$. Suppose $\mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots$ is a sequence of type $I_{n_r}$ subfactors of $\mathcal{M}$ such that $\mathcal{N}_r \cong M_{n_r}(\mathbb{C})$ and $\lim_{r\to\infty} n_r = \infty$. If the restriction of $|||\cdot|||$ to $\mathcal{N}_r$ is an extreme point of $\mathfrak{N}(\mathcal{N}_r)$ for all $r = 1, 2, \ldots,$ then $|||\cdot|||$ is an extreme point of $\mathfrak{N}(\mathcal{M})$.*

**Proof.** Suppose $0 < \alpha < 1$ and $|||\cdot|||_1, |||\cdot|||_2 \in \mathfrak{N}(\mathcal{M})$ such that $|||\cdot||| = \alpha|||\cdot|||_1 + (1-\alpha)|||\cdot|||_2$ on $\mathcal{M}$. Then for every $r = 1, 2, \ldots,$ $|||\cdot||| = \alpha|||\cdot|||_1 + (1-\alpha)|||\cdot|||_2$ on $\mathcal{N}_r$. By the assumption of the lemma, $|||\cdot||| = |||\cdot|||_1 = |||\cdot|||_2$ on $\mathcal{N}_r$. By Corollary 4.7, $|||\cdot||| = |||\cdot|||_1 = |||\cdot|||_2$ on $\mathcal{M}$. So $|||\cdot|||$ is an extreme point of $\mathfrak{N}(\mathcal{M})$. $\square$

**Proof of Theorem K.** By the assumption of the theorem, $t = \frac{k}{n}$ is a rational number. Then we can construct a sequence of type $\mathrm{I}_{rn}$ subfactor $\mathcal{M}_{rn}$ of $\mathcal{M}$ such that $\mathcal{M}_n \subseteq \mathcal{M}_{2n} \subseteq \cdots$. Then the restriction of $\|\cdot|||_{(t)}$ on $\mathcal{M}_{rn}$ is $\|\cdot|||_{(\frac{rk}{rn})}$. By Proposition 10.4, the restriction of $\|\cdot|||_{(t)}$ on $\mathcal{M}_{rn}$ is an extreme point of $\mathfrak{N}(M_{rn}(\mathbb{C}))$. By Lemma 10.15, $\|\cdot|||_{(t)}$ is an extreme point of $\mathfrak{N}(\mathcal{M})$. $\square$

**Remark 10.16.** Here we point out other interesting examples of extreme points of $\mathfrak{N}(\mathcal{M})$. For $0 \leqslant t \leqslant 1$, recall that $\|\cdot|||_{(t)}$ is the $t$th Ky Fan norm on $\mathcal{M}$. For any non-negative function $c(t)$ on $[0,1]$ such that $\|c(t)\|_\infty = 1$ and $T \in \mathcal{M}$, define

$$\|T|||_{[c(t)]} = \big\|c(t)\|T|||_{(t)}\big\|_\infty.$$

Then it is easy to see that $\|\cdot|||_{[c(t)]}$ is a normalized unitarily invariant norm on $\mathcal{M}$. It can be proved that if $c(t)$ is a simple function or if $tc(t)$ is a simple function, then $\|\cdot|||_{[c(t)]}$ is an extreme point of $\mathfrak{N}(\mathcal{M})$.

## 11. Proof of Theorem G

In this section, we assume that $\mathcal{M}$ is a type $\mathrm{II}_1$ factor with the unique tracial state $\tau$ and $|||\cdot|||$ is a unitarily invariant norm on $\mathcal{M}$. For two projections $E, F$ in $\mathcal{M}$, $\tau(E) \leqslant \tau(F)$ if and only if there is a unitary operator $U \in \mathcal{M}$ such that $UEU^* \leqslant F$. By Corollary 3.3, if $\tau(E) \leqslant \tau(F)$, $|||E||| \leqslant |||F|||$. So we can define

$$r\big(|||\cdot|||\big) = \lim_{\tau(E)\to 0+} |||E|||.$$

**Definition 11.1.** A unitarily invariant norm $|||\cdot|||$ on $\mathcal{M}$ is *singular* if $r(|||\cdot|||) > 0$ and *continuous* if $r(|||\cdot|||) = 0$.

**Example 11.2.** The operator norm is singular since $r(\|\cdot\|) = \lim_{\tau(E)\to 0+} \|E\| = 1$. If $0 < t \leqslant 1$, the Ky Fan $t$th norm $|||\cdot|||_{(t)}$ is continuous since $r(|||\cdot|||_{(1)}) = r(\|\cdot\|_1) = \lim_{\tau(E)\to 0+} \tau(E) = 0$ and $r(|||\cdot|||_{(t)}) \leqslant \frac{1}{t} \cdot r(|||\cdot|||_{(1)}) = 0$. If $1 \leqslant p < \infty$, it is easy to see that the $L^p$-norm on $\mathcal{M}$ is also continuous.

