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Automorphic spectral identities and applications to automorphic L-functions on GL_2 ☆

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ABSTRACT

We prove an automorphic spectral identity on GL_2 involving second moments. From it we obtain an asymptotic, with power-saving error term, for (non-archimedean) conductor-aspect integral moments, twisting by GL_1 characters ramifying at a fixed finite place. The strength of the spectral identity, and of the resulting asymptotics, is illustrated by extracting a subconvex bound in conductor aspect at a fixed finite prime.

1. Introduction

On GL_2 over a fixed number field we construct a Poincaré series which, on one hand, produces integral second moments of GL_2 automorphic L-functions $L(\frac{1}{2}+it, f\otimes\chi)$ attached to newforms f on GL_2, averaged over twisting against Hecke characters χ ramifying at a fixed finite place. On the other hand, we show that the Poincaré series has an explicable spectral expansion allowing a meromorphic continuation in an auxiliary complex parameter. Standard devices then produce an asymptotic with power-saving error term for the integral moments of $L(\frac{1}{2}+it, f\otimes\chi)$ over twists by χ, in terms of the finite-prime conductor of χ.

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It is well known that moments of zeta and L-functions capture subtle information [5,11–13,28, 31]. If one can prove even a *bound* for moments, it must be a sort of Lindelöf-on-average, since, presumably, one does not disprove the corresponding Lindelöf Hypothesis [8,14–17,20,21]. However, it is noteworthy that the proof that an asymptotic exists, and production of the leading constant in the asymptotic, goes beyond what would follow from Lindelöf. Further, similarly, the fact that the asymptotic has a power-saving error term does not follow from Lindelöf. Random matrix theory (see [7]) gives a heuristic for the leading constant and other parts of asymptotics for moments, but does not suggest proof mechanisms.

The present results demonstrate that asymptotics for integral moments, with power-saving error terms, can be extracted by standard methods from *automorphic spectral identities*, and show how to produce the relevant identities. Specifically, we produce asymptotics for integral moments for GL_2 automorphic L-functions summed over GL_1 twists ramifying at a single, fixed finite place. The method applies to GL_2 over an arbitrary number field. Until the recent works [9,10], which exclusively address t-aspect moments, there were no results on moments over arbitrary number fields. Here we use spectral methods to address non-archimedean conductor aspect moments.

Asymptotics for integral moments, with power-saving error terms, imply corresponding subconvexity bounds for the individual values in the moment, when the averaging family is not too large. The family of GL_1 twists we consider here is small enough to allow us to extract a subconvexity corollary. Until recently, there were few subconvexity results over general number fields, but this has changed in the last few years: [6] treats totally real fields by the shifted-sums method of [29], [10] treats t-aspect subconvexity over arbitrary number fields via integral moments by extending Good's [18,19] spectral idea, [27] gives a hybrid bound over arbitrary number fields (using methods involving ergodic theory and regularization of integrals of automorphic forms), and [3] treats totally real fields by shifted sums. Until Diaconu–Garrett's two papers and the most recent Michel–Venkatesh paper, subconvexity was discussed at most for totally real fields, often assuming narrow class number one.

Subconvex bounds have significant applications, such as the ternary quadratic forms problem treated in [4] and [6] (and see the survey [22]).

More specifically, about what is done here: Diaconu–Garrett [9] over an arbitrary number field, produced a spectral identity from which was extracted asymptotics with power-saving error term for integral moments of $L(\frac{1}{2}+it, f\otimes\chi)$, averaged not only over the critical line but also over twists by *unramified* Hecke characters χ. Here f is a newform on GL_2. Further analysis of the archimedean places gave a t-aspect subconvexity result there. In the present paper, we consider a *larger* family of twisting data χ, allowing arbitrary ramification of χ at a single fixed finite place. *Freezing* the archimedean data isolates dependence on the non-archimedean data, and we extract an asymptotic with power-saving error in terms of the (non-archimedean) conductor of χ. (See [25].)

The structure of this paper is as follows. Section 2 states the main result on asymptotics with power-saving error term for the integral moment of $L(\frac{1}{2}+it, f\otimes\chi)$ over χ unramified except for arbitrary ramification at a fixed finite prime. The subconvexity corollary is obtained immediately. Section 3 computes the integral of the Poincaré series against $|f|^2$ for cuspform f, obtaining the integral moment expansion. Section 4 determines the spectral expansion of the Poincaré series itself, thus providing the meromorphic continuation in the auxiliary complex parameter. Section 5 verifies the meromorphic continuation and vertical polynomial growth and obtains the asymptotics-with-error for the integral moment. Section 6 obtains the subconvexity corollary from the asymptotic-with-error.

Remarks.

- The relative weak exponent of our present subconvexity corollary is inevitable, since proving subconvexity by proving asymptotics with power-saving error is inefficient.
- A contrast between asymptotics for moments and subconvex bounds concerns the size of the family of L-functions. Since asymptotics for integral moments are essentially assertions that the corresponding Lindelöf Hypothesis is true *on average*, in our present state of knowledge integral moments over *smaller* families of twists give more information than averages over *larger* families. Thus, producing asymptotics for integral moments restricting to twisting by characters ramifying at a *single* finite place, as we do here, is stronger than obtaining an asymptotic for moments

allowing ramification at *all* finite places. However, asymptotics for moments over smaller families, while giving sharper Lindelöf-on-average results, typically produce weaker subconvexity results for individuals, exactly because they refer to smaller families of individuals.

- Not every asymptotic for integral moments (with or without power-saving error term) can give a subconvexity corollary. P. Sarnak has called asymptotics *pregnant* if a power-saving error term would give a subconvex corollary. It is easy to distinguish these cases.
- Good, Diaconu–Goldfeld, and Diaconu–Garrett all emphasized t-aspect moments (and subconvexity corollaries), and no one had viewed conductor-aspect moments from a spectral-identity viewpoint [1,2,26]. Indeed, from a neo-classical viewpoint it is less easy to see how a spectral identity could or would produce a suitable finite-prime moment asymptotic, and part of the point of the present paper is to make clear that, especially in an adele-group setting, a completely analogous line of argument is possible. Thus, there is some prospect of extending to larger groups this spectral argument for asymptotics of moments.

2. The main result

In this paper we use spectral identities to obtain a moment asymptotic in the finite prime conductor-aspect for a family of L-functions $L(\frac{1}{2}+it, f\otimes\chi)$, where χ has arbitrary ramification at a fixed finite prime v_1. The moment expansion is a sum of weighted integrals of L-functions $L(s, f\otimes\chi)$ of twists of f by idele class characters χ. The weight functions depend on archimedean data and data associated with the finite place v_1. We make a non-trivial choice of data at v_1 and at the archimedean place, and obtain asymptotics with power-saving error term for $L(\frac{1}{2}+it, f\otimes\chi)$. We isolated the non-archimedean part by freezing the archimedean part. This yields our main theorem in Section 5:

Main Theorem. *For a cuspform f on $GL_2(k)$, where k is a number field of degree d over $\mathbb{Q}$, the finite prime conductor-aspect moment asymptotic for the twisted L-function $L(\frac{1}{2}+it, f\otimes\chi)$ is*

$$\sum_{\chi:\, q^N\leqslant T}\int_{-\infty}^{\infty}\left|L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2\cdot\mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right)dt \ll_\epsilon T^{1+\epsilon}\quad (\textit{for all } \epsilon>0)$$

where q^N, with $N\geqslant 1$, is the finite prime conductor of χ and $\mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)$ refers to the archimedean data.

We applied these ideas and methods from analytic number theory to obtain a subconvexity corollary:

Example application. Fix a number field k of degree d over $\mathbb{Q}$ and a cuspform f on $GL_2(k)$. For a computable constant $\vartheta<1$,

$$L\left(\frac{1}{2}+it, f\otimes\chi\right)\ll_\epsilon \left(q^N\right)^{\frac{d-1+\vartheta}{2}+\epsilon}\quad (\text{for all } \epsilon>0)$$

3. The moment expansion

In this section, the integral moment expansion is obtained by unwinding the integral representation

$$\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} P\acute{e}\cdot|f|^2$$

where *Pé* is a Poincaré series and f is a cuspform on GL_2. We then obtain asymptotics from the weight functions.

3.1. Unwinding to an Euler product

Define the following subgroups of $G = GL_2$:

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \qquad N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}, \qquad H = \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$Z = \text{center of } G, \qquad M = ZH = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$$

For any place v of k, let $K_v^{\max}$ be the standard maximal compact subgroup. So for finite v,

$$K_v^{\max} = GL_2(\mathfrak{o}_v)$$

and for infinite v,

$$K_v^{\max} = \begin{cases} O_2 & (v \approx \mathbb{R}) \\ U_2 & (v \approx \mathbb{C}) \end{cases}$$

The Poincaré series *Pé* is of the form

$$P\acute{e}(g) = \sum_{\gamma \in M_k \backslash G_k} \varphi(\gamma g) \quad (\text{where } g \in G_{\mathbb{A}}) \tag{3.1}$$

for suitable functions φ on $G_{\mathbb{A}}$ defined as follows. Let

$$\varphi = \bigotimes_v \varphi_v$$

where for finite primes $v \neq v_1$,

$$\varphi_v(g) = \begin{cases} \chi_{0,v}(m) = |\frac{a}{d}|_v^{s'} & (\text{for } g = mk,\ m = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in M,\ s' \in \mathbb{C},\ k \in K_v^{\max}) \\ 0 & (\text{otherwise}) \end{cases}$$

For finite $v = v_1$ (at which χ is allowed to be ramified)

$$\varphi_v(mg) = \left|\frac{a}{d}\right|_v^{s'} \cdot \varphi_v(g) \quad (m \in M_v,\ g \in G_v)$$

The data determining φ_v for $v = v_1$ consists of its values on N_v where our simple choice is

$$\varphi_v \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{cases} 1 & (\text{for } x \in \mathfrak{o}_v) \\ |x|_v^{-w'} & (\text{for } w' \in \mathbb{C},\ x \notin \mathfrak{o}_v) \end{cases} \tag{3.2}$$

For infinite v require right K_v-invariance and left equivariance:

$$\varphi_v(mg) = \left|\frac{a}{d}\right|_v^{s'} \cdot \varphi_v(g) \quad (m \in M_v,\ g \in G_v)$$

where

$$\varphi_v\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}=\begin{cases}(1+|x|^2)^{-\frac{w}{2}} & (\text{for } v\approx\mathbb{R},\ w\in\mathbb{C})\\ (1+x\bar{x})^{-w} & (\text{for } v\approx\mathbb{C})\end{cases}$$

The Poincaré series $P\acute{e}$ converges absolutely and locally uniformly for $\Re(s')>1$, $\Re(w)>1$ for all $v|\infty$, and for $\Re(w')>1$ (see Proposition 2.6 in [9]).

Lemma 3.1.

$$\int\limits_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} P\acute{e}\cdot|f|^2\,dg$$

is an integral of products of local factors of standard L-functions.

Proof. The Fourier expansion of a cuspform f on $G_{\mathbb{A}}$ is

$$f(g)=\sum_{\xi\in Z_k\backslash M_k} W_f(\xi g)$$

where W_f is the Whittaker function of f and $W_f=\bigotimes_v W_{f,v}$ is the factorization of W_f into local data. So

$$\begin{aligned}\int\limits_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} P\acute{e}\cdot|f|^2\,dg &= \int\limits_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}}\sum_{\gamma\in M_k\backslash G_k}\varphi(\gamma g)\big|f(g)\big|^2\,dg=\int\limits_{Z_{\mathbb{A}}M_k\backslash G_{\mathbb{A}}}\varphi(g)\big|f(g)\big|^2\,dg\\ &=\int\limits_{Z_{\mathbb{A}}M_k\backslash G_{\mathbb{A}}}\varphi(g)\sum_{\xi\in Z_k\backslash M_k}W_f(\xi g)\overline{f}(g)\,dg=\int\limits_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\varphi(g)W_f(g)\overline{f}(g)\,dg\end{aligned}$$

Let C be the idele class group $GL_1(k)\backslash GL_1(\mathbb{A})$ and $\hat{C}$ its dual. $\hat{C}\approx\mathbb{R}\times\hat{C}_0$ where $\hat{C}_0$ is discrete. The Mellin transform and inversion are

$$f(x)=\int\limits_{\hat{C}}\int\limits_{C} f(y)\chi^{-1}(y)\,dy\,\chi(x)\,d\chi=\sum_{\chi'\in\hat{C}_0}\frac{1}{2\pi i}\int\limits_{\Re(s)=\sigma}\int\limits_{C} f(y)\chi'^{-1}(y)|y|^{-s}\,dy\,\chi'(x)|x|^s\,ds$$

With $Z_{\mathbb{A}}M_k\backslash M_{\mathbb{A}}\approx C$, and for finite $v\neq v_1$,

$$\begin{aligned}&\int\limits_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\varphi(g)W_f(g)\overline{f}(g)\,dg\\ &=\int\limits_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\varphi(g)W_f(g)\Bigg(\int\limits_{\hat{C}}\int\limits_{Z_{\mathbb{A}}M_k\backslash M_{\mathbb{A}}}\overline{f}\big(m'g\big)\chi\big(m'\big)\,dm'\,d\chi\Bigg)\,dg\\ &=\int\limits_{\hat{C}}\Bigg(\int\limits_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}}\varphi(g)W_f(g)\int\limits_{Z_{\mathbb{A}}M_k\backslash M_{\mathbb{A}}}\sum_{\xi\in Z_k\backslash M_k}\overline{W}_f\big(\xi m'g\big)\chi\big(m'\big)\,dm'\,dg\Bigg)\,d\chi\end{aligned}$$

$$= \int_{\hat{C}} \left(\int_{Z_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi(g) W_f(g) \int_{Z_{\mathbb{A}} \backslash M_{\mathbb{A}}} \overline{W}_f\big(m'g\big) \chi\big(m'\big)\, dm'\, dg \right) d\chi$$

$$= \int_{\hat{C}} \prod_{v} \left(\int_{Z_v \backslash G_v} \int_{Z_v \backslash M_v} \varphi_v(g_v) W_{f,v}(g_v) \overline{W}_{f,v}\big(m'_v g_v\big) \chi_v\big(m'_v\big)\, dm'_v\, dg_v \right) d\chi$$

