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The Rainbow (Vertex) Connection Number of Pencil Graphs

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Abstract

An edge colored graph $G = (V(G), E(G))$ is said rainbow connected, if any two vertices are connnected by a path whose edges have distinct colors. The rainbow connection number of G, denoted by $rc(G)$, is the smallest positive integer of colors needed in order to make G rainbow connected. The vertex-colored graph G is said rainbow vertex-connected, if for every two vertices u and v in $V(G)$, there is a u-v path with all internal vertices have distinct color. The rainbow vertex connection number of G, denoted by $rvc(G)$, is the smallest number of colors needed in order to make G rainbow vertex-connected. In this paper, we determine rainbow (vertex) connection number of pencil graphs.

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1. Introduction

Let G be a simple, finite, and connected graph, and $c : E(G) \longrightarrow \{1, 2, ..., k\}$ be an edge k-coloring, for some $k \in \mathbb{N}$. A path P in G with an edge k-coloring is said *rainbow path*, if no colors repeated. The graph G is said *rainbow connected*, if for any two vertices u and v in $V(G)$ there exist a rainbow u-v path. An edge k-coloring of G is said *rainbow coloring*, if G rainbow connected under c. The *rainbow connection number*, denoted by $rc(G)$, is the smallest positive integer k such that G has rainbow k-coloring. The concept of rainbow connection in graphs was introduced by Chartrand et al[1]. Let G be a connected graph with size m and diameter $diam(G)$, then they stated that

$$diam(G) \leq rc(G) \leq m. \tag{1}$$

The concept of rainbow connection has several interesting variants, one of them is rainbow vertex-connection. It was introduced by Krivelevich and Yuster[2]. Let $c' : V(G) \longrightarrow \{1, 2, ..., k\}$ be a vertex k-coloring, for some $k \in \mathbb{N}$. A path P in G with a vertex k-coloring is said *rainbow vertex-path*, if all internal vertices of P have distinct colors. The graph G is said *rainbow vertex-connected*, if for any two vertices u and v in $V(G)$ there is a rainbow vertex-path. The

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rainbow vertex-connection number of a graph G, denoted by $rvc(G)$, is the smallest positive integer k such that G is rainbow vertex connected under the c' coloring. Krivelevich and Yuster[2] gave the lower bound for $rvc(G)$, namely

$$rvc(G) \geq diam(G) - 1. \tag{2}$$

In some cases $rvc(G)$ is not always larger than $rc(G)$. For example (see[2]), take n vertex-disjoint triangles and designate one vertex from each of them, create a complete graph on designated vertices. The graph has n cut vertices and hence $rvc(G) \geq n$. In fact, by coloring the cut vertices with distinct colors, we obtain $rvc(G) = n$. In other hand, to determine $rc(G)$, we just color the edges of K_n with 1, and color the edges of each triangle with 2, 3, 4. We obtain $rc(G) \leq 4$. Meanwhile, $rvc(G)$ may also be smaller than $rc(G)$. For example, let S_n be a star graph on $n + 1$ vertices. We have $rc(S_n) = n$ and $rvc(S_n) = 1$.

There are many interesting results about rainbow connection numbers and rainbow vertex-connection numbers. Some of them are stated by Li and Liu[3] and Estikasari and Syafrizal[4]. Li and Liu[3] determined the rainbow vertex-connection number of cycle C_n of order $n \geq 3$. Based on it, they prove that for any 2-connected graph G, $rvc(G) \leq rvc(C_n)$. In 2013, Estikasari and Syafrizal[4] determined the rainbow connection number for some corona graphs.

In this paper, we introduce a new cubic graph that we called a pencil graphs. We derive the rainbow (vertex) connection number of pencil graphs. For simplifying, we define $[a, b] = \{x \in \mathbb{Z} | a \leq x \leq b\}$ and $pq \ mod \ p = p$, for any two integers p and q.