**Lemma 11.3.** *If $|||\cdot|||$ is singular, then $|||\cdot|||$ is equivalent to the operator norm $\|\cdot\|$. Indeed, for every $T \in \mathcal{M}$, we have*

$$r\big(|||\cdot|||\big)\|T\| \leqslant |||T||| \leqslant |||1||| \cdot \|T\|.$$

**Proof.** By Lemma 3.2, $|||T||| \leqslant |||1||| \cdot \|T\|$. We need to prove $r(|||\cdot|||)\|T\| \leqslant |||T|||$. We may assume that $T > 0$. For any $\epsilon > 0$, let $E = \chi_{[\|T\|-\epsilon, \|T\|]}(T) > 0$. Then $T \geqslant (\|T\| - \epsilon)E$. By Corollary 3.3 and Lemma 3.2, $|||T||| \geqslant |||(\|T\| - \epsilon)E||| \geqslant (\|T\| - \epsilon) \cdot |||E||| \geqslant (\|T\| - \epsilon)r(|||\cdot|||)$. Since $\epsilon > 0$ is arbitrary, $r(|||\cdot|||)\|T\| \leqslant |||T|||$. $\square$

Recall that a neighborhood $N(\epsilon,\delta)$ of $0\in\mathcal{M}$ in the measure topology (see [14]) is

$$N(\epsilon,\delta)=\{T\in\mathcal{M},\ \text{there is a projection } E\in\mathcal{M} \text{ such that } \tau(E)<\delta \text{ and } \|TE^\perp\|<\epsilon\}$$

**Proof of Theorem G.** By Lemma 11.3, if $|||\cdot|||$ is singular, then $\mathcal{T}$ is the operator topology on $\mathcal{M}_{1,\|\cdot\|}$. Suppose $|||\cdot|||$ is continuous. For $\epsilon,\delta>0$ and $T\in\mathcal{M}$ such that $\|T\|\leqslant 1$ and $|||T|||<\epsilon\delta$, by Corollary 3.31, $\tau(\chi_{[\epsilon,1]}(|T|)\leqslant\frac{\|T\|_1}{\epsilon}\leqslant\frac{|||T|||}{\epsilon}<\delta$ and $\|T\cdot\chi_{[0,\epsilon)}(|T|)\|<\epsilon$. This implies that $\{T\in\mathcal{M}_{1,\|\cdot\|}\colon |||T|||<\epsilon\delta\}\subseteq N(\epsilon,\delta)$. Conversely, let $\omega>0$. Since $r(|||\cdot|||)=0$, there is an $\epsilon$, $0<\epsilon<\omega/2$, such that if $\tau(E)<\epsilon$ then $|||E|||<\omega/2$. For every $T\in N(\epsilon,\omega/2)$ and $\|T\|\leqslant 1$, choose $E\in\mathcal{M}$ such that $\tau(E)<\epsilon$ and $\|TE^\perp\|<\omega/2$. By Proposition 3.18 and Corollary 3.5, $|||T|||\leqslant|||TE|||+|||TE^\perp|||<\|T\|\cdot|||E|||+\|TE^\perp\|<\omega/2+\omega/2=\omega$. Hence $\{T\in N(\epsilon,\omega/2)\colon \|T\|\leqslant 1\}\subseteq\{T\in\mathcal{M}\colon |||T|||<\omega\}$. □

**Corollary 11.4.** *Topologies induced by the $L^p$-norms, $1\leqslant p<\infty$, on the unit ball of a type* $\mathrm{II}_1$ *factor are the same.*

## 12. Completion of type $\mathrm{II}_1$ factors with respect to unitarily invariant norms

In this section, we assume that $\mathcal{M}$ is a type $\mathrm{II}_1$ factor with the unique tracial state $\tau$ and $|||\cdot|||$ is a unitarily invariant norm on $\mathcal{M}$. The completion of $\mathcal{M}$ with respect to $|||\cdot|||$ is denoted by $\overline{\mathcal{M}_{|||\cdot|||}}$. We will use the traditional notation $L^p(\mathcal{M},\tau)$ to denote the completion of $\mathcal{M}$ with respect to the $L^p$-norm defined as in Example 11.2. Note that $L^\infty(\mathcal{M},\tau)=\mathcal{M}$. Let $\widetilde{\mathcal{M}}$ be the completion of $\mathcal{M}$ in the measure topology in the sense of [14].