Suppress finite $v \neq v_1$, invoke the v-adic Iwasawa decomposition $G = MNK$ and write the vth local integral as

$$\int_{Z \backslash MNK} \int_{Z \backslash M} \varphi(mnk) W_f(mnk) \overline{W}_f\big(m'mnk\big) \chi\big(m'\big)\, dm'\, dm\, dn\, dk$$

For simplicity, take φ and f to be right $K_v^{\max}$-invariant for finite $v \neq v_1$. This gives

$$\int_{Z \backslash MN} \int_{Z \backslash M} \varphi(mn) W_f(mn) \overline{W}_f\big(m'mn\big) \chi\big(m'\big)\, dm'\, dm\, dn$$

Replace m' by $m'm^{-1}$ to get

$$\int_{Z \backslash MN} \int_{Z \backslash M} \varphi(mn) W_f(mn) \overline{W}_f\big(m'n\big) \chi\big(m'\big) \chi^{-1}(m)\, dm'\, dm\, dn$$

The Whittaker function has the equivariance

$$W_f(ng) = \psi(n) W_f(g) \quad (n \in N_{\mathbb{A}})$$

Thus,

$$W_f(mn) = W_f\big(mnm^{-1}m\big) = \psi\big(mnm^{-1}\big) W_f(m) \quad \big(\text{since } mnm^{-1} \in N\big)$$

and

$$\overline{W}_f\big(m'n\big) = \overline{W}_f\big(m'nm^{-1}m\big) = \overline{\psi}\big(m'nm'^{-1}\big) \overline{W}_f\big(m'\big)$$

so obtaining

$$\int_{Z \backslash MN} \int_{Z \backslash M} \varphi(mn) W_f(m) \overline{W}_f\big(m'\big) \chi\big(m'\big) \chi^{-1}(m) \psi\big(mnm^{-1}\big) \overline{\psi}\big(m'nm'^{-1}\big)\, dm'\, dm\, dn$$

Let

$$X\big(m, m'\big) = \int_N \varphi(n) \psi\big(mnm^{-1}\big) \overline{\psi}\big(m'nm'^{-1}\big)\, dn$$

We get

$$\int_{Z\backslash M}\int_{Z\backslash M} \chi_0(m)W_f(m)\overline{W}_f\big(m'\big)\chi\big(m'\big)\chi^{-1}(m)X\big(m,m'\big)\,dm'\,dm$$

Now

$$W_f(mn)=\psi\big(mnm^{-1}\big)\cdot W_f(m)$$

and

$$W_f(mn)=W_f(m)\cdot 1$$

by the right K-invariance of W_f. So for $W_f(m)\neq 0$, $\psi(mnm^{-1})=1$, and $X(m,m')=1$ for m, m' in the support of W_f. So

$$\begin{aligned}&\int_{Z\backslash M}\big(\chi_0\cdot\chi^{-1}\big)(m)W_f(m)\,dm\cdot\int_{Z\backslash M}\chi\big(m'\big)\overline{W}_f\big(m'\big)\,dm'\\ &\quad= L_v\big(\chi_{0,v}\cdot\chi_v^{-1}|y|_v^{\frac{1}{2}},f\big)\cdot L_v\big(\chi_v\big|y'\big|_v^{\frac{1}{2}},\overline{f}\big)\quad\left(\text{where } m=\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix},\ m'=\begin{pmatrix} y' & 0\\ 0 & 1\end{pmatrix}\right)\end{aligned}$$

is a product of local factors of L-functions at finite primes $v\neq v_1$. $\square$

Thus the integral can be written as

$$\begin{aligned}I(\chi_0)=\sum_{\chi\in\hat{C}_0}\frac{1}{2\pi i}\int_{\Re(s)=\sigma} &L\big(\chi_0\cdot\chi^{-1}|y|^{1-s},f\big)\cdot L\big(\chi\big|y'\big|^{s},\overline{f}\big)\cdot\mathcal{K}_{v_1}\big(w',\chi_{v_1}\big)\\ &\cdot\mathcal{K}_\infty(s,\chi_0,\chi)\,ds\end{aligned}\tag{3.3}$$

where

$$\mathcal{K}_\infty(s,\chi_0,\chi)=\prod_{v|\infty}\mathcal{K}_v(s,\chi_{0,v},\chi_v)$$

and

$$\begin{aligned}\mathcal{K}_v(s,\chi_{0,v},\chi_v)=\int_{Z_v\backslash M_vN_v}\int_{Z_v\backslash M_v}&\varphi_v(m_vn_v)W_{f,v}(m_vn_v)\overline{W}_{f,v}\big(m'_vn_v\big)\\ &\cdot\chi_v\big(m'_v\big)\big|m'_v\big|_v^{s-\frac{1}{2}}\chi_v^{-1}(m_v)|m_v|_v^{\frac{1}{2}-s}\,dm'_v\,dm_v\,dn_v\end{aligned}$$

$$\begin{aligned}\mathcal{K}_{v_1}\big(w',\chi_{v_1}\big)=\int_{k_v^\times}\int_{k_v^\times}&\chi(y)|y|_{v_1}^{s}\chi^{-1}\big(y'\big)\big|y'\big|_{v_1}^{1-s}W\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\overline{W}\begin{pmatrix} y' & 0\\ 0 & 1\end{pmatrix}\\ &\cdot\int_{k_v}\overline{\psi}\big(x\cdot\big(y-y'\big)\big)\varphi_{v_1}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}dx\,dy\,dy'\end{aligned}$$

The non-decoupled integrals $\mathcal{K}_v(s, \chi_{0,v}, \chi_v)$ and $\mathcal{K}_{v_1}(w', \chi_{v_1})$, which represent the weight functions, will be subsequently computed. For $\Re(s')$ and $\Re(w')$ sufficiently large, the integral $I(\chi_0) = I(s', w')$ is

$$I(s', w') = \sum_{\chi \in \hat{C}_{0,S}} \frac{1}{2\pi i} \int_{\Re(s)=\sigma} L\left(\chi^{-1}|.|^{s'+1-s}, f\right) \cdot L\left(\chi|.|^{s}, \overline{f}\right) \cdot \mathcal{K}_{v_1}\left(w', \chi_{v_1}\right) \cdot \mathcal{K}_\infty\left(s, s', w, \chi\right) ds \tag{3.4}$$

where S is a finite set of places including archimedean places, and the sum is over the set $\hat{C}_{0,S}$ of characters ramified at the finite place v_1. $I(s', w')$ has meromorphic continuation to a region in $\mathbb{C}^2$ containing the point $s' = 0$, $w' = 1$, and $I(0, w')$ is holomorphic for $\Re(w') > \frac{11}{18}$ except for $w' = 1$ where it has a pole of order 1. We will find asymptotics for $\mathcal{K}_{v_1}(w', \chi_{v_1})$ and $\mathcal{K}_\infty(s, s', w, \chi)$, shifting the line of integration to $\Re(s) = \frac{1}{2}$ and setting $s' = 0$. Thus for $\Re(w')$ sufficiently large

$$\begin{aligned} I(0, w') &= \sum_{\chi \in \hat{C}_0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} L\left(\chi^{-1}|.|^{\frac{1}{2}-it}, f\right) \cdot L\left(\chi|.|^{\frac{1}{2}+it}, \overline{f}\right) \cdot \mathcal{K}_{v_1}\left(w', \chi_{v_1}\right) \\ &\quad \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it, 0, w, \chi\right) dt \\ &= \sum_{\chi} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \mathcal{K}_{v_1}\left(w', \chi_{v_1}\right) \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it, 0, w, \chi\right) dt \end{aligned} \tag{3.5}$$

3.2. The non-decoupled integrals

Lemma 3.2. *The non-decoupled integral*:

$$\mathcal{K}_{v_1}\left(w', \chi_{v_1}\right) = \frac{q^{1-Nw'}}{q-1} \cdot \frac{1 - |\alpha|^2|\beta|^2 q^{-2w'}}{(1 - |\alpha|^2 q^{-w'})(1 - |\beta|^2 q^{-w'})(1 - \overline{\alpha}\beta q^{-w'})(1 - \alpha\overline{\beta} q^{-w'})}$$

Proof. Henceforth, we will suppress the v for ease of notation. ψ is the standard additive character which is trivial on the local integers $\mathfrak{o}$ and non-trivial on $\varpi^{-1}\mathfrak{o}$. χ is a ramified multiplicative character, i.e. χ is non-trivial on $\mathfrak{o}^\times$. W is a Whittaker function which is invariant on $\mathfrak{o}^\times$ since it is spherical. The spherical Whittaker function is of the form

$$W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, & n \geqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

where α, β are Satake's parameters and $ord(y) = n$. We will first compute the integral in y and y', and then compute the integral in x. Now

$$\overline{\psi}\left(x\left(y - y'\right)\right) = \overline{\psi}\left(xy - xy'\right) = \overline{\psi}(xy) \cdot \psi\left(xy'\right)$$

Thus the integrals in y and y' are as follows:

$$\int_{k^\times} \overline{\psi}(xy)\chi(y)|y|^s W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy \cdot \int_{k^\times} \psi\left(xy'\right)\chi^{-1}\left(y'\right)\left|y'\right|^{1-s} \overline{W}\begin{pmatrix} y' & 0 \\ 0 & 1 \end{pmatrix} dy'$$

Consider the integral in y:

$$\int_{k^\times} \overline{\psi}(xy)\chi(y)|y|^s W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy$$

Let $\eta \in \mathfrak{o}^\times$. Replace y with $y\eta$ to get

$$\int_{k^\times}\left(\int_{\mathfrak{o}^\times} \overline{\psi}(xy\eta)\chi(y\eta)\,d\eta\right)|y|^s W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy$$

Consider the inner integral:

$$\int_{\mathfrak{o}^\times} \overline{\psi}(xy\eta)\chi(y\eta)\,d\eta$$

Recall that χ is a ramified character. Let N be the conductor of χ. So χ is trivial on some subgroup $1+\mathfrak{m}^N$ of $k^\times$ and non-trivial on $1+\mathfrak{m}^{N-1}$ where $N \geqslant 1$ is the smallest such an integer. A standard computation [30] shows that

$$\int_{k^\times} \psi(xy)\chi(y)\,dy = 0 \quad \text{unless } ord(x) = -N$$

We claim that our inner integral is zero unless $ord(xy) = -N$. So $ord(y) = -ord(x) - N$. The integral

$$\int_{\mathfrak{o}^\times} \overline{\psi}(xy\eta)\chi(y\eta)\,d\eta$$

is a Gauss sum. A Gauss sum, $\mathfrak{g}(\chi,\psi)$, where χ is a ramified multiplicative character with conductor N, is

$$\mathfrak{g}(\chi,\psi) = \int_{\mathfrak{o}^\times} \chi(x)\cdot\overline{\psi}\left(\frac{x}{\varpi^N}\right)dx = \frac{q^{2-N}}{(q-1)^2}$$

The integral in y' is:

$$\int_{k^\times}\left(\int_{\mathfrak{o}^\times} \psi(xy't)\overline{\chi}(y't)\,dt\right)|y'|^{1-s}\,\overline{W}\begin{pmatrix} y' & 0 \\ 0 & 1 \end{pmatrix} dy', \quad t \in \mathfrak{o}^\times$$

the conjugate of the integral in y. Thus, by replacing $y\eta$ with u, and x with m, the integrals over $\mathfrak{o}^\times$ in y and y' are

$$\left|\int_{\mathfrak{o}^\times} \overline{\psi}(xy\eta)\chi(y\eta)\,d\eta\right|^2 = \left|\int_{\mathfrak{o}^\times} \overline{\psi}\left(\frac{mu}{\varpi^N}\right)\chi(u)\,du\right|^2 = \frac{q^{2-N}}{(q-1)^2}$$

So the entire non-decoupled local integral becomes

$$\frac{q^{2-N}}{(q-1)^2}\left[\int_{k^\times} |y|^s W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy \cdot \int_{k^\times} \left|y'\right|^{1-s} \overline{W}\begin{pmatrix} y' & 0 \\ 0 & 1 \end{pmatrix} dy' \cdot \int_k \varphi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} dx\right]$$

Recall that the integral is zero unless $ord(y) = -ord(x) - N$. So rewrite the integral as:

$$\begin{aligned} &\frac{q^{2-N}}{(q-1)^2} \cdot \int_k \varphi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \int_{ord(y)=-ord(x)-N} |y|^s W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy \\ &\quad \cdot \int_{ord(y')=-ord(x)-N} \left|y'\right|^{1-s} \overline{W}\begin{pmatrix} y' & 0 \\ 0 & 1 \end{pmatrix} dy'\, dx \\ &= \frac{q^{2-N}}{(q-1)^2} \cdot \int_k \varphi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \int_{ord(y)=-ord(x)-N} |y| \left|W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right|^2 dy\, dx \end{aligned}$$

Since $ord(y) = -ord(x) - N$, then y can be written as

$$y = \frac{t}{\varpi^N x}, \quad t \in \mathfrak{o}^\times$$

So the entire integral is:

$$\frac{q^{2-N}}{(q-1)^2} \cdot \int_k \varphi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \left|\frac{1}{\varpi^N x}\right| \left|W\begin{pmatrix} \frac{1}{\varpi^N x} & 0 \\ 0 & 1 \end{pmatrix}\right|^2 dx$$

Now $y \to W\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is supported on $\mathfrak{o} \cap k^\times$, so $ord(x) \leqslant -N$. Then $x \notin \mathfrak{o}$. Thus, we integrate over $k^\times$, and by a change in Haar's measure, the integral becomes

$$\begin{aligned} &\frac{q^{2-N}}{(q-1)^2} \cdot \int_k |x|^{-w'} \cdot \left|\frac{1}{\varpi^N x}\right| \left|W\begin{pmatrix} \frac{1}{\varpi^N x} & 0 \\ 0 & 1 \end{pmatrix}\right|^2 dx \\ &= \frac{q^{1-N}}{q-1} \cdot \int_{k^\times} |x|^{1-w'} \cdot \left|\frac{1}{\varpi^N x}\right| \left|W\begin{pmatrix} \frac{1}{\varpi^N x} & 0 \\ 0 & 1 \end{pmatrix}\right|^2 dx \end{aligned}$$

