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A Simple Proof of Gabriel and Popesco's Theorem

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The purpose of this note is to show a simple proof of the following theorem in [1].

THEOREM *Let C be a Grothendieck category. Let $U \in C$, $A = C(U, U)$,* Mod(A) *be the category of right A-modules and S :*

$$C \to \mathrm{Mod}(A) : M \mapsto C(U, M).$$

Let T be a left adjoint of S. If U is a generator of C, then T is exact and the adjunction $TS \to 1$ is an isomorphism.

LEMMA 1. *If $Y \subset SX$, then $TY \to X$ is mono.*

Proof. Let I be the image of $\oplus_Y U \to X$. Then $Y \subset SI \subset SX$. Applying the method of the proof "(i) → (ii)" in [1], one sees that the composite

$$C(I, Z) \xrightarrow{S} \mathrm{Hom}_A(SI, SZ) \longrightarrow \mathrm{Hom}_A(Y, SZ)$$

is a bijection for any $Z \in C$. Hence I can be identified with TY and $I \subset X$ means that $TY \to X$ is mono.

COROLLARY. *$TS \to 1$ is an isomorphism.*

Proof. Take $Y = SX$ in the proof above. Then $TSX = I = X$.

LEMMA 2. *If $Y \subset F \in$ Mod(A) and F is a free module, then $TY \to TF$ is mono.*

Proof. There exists a directed family $\{F_\alpha\}$ of free finitely generated submodules of F, such that $F = \bigcup F_\alpha$. Then $Y = \bigcup (Y \cap F_\alpha)$. F_α being of the form $(SU)^n = S(U^n)$, the lemma 1 shows that $T(Y \cap F_\alpha) \to TF_\alpha$ is mono. Since T commutes with $\varinjlim$ and $\varinjlim$ is exact in C,

$$TY = \varinjlim T(Y \cap F_\alpha) \to \varinjlim TF_\alpha = TF$$

is mono.

LEMMA 3. *If* $0 \leftarrow Y \leftarrow F_0 \leftarrow F_1 \cdots$ *is a free resolution of* $Y \in \mathrm{Mod}(A)$, *then* $0 \leftarrow TY \leftarrow TF_0 \leftarrow TF_1 \leftarrow \cdots$ *is exact.*

Proof. Since T is right exact, this is an immediate consequence of the Lemma 2.

COROLLARY. T *is exact.*

Proof. The Lemma 3 means that the n-th left derived functor of T is zero for $n > 0$.

REFERENCE

1. N. POPESCO AND P. GABRIEL, Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes, *C. R. Acad. Sci. Paris* **258** (1964), 4188–4190.