### 12.1. Embedding of $\overline{\mathcal{M}_{|||\cdot|||}}$ into $\widetilde{\mathcal{M}}$

**Lemma 12.1.** *Let $|||\cdot|||$ be a continuous unitarily invariant norm on $\mathcal{M}$ and $T\in\mathcal{M}$. For every $\epsilon>0$, there is a $\delta>0$ such that if $\tau(E)<\delta$, then $|||TE|||<\epsilon$.*

**Proof.** Since $|||\cdot|||$ is continuous, $\lim_{\tau(E)\to 0}|||E|||=0$. Hence, for every $\epsilon>0$, there is a $\delta>0$ such that if $\tau(E)<\delta$, then $|||E|||<\frac{\epsilon}{1+\|T\|}$. By Proposition 3.18, $|||TE|||\leqslant\|T\|\cdot|||E|||<\epsilon$. □

**Lemma 12.2.** *Let $|||\cdot|||$ be a continuous unitarily invariant norm on $\mathcal{M}$ and let $\{T_n\}$ in $\mathcal{M}$ be a Cauchy sequence with respect to $|||\cdot|||$. For every $\epsilon>0$, there is a $\delta>0$ such that if $\tau(E)<\delta$, then $|||T_nE|||<\epsilon$ for all $n$.*

**Proof.** Since $\{T_n\}$ is a Cauchy sequence with respect to $|||\cdot|||$, there is an $N$ such that for all $n\geqslant N$, $|||T_n-T_N|||<\epsilon/2$. By Lemma 12.1, there is a $\delta_1$ such that if $\tau(E)<\delta_1$ then $|||T_NE|||<\epsilon/2$. By Proposition 3.18, for $n\geqslant N$, $|||T_nE|||\leqslant|||(T_n-T_N)E|||+|||T_NE|||<|||(T_n-T_N)|||\cdot\|E\|+\epsilon/2<\epsilon$. A simple argument shows that we can choose $0<\delta<\delta_1$ such that if $\tau(E)<\delta$ then $|||T_nE|||<\epsilon$ for all $n$. □

The following proposition generalizes Theorem 5 of [14].

**Proposition 12.3.** *Let $\mathcal{M}$ be a type* $\mathrm{II}_1$ *factor and let $|||\cdot|||$ be a unitarily invariant norm on $\mathcal{M}$. There is an injective map from $\overline{\mathcal{M}_{|||\cdot|||}}$ to $\widetilde{\mathcal{M}}$ that extends the identity map from $\mathcal{M}$ to $\mathcal{M}$.*

**Proof.** If $|||\cdot|||$ is singular, by Lemma 11.3, $\overline{\mathcal{M}_{|||\cdot|||}} = \mathcal{M}$. So we will assume that $|||\cdot|||$ is continuous. If $\{T_n\}$ in $\mathcal{M}$ is a Cauchy sequence with respect to $|||\cdot|||$, then $\{T_n\}$ is a Cauchy sequence in the $L^1$-norm by Corollary 3.31. For every $\delta > 0$ and $T \in \mathcal{M}$, $\tau(\chi_{(\delta,\infty)}(|T|) \leqslant \frac{\tau(|T|)}{\delta}$. Hence, if $\{T_n\}$ is a Cauchy sequence in $\mathcal{M}$ in the $L^1$-norm, then $\{T_n\}$ is a Cauchy sequence in the measure topology. So there is a natural map $\Phi$ from $\overline{\mathcal{M}_{|||\cdot|||}}$ to $\widetilde{\mathcal{M}}$ that extends the identity map from $\mathcal{M}$ to $\mathcal{M}$. To prove that $\Phi$ is injective, we need to prove that if $\{T_n\}$ in $\mathcal{M}$ is a Cauchy sequence with respect to $|||\cdot|||$ and $T_n \to 0$ in the measure topology, then $\lim_{n\to\infty} |||T_n||| = 0$. Let $\epsilon > 0$. By Lemma 12.2, there is a $\delta > 0$ such that if $\tau(E) < \delta$ then $|||T_n E||| < \epsilon/2$ for all $n$. Since $T_n \to 0$ in the measure topology, there are $N$ and $\delta_1$, $0 < \delta_1 < \delta$, such that for all $n \geqslant N$, there is a projection $E_n$ such that $\tau(E_n) < \delta_1$ and $\|T_n E_n^\perp\| < \epsilon/2$. By Corollary 3.31, $|||T_n||| \leqslant |||T_n E_n^\perp||| + |||T_n E_n||| < \|T_n E_n^\perp\| + \epsilon/2 < \epsilon$. This proves that $\lim_{n\to\infty} |||T_n||| = 0$ and hence $\Phi$ is an injective map from $\overline{\mathcal{M}_{|||\cdot|||}}$ to $\widetilde{\mathcal{M}}$ that extends the identity map from $\mathcal{M}$ to $\mathcal{M}$. $\square$

By the proof of Proposition 12.3, we have the following.