Invert x to get

$$\frac{q^{1-N}}{q-1} \cdot \int_{k^\times} |x|^{w'-1} \cdot \left|\frac{x}{\varpi^N}\right| \left|W\begin{pmatrix} \frac{x}{\varpi^N} & 0 \\ 0 & 1 \end{pmatrix}\right|^2 dx$$

Replace x by $\varpi^N x$ and let $ord(x) = \ell$ to get

$$\mathcal{K}_{v_1}\big(w', \chi_{v_1}\big) = \frac{q^{1-N}}{q-1} \cdot q^{-Nw'} \cdot q^N \cdot \int\limits_{k^\times} |x|^{w'-1} \cdot |x| \cdot \left| W \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right|^2 dx$$

$$= \frac{q}{q-1} \cdot q^{-Nw'} \cdot \sum_{\ell=0}^{\infty} q^{-\ell w'} \cdot \frac{\alpha^{\ell+1} - \beta^{\ell+1}}{\alpha - \beta} \cdot \frac{\overline{\alpha}^{\ell+1} - \overline{\beta}^{\ell+1}}{\overline{\alpha} - \overline{\beta}}$$

$$= \frac{q^{1-Nw'}}{q-1} \cdot \frac{1 - |\alpha|^2|\beta|^2 q^{-2w'}}{(1-|\alpha|^2 q^{-w'})(1-|\beta|^2 q^{-w'})(1-\overline{\alpha}\beta q^{-w'})(1-\alpha\overline{\beta} q^{-w'})} \qquad \square$$

3.3. Asymptotics

Lemma 3.3.

$$\mathcal{K}_{v_1}\big(w', \chi_{v_1}\big) \ll \big(q^N\big)^{-w'}$$

Proof.

$$\frac{q}{q-1} \cdot \frac{1 - |\alpha|^2|\beta|^2 q^{-2w'}}{(1-|\alpha|^2 q^{-w'})(1-|\beta|^2 q^{-w'})(1-\overline{\alpha}\beta q^{-w'})(1-\alpha\overline{\beta} q^{-w'})}$$

is independent of the conductor q^N of χ. $\square$

The non-decoupled integral

$$\mathcal{K}_\infty(s, \chi_0, \chi) = \prod_v \mathcal{K}_v(s, \chi_{0,v}, \chi_v) = \prod_v \mathcal{K}_v\big(s, s', w, \chi_v\big)$$

has the following asymptotic formula.

For v complex,

$$\mathcal{K}_v\big(s, s', w, \chi_v\big) = \pi^{-2s'+1} A\big(s', w, \mu_1, \mu_2\big) \cdot \big(1 + \ell_v^2 + 4(t+t_v)^2\big)^{-w} \cdot \Big[1 + O\Big(\Big(\sqrt{1 + \ell_v^2 + 4(t+t_v)^2}\,\Big)^{-1}\Big)\Big]$$

where $A(s', w, \mu_1, \mu_2)$ is the ratio of products of gamma functions

$$2^{4w-4s'-4} \frac{\Gamma(w+s'+i\mu_1+i\overline{\mu}_2)\Gamma(w+s'-i\mu_1+i\overline{\mu}_2)\Gamma(w+s'+i\mu_1-i\overline{\mu}_2)\Gamma(w+s'-i\mu_1-i\overline{\mu}_2)}{\Gamma(2w+2s')}$$

and it_v, ℓ_v are the parameters of the local component χ_v of χ.

For v real,

$$\mathcal{K}_v\big(s, s', w, \chi_v\big) = B\big(s', w, \mu_1, \mu_2\big) \cdot \big(1 + |t+t_v|\big)^{-w} \cdot \Big[1 + O\Big(\big(1 + |t+t_v|\big)^{-\frac{1}{2}}\Big)\Big]$$

where $B(s', w, \mu_1, \mu_2)$ is a similar ratio of products of gamma functions. (See Section 5 in [9].)

4. Spectral decomposition of the Poincaré series

In this section, we spectrally decompose the Poincaré series. This is central to the ideas underlying the integral moments of automorphic L-functions on GL_2 to prove the meromorphic continuation of the Poincaré series. The decomposition consists of a leading (non-L^2) term, cuspidal part and continuous part.

4.1. The cuspidal part

Let F be a cuspform on $G_{\mathbb{A}}$ generating a spherical representation locally everywhere, and suppose F corresponds to a spherical vector everywhere locally. The Fth (cuspidal) component of the spectral decomposition of the Poincaré series is $\langle Pé, F\rangle \cdot F$. So

$$\begin{aligned}\langle Pé, F\rangle &= \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} Pé(g)\cdot \overline{F}(g)\,dg = \int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}} \sum_{\gamma\in M_k\backslash G_k} \varphi(\gamma g)\,\overline{F}(g)\,dg \\ &= \int_{Z_{\mathbb{A}}M_k\backslash G_{\mathbb{A}}} \varphi(g)\overline{F}(g)\,dg = \int_{Z_{\mathbb{A}}M_k\backslash G_{\mathbb{A}}} \varphi(g) \sum_{\xi\in Z_k\backslash M_k} \overline{W}_F(\xi g)\,dg = \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \varphi(g)\overline{W}_F(g)\,dg \\ &= \prod_{v<\infty, v\neq v_1} \int_{Z_v\backslash G_v} \varphi_v(g)\overline{W}_{F,v}(g)\,dg \cdot \int_{Z_{v_1}\backslash G_{v_1}} \varphi_{v_1}(g)\overline{W}_{F,v_1}(g)\,dg \\ &\quad \cdot \prod_{v|\infty} \int_{Z_v\backslash G_v} \varphi_v(g)\overline{W}_{F,v}(g)\,dg \end{aligned} \tag{4.1}$$

Lemma 4.1. *For $v\neq v_1$, the vth local factor of $\langle Pé, F\rangle$ is $L_v(s'+\frac{1}{2}, \overline{F})$.*

Proof. Suppress v. By the Iwasawa decomposition and right K_v-invariance, we get

$$\int_{Z\backslash MN} \varphi(mn)\overline{W}_F(mn)\,dm\,dn$$

Further, with $Z\backslash MN \approx HN$ and

$$W_F(mn) = \psi\big(mnm^{-1}\big)W_F(m)$$

we get

$$\int_H\int_N \chi_0(m)\varphi(n)\overline{\psi}\big(mnm^{-1}\big)\overline{W}_F(m)\,dm\,dn$$

Again, for m in the support of W_F and $n\in N\cap K$

$$\int_N \varphi(n)\overline{\psi}\big(mnm^{-1}\big)\,dn = 1$$

So the integral becomes

$$\int_H \chi_0(m)\overline{W}_F(m)\,dm = \int_{k^\times} |y|^{s'}\overline{W}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy = \left(1-\bar{\alpha}q^{-s'}\right)^{-1}\left(1-\bar{\beta}q^{-s'}\right)^{-1}$$
$$= L_v\left(s'+\frac{1}{2}, \overline{F}\right) \qquad \square$$

Lemma 4.2. *For $v = v_1$, the vth local factor of $\langle Pé, F\rangle$ is*

$$L_v\left(s'+\frac{1}{2}, \overline{F}\right) + \frac{(q-1)(q^{-w'} - \frac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})(1-\beta^{-1}q^{1-w'+s'})} \cdot L_v\left(s'+\frac{1}{2}, \overline{F}\right)$$
$$- q^{-w'} L_v\left(s'+w'-\frac{1}{2}, \overline{F}\right)$$

Proof. For $v = v_1$, $\langle Pé, F\rangle$ unwinds to

$$\int_H \int_N \chi_0(m)\varphi(n)\overline{\psi}\left(mnm^{-1}\right)\overline{W}_F(m)\,dm\,dn$$

Now

$$mnm^{-1} = \begin{pmatrix} 1 & xy \\ 0 & 1 \end{pmatrix}$$

So the integral becomes

$$\int_k \int_{k^\times} \overline{\psi}(xy)|y|^{s'}\overline{W}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\varphi(x)\,dy\,dx$$

We will first consider the integral in y

$$\int_{k^\times} \overline{\psi}(xy)|y|^{s'}\overline{W}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy$$

where $ord(y) = n$. For $x \in \mathfrak{o}$, since ψ is trivial on $\mathfrak{o}$, the integral is

$$\sum_{n=0}^{\infty} q^{-ns'}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} = \left(1-\alpha q^{-s'}\right)^{-1}\left(1-\beta q^{-s'}\right)^{-1} = L_v\left(s'+\frac{1}{2}, \overline{F}\right)$$

For $x \notin \mathfrak{o}$ (i.e. for $ord(x) < 0$), we first evaluate

$$\int_{\mathfrak{o}^\times} \overline{\psi}(xy)\,dy$$

Let $ord(x) = m$. Write

$$y = \varpi^n t, \qquad x = \varpi^m \eta, \quad t, \eta \in \mathfrak{o}^\times$$

Then, replacing $t\eta$ by u, the integral becomes

$$\int_{\mathfrak{o}^\times} \overline{\psi}\big(\varpi^{m+n}u\big)\,du$$

Now

$$\int_{\mathfrak{o}^\times} \overline{\psi}\big(\varpi^{\ell}u\big)\,du = \int_{\mathfrak{o}} \overline{\psi}\big(\varpi^{\ell}u\big)\,du - \int_{\mathfrak{m}} \overline{\psi}\big(\varpi^{\ell}u\big)\,du$$

For $\ell \geqslant 0$, the integrand is 1 so

$$\int_{\mathfrak{o}^\times} \overline{\psi}\big(\varpi^{\ell}u\big)\,du = meas\big(\mathfrak{o}^\times\big) = 1$$

For $\ell = -1$, $\overline{\psi}$ is non-trivial on $\mathfrak{o}$ and trivial on $\mathfrak{m}$, so

$$\int_{\mathfrak{o}^\times} \overline{\psi}\big(\varpi^{\ell}u\big)\,du = -meas(\mathfrak{m}) = -\frac{1}{q-1}$$

For $\ell \leqslant -2$, $\overline{\psi}$ is non-trivial on $\mathfrak{o}$ and on $\mathfrak{m}$, so

$$\int_{\mathfrak{o}^\times} \overline{\psi}\big(\varpi^{\ell}u\big)\,du = 0$$

So keeping in mind that $ord(y) = n$ and $ord(x) = m$,

$$\int_{\mathfrak{o}^\times} \overline{\psi}(xy)\,dy = \int_{\mathfrak{o}^\times} \overline{\psi}\big(\varpi^{m+n}u\big)\,du = \begin{cases} 1 & (\text{for } ord(y) \geqslant -ord(x)) \\ -\frac{1}{q-1} & (\text{for } ord(y) = -ord(x) - 1) \\ 0 & (\text{otherwise}) \end{cases}$$

So, for $x \notin \mathfrak{o}$,

$$\int_{k^\times} \overline{\psi}(xy)|y|^{s'} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy = \int_{ord(y) \geqslant -ord(x)} |y|^{s'} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy - \frac{1}{q-1} \int_{ord(y) = -ord(x)-1} |y|^{s'} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy$$

Now the whole integral is

$$\int_{k}\int_{k^\times} \overline{\psi}(xy)|y|^{s'} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \varphi(x)\,dy\,dx$$

Again, the sub-integral over $x \in \mathfrak{o}$ evaluates to

$$\left(1-\alpha q^{-s'}\right)^{-1}\left(1-\beta q^{-s'}\right)^{-1} = L_v\left(s'+\frac{1}{2}, f\right)$$

The sub-integral over $x \notin \mathfrak{o}$ becomes

$$\int\limits_{ord(x)<0} |x|^{-w'} \int\limits_{ord(y)\geqslant -ord(x)} |y|^{s'} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy\,dx - \frac{1}{q-1}\int\limits_{ord(x)<0} |x|^{-w'} \int\limits_{ord(y)=-ord(x)-1} |y|^{s'} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy\,dx$$

First,

$$\begin{aligned}
&\int\limits_{ord(x)<0} |x|^{-w'} \int\limits_{ord(y)\geqslant -ord(x)} |y|^{s'} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy\,dx \\
&= \frac{q-1}{q}\int\limits_{ord(x)<0} |x|^{1-w'} \int\limits_{ord(y)\geqslant -ord(x)} |y|^{s'} W\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} dy\,dx \\
&= \frac{q-1}{q}\sum_{m=1}^{\infty} q^{m(1-w')}\cdot \sum_{n\geqslant -m} q^{-ns'}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} \\
&= \frac{q-1}{q}\sum_{m=1}^{\infty} q^{m(1-w')}\cdot(\alpha-\beta)^{-1}\left[\frac{\alpha^{1-m}q^{s'm}}{1-\alpha q^{-s'}} - \frac{\beta^{1-m}q^{s'm}}{1-\beta q^{-s'}}\right] \\
&= \frac{q-1}{q}\cdot(\alpha-\beta)^{-1}\Bigg[\left(1-\alpha q^{-s'}\right)^{-1}\alpha\sum_{m=1}^{\infty}\left(\alpha^{-1}q^{1-w'+s'}\right)^m \\
&\quad - \left(1-\beta q^{-s'}\right)^{-1}\beta\sum_{m=1}^{\infty}\left(\beta^{-1}q^{1-w'+s'}\right)^m\Bigg] \\
&= \frac{q-1}{q}\cdot(\alpha-\beta)^{-1}\left[\frac{q^{1-w'+s'}}{(1-\alpha q^{-s'})(1-\alpha^{-1}q^{1-w'+s'})} - \frac{q^{1-w'+s'}}{(1-\beta q^{-s'})(1-\beta^{-1}q^{1-w'+s'})}\right] \\
&= \frac{(q-1)(q^{-w'}-\frac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha q^{-s'})(1-\beta q^{-s'})(1-\alpha^{-1}q^{1-w'+s'})(1-\beta^{-1}q^{1-w'+s'})} \\
&= L_{v_1}\left(s'+\frac{1}{2}, f\right)\cdot\frac{(q-1)(q^{-w'}-\frac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})(1-\beta^{-1}q^{1-w'+s'})}
\end{aligned}$$