**Corollary 12.4.** *There is an injective map from $\overline{\mathcal{M}_{|||\cdot|||}}$ to $L^1(\mathcal{M}, \tau)$ that extends the identity map from $\mathcal{M}$ to $\mathcal{M}$.*

By Proposition 12.3, we will consider $\overline{\mathcal{M}_{|||\cdot|||}}$ as a subset of $\widetilde{\mathcal{M}}$. The following corollary is very useful.

**Corollary 12.5.** *Let $\mathcal{M}$ be a type $\mathrm{II}_1$ factor and let $|||\cdot|||$ be a unitarily invariant norm on $\mathcal{M}$. If $\{T_n\} \subset \mathcal{M}$ is a Cauchy sequence with respect to $|||\cdot|||$ and $\lim_{n\to\infty} T_n = T$ in the measure topology, then $T \in \overline{\mathcal{M}_{|||\cdot|||}}$ and $\lim_{n\to\infty} T_n = T$ in the topology induced by $|||\cdot|||$.*

**Corollary 12.6.** *$\overline{\mathcal{M}_{|||\cdot|||}}$ is a linear subspace of $\widetilde{\mathcal{M}}$ satisfying the following conditions*:

1. *if $T \in \overline{\mathcal{M}_{|||\cdot|||}}$, then $T^* \in \overline{\mathcal{M}_{|||\cdot|||}}$;*
2. *$T \in \overline{\mathcal{M}_{|||\cdot|||}}$ if and only if $|T| \in \overline{\mathcal{M}_{|||\cdot|||}}$;*
3. *if $T \in \overline{\mathcal{M}_{|||\cdot|||}}$ and $A, B \in \mathcal{M}$, then $ATB \in \overline{\mathcal{M}_{|||\cdot|||}}$ and $|||ATB||| \leqslant \|A\| \cdot |||T||| \cdot \|B\|$.*

*In particular, $|||\cdot|||$ can be extended to a unitarily invariant norm, also denoted by $|||\cdot|||$, on $\overline{\mathcal{M}_{|||\cdot|||}}$.*

### *12.2. $\widetilde{\mathcal{M}}$ and $L^1(\mathcal{M}, \tau)$*

The following theorem is due to Nelson [14].

**Theorem 12.7.** *(See Nelson [14].) $\widetilde{\mathcal{M}}$ is a $*$-algebra and $T \in \widetilde{\mathcal{M}}$ if and only if $T$ is a closed, densely defined operator affiliated with $\mathcal{M}$. Furthermore, if $T \in \widetilde{\mathcal{M}}$ is a positive operator, then $\lim_{n\to\infty} \chi_{[0,n]}(T) = T$ in the measure topology.*

In the following, we define $s$-numbers for unbounded operators in $\widetilde{\mathcal{M}}$ as in [5].

**Definition 12.8.** For $T \in \widetilde{\mathcal{M}}$ and $0 \leqslant s \leqslant 1$, define the *sth numbers* of $T$ by

$$\mu_s(T) = \inf\bigl\{\|TE\|\colon\ E \in \mathcal{M} \text{ is a projection such that } \tau\bigl(E^\perp\bigr) = s\bigr\}.$$

**Theorem 12.9.** *(See Fack and Kosaki [5].) Let $T$ and $T_n$ be a sequence of operators in $\widetilde{\mathcal{M}}$ such that $\lim_{n\to\infty} T_n = T$ in the measure topology. Then for almost all $s \in [0,1]$, $\lim_{n\to\infty} \mu_s(T_n) = \mu_s(T)$.*

Let $\{T_n\}$ be a sequence of operators in $\mathcal{M}$ such that $T = \lim_{n\to\infty} T_n$ in the $L^1$-norm. By Lemma 3.8, $\{\tau(T_n)\}$ is a Cauchy sequence in $\mathbb{C}$. Define $\tau(T) = \lim_{n\to\infty} \tau(T_n)$. It is obvious that $\tau(T)$ does not depend on the sequence $\{T_n\}$. In this way, $\tau$ is extended to a linear functional on $L^1(\mathcal{M}, \tau)$.

**Lemma 12.10.** *Let $|||\cdot|||$ be a normalized unitarily invariant norm on a type $\mathrm{II}_1$ factor $\mathcal{M}$. If $T \in \overline{\mathcal{M}}_{|||\cdot|||}$ and $X \in \mathcal{M}$, then $TX \in L^1(\mathcal{M}, \tau)$.*

**Proof.** By the proof of Proposition 12.3, $\lim_{n\to\infty} T_n = T$ in the measure topology. Hence $\lim_{n\to\infty} T_n X = TX$ in the measure topology (see [14, Theorem 1]). By Corollary 6.4, $\|T_n X - T_m X\|_1 \leqslant |||T_n - T_m||| \cdot |||X|||^{\#}$. So $\{T_n X\}$ is a Cauchy sequence in the $L^1$-norm. By Corollary 12.5, $TX \in L^1(\mathcal{M}, \tau)$ and $\lim_{n\to\infty} T_n X = TX$ in the $L^1$-norm. $\square$

*12.3. Elements in $\overline{\mathcal{M}}_{|||\cdot|||}$*

**Lemma 12.11.** *For all $T \in \overline{\mathcal{M}}_{|||\cdot|||}$,*

$$|||T||| = \sup\bigl\{\bigl|\tau(TX)\bigr|:\ X \in \mathcal{M},\ |||X|||^{\#} \leqslant 1\bigr\}.$$

**Proof.** Let $\{T_n\}$ be a sequence of operators in $\mathcal{M}$ such that $\lim_{n\to\infty} T_n = T$ with respect to $|||\cdot|||$. By Corollary 6.4, if $X \in \mathcal{M}$ and $|||X|||^{\#} \leqslant 1$, then $|\tau(TX)| = \lim_{n\to\infty} |\tau(T_n X)| \leqslant \lim_{n\to\infty} |||T_n||| = |||T|||$. Therefore, $|||T||| \geqslant \sup\{|\tau(TX)|:\ X \in \mathcal{M},\ |||X|||^{\#} \leqslant 1\}$.

We need to prove that $|||T||| \leqslant \sup\{|\tau(TX)|:\ X \in \mathcal{M},\ |||X|||^{\#} \leqslant 1\}$. Let $\epsilon > 0$. Since $\lim_{n\to\infty} T_n = T$ with respect to $|||\cdot|||$, there is an $N$ such that $|||T - T_N||| < \epsilon/3$. For $T_N$, there is an $X \in \mathcal{M}$, $|||X|||^{\#} \leqslant 1$, such that $|||T_N||| \leqslant |\tau(T_N X)| + \epsilon/3$. By the proof of Lemma 12.10 and Corollary 6.4,

$$\begin{aligned}\bigl|\tau(TX) - \tau(T_N X)\bigr| &= \lim_{n\to\infty} \bigl|\tau(T_n X) - \tau(T_N X)\bigr| \\ &\leqslant \lim_{n\to\infty} |||T_n - T_N||| \cdot |||X|||^{\#} \leqslant |||T - T_N||| < \epsilon/3.\end{aligned}$$

So $|\tau(TX)| \geqslant |\tau(T_N X)| - |\tau((T_N - T)X)| \geqslant |||T_N||| - \epsilon/3 - \epsilon/3 \geqslant |||T||| - \epsilon$. Therefore, $|||T||| \leqslant \sup\{|\tau(TX)|:\ X \in \mathcal{M},\ |||X|||^{\#} \leqslant 1\}$. $\square$

The following theorem generalizes Theorem A. Its proof is based on Lemma 12.11 and is similar to the proof of Theorem A. So we omit the proof.

**Theorem 12.12.** *If $|||\cdot|||$ is a unitarily invariant norm on a type $\mathrm{II}_1$ factor $\mathcal{M}$, then there is a subset $\mathcal{F}'$ of $\mathcal{F}$ containing the constant 1 function on $[0,1]$ such that for all $T \in \overline{\mathcal{M}}_{|||\cdot|||}$,*

$$|||T||| = \sup\bigl\{|||T|||_f:\ f \in \mathcal{F}'\bigr\},$$

*where* $|||T|||_f$ *is defined in Lemma* 7.1 *by Eq.* (3) *or by Eq.* (4) *and* $\mathcal{F} = \{f(x) = a_1\chi_{[0,\frac{1}{n})}(x) + a_2\chi_{[\frac{1}{n},\frac{2}{n})}(x) + \cdots + a_n\chi_{[\frac{n-1}{n},1]}(x)\colon a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0, \frac{a_1+\cdots+a_n}{n} \leqslant 1, n = 1, 2, \ldots\}$.

Combining Theorems 12.12 and 12.9, we have the following corollary.