For $ord(y) = -ord(x) - 1$, write y as

$$y = \frac{t}{\varpi x} \quad \left(t \in \mathfrak{o}^{\times}\right)$$

Then

$$\frac{1}{q-1}\cdot\int\limits_{ord(x)<0}|x|^{-w'}\cdot\int\limits_{ord(y)=-ord(x)-1}|y|^{s'}W\begin{pmatrix}y&0\\0&1\end{pmatrix}dy\,dx$$
$$=\frac{1}{q-1}\cdot\int\limits_{ord(x)<0}|x|^{-w'}\cdot\left|\frac{1}{\varpi x}\right|^{s'}W\begin{pmatrix}\frac{1}{\varpi x}&0\\0&1\end{pmatrix}dx$$

Now $y\to W\left(\begin{smallmatrix}y&0\\0&1\end{smallmatrix}\right)$ is supported on $\mathfrak{o}\cap k^\times$. Thus, integrate over $k^\times$, and by changing to multiplicative Haar's measure, the integral becomes

$$\frac{q-1}{q}\cdot\frac{1}{q-1}\cdot\int\limits_{ord(x)<0}|x|^{1-w'}\cdot\left|\frac{1}{\varpi x}\right|^{s'}W\begin{pmatrix}\frac{1}{\varpi x}&0\\0&1\end{pmatrix}dx$$

Invert x to obtain

$$\frac{1}{q}\cdot\int\limits_{ord(x)>0}|x|^{w'-1}\cdot\left|\frac{x}{\varpi}\right|^{s'}W\begin{pmatrix}\frac{x}{\varpi}&0\\0&1\end{pmatrix}dx$$

Replace x by ϖx and with $ord(x)=m$

$$\frac{1}{q}\cdot q^{1-w'}\int\limits_{ord(x)\geqslant 0}|x|^{w'-1}\cdot|x|^{s'}\cdot W\begin{pmatrix}x&0\\0&1\end{pmatrix}dx$$
$$=q^{-w'}\cdot\sum_{m=0}^{\infty}q^{-m(w'-1+s')}\cdot\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}$$
$$=q^{-w'}\cdot\left(1-\alpha q^{1-w'-s'}\right)^{-1}\left(1-\beta q^{1-w'-s'}\right)^{-1}=q^{-w'}L\left(s'+w'-\frac{1}{2},F\right)$$

So, for $v=v_1$, the vth local factor of $\langle P\acute{e},F\rangle$ is

$$\frac{1}{(1-\alpha q^{-s'})(1-\beta q^{-s'})}+\frac{(q-1)(q^{-w'}-\frac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha q^{-s'})(1-\beta q^{-s'})(1-\alpha^{-1}q^{1-w'+s'})(1-\beta^{-1}q^{1-w'+s'})}$$
$$-\frac{1}{q^{w'}(1-\alpha q^{1-w'-s'})(1-\beta q^{1-w'-s'})}$$
$$=L_v\left(s'+\frac{1}{2},\overline{F}\right)+\frac{(q-1)(q^{-w'}-\frac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})(1-\beta^{-1}q^{1-w'+s'})}\cdot L_v\left(s'+\frac{1}{2},\overline{F}\right)$$
$$-q^{-w'}L_v\left(s'+w'-\frac{1}{2},\overline{F}\right)\qquad\square$$

For infinite v, the vth local factor of $\langle P\acute{e},F\rangle$ is $\mathcal{G}(\frac{1}{2}+i\overline{\mu}_{F,v};s',w)$, where up to a constant, for $v\approx\mathbb{R}$,

$$\mathcal{G}_v\big(s; s', w\big) = \pi^{-s'} \frac{\Gamma(\frac{s'+1-s}{2})\Gamma(\frac{s'+w-s}{2})\Gamma(\frac{s'+s}{2})\Gamma(\frac{s'+w+s-1}{2})}{\Gamma(\frac{w}{2})\Gamma(s'+\frac{w}{2})}$$

and at $v \approx \mathbb{C}$,

$$\mathcal{G}_v\big(s; s', w\big) = 2\pi^{-2s'} \frac{\Gamma(s'+1-s)\,\Gamma(s'+w-s)\Gamma(s'+s)\Gamma(s'+w+s-1)}{\Gamma(w)\Gamma(2s'+w)}$$

(See formulas (4.2) and (4.3) in [9].) Group the archimedean factors as

$$\mathcal{G}_{F_\infty}\big(s', w\big) = \prod_{v|\infty} \mathcal{G}_v\left(\frac{1}{2} + i\overline{\mu}_{F,s'}; s', w\right)$$

and let all ambiguous constants be absorbed into $\overline{\rho}_F$. Then, for cuspforms F, the cuspidal part of the spectral decomposition of the Poincaré series is

$$\begin{aligned}\sum_F \langle P\acute{e}, F\rangle \cdot F = \sum_F \overline{\rho}_F \mathcal{G}_{F_\infty}\big(s', w\big) \cdot \Bigg[& L_v\left(s' + \frac{1}{2}, \overline{F}\right) + L_{v_1}\left(s' + \frac{1}{2}, \overline{F}\right) \\ & + \frac{(q-1)(q^{-w'} - \frac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})(1-\beta^{-1}q^{1-w'+s'})} \cdot L_{v_1}\left(s' + \frac{1}{2}, \overline{F}\right) \\ & - q^{-w'} L_{v_1}\left(s' + w' - \frac{1}{2}, \overline{F}\right)\Bigg] \cdot F \end{aligned} \tag{4.2}$$

There is no residual spectrum since residual automorphic forms on $GL(2)$ are associated to one-dimensional representations which have no Whittaker models.

4.2. The leading term

Lemma 4.3. *The leading term*

$$\int_{N_{\mathbb{A}}} \varphi = \int_{N_\infty} \varphi_\infty \cdot \frac{1-q^{-w'}}{1-q^{1-w'}} \cdot E_{s'+1,1}$$

Proof.

$$\int_{N_{\mathbb{A}}} \varphi = \left(\int_{N_\infty} \varphi_\infty \cdot \left[\int_{N_{v \neq v_1}} \varphi_{v \neq v_1} \cdot \int_{N_{v_1}} \varphi_{v_1} \right] \right) \cdot E_{s'+1,1}$$

where an elementary computation shows

$$\int_{N_v} \varphi_v = \begin{cases} \sqrt{\pi}\, \frac{\Gamma(\frac{w-1}{2})}{\Gamma(\frac{w}{2})} & (v \approx \mathbb{R}) \\ 2\pi (w-1)^{-1} & (v \approx \mathbb{C}) \end{cases}$$

(See (4.16) in [9].) Now for $v \neq v_1$,

$$\int_{N_v} \varphi_v \, dn = \int_{k_v} 1 \, dx = 1$$

and

$$\int_{N_{v_1}} \varphi_{v_1} \, dn = \int_{x \in \mathfrak{o}_v} 1 \, dx + \int_{x \notin \mathfrak{o}_v} |x|^{-w'} \, dx$$

$$= 1 + \frac{q-1}{q} \cdot \sum_{m=1}^{\infty} (q^m)^{1-w'} = 1 + \frac{q-1}{q} \cdot \frac{q^{1-w'}}{1-q^{1-w'}} = \frac{1-q^{-w'}}{1-q^{1-w'}} \qquad \square$$

4.3. *The continuous part*

Subtract an Eisenstein series from the Poincaré series and denote the resulting function by $P\acute{e}^*$. This function is L^2 and has sufficient decay so that it can be integrated against an Eisenstein series (see Section 4 in [9]). The continuous part of the spectral decomposition of $P\acute{e}$ is

$$\frac{1}{4\pi i\kappa} \sum_{\chi} \int_{Re(s)=\frac{1}{2}} \langle P\acute{e}^*, E_{s,\chi} \rangle \cdot E_{s,\chi} \, ds \quad \bigl(\text{where } \kappa = meas\bigl(\mathbb{J}^1/k^\times\bigr)\bigr)$$

$$\langle P\acute{e}^*, E_{s,\chi} \rangle = \left(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{s,\chi,\infty} \right) \cdot \left(\prod_{v<\infty} \int_{Z_v \backslash G_v} \varphi_v(g_v) \cdot \overline{W}^E_{s,\chi,v}(g_v) \right) dg_v$$

where

$$\int_{Z_v \backslash G_v} \varphi_\infty \cdot \overline{W}^E_{s,\chi,v} = \begin{cases} \frac{\mathcal{G}_v(s,s',w)}{\pi^{-s}\Gamma(s)} & (v \approx \mathbb{R}) \\ \frac{\mathcal{G}_v(s,s',w)}{2\pi^{-2s-1}\Gamma(2s)} & (v \approx \mathbb{C}) \end{cases}$$

and for finite $v \neq v_1$,

$$\int_{Z_v \backslash G_v} \varphi_v(g_v) \cdot \overline{W}^E_{s,\chi,v}(g_v) = |\mathfrak{d}_v|_v^{\frac{1}{2}} \cdot \frac{L_v(s'+\bar{s}, \overline{\chi}_v) \cdot L_v(s'+1-\bar{s}, \chi_v)}{L_v(2\bar{s}, \overline{\chi}_v^2)} \cdot |\mathfrak{d}_v|_v^{-(s'+1-\bar{s})} \cdot \overline{\chi}_v(\mathfrak{d}_v)$$

where $\mathfrak{d}$ is the idele with vth component $\mathfrak{d}_v$ at finite place v and component 1 at archimedean places. (See Section 4 in [9].)

Lemma 4.4. *For finite* $v = v_1$,

$$\int_{Z_v \backslash G_v} \varphi_v(g_v) \cdot \overline{W}^E_{s,\chi,v}(g_v)$$

$$= \frac{L_v(s'+s, \chi) \cdot L_v(s'+1-s, \overline{\chi}) \cdot |\mathfrak{d}_v|^{-(s'+s-\frac{1}{2})} \cdot \chi(\mathfrak{d}_v)}{L_v(2-2s, \overline{\chi}^2)}$$

$$+ \frac{L_v(2s-1, \overline{\chi}) \cdot [\frac{(q-1)\, q^{1-w'+s+s'}}{(1-q^{2-w'-s+s'})(1-q^{-1+s-s'})} - \frac{q^{-w'}}{1-q^{-w'+s-s'}}] \cdot |\mathfrak{d}_v|^{\frac{3}{2}-2s} \cdot \chi(\mathfrak{d}_v)}{L_v(2-2s, \chi^2)}$$

Proof. For finite $v = v_1$,

$$\int\limits_{Z_v\backslash G_v} \varphi_v(g_v) \cdot \overline{W}^E_{s,\chi,v}(g_v) = \int\limits_k \int\limits_{k^\times} |y|^{s'} \overline{\psi}(xy) \overline{W}^E_{s,\chi} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot \varphi(x)\, dy\, dx$$

Define an Eisenstein series by

$$E(g) = \sum_{\lambda \in P_k \backslash G_k} \eta(\lambda g)$$

for η left P_k-invariant, left M_k-invariant and left $N_{\mathbb{A}}$-invariant. Present the vectors η_v in a different form, namely

$$\eta_v(pk) = \left|\frac{a}{d}\right|_v^s \cdot \chi_v\left(\frac{a}{d}\right) \quad \left(\text{for } p = \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \in P_v,\ k \in K_v\right)$$

Let ϕ_v be any Schwartz function on k_v^2, invariant under k_v and put

$$\eta'_v(g) = \chi_v(\det g)|\det g|_v^s \cdot \int\limits_{k_v^\times} \chi_v^2(t)|t|_v^{2s} \cdot \phi_v(t \cdot e_2 \cdot g)\, dt$$

where $e_2 = e_{2,v}$ is the second basis element in k_v^2. η'_v has the same left P_v-equivariance as η_v:

$$\eta'_v\left(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \cdot g\right) = \left|\frac{a}{d}\right|_v^s \cdot \chi_v\left(\frac{a}{d}\right) \cdot \eta'_v(g)$$

For ϕ_v invariant under K_v, the function η'_v is right K_v-invariant.

$$\eta'_v(g) = \eta'_v(1) \cdot \eta_v(g) \quad \left(\text{since } \eta_v(1) = 1\right)$$

and

$$\eta'_v(1) = \int\limits_{k_v^2} \chi_v^2(t)|t|_v^{2s} \cdot \phi_v(t \cdot e_2 \cdot 1)\, dt = \zeta_v\left(2s, \chi^2, \phi(0,*)\right)$$

(See Appendix 2 in [9].) Thus, it suffices to compute the local Mellin transform of

$$\begin{aligned} \eta'_v(1) \cdot W^E_{s,\chi,v}(m) &= \int\limits_{N_v} \overline{\psi}(n) \cdot \eta'_v(w_0 n m)\, dn \\ &= \chi(y)|y|^s \cdot \int\limits_{N_v} \overline{\psi}(n) \int\limits_{k_v^\times} \chi_v^2(t)|t|_v^{2s} \cdot \phi_v(t \cdot e_2 \cdot w_0 \cdot nm)\, dt\, dn \\ &= \chi(y)|y|^s \cdot \int\limits_{k_v} \overline{\psi}(x') \int\limits_{k_v^\times} \chi_v^2(t)|t|_v^{2s} \cdot \phi_v(tx', ty)\, dt\, dx' \quad \left(\text{with } m = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \end{aligned}$$

At finite primes, take

$$\phi\big(t, x'\big) = ch_{\mathfrak{o}_v}(t) \cdot ch_{\mathfrak{o}_v}\big(x'\big) \quad (ch_X = \text{characteristic function of a set } X)$$