**Corollary 12.13.** *Let* $|||\cdot|||$ *be a unitarily invariant norm on a type* $\mathrm{II}_1$ *factor* $\mathcal{M}$ *and let* $|||\cdot|||'$ *be the corresponding symmetric gauge norm on* $(L^\infty[0,1], \int_0^1 dx)$ *as in Corollary* 2. *If* $T \in \widetilde{\mathcal{M}}$, *then* $T \in \overline{\mathcal{M}_{|||\cdot|||}}$ *if and only if* $\mu_s(T) \in \overline{L^\infty[0,\infty)_{|||\cdot|||'}}$. *In this case,* $|||T||| = |||\mu_s(T)|||'$.

**Example 12.14.** Let $T \in \widetilde{\mathcal{M}}$ and $1 \leqslant p \leqslant \infty$. Then $T \in L^p(\mathcal{M},\tau)$ if and only if $\mu_s(T) \in L^p([0,1])$. In this case, $\|T\|_p = (\int_0^1 \mu_s(T)^p\,ds)^{1/p} = (\int_0^\infty \lambda^p\,d\mu_{|T|}(\lambda))^{1/p}$.

### *12.4. A generalization of Hölder's inequality*

**Lemma 12.15.** *Let* $|||\cdot|||$ *be a unitarily invariant norm on a finite factor* $\mathcal{M}$ *and let* $T \in \overline{\mathcal{M}_{|||\cdot|||}}$ *be a positive operator. Then* $\lim_{n\to\infty}\chi_{[0,n]}(T) = T$ *with respect to* $|||\cdot|||$.

**Proof.** If $|||\cdot|||$ is singular, then $T \in \mathcal{M}$ by Lemma 11.3 and the lemma is obvious. We may assume that $|||\cdot|||$ is continuous. Let $T_n = \chi_{[0,n]}(T))$ and $\epsilon > 0$. By Lemma 12.2, there is a $\delta > 0$ such that if $\tau(E) < \delta$ then $|||TE||| < \epsilon$. There is an $N$ such that $\mu_s([N,\infty)) < \delta$. So for $m > n \geqslant N$, $|||T_m - T_n||| = |||T\cdot\chi_{(m,n]}(T)||| < \epsilon$. This implies that $\{T_n\}$ is a Cauchy sequence of $\mathcal{M}$ with respect to $|||\cdot|||$. Since $\lim_{n\to\infty} T_n = T$ in the measure topology, by Corollary 12.5, $\lim_{n\to\infty} T_n = T$ in the topology induced by $|||\cdot|||$. □

The following theorem is a generalization of Hölder's inequality.

**Theorem 12.16.** *Let* $|||\cdot|||$ *be a normalized unitarily invariant norm on a finite factor* $\mathcal{M}$. *If* $T \in \overline{\mathcal{M}_{|||\cdot|||}}$ *and* $S \in \overline{\mathcal{M}_{|||\cdot|||^\#}}$, *then* $TS \in L^1(\mathcal{M},\tau)$ *and* $\|TS\|_1 \leqslant |||T|||\cdot|||S|||^\#$.

**Proof.** By the polar decomposition and Corollary 12.6, we may assume that $S$ and $T$ are positive operators. Let $T_n = \chi_{[0,n]}(T)$ and $S_n = \chi_{[0,n]}(S)$. By Lemma 12.15, $\lim_{n\to\infty}|||T - T_n||| = \lim_{n\to\infty}|||S - S_n|||^\# = 0$. Let $K$ be a positive number such that $|||T_n||| \leqslant K$ and $|||S_n|||^\# \leqslant K$ for all $n$ and $\epsilon > 0$. Then there is an $N$ such that for all $m > n \geqslant N$, $|||T_m - T_n||| < \epsilon/(2K)$ and $|||S_m - S_n|||^\# < \epsilon/(2K)$. By Corollary 6.4, $\|T_mS_m - T_nS_n\|_1 \leqslant \|(T_m - T_n)S_m\|_1 + \|T_n(S_m - S_n)\|_1 \leqslant |||T_m - T_n|||\cdot|||S_m|||^\# + |||T_n|||\cdot|||S_m - S_n|||^\# < \epsilon$. This implies that $\{T_nS_n\}$ is a Cauchy sequence in $\mathcal{M}$ with respect to $\|\cdot\|_1$. Since $\lim_{n\to\infty} T_nS_n = TS$ in the measure topology, by Proposition 12.3, $\lim_{n\to\infty} T_nS_n = TS$ in $\|\cdot\|_1$. By Corollary 6.4, $\|T_nS_n\|_1 \leqslant |||T_n|||\cdot|||S_n|||^\#$ for every $n$. Hence, $\|TS\|_1 \leqslant |||T|||\cdot|||S|||^\#$. □

Combining Example 8.5 and Theorem 12.16, we obtain the non-commutative Hölder's inequality.