Then

$$\eta'_v(1) = \zeta_v\big(2s, \chi^2, ch_{\mathfrak{o}_v}\big) = L_v\big(2s, \chi^2\big)$$

and

$$\begin{aligned}\eta'_v(1) \cdot W^E_{s,\chi,v}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} &= \chi(y)|y|^s \cdot \int_{k_v} \overline{\psi}\big(x'\big) ch_{\mathfrak{o}_v}\big(tx'\big) \cdot \int_{k_v^\times} \chi_v^2(t)|t|_v^{2s} ch_{\mathfrak{o}_v}(ty)\, dt\, dx' \\ &= \chi(y)|y|^s\, meas(\mathfrak{o}_v) \int_{k_v^\times} ch_{\mathfrak{o}_v^*}\left(\frac{1}{t}\right) \chi_v^2(t)|t|_v^{2s-1} ch_{\mathfrak{o}_v}(ty)\, dt \\ &= |\mathfrak{d}_v|^{\frac{1}{2}} \cdot \chi(y)|y|^s \int_{k_v^\times} ch_{\mathfrak{o}_v^*}\left(\frac{1}{t}\right) \chi_v^2(t)|t|_v^{2s-1} ch_{\mathfrak{o}_v}(ty)\, dt\end{aligned}$$

where $\mathfrak{d}_v \in k_v^\times$ is such that $(\mathfrak{o}_v^*)^{-1} = \mathfrak{d}_v \cdot \mathfrak{o}_v$. So, omitting $|\mathfrak{d}_v|^{\frac{1}{2}}$ for now,

$$\begin{aligned}&\int_k \int_{k^\times} |y|^{s'} \overline{\psi}(xy) \overline{W}^E_{s,\chi}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot \varphi(x)\, dy\, dx \\ &= \int_k \int_{k^\times} |y|^{s'} \overline{\psi}(xy) \cdot \left(\chi(y)|y|^s \int_{k_v^\times} ch_{\mathfrak{o}_v^*}\left(\frac{1}{t}\right) \chi_v^2(t)|t|_v^{2s-1} ch_{\mathfrak{o}_v}(ty)\, dt\right) \cdot \varphi(x)\, dy\, dx\end{aligned}$$

Consider the integrals in y and t. Replace y by $\frac{y}{t}$ and then t by $\frac{1}{t}$ to get

$$\int_{k^\times} \int_{k^\times} \overline{\psi}(xyt) \cdot \chi(y)|y|^{s+s'} ch_{\mathfrak{o}_v}(y) \cdot ch_{\mathfrak{d}_v \mathfrak{o}_v}(t) \cdot \overline{\chi}(t)|t|^{s'+1-s}\, dt\, dy$$

First consider the integral in y:

$$\int_{k^\times} \overline{\psi}(xty) \cdot \chi(y)|y|^{s+s'} ch_{\mathfrak{o}_v}(y)\, dy$$

For $x \in \mathfrak{o}^\times$, ψ is trivial on $\mathfrak{o}$, so we get

$$\begin{aligned}&\int_{\mathfrak{o}^\times} \chi(y)|y|^{s+s'}\, dy \cdot \int_{k^\times} ch_{\mathfrak{d}_v \mathfrak{o}_v}(t) \cdot \overline{\chi}(t)|t|^{s'+1-s}\, dt \\ &= L_{v_1}\big(s+s', \chi\big) \cdot L_{v_1}\big(s'+1-s, \overline{\chi}\big) \cdot |\mathfrak{d}_v|^{-(s'+1-s)} \chi(\mathfrak{d}_v)\end{aligned}$$

For $x \notin \mathfrak{o}^\times$,

$$\int_{\mathfrak{o}^\times} \overline{\psi}(xty)\,dy = \begin{cases} 1 & (ord(y) \geqslant -ord(x) - ord(t)) \\ -\frac{1}{q-1} & (ord(y) = -ord(x) - ord(t) - 1) \\ 0 & (\text{otherwise}) \end{cases}$$

So

$$\begin{aligned} &\int_{k^\times} \overline{\psi}(xty) \cdot \chi(y)|y|^{s+s'} ch_{\mathfrak{o}_v}(y)\,dy \\ &\quad = \int_{\mathfrak{o}^\times} \overline{\psi}(xty) \cdot \chi(y)|y|^{s+s'}\,dy \\ &\quad = \int_{ord(y)\geqslant -ord(x)-ord(t)} \chi(y)|y|^{s+s'}\,dy - \frac{1}{q-1} \int_{ord(y)=-ord(x)-ord(t)-1} \chi(y)|y|^{s+s'}\,dy \end{aligned}$$

The entire integral in t and y is:

$$\begin{aligned} &\int_{x\notin\mathfrak{o}} \varphi(x) \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)|t|^{s'+1-s}\,dt \\ &\quad \cdot \left[\int_{ord(y)\geqslant -ord(x)-ord(t)} \chi(y)|y|^{s+s'}\,dy - \frac{1}{q-1} \int_{ord(y)=-ord(x)-ord(t)-1} \chi(y)|y|^{s+s'}\,dy \right] \end{aligned}$$

First take

$$\begin{aligned} &\int_{x\notin\mathfrak{o}} \varphi(x) \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)|t|^{s'+1-s}\,dt \cdot \int_{ord(y)\geqslant -ord(x)-ord(t)} \chi(y)|y|^{s+s'}\,dy\,dx \\ &\quad = \frac{q-1}{q} \sum_{m=1}^{\infty} \left(q^m\right)^{1-w'} \cdot \sum_{n\geqslant -m-r}^{\infty} \left(q^{-n}\right)^{s+s'} \cdot \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)|t|^{s'+1-s}\,dt \end{aligned}$$

(where $ord(t) = r$ and $\chi(y)$ is omitted for now)

$$\begin{aligned} &= \frac{q-1}{q} \sum_{m=1}^{\infty} \frac{(q^{1-w'})^m \cdot (q^{s+s'})^m \cdot (q^r)^{s+s'}}{1-q^{-(s+s')}} \cdot \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)|t|^{s'+1-s}\,dt \\ &= \frac{(q-1)q^{1-w'+s+s'}}{q(1-q^{1-w'+s+s'})(1-q^{-s-s'})} \cdot \int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t) \cdot \overline{\chi}(t)|t|^{s'+1-s}|t|^{-s-s'}\,dt \\ &= \frac{(q-1)q^{-w'+s+s'}}{(1-q^{1-w'+s+s'})(1-q^{-s-s'})} \cdot L_{v_1}(1-2s, \overline{\chi}) \end{aligned}$$

Next we take

$$-\frac{1}{q-1}\int_{x\notin\mathfrak{o}}\varphi(x)\int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t)\cdot\overline{\chi}(t)|t|^{s'+1-s}\,dt\cdot\int_{ord(y)=-ord(x)-ord(t)-1}\chi(y)|y|^{s+s'}\,dy\,dx$$

Since $ord(y)=-ord(x)-ord(t)-1$, y can be written as $y=\frac{1}{\varpi tx}$. So the entire integral becomes an integral in t and x as follows:

$$\begin{aligned}
&-\frac{1}{q-1}\int_{x\notin\mathfrak{o}}\varphi(x)\int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t)\cdot\overline{\chi}(t)|t|^{s'+1-s}\,dt\cdot\int_{ord(y)=-ord(x)-ord(t)-1}\chi(y)|y|^{s+s'}\,dy\,dx\\
&=-\frac{1}{q-1}\cdot\frac{q-1}{q}\int_{x\notin\mathfrak{o}}|x|^{1-w'}\int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t)\cdot\overline{\chi}(t)|t|^{s'+1-s}\,dt\cdot\left|\frac{1}{\varpi xt}\right|^{s+s'}dx\\
&=-q^{s+s'-1}\int_{|x|>1}|x|^{1-w'-s-s'}\int_{k^\times} ch_{\mathfrak{d}_v\mathfrak{o}_v}(t)\cdot\overline{\chi}(t)|t|^{1-2s}\,dt\,dx\\
&=-q^{s+s'-1}\sum_{m=1}^{\infty}\left(q^m\right)^{1-w'-s-s'}\cdot L_{v_1}(1-2s,\overline{\chi})=-\frac{q^{s+s'-1}\cdot q^{1-w'-s-s'}}{1-q^{1-w'-s-s'}}\cdot L_{v_1}(1-2s,\overline{\chi})\\
&=-\frac{q^{-w'}}{1-q^{1-w'-s-s'}}\cdot L_{v_1}(1-2s,\overline{\chi})
\end{aligned}$$

Adding up we get

$$L_{v_1}(1-2s,\overline{\chi})\cdot\left[\frac{(q-1)q^{-w'+s+s'}}{(1-q^{1-w'+s+s'})(1-q^{-s-s'})}-\frac{q^{-w'}}{1-q^{1-w'-s-s'}}\right]$$

Thus at finite primes $v=v_1$, the integral evaluates to:

$$\begin{aligned}
&L_{v_1}\left(s+s',\chi\right)\cdot L_{v_1}\left(s'+1-s,\overline{\chi}\right)\cdot|\mathfrak{d}_{v_1}|^{-(s'+1-s)}\chi(\mathfrak{d}_{v_1})\\
&\quad+L_{v_1}(1-2s,\overline{\chi})\cdot\left[\frac{(q-1)q^{-w'+s+s'}}{(1-q^{1-w'+s+s'})(1-q^{-s-s'})}-\frac{q^{-w'}}{1-q^{1-w'-s-s'}}\right]\cdot|\mathfrak{d}_{v_1}|^{-(1-2s)}\chi(\mathfrak{d}_{v_1})
\end{aligned}$$

Then dividing through by η'_v and putting back the measure constant $|\mathfrak{d}_v|^{\frac{1}{2}}$, we get for $v=v_1$,

$$\begin{aligned}
&|\mathfrak{d}_v|^{\frac{1}{2}}\cdot\int_{k^\times}\int_{k}|y|^{s'}\overline{\psi}(xy)\overline{W}^{E}_{s,\chi}\begin{pmatrix}y&0\\0&1\end{pmatrix}\cdot\varphi(x)\,dy\,dx\\
&=\frac{L_v(s'+s,\chi)\cdot L_v(s'+1-s,\overline{\chi})\cdot|\mathfrak{d}_v|^{-(s'+1-s)}\cdot|\mathfrak{d}_v|^{\frac{1}{2}}\cdot\chi(\mathfrak{d}_v)}{L_v(2s,\chi^2)}\\
&\quad+\frac{L_v(1-2s,\overline{\chi})\cdot[\frac{(q-1)q^{-w'+s+s'}}{(1-q^{1-w'+s+s'})(1-q^{-s-s'})}-\frac{q^{-w'}}{1-q^{1-w'-s-s'}}]\cdot|\mathfrak{d}_v|^{-(1-2s)}\cdot|\mathfrak{d}_v|^{\frac{1}{2}}\cdot\chi(\mathfrak{d}_v)}{L_v(2s,\chi^2)}
\end{aligned}$$

Replacing s by $1-s$ and χ by $\overline{\chi}$ we get

$$|\mathfrak{d}_v|^{\frac{1}{2}} \cdot \int_{k^\times} \int_k |y|^{s'} \overline{\psi}(xy) \overline{W}^E_{1-s,\overline{\chi}} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot \varphi(x)\, dy\, dx$$

$$= \frac{L_v(s'+s,\chi) \cdot L_v(s'+1-s,\overline{\chi}) \cdot |\mathfrak{d}_v|^{-(s'+s-\frac{1}{2})} \cdot \chi(\mathfrak{d}_v)}{L_v(2-2s,\overline{\chi}^2)}$$

$$+ \frac{L_v(2s-1,\overline{\chi}) \cdot [\frac{(q-1)q^{1-w'+s+s'}}{(1-q^{2-w'-s+s'})(1-q^{-1+s-s'})} - \frac{q^{-w'}}{1-q^{-w'+s-s'}}] \cdot |\mathfrak{d}_v|^{\frac{3}{2}-2s} \cdot \chi(\mathfrak{d}_v)}{L_v(2-2s,\chi^2)} \qquad \square$$

So the spectral decomposition of the Poincaré series is:

$$Pé = \left(\int_{N_\infty} \varphi_\infty\right) \cdot \frac{1-q^{-w'}}{1-q^{1-w'}} \cdot E_{s'+1,1} + \sum_F \overline{\rho}_F \mathcal{G}_{F_\infty}(s',w) \cdot \Bigg[L_v\left(s'+\frac{1}{2},\overline{F}\right) + L_{v_1}\left(s'+\frac{1}{2},\overline{F}\right)$$

$$+ \frac{(q-1)(q^{-w'} - \frac{q^{1-2w'+2s'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'+s'})(1-\beta^{-1}q^{1-w'+s'})} \cdot L_{v_1}\left(s'+\frac{1}{2},\overline{F}\right) - q^{-w'} L_{v_1}\left(s'+w'-\frac{1}{2},\overline{F}\right)\Bigg] \cdot F$$

$$+ \frac{1}{4\pi i\kappa} \sum_\chi \int_{\Re(s)=\frac{1}{2}} \left(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{s,\chi,\infty}\right)$$

$$\cdot \Bigg(\frac{L_v(s'+s,\chi) \cdot L_v(s'+1-s,\overline{\chi}) \cdot |\mathfrak{d}_v|^{-(s'+s-\frac{1}{2})} \cdot \overline{\chi}(\mathfrak{d}_v)}{L_v(2-2s,\overline{\chi}^2)}$$

$$+ \frac{L_v(s'+s,\chi) \cdot L_v(s'+1-s,\overline{\chi}) \cdot |\mathfrak{d}_v|^{-(s'+s-\frac{1}{2})} \cdot \overline{\chi}(\mathfrak{d}_v)}{L_v(2-2s,\overline{\chi}^2)}$$

$$+ \frac{L_v(2s-1,\overline{\chi}) \cdot [\frac{(q-1)q^{1-w'-s+s'}}{(1-q^{2-w'-s+s'})(1-q^{-1+s-s'})} - \frac{q^{-w'}}{1-q^{-w'+s-s'}}] \cdot |\mathfrak{d}_v|^{\frac{3}{2}-2s} \cdot \chi(\mathfrak{d}_v)}{L_v(2-2s,\chi^2)}\Bigg) \cdot E_{s,\chi}\, ds \qquad (4.3)$$

From the spectral decomposition, the Poincaré series has meromorphic continuation to a region in $\mathbb{C}^2$ containing $s'=0$, $w'=1$ (see Section 4 in [9]). As a function of w', for $s'=0$, it is holomorphic in the half-plane $\Re(w') = \frac{11}{18}$ ([24] and [23]), except for $w'=1$ where it has a pole of order 1.