**Corollary 12.17.** *Let* $\mathcal{M}$ *be a finite factor with the faithful normal tracial state* $\tau$. *If* $T \in L^p(\mathcal{M},\tau)$ *and* $S \in L^q(\mathcal{M},\tau)$, *then* $TS \in L^1(\mathcal{M},\tau)$ *and*

$$\|TS\|_1 \leqslant \|T\|_p \cdot \|S\|_q,$$

*where* $1 \leqslant p, q \leqslant \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

## 13. Proof of Theorems H and I

In this section, we assume that $\mathcal{M}$ is a type $\mathrm{II}_1$ factor with the unique tracial state $\tau$, $|||\cdot|||$ is a unitarily invariant norm on $\mathcal{M}$ and $|||\cdot|||^{\#}$ is the dual unitarily invariant norm on $\mathcal{M}$ (see Definition 6.2). Let $|||\cdot|||_1$ be the corresponding symmetric gauge norm on $(L^{\infty}[0,1], \int_0^1 dx)$ as in Theorem D and $|||\cdot|||_1^{\#}$ be the dual norm on $(L^{\infty}[0,1], \int_0^1 dx)$.

**Lemma 13.1.** *If $\overline{\mathcal{M}}_{|||\cdot|||^{\#}}$ is the dual space of $\overline{\mathcal{M}}_{|||\cdot|||}$ in the sense of Question* 1*, then $\overline{L^{\infty}[0,1]}_{|||\cdot|||_1^{\#}}$ is the dual space of $\overline{L^{\infty}[0,1]}_{|||\cdot|||_1}$ in the sense of Question* 1.

**Proof.** By Corollary 2 and Lemma 2.6, there is a separable diffuse abelian von Neumann subalgebra $\mathcal{A}$ of $\mathcal{M}$ and a $*$-isomorphism $\alpha$ from $\mathcal{A}$ onto $L^{\infty}[0,1]$ such that $\tau = \int_0^1 dx \circ \alpha$ and $|||\alpha(T)|||_1 = |||T|||$ for each $T \in \mathcal{A}$. By Theorem E, $|||\alpha(T)|||_1^{\#} = |||T|||^{\#}$ for each $T \in \mathcal{A}$. So we need only prove that $\overline{\mathcal{A}}_{|||\cdot|||^{\#}}$ is the dual space of $\overline{\mathcal{A}}_{|||\cdot|||}$ in the sense of Question 1. Let $\phi \in \overline{\mathcal{A}_{|||\cdot|||}}^{\#}$. By the Hahn–Banach extension theorem, $\phi$ can be extended to a bounded linear functional $\psi$ on $\overline{\mathcal{M}}_{|||\cdot|||}$ such that $\|\psi\| = \|\phi\|$. By the assumption of the lemma, there is an operator $X \in \mathcal{M}^{\#}_{|||\cdot|||}$ such that $\psi(S) = \tau(SX)$ for all $S \in \overline{\mathcal{M}}_{|||\cdot|||}$ and $\|\psi\| = |||X|||^{\#}$. Let $X = U|X|$ be the polar decomposition of $X$ and $X_n = U \cdot \chi_{[0,n]}(|X|)$. By Lemma 12.15, $\lim_{n\to\infty} X_n = X$ with respect to the norm $|||\cdot|||^{\#}$. Let $Y_n = \mathbf{E}_{\mathcal{A}}(X_n)$ for $n = 1, 2, \ldots$. By Corollary 1, $\{Y_n\}$ is a Cauchy sequence in $\mathcal{A}$ with respect to the norm $|||\cdot|||^{\#}$ and $|||Y_n|||^{\#} \leqslant |||X_n|||^{\#}$. Let $Y = \lim_{n\to\infty} Y_n$ with respect to the norm $|||\cdot|||^{\#}$. Then $Y \in \mathcal{A}^{\#}_{|||\cdot|||}$ and $|||Y|||^{\#} \leqslant |||X|||^{\#} = \|\psi\| = \|\phi\|$. For $T \in \overline{\mathcal{A}}_{|||\cdot|||}$, $\phi(T) = \psi(T) = \tau(TX) = \lim_{n\to\infty} \tau(TX_n) = \lim_{n\to\infty} \tau(\mathbf{E}_{\mathcal{A}}(TX_n)) = \lim_{n\to\infty} \tau(TY_n) = \tau(TY)$. By Lemma 12.11, $\|\phi\| = |||Y|||^{\#}$. $\square$

Recall that $|||\cdot|||$ is a singular norm on $\mathcal{M}$ if $\lim_{\tau(E)\to 0+} |||E||| > 0$ and is a continuous norm on $\mathcal{M}$ if $\lim_{\tau(E)\to 0+} |||E||| = 0$ (see Section 11).