5. Preliminaries to subconvexity

Fix a non-archimedean place v_1, and take $1 < \beta' < 2$. At the archimedean place, fix $t = t_0$. Define

$$Z(w') = \sum_{\chi \in \hat{C}_{0,S}} \int_{-\infty}^{\infty} \left|L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot \left(q^N\right)^{-w'} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt \qquad (5.1)$$

This is a modified function obtained from (3.5) by taking the asymptotic formula for $\mathcal{K}_{v_1}(w', \chi_{v_1})$ in Lemma 3.3. $Z(w')$ is absolutely convergent for $\Re(w') > 1$ (see Section 5 in [9]). In this section, we will obtain the moment asymptotic, and prove the meromorphic continuation and polynomial growth of $Z(w')$. This will enable us to obtain subconvexity bounds in the finite prime conductor-aspect.

5.1. Meromorphic continuation of $Z(w')$

Theorem 5.1. *The function*

$$Z(w') = \sum_{\chi \in \hat{C}_{0,S}} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot (q^N)^{-w'} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt$$

where the sum is over a set $\hat{C}_{0,S}$ of characters ramified at the finite prime v_1 with conductor q^N, and $1 < \beta' \leqslant 2$, $\Re(w') > 1$, has analytic continuation to the half-plane $\Re(w') > \frac{11}{18}$, except for $w' = 1$ where it has a pole of order 1.

Proof. Let $w' = \delta + i\eta$. Split Z into Z_1 and Z_2 as follows:

$$Z(w') = Z_1(w') + Z_2(w') \tag{5.2}$$

Choose a positive constant C and define

$$Z_1(w') = \sum_{\chi \in \hat{C}_{0,S}:\, q^N \ll C} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot (q^N)^{-w'} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt \tag{5.3}$$

We first show that $Z_1(w')$ has analytic continuation by showing that it is holomorphic for $\delta > 0$. Now

$$\left|Z_1(w')\right| \leqslant \sum_{\chi:\, q^N \ll C} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot \left|(q^N)^{-w'}\right| \cdot \left|\mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right)\right| dt$$

$\mathcal{K}_\infty(\frac{1}{2}+it_0, 0, \beta', \chi)$ is positive (see Section 4 in [10]). So

$$\left|Z_1(w')\right| \leqslant \sum_{\chi:\, q^N \ll C} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot (q^N)^{-\delta} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt$$

Since

$$(q^N)^{-\delta} \ll_{\beta', C} (q^N)^{-\beta'}$$

then

$$\begin{aligned}\left|Z_1(w')\right| &\ll \sum_{\chi:\, q^N \ll C} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot (q^N)^{-\beta'} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt \\ &< \sum_{\chi \in \hat{C}_{0,S}} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot (q^N)^{-\beta'} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt = Z(\beta') \end{aligned} \tag{5.4}$$

which is convergent for $\Re(w') > \frac{2}{9}$. Thus, $Z_1(w')$ is holomorphic for $\Re(w') = \delta > 0$ (in particular for $\Re(w') > \frac{11}{18}$). Now we prove that $Z_2(w')$ has analytic continuation. Consider

$$I\big(s', w', \beta'\big) = \sum_{\chi \in \hat{C}_{0,S}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} L(s, f \otimes \chi) \cdot L\big(s' + 1 - s, \overline{f} \otimes \overline{\chi}\big) \cdot \mathcal{K}_{v_1}\big(w', \chi_{v_1}\big) \cdot \mathcal{K}_\infty\big(s, s', \beta', \chi\big)\, dt \tag{5.5}$$

$$I\big(0, w', \beta'\big) = \sum_{\chi \in \hat{C}_{0,S}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \mathcal{K}_{v_1}\big(w', \chi_{v_1}\big) \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it_0, 0, \beta', \chi\right) dt \tag{5.6}$$

is holomorphic for $\Re(w') > \frac{11}{18}$ except at $w' = 1$ where there is a pole of order 1. In the region of absolute convergence for $\Re(w') = \delta > 1$, write

$$I\big(0, w', \beta'\big) = I_1\big(0, w', \beta'\big) + I_2\big(0, w', \beta'\big) \tag{5.7}$$

where

$$I_1\big(0, w', \beta'\big) = \sum_{\chi:\, q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \mathcal{K}_{v_1}\big(w', \chi_{v_1}\big) \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it_0, 0, \beta', \chi\right) dt \tag{5.8}$$

Now

$$I\big(0, w', \beta'\big) = I_1\big(0, w', \beta'\big) + \sum_{\chi:\, q^N \gg C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot C' \cdot \big(q^N\big)^{-w'} \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it_0, 0, \beta', \chi\right) dt \tag{5.9}$$

where the constant

$$C' = \frac{q}{q-1} \cdot \frac{1 - |\alpha|^2|\beta|^2 q^{-2w'}}{(1 - |\alpha|^2 q^{-w'})(1 - |\beta|^2 q^{-w'})(1 - \overline{\alpha}\beta q^{-w'})(1 - \alpha\overline{\beta} q^{-w'})} \tag{5.10}$$

is obtained from Lemma 3.2. So

$$I\big(0, w', \beta'\big) = I_1\big(0, w', \beta'\big) + C' \cdot Z_2\big(w'\big) \tag{5.11}$$

Thus, to show that $Z_2(w')$ has analytic continuation, it suffices to show that $I_1(0, w', \beta')$ is absolutely convergent for $\Re(w') > \frac{11}{18}$.

$$I_1\big(0, w', \beta'\big) = \sum_{\chi:\, q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \mathcal{K}_{v_1}\big(w', \chi_{v_1}\big) \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it_0, 0, \beta', \chi\right) dt$$

$$\begin{aligned}
&\ll \sum_{\chi:\, q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \left|\left(q^N\right)^{-w'}\right| \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt \\
&\ll \sum_{\chi:\, q^N \ll C} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \left(q^N\right)^{-\beta} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt \\
&< \sum_{\chi\in\hat{C}_{0,S}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \left(q^N\right)^{-\beta} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt \\
&= Z\left(\beta'\right)
\end{aligned} \tag{5.12}$$

which is convergent for $\Re(w') > \frac{2}{9}$. Thus $Z_2(w')$ is absolutely convergent for $\Re(w') > \frac{11}{18}$, proving the theorem. □

5.2. Polynomial growth of $Z(w')$

Theorem 5.2. *For every fixed small positive ϵ, the generating function*

$$Z\left(w'\right) = \sum_{\chi\in\hat{C}_{0,S}} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \left(q^N\right)^{-w'} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt$$

has polynomial growth in the conductor q^N for $\frac{11}{18}+\epsilon \leqslant \Re(w') \leqslant 1+\epsilon$. That is, for a computable $\gamma > 0$ independent of β', on the vertical line $\Re(w') = \frac{11}{18}+\epsilon$,

$$Z\left(w'\right) \ll_{\epsilon,\beta'} \left(q^N\right)^{\gamma}$$

Before giving details of the proof, we find it useful to first present the main ideas of our argument.

Idea of the proof. From the definition of $I(s', w', \beta')$ in (5.5) split $I = I_1 + I_2$ with

$$I_2\left(0, w', \beta'\right) = C' \cdot Z_2\left(w'\right) = C'\left[Z\left(w'\right) - Z_1\left(w'\right)\right]$$

where C' is a constant obtained in (5.10). $Z_1(w')$ has polynomial growth in the conductor q^N so it suffices to show that $I_2(0, w', \beta')$ has polynomial growth. Using the spectral decomposition of the Poincaré series, we rewrite $I(w')$, define an auxiliary function $I^{aux}(w')$ and show that $I(w') - I^{aux}(w')$ extends holomorphically to the vertical strip $-\epsilon \leqslant \Re(w') \leqslant 1+\epsilon$. We then prove that $I^{aux}(w')$ has polynomial growth in $\frac{11}{18}+\epsilon \leqslant \Re(w') \leqslant 1+\epsilon$, and apply the Phragmen–Lindelöf principle to prove polynomial growth of $I_2(w') - I^{aux}(w')$ within the strip.

Proof of Theorem 5.2. From (5.2), (5.7) and (5.11), deduce that

$$I_2\left(0, w', \beta'\right) = C'\left[Z\left(w'\right) - Z_1\left(w'\right)\right] \tag{5.13}$$

with $Z(w)$ and $Z(w')$ defined in (5.1) and (5.3). $Z_1(w')$ has polynomial growth in q^N. Thus, the polynomial bound of $Z(w')$ will be deduced from that of $I_2(0, w', \beta')$. In the spectral decomposition of *Pé*, set $s' = 0$ and obtain

$$Pé = \lim_{s'\to 0}\left(\int_{N_\infty}\varphi_\infty\right)\cdot\frac{1-q^{-w'}}{1-q^{1-w'}}\cdot E_{s'+1,1}+\sum_F\overline{\rho}_F\mathcal{G}_{F_\infty}(\beta')\cdot\left[L_v\left(\frac{1}{2},\overline{F}\right)+L_{v_1}\left(\frac{1}{2},\overline{F}\right)\right.$$
$$\left.+\frac{(q-1)(q^{-w'}-\frac{q^{1-2w'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'})(1-\beta^{-1}q^{1-w'})}\cdot L_{v_1}\left(\frac{1}{2},\overline{F}\right)-q^{-w'}L_{v_1}\left(\frac{2w'-1}{2},\overline{F}\right)\right]\cdot F$$
$$+\sum_\chi\frac{\overline{\chi}(\mathfrak{d}_v)}{4\pi i\kappa}\int_{\Re(s)=\frac{1}{2}}\left(\int_{Z_\infty\backslash G_\infty}\varphi_\infty\cdot\overline{W}^E_{s,\overline{\chi},\infty}\right)$$
$$\cdot\left(\frac{L_v(s,\chi)\cdot L_v(1-s,\overline{\chi})\cdot|\mathfrak{d}_v|^{-(s-\frac{1}{2})}}{L_v(2-2s,\overline{\chi}^2)}+\frac{L_{v_1}(s,\chi)\cdot L_{v_1}(1-s,\overline{\chi})\cdot|\mathfrak{d}_{v_1}|^{-(s-\frac{1}{2})}}{L_{v_1}(2-2s,\overline{\chi}^2)}\right.$$
$$\left.+\frac{L_{v_1}(2s-1,\overline{\chi})\cdot[\frac{(q-1)q^{1-w'-s}}{(1-q^{2-w'-s})(1-q^{s-1})}-\frac{q^{-w'}}{1-q^{-w'+s}}]\cdot|\mathfrak{d}_{v_1}|^{\frac{3}{2}-2s}}{L_{v_1}(2-2s,\chi^2)}\right)\cdot E_{s,\chi}\,ds \tag{5.14}$$

We write

$$I(w')=I_{sing}(w')+I_{cusp}(w')+I_{cont}(w') \tag{5.15}$$

where

$$I_{sing}(w')=\lim_{s'\to 0}\left(\int_{N_\infty}\varphi_\infty\right)\cdot\frac{1-q^{-w'}}{1-q^{1-w'}}\cdot\langle E_{s'+1,1},|f|^2\rangle$$

$$I_{cusp}(w')=\sum_F\overline{\rho}_F\mathcal{G}_{F_\infty}(\beta')\cdot\left[2L_v\left(\frac{1}{2},\overline{F}\right)+\frac{(q-1)(q^{-w'}-\frac{q^{1-2w'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'})(1-\beta^{-1}q^{1-w'})}\cdot L_v\left(\frac{1}{2},\overline{F}\right)\right.$$
$$\left.-q^{-w'}L_v\left(\frac{2w'-1}{2},\overline{F}\right)\right]\cdot\langle F,|f|^2\rangle$$

$$I_{cont}(w')=\sum_\chi\frac{\overline{\chi}(\mathfrak{d}_v)}{4\pi i\kappa}\int_{\Re(s)=\frac{1}{2}}\left(\int_{Z_\infty\backslash G_\infty}\varphi_\infty\cdot\overline{W}^E_{1-s,\overline{\chi},\infty}\right)\cdot\left(\frac{2L_v(s,\chi)\cdot L_v(1-s,\overline{\chi})\cdot|\mathfrak{d}_v|^{-(s-\frac{1}{2})}}{L_v(2-2s,\overline{\chi}^2)}\right.$$
$$\left.+\frac{L_v(2s-1,\overline{\chi})\cdot[\frac{(q-1)q^{1-w'-s}}{(1-q^{2-w'-s})(1-q^{s-1})}-\frac{q^{-w'}}{1-q^{-w'+s}}]\cdot|\mathfrak{d}_v|^{\frac{3}{2}-2s}}{L_v(2-2s,\chi^2)}\right)\cdot\langle E_{s,\chi},|f|^2\rangle ds$$

Then define the auxiliary function $I^{aux}(w')$ by

$$I^{aux}(w')=\sum_F\overline{\rho}_F\mathcal{G}_{F_\infty}(\beta')\cdot\left[2L\left(\frac{1}{2},\overline{F}\right)+\mathcal{M}_1^{aux}(w')\right]\cdot\langle F,|f|^2\rangle$$
$$+\sum_\chi\frac{\overline{\chi}(\mathfrak{d})}{4\pi i\kappa}\int_{\Re(s)=\frac{1}{2}}\left(\int_{Z_\infty\backslash G_\infty}\varphi_\infty\cdot\overline{W}^E_{1-s,\overline{\chi},\infty}\right)\cdot\frac{2L(s,\chi)\cdot L(1-s,\overline{\chi})\cdot|\mathfrak{d}|^{-(s-\frac{1}{2})}}{L(2-2s,\overline{\chi}^2)}$$
$$+\frac{L(2s-1,\overline{\chi})\cdot\mathcal{M}_2^{aux}(w')\cdot|\mathfrak{d}|^{\frac{3}{2}-2s}}{L(2-2s,\chi^2)}\cdot\langle E_{s,\chi},|f|^2\rangle ds \tag{5.16}$$

where $\mathcal{M}_1^{aux}(w')$ and $\mathcal{M}_2^{aux}(w')$ are defined by

$$\mathcal{M}_1^{aux}(w') = \mathcal{M}_1(w') \cdot (q^N)^\gamma \quad \text{and} \quad \mathcal{M}_2^{aux}(w') = \mathcal{M}_2(w') \cdot (q^N)^\gamma$$

where $\gamma > 0$ is independent of β'. $\mathcal{M}_1(w')$ and $\mathcal{M}_2(w')$ are expressions in $I_{cusp}(w')$ and $I_{cont}(w')$ given by:

$$\mathcal{M}_1(w') = \frac{(q-1)(q^{-w'} - \frac{q^{1-2w'}}{\alpha\beta})}{(1-\alpha^{-1}q^{1-w'})(1-\beta^{-1}q^{1-w'})} \cdot L\left(\frac{1}{2}, \overline{F}\right) - q^{-w'} L\left(\frac{2w'-1}{2}, \overline{F}\right)$$

and

$$\mathcal{M}_2(w') = \frac{(q-1)q^{1-w'-s}}{(1-q^{2-w'-s})(1-q^{s-1})} - \frac{q^{-w'}}{1-q^{-w'+s}}$$

Define

$$H(w') = I(w') - I^{aux}(w')$$

(See Proposition 3.6 in [10].)

$$\begin{aligned} H(w') = {} & \lim_{s'\to 0} \left(\int_{N_\infty} \varphi_\infty \right) \cdot \frac{1-q^{-w'}}{1-q^{1-w'}} \cdot \langle E_{s'+1,1}, |f|^2 \rangle \\ & + \sum_F \overline{\rho}_F \mathcal{G}_{F_\infty}(\beta') \cdot [\mathcal{M}_1(w') - \mathcal{M}_1^{aux}(w')] \cdot \langle F, |f|^2 \rangle \\ & + \sum_\chi \frac{\overline{\chi}(\mathfrak{d})}{4\pi i \kappa} \int_{\Re(s)=\frac{1}{2}} \left(\int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \overline{W}^E_{1-s,\overline{\chi},\infty} \right) \\ & \cdot \frac{L(2s-1,\overline{\chi}) \cdot [\mathcal{M}_2(w') - \mathcal{M}_2^{aux}(w')] \cdot |\mathfrak{d}|^{\frac{3}{2}-2s}}{L(2-2s,\chi^2)} \cdot \langle E_{s,\chi}, |f|^2 \rangle \, ds \end{aligned} \tag{5.17}$$

Proposition 5.3. *For ϵ sufficiently small,*

$$H(w') = I(w') - I^{aux}(w')$$

restricted to $\frac{11}{18} < \Re(w') \leqslant 1+\epsilon$, extends holomorphically to the whole vertical strip $-\epsilon \leqslant \Re(w') \leqslant 1+\epsilon$.

Proof. The first term in $H(w')$ is holomorphic in the strip $-\epsilon \leqslant \Re(w') \leqslant 1+\epsilon$, except at $w' = 0, 1$ where there are poles.

$$\mathcal{M}_1(w') - \mathcal{M}_1^{aux}(w') = \mathcal{M}_1(w') - \mathcal{M}_1(w') \cdot (q^N)^\gamma = \mathcal{M}_1(w')\left[1 - (q^N)^\gamma\right]$$

Similarly

$$\mathcal{M}_2(w') - \mathcal{M}_2^{aux}(w') = \mathcal{M}_2(w')\left[1 - (q^N)^\gamma\right]$$

Since both $\mathcal{M}_1(w')$ and $\mathcal{M}_2(w')$ are holomorphic in the strip, then $H(w')$ is also holomorphic in the strip. □

Proposition 5.4. *Fix a small positive ϵ. For $\frac{11}{18}+\epsilon \leqslant \Re(w') \leqslant 1+\epsilon$, or $\Re(w') = -\epsilon$,*

$$I^{aux}(w') \ll_{\epsilon,\beta'} (q^N)^{\gamma}$$

Proof. All other terms in $I^{aux}(w')$ defined in (5.16) are independent of the conductor q^N. □

Recall we are trying to prove a polynomial bound for $I_2(w')$ in the conductor q^N. Now

$$I_2(w') = I_2(w') - I^{aux}(w') + I^{aux}(w') \tag{5.18}$$

We have proven a polynomial bound for $I^{aux}(w')$, so we now prove a polynomial bound for $I_2(w') - I^{aux}(w')$.

$$H(w') - I_1(w') = I_2(w') - I^{aux}(w') \tag{5.19}$$

Thus it suffices to prove a polynomial bound for $H(w') - I_1(w')$ on the line $\Re(w') = \frac{11}{18}+\epsilon$. From (5.12) recall that

$$I_1(w') \ll Z(\beta') < \infty$$

So $I_1(w')$ is holomorphic throughout the strip. Thus $H(w') - I_1(w')$ is also holomorphic throughout the strip. For $\Re(w') = 1+\epsilon$, since $I^{aux}(w') \ll (q^N)^{\gamma}$, for $\gamma > 0$, $Z(w') = O(1)$ and $Z_1(w')$ already has polynomial growth in q^N, we conclude that

$$H(w') - I_1(w') = I_2(w') - I^{aux}(w')$$

has polynomial growth in q^N for $\Re(w') = 1+\epsilon$. Now assume $\Re(w') = -\epsilon$.

$$H(w') - I_1(w') = I(w') - I^{aux}(w') - I_1(w')$$

Again, $I^{aux}(w')$ has polynomial growth for $\Re(w') = -\epsilon$, and $I_1(w') \ll Z(\beta')$. The spectral expansion of $I(w')$ and $I_1(w')$ shows that $I(w')$ and $I_1(w')$ also have polynomial growth for $\Re(w') = -\epsilon$. Thus $H(w') - I_1(w')$ has polynomial growth in q^N for $\Re(w') = -\epsilon$. We now apply Phragmen–Lindelöf and conclude that $I_2(w') - I^{aux}(w')$ has polynomial growth in q^N within the strip $\frac{11}{18}+\epsilon \leqslant \Re(w') \leqslant 1+\epsilon$, and hence, so has $I_2(w')$. □

5.3. The moment asymptotic

Main Theorem. *For a cuspform f on $GL_2(k)$, where k is a number field of degree d over $\mathbb{Q}$, the finite prime conductor-aspect moment asymptotic for the twisted L-function $L(\frac{1}{2}+it, f\otimes\chi)$ is*

$$\sum_{\chi:\, q^N \leqslant T} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt \ll_\epsilon T^{1+\epsilon} \quad (\textit{for all } \epsilon > 0)$$

where q^N, with $N \geqslant 1$, is the finite prime conductor of χ and $\mathcal{K}_\infty(\frac{1}{2}+it, 0, \beta', \chi)$ refers to the archimedean data.

Proof. From the definition of $Z(w')$ in (5.3), set $w' = 1 + \epsilon$, then for arbitrary $T > 1$

$$\sum_{\chi: q^N \leqslant T} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it_0, 0, \beta', \chi\right) \cdot T^{-1-\epsilon}\, dt \ll 1 \qquad \square \tag{5.20}$$

6. Subconvexity bounds

Our goal is to break convexity in the finite prime conductor-aspect for a family of L-functions $L(\frac{1}{2} + it, f \otimes \chi)$, where χ has arbitrary ramification at a fixed finite prime v_1. For a cuspform f on $GL_2(k)$, the finite prime conductor-aspect convexity bound for the twisted L-function $L(\frac{1}{2} + it, f \otimes \chi)$ is

$$L\left(\frac{1}{2} + it, f \otimes \chi\right) \ll_\epsilon q^{N(\frac{d}{2}+\epsilon)} \quad (\text{for all } \epsilon > 0) \tag{6.1}$$

where q^N with $N \geqslant 1$ is the conductor of χ allowed to be ramified at the finite place v_1, and d is the degree of the number field k over $\mathbb{Q}$. We will break convexity at the finite place v_1 by decreasing the exponent, proving:

Example application. Fix a number field k of degree d over $\mathbb{Q}$ and a cuspform f on $GL_2(k)$. For a computable constant $\vartheta < 1$,

$$L\left(\frac{1}{2} + it, f \otimes \chi\right) \ll_\epsilon \left(q^N\right)^{\frac{d-1+\vartheta}{2}+\epsilon} \quad (\text{for all } \epsilon > 0)$$

Proof. Fix a non-archimedean place v_1, take $1 < \beta' < 2$ and fix $t = t_0$ such that $0 < t < 1$ in the non-decoupled integral at the archimedean places. $Z(w')$ has analytic continuation to $\Re(w') > \frac{11}{18}$ with a pole of order 1 at $w' = 1$, and has polynomial growth on every vertical strip inside $\frac{11}{18} + \epsilon \leqslant \Re(w') \leqslant 1 + \epsilon$. Choose $\frac{11}{18} < \delta_0 < 1$. For $\delta_0 \leqslant \Re(w') \leqslant 1 + \epsilon$, and by Phragmen–Lindelöf, $Z(\delta_0 + i\eta)$ has polynomial growth of exponent less than $\frac{1}{2}$ (see Section 4 in [10]). Consider the rectangle R with vertices at $\delta_0 - iS$, $\beta' - iS$, $\beta' + iS$, $\delta_0 + iS$. Recall Perron's formula: for $\beta' > 1$,

$$\frac{1}{2\pi i} \int_{\beta'-iS}^{\beta'+iS} \frac{x^w}{w}\, dw = \begin{cases} 1 & (\text{for } x > 1) \\ 0 & (\text{for } x < 1) \end{cases} + x^{\beta'} O_{\beta'}\left(\min\left\{1, \frac{1}{S|\log x|}\right\}\right)$$

Applying Perron's formula to the integral

$$\frac{1}{2\pi i} \int_{\beta'-iS}^{\beta'+iS} \frac{Z(w')x^{w'}}{w'}\, dw'$$

gives

$$\frac{1}{2\pi i} \int_{\beta'-iS}^{\beta'+iS} \frac{Z(w')\,x^{w'}}{w'}\, dw'$$
$$= \frac{1}{2\pi i} \sum_\chi \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \left(\int_{\beta-iS}^{\beta+iS} \frac{(x/q^N)^{w'}}{w'}\, dw' \right) \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it_0, 0, \beta', \chi\right) dt$$

$$= \sum_{\chi:\, q^N \leqslant x} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot 1 \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt + E(x,S)$$

where the error term $E(x,S)$ is

$$E(x,S) \ll \sum_{\chi} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \left(\frac{x}{q^N}\right)^{\beta'} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) \cdot \min\left\{1, \frac{1}{S|\log(\frac{x}{q^N})|}\right\} dt \qquad \square$$

Theorem 6.1.

$$\lim_{S\to\infty} E(x,S) = 0 \quad (\textit{for } x > 0)$$

Proof. We first show that

$$\lim_{S\to\infty} \int_{\delta_0+iS}^{\beta'+iS} \frac{Z(w')x^{w'}}{w'}\,dw' = 0 \quad \text{and} \quad \lim_{S\to\infty} \int_{\delta_0-iS}^{\beta'-iS} \frac{Z(w')x^{w'}}{w'}\,dw' = 0$$

Let $w' = \delta + iS$. Then

$$Z(w') \ll S^m \quad \left(\text{for } m < \frac{1}{2} \text{ and } |w'| = \sqrt{\delta^2 + S^2} \ll S\right)$$

Thus the integrals above approach 0 as $S \to \infty$.

Consider the sets:

$$A = \left\{N\colon \frac{1}{S|\log(\frac{x}{q^N})|} \leqslant \frac{1}{\sqrt{S}}\right\} \quad \text{and} \quad B = \left\{N\colon \frac{1}{S|\log(\frac{x}{q^N})|} \geqslant \frac{1}{\sqrt{S}}\right\}$$

On A,

$$E(x,S) \ll \frac{1}{\sqrt{S}} \sum_{\chi} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \left(\frac{x}{q^N}\right)^{\beta'} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt$$
$$= \frac{x^{\beta'}}{\sqrt{S}} \sum_{\chi} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \left(q^N\right)^{-\beta'} \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt = \frac{x^{\beta'}}{\sqrt{S}} Z\left(\beta'\right)$$

where $Z(\beta')$ is independent of S. So

$$\lim_{S\to\infty} E(x,S) = 0$$

On B, $\mathcal{K}_\infty(\frac{1}{2}+it,0,\beta',\chi)$ can be estimated by the analytic conductor:

$$Q(\chi,t)=\prod_{v\approx\mathbb{R}}\bigl(1+|t+t_v|\bigr)\cdot\prod_{v\approx\mathbb{C}}\bigl(1+\ell_v^2+4(t+t_v)^2\bigr)$$

Break up $E(x,S)$ into two sums over $q^N\leqslant\log S$ and $q^N\geqslant\log S$. Since $Z(w')$ converges absolutely for $\Re(w')>1$, the second sum over $q^N\geqslant\log S$ approaches 0. So consider

$$\sum_{\chi:\,q^N\leqslant\log S}\int_{-\infty}^{\infty}\left|L\left(\frac{1}{2}+it,f\otimes\chi\right)\right|^2\cdot\left(\frac{x}{q^N}\right)^{\beta'}\cdot\mathcal{K}_\infty\left(\frac{1}{2}+it_0,0,\beta',\chi\right)\cdot\min\left\{1,\frac{1}{S|\log(\frac{x}{q^N})|}\right\}dt$$

Now in B,

$$\sum_{q^N\leqslant\log S}\int_{-\infty}^{\infty}1\ll(\log S)^k,\quad k>0$$

The convexity bound in the non-archimedean aspect gives

$$L\left(\frac{1}{2}+it,f\otimes\chi\right)\ll\bigl(q^N\bigr)^{\frac{1}{2}}\leqslant(\log S)^{\frac{1}{2}}$$

Fix $\chi=1$ and choose $0<t_0<1$ for $v|\infty$. Then

$$\mathcal{K}_\infty\left(\frac{1}{2}+it_0,0,\beta',\chi\right)\ll 1$$

Also

$$\frac{1}{S|\log(\frac{x}{q^N})|}\geqslant\frac{1}{\sqrt{S}}\quad\Longrightarrow\quad\frac{1}{\sqrt{S}|\log(\frac{x}{q^N})|}\geqslant 1\quad\Longrightarrow\quad xe^{-\frac{1}{\sqrt{S}}}\leqslant q^N\leqslant xe^{\frac{1}{\sqrt{S}}}$$

This restricts N to a set of measure $\ll\frac{1}{\sqrt{S}}$. So in the second case

$$\lim_{S\to\infty}E(x,S)=0\qquad\square$$

By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_R\frac{Z(w')x^{w'}}{w'}\,dw'=xP(\log x)$$