**Corollary 13.2.** *If $|||\cdot|||$ is a singular unitarily invariant norm on $\mathcal{M}$, then $\overline{\mathcal{M}}_{|||\cdot|||^{\#}}$ is not the dual space of $\overline{\mathcal{M}}_{|||\cdot|||}$ in the sense of Question* 1.

**Proof.** Since $|||\cdot|||$ is a singular norm on $\mathcal{M}$, by Lemma 11.3, $|||\cdot|||$ is equivalent to the operator norm on $\mathcal{M}$ and $\overline{\mathcal{M}}_{|||\cdot|||} = \mathcal{M}$. By Corollary 6.16 and Theorem 6.17, $|||\cdot|||^{\#}$ is equivalent to the $L^1$-norm on $\mathcal{M}$. So $|||\cdot|||_1$ is equivalent to the $L^{\infty}$-norm on $L^{\infty}[0,1]$ and $|||\cdot|||_1^{\#}$ is equivalent to the $L^1$-norm on $L^{\infty}[0,1]$ by Theorem E. Note that $\overline{L^{\infty}[0,1]}_{|||\cdot|||_1} = L^{\infty}[0,1]$ is not separable with respect to $|||\cdot|||_1$ but $\overline{L^{\infty}[0,1]}_{|||\cdot|||_1^{\#}}$ is separable with respect to $|||\cdot|||_1^{\#}$. So $\overline{L^{\infty}[0,1]}_{|||\cdot|||_1^{\#}}$ is not the dual space of $\overline{L^{\infty}[0,1]}_{|||\cdot|||_1}$ in the sense of Question 1. By Lemma 13.1, $\overline{\mathcal{M}}_{|||\cdot|||^{\#}}$ is not the dual space of $\overline{\mathcal{M}}_{|||\cdot|||}$ in the sense of Question 1. $\square$

**Lemma 13.3.** *If $|||\cdot|||$ is a continuous unitarily invariant norm on $\mathcal{M}$, then $\overline{\mathcal{M}}_{|||\cdot|||^{\#}}$ is the dual space of $\overline{\mathcal{M}}_{|||\cdot|||}$ in the sense of Question* 1.

**Proof.** We may assume that $|||1||| = 1$. By Theorem 12.16, $\overline{\mathcal{M}}_{|||\cdot|||^{\#}}$ is a subspace of the dual space of $\overline{\mathcal{M}}_{|||\cdot|||}$ in the sense of Question 1. Let $\phi$ be a linear functional in the dual space of $\overline{\mathcal{M}}_{|||\cdot|||}$. Then for every $T \in \overline{\mathcal{M}}_{|||\cdot|||}$, $|\phi(T)| \leqslant \|\phi\| \cdot |||T|||$. By Corollary 3.31, for every $T \in \mathcal{M}$,

$|\phi(T)| \leqslant \|\phi\| \cdot \|T\|$. So $\phi$ is a bounded linear functional on $\mathcal{M}$. Since $|||\cdot|||$ is a continuous norm on $\mathcal{M}$, $\lim_{\tau(E)\to 0} |||E||| = 0$. Hence, $\lim_{\tau(E)\to 0} \phi(E) = 0$. This implies that $\phi$ is an ultraweakly continuous linear functional on $\mathcal{M}$ and hence in the predual space of $\mathcal{M}$. So there is an operator $X \in L^1(\mathcal{M}, \tau)$ such that for all $T \in \mathcal{M}$, $\phi(T) = \tau(TX)$. By Lemma 12.11, $|||X|||^{\#} = \|\phi\| < \infty$. This implies that $X \in \overline{\mathcal{M}_{|||\cdot|||^{\#}}}$. So $\phi(T) = \tau(TX)$ for all $T \in \overline{\mathcal{M}_{|||\cdot|||}}$ and $\|\phi\| = |||X|||^{\#}$. This proves the lemma. □

**Proof of Theorems H and I.** Combining Lemmas 13.1, 13.3 and Theorem A gives the proof of Theorems H and I. □

**Example 13.4.** If $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $L^q(\mathcal{M}, \tau)$ is the dual space of $L^p(\mathcal{M}, \tau)$. $L^1(\mathcal{M}, \tau)$ is not the dual space of $\mathcal{M}$.

**Example 13.5.** For $1 < p < \infty$, $L^p(\mathcal{M}, \tau)$ is a reflexive space. $L^1(\mathcal{M}, \tau)$ and $\mathcal{M}$ are not reflexive spaces. By Theorem 6.17, for $0 \leqslant t \leqslant 1$, $\overline{\mathcal{M}_{|||\cdot|||_{(t)}}}$ is not a reflexive space.

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