Indeed, $Z(w')$ has a pole of order 1 at $w'=1$, so by the residue theorem:

$$\frac{1}{2\pi i}\int_R\frac{Z(w')x^{w'}}{w'}\,dw'=Res_{w'=1}\left(Z\bigl(w'\bigr)\cdot\frac{x^{w'}}{w'}\right)$$

Consider the Laurent expansion

$$Z(w') = \sum_{n=-\infty}^{\infty} a_n (w'-1)^n$$

and

$$x^{w'} = xe^{(w'-1)\log x} = x\sum_{n=0}^{\infty} \frac{(w'-1)^n \log^n x}{n!}$$

Then the coefficient of $(w'-1)^{-1}$ in the product $Z(w') \cdot \frac{x^{w'}}{w'}$ is $xP(\log x)$, where $P(\log x)$ is a polynomial in $\log x$. So

$$\frac{1}{2\pi i}\int_R \frac{Z(w')x^{w'}}{w'}\,dw' = \frac{1}{2\pi i}\int_{\beta'-iS}^{\beta'+iS} \frac{Z(w')x^{w'}}{w'}\,dw' - \frac{1}{2\pi i}\int_{\delta_0-iS}^{\delta_0+iS} \frac{Z(w')x^{w'}}{w'}\,dw' = xP(\log x)$$

Now Perron's formula showed that

$$\frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty} \frac{Z(w')x^{w'}}{w'}\,dw' = \sum_{q^N \leqslant x}\int_{-\infty}^{\infty} \left|L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt$$

Thus as $S \to \infty$,

$$\begin{aligned}\sum_{q^N \leqslant x}\int_{-\infty}^{\infty} \left|L\left(\frac{1}{2}+it, f\otimes\chi\right)\right|^2 \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt \\ = xP(\log x) + \frac{1}{2\pi i}\int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w')x^{w'}}{w'}\,dw' \end{aligned} \tag{6.2}$$

Theorem 6.2.

$$\frac{1}{2\pi i}\int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w')x^{w'}}{w'}\,dw' \ll x^{\frac{2\delta_0+1}{3}} \cdot \log x \quad \left(\frac{11}{18} < \delta_0 < 1\right)$$

Proof. By the choice of δ_0,

$$\frac{Z(w')}{w'} = \frac{Z(\delta_0+i\eta)}{\delta_0+i\eta}$$

is a square integrable function on $\mathbb{R}$. Let

$$E(x) = \frac{1}{2\pi i}\int_{\delta_0-i\infty}^{\delta_0+i\infty} \frac{Z(w')x^{w'}}{w'}\,dw' \tag{6.3}$$

Lemma 6.3.

$$\int_0^x \left|E(t)\right|^2 dt \ll x^{2\delta_0+1}$$

Proof. Let $x = e^{-2\pi u}$ and again $w' = \delta + i\eta$. So

$$E\left(e^{-2\pi u}\right) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Z(\delta_0 + i\eta)}{\delta_0 + i\eta} \cdot e^{-2\pi u(\delta_0+i\eta)} \cdot i\,d\eta$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\pi i u\eta} \cdot f(\eta) \cdot e^{-2\pi u\delta_0}\,d\eta \quad \left(\text{where } f(\eta) = \frac{Z(\delta_0 + i\eta)}{\delta_0 + i\eta}\right)$$

Now

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(\eta) e^{-2\pi i\eta u}\,d\eta$$

Thus

$$e^{2\pi u\delta_0} \cdot 2\pi \cdot E\left(e^{-2\pi u}\right) = \hat{f}(u)$$

Using Plancherel's theorem:

$$\int_{-\infty}^{\infty} \left|\hat{f}(u)\right|^2 du = \int_{-\infty}^{\infty} \left|f(\eta)\right|^2 d\eta \ll 1$$

So

$$1 \gg 4\pi^2 \int_{-\infty}^{\infty} \left|e^{2\pi u\delta_0} \cdot E\left(e^{-2\pi u}\right)\right|^2 du$$

Replace $e^{-2\pi u}$ by y to get

$$1 \gg \frac{4\pi^2}{2\pi} \int_0^{\infty} y^{-2\delta_0} \cdot \left|E(y)\right|^2 \frac{dy}{y} = 2\pi \int_0^{\infty} y^{-(2\delta_0+1)} \cdot \left|E(y)\right|^2 dy$$
$$\geqslant \int_0^x y^{-(2\delta_0+1)} \cdot \left|E(y)\right|^2 dy \geqslant x^{-(2\delta_0+1)} \int_0^x \left|E(y)\right|^2 dy \quad \text{for } 0 \leqslant y \leqslant x$$

Thus

$$\int_0^x \left|E(y)\right|^2 dy \ll x^{2\delta_0+1}, \quad 0 < \delta_0 < 1 \qquad \square$$

We now prove Theorem 6.2, that

$$E(x) \ll x^{\frac{2\delta_0+1}{3}} \cdot \log x$$

For $x \leqslant y$, $\{N\colon\ q^N \leqslant x\} \subseteq \{N\colon\ q^N \leqslant y\}$. From (6.2),

$$\begin{aligned} E(y) - E(x) = \sum_{q^N \leqslant y} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it_0, 0, \beta', \chi\right) dt \\ - \sum_{q^N \leqslant x} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \mathcal{K}_\infty\left(\frac{1}{2} + it_0, 0, \beta', \chi\right) dt \\ - \big(yP(\log y) - xP(\log x)\big) \qquad (6.4) \end{aligned}$$

Since $\mathcal{K}_\infty(\frac{1}{2} + it_0, 0, \beta', \chi)$ is positive,

$$E(y) - E(x) \geqslant -\big(yP(\log y) - xP(\log x)\big) \qquad (6.5)$$

Fix $x \geqslant 3$.

(a) Replace y with $x + u$ for $0 \leqslant u \leqslant x$:

$$E(x) \leqslant E(x+u) + (x+u)P\log(x+u) - xP(\log x)$$

Now P is a linear polynomial, so rewrite

$$\begin{aligned} (x+u)P\log(x+u) - xP(\log x) &= (x+u)\big[A\log(x+u) + B\big] - x\big[A\log x + B\big] \\ &= Ax\left(\log\frac{x+u}{x}\right) + Au\log(x+u) + Bu \\ &= Ax\left(\log\left(1 + \frac{u}{x}\right)\right) + Au\log(x+u) + Bu \end{aligned}$$

Since $\log(1 + \frac{u}{x}) \leqslant \frac{u}{x} \leqslant 1$, then

$$\begin{aligned} Ax\left(\log\left(1 + \frac{u}{x}\right)\right) + Au\log(x+u) + Bu &\leqslant Ax \cdot \frac{u}{x} + Au\log(x+u) + Bu \\ &= Au + Bu + Au\log(x+u) \\ &\leqslant Du\log x + Au(\log x + \log 2) \quad \text{since } u \leqslant x \end{aligned}$$

Thus

$$E(x) \leqslant E(x+u) + Cu\log x \quad \text{for some constant } C$$

(b) Replace x with $x - u$ and y with x for $0 \leqslant u < x$. Then

$$E(x) \geqslant E(x-u) - Cu\log x$$

Let $0 \leqslant H \leqslant x$. Integrate the inequalities over $0 \leqslant u \leqslant H$:

$$\int_0^H E(x)\,du \leqslant \int_0^H \bigl(E(x+u) + Cu\log x\bigr)\,du = H \cdot E(x) \leqslant \int_0^H E(x+u)\,du + \frac{C}{2}H^2\log x \tag{6.6}$$

and

$$H \cdot E(x) \geqslant \int_0^H E(x-u)\,du - \frac{C}{2}H^2\log x$$

So

$$\int_0^H E(x-u)\,du - \frac{C}{2}H^2\log x \leqslant H \cdot E(x) \leqslant \int_0^H E(x+u)\,du + \frac{C}{2}H^2\log x$$

Change variables and replace $\frac{C}{2}$ with C to get

$$\frac{1}{H}\int_{x-H}^{x} E(t)\,dt - CH\log x \leqslant E(x) \leqslant \frac{1}{H}\int_{x}^{x+H} E(t)\,dt + CH\log x$$

For $E(x) \geqslant 0$, apply the second inequality, otherwise apply the first one. So for $E(x) \geqslant 0$,

$$E(x)^2 \ll \frac{1}{H^2}\left(\int_x^{x+H} E(t)\,dt\right)^2 + C^2H^2\log^2 x$$

Apply Cauchy–Schwarz:

$$\begin{aligned} E(x)^2 &\ll \frac{1}{H^2}\int_x^{x+H} \bigl|E(t)\bigr|^2\,dt \cdot \int_x^{x+H} 1\,dt + H^2\log^2 x \\ &= \frac{1}{H}\int_x^{x+H} \bigl|E(t)\bigr|^2\,dt + H^2\log^2 x \ll \frac{1}{H}\cdot x^{2\delta_0+1} + H^2\log x \end{aligned}$$

since

$$\int_0^x \bigl|E(t)\bigr|^2\,dt \ll x^{2\delta_0+1} \quad \text{and} \quad H \leqslant x$$

We want $\frac{1}{H}\cdot x^{2\delta_0+1} = H^2$, so take

$$H = x^{\frac{2\delta_0+1}{3}}$$

Then

$$E(x) \ll H \log x = x^{\frac{2\delta_0+1}{3}} \cdot \log x \qquad \square$$

Let us now use the results obtained above to break convexity. Choose H such that

$$x^{\frac{2\delta_0+1}{3}} \ll H \ll x^{\frac{2\delta_0+1}{3}} \tag{6.7}$$

Let

$$\begin{aligned} S(x) &= \sum_{q^N \leqslant x} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot \mathcal{K}_\infty\left(\frac{1}{2}+it_0, 0, \beta', \chi\right) dt \\ &= xP(\log x) + O\left(x^{\frac{2\delta_0+1}{3}} \log x\right) = xP(\log x) + E(x) \end{aligned} \tag{6.8}$$

Now for $H > 0$, $\{N\colon\ q^N \leqslant x\} \subset \{N\colon\ q^N \leqslant x+H\}$ and $\mathcal{K}_\infty(\frac{1}{2}+it_0, 0, \beta', \chi)$ is *positive*. So for trivial χ,

$$S(x+H+1) - S(x) \geqslant \sum_{x \leqslant q^N \leqslant x+H} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot \prod_{v|\infty} \mathcal{K}_v\left(\frac{1}{2}+it_0, 0, \beta', 1\right) dt \tag{6.9}$$

Now

$$S(x+H+1) - S(x) = (x+H+1)P\bigl(\log(x+H+1)\bigr) - xP(\log x) + E(x+H+1) - E(x) \tag{6.10}$$

From (6.7) and Theorem 6.2,

$$E(x+H+1) - E(x) \ll x^{\frac{2\delta_0+1}{3}} \log x \tag{6.11}$$

and

$$(x+H+1)P\bigl(\log(x+H+1)\bigr) - xP(\log x) \leqslant C(H+1)\log x$$

So

$$S(x+H+1) - S(x) \ll x^{\frac{2\delta_0+1}{3}} \cdot \log x \tag{6.12}$$

and

$$\sum_{x \leqslant q^N \leqslant x+H} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2}+it, f \otimes \chi\right)\right|^2 \cdot \prod_{v|\infty} \mathcal{K}_v\left(\frac{1}{2}+it_0, 0, \beta', 1\right) dt \ll x^{\frac{2\delta_0+1}{3}} \cdot \log x$$

Now

$$Q(\chi, t)^{-\beta'} \ll \mathcal{K}_v\left(\frac{1}{2}+it, 0, w, \chi\right) \ll Q(\chi, t)^{-\beta'} \quad (\text{for } v|\infty)$$

where

$$Q(\chi, t) = \prod_{v \approx \mathbb{R}} \left(1 + |t + t_v|\right) \cdot \prod_{v \approx \mathbb{C}} \left(1 + \ell_v^2 + 4(t + t_v)^2\right)$$

For trivial χ, $t_v = l_v = 0$. Also recalling for $v|\infty$, fix $0 < t < 1$. Then

$$\mathcal{K}_v\left(\frac{1}{2} + it_0, 0, \beta', 1\right) \gg (1)^{-(d-1)\beta'}$$

So

$$\begin{aligned} x^{\frac{2\delta_0+1}{3}} \log x &\gg \sum_{x \leqslant q^N \leqslant x+H} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \prod_{v|\infty} \mathcal{K}_v\left(\frac{1}{2} + it_0, 0, \beta', 1\right) dt \\ &\gg \sum_{x \leqslant q^N \leqslant x+H} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot (1)^{-(d-1)\beta'} dt \\ &\gg \sum_{x \leqslant q^N \leqslant x+H} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot \left(q^N\right)^{-(d-1)\beta'} dt \\ &\geqslant \sum_{x \leqslant q^N \leqslant x+H} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 \cdot (x+H)^{-(d-1)\beta'} \end{aligned} \tag{6.13}$$

Then

$$\begin{aligned} \sum_{x \leqslant q^N \leqslant x+H} \left| L\left(\frac{1}{2} + it, f \otimes \chi\right)\right|^2 dt &\ll (x+H)^{(d-1)\beta'} \cdot x^{\frac{2\delta_0+1}{3}} \cdot \log x \\ &\ll x^{d-1+\frac{2\delta_0+1}{3}+\frac{\epsilon}{2}} \cdot \log x \quad \text{where } \beta' = 1 + \frac{\epsilon}{2d-2} \\ &\ll x^{d-1+\frac{2\delta_0+1}{3}+\epsilon} \end{aligned} \tag{6.14}$$

A pointwise estimate is then obtained from the short-interval integral estimate using Cauchy's theorem and the functional equation of $L(s, f)$. This is a standard argument analogous to that in [18] applied to $GL_2(\mathbb{Q})$ in which case, an asymptotic result was obtained for individual L-functions, rather than a sum. In our present case, we adapt this for general number fields. Thus, this short-interval moment bound implies the pointwise bound

$$L\left(\frac{1}{2} + it, f \otimes \chi\right) \ll \left(q^N\right)^{\frac{d-1}{2}+\frac{2\delta_0+1}{6}+\epsilon} \ll_\epsilon \left(q^N\right)^{\frac{d-1+\vartheta}{2}+\epsilon} \quad (\text{for all } \epsilon > 0) \qquad \square$$

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