2. Main Results

Definition 1. *Let n be a positive integer with $n \geq 2$. A pencil graph with $2n + 2$ vertices, denoted by Pc_n, is a graph with the vertex set and the edge set as follows.*

$$V(Pc_n) = \{u_i, v_i | i \in [0, n]\}$$
$$E(Pc_n) = \{u_i u_{i+1}, v_i v_{i+1} | i \in [1, n-1]\} \cup \{u_i v_i | i \in [0, n]\} \cup \{u_1 u_0, v_1 u_0, u_n v_0, v_n v_0\}.$$

It is easy to check that the diameter of Pc_n is $diam(Pc_n) = d = \left\lceil \frac{n}{2} \right\rceil + 1$, for $n \geq 2$.

Theorem 2. *Let n be an integer at least 2, then*

$$rc(Pc_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Proof. By using (1), we obtain

$$rc(Pc_n) \geq \left\lceil \frac{n}{2} \right\rceil + 1. \tag{3}$$

In order to show that $rc(Pc_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1$, we construct a coloring $c : E(Pc_n) \to [1, d]$ as follows :

$$c(u_0 u_1) = d$$
$$c(u_i u_{i+1}) = i \ mod \ d, i \in [1, n-1]$$
$$c(v_0 v_n) = d - 2$$
$$c(v_i v_{i+1}) = i \ mod \ d, i \in [1, n-1]$$
$$c(u_0 v_0) = d - 1$$
$$c(u_i v_1) = d, i \in \{0, 1\}$$
$$c(u_i v_i) = (i-1) \ mod \ d, i \in [2, n-1]$$
$$c(u_n v_i) = d - 2, i \in \{0, n\}.$$

Futhermore, we can evaluate that Pc_n is rainbow connected under c. Let u and v be two vertices of Pc_n. It is obvious that there exist a rainbow $u - v$ path if u is adjacent to v. In order to show a rainbow $u - v$ path if u is not adjacent to v, we shall devide the proof into 14 cases as shown in Table 1.

So, we conclude that c is a rainbow coloring. We obtain

Table 1. $u - v$ rainbow path in Pc_n

Case	u	v	Condition	Rainbow path
1	u_0	u_i	$i \in [1, d]$	$u_0, u_1, u_2, ..., u_i$
2	u_0	u_i	$i \in [d+1, n]$	$u_0, v_0, u_n, u_{n-1}, ..., u_i$
3	u_0	v_j	$j \in [1, d]$	$u_0, v_1, v_2, ..., v_j$
4	u_0	v_j	$j \in [d+1, n]$	$u_0, v_0, v_n, v_{n-1} ..., v_j$
5	u_i	u_j	$i < j$, $i \in [1, n-1]$, and $j \leq d+i$	$u_i, u_{i+1}, u_{i+2} ..., u_j$
6	u_i	u_j	$i < j$, $i \in [1, n-1]$, and $j > d+i$	$u_i, u_{i-1}, ..., u_0, v_0, u_n, u_{n-1}, ..., u_j$
7	v_i	v_j	$i < j$, $i \in [1, n-1]$, and $j \leq d+i$	$v_i, v_{i+1}, v_{i+2} ..., v_j$
8	v_i	v_j	$i < j$, $i \in [1, n-1]$, and $j > d+i$	$v_i, v_{i-1}, ..., u_0, v_0, v_n, v_{n-1}, ..., v_j$
9	v_0	u_i	$i \in [1, d-1]$	$v_0, u_0, u_1, u_2, ..., u_i$
10	v_0	u_i	$i \in [d, n]$	$v_0, u_n, u_{n-1}, ..., u_i$
11	v_0	v_i	$i \in [1, d-1]$	$v_0, u_0, v_1, v_2, ..., v_i$
12	v_0	v_i	$i \in [d, n]$	$v_0, v_n, v_{n-1}, ..., u_i$
13	u_i	v_j	$i < j$	
			$i \in [2, n]$ and $j \leq d+i-1$	$u_i, v_i, v_{i+1}, ..., v_j$
			$i \in [2, n]$ and $j > d+i-1$	$u_i, u_{i-1}, ..., u_0, v_0, v_n, v_{n-1} ..., v_j$
14	u_i	v_j	$i > j$ and	
			($i \in [1, d-1]$ or	
			$i \in [d, n]$ and $j \in [d-1, n]$)	$u_i, u_{i-1}, u_{i-2}, ..., u_j, v_j$
			$i > j$, $i \in [d, n]$, and $j \in [1, d-1]$	$u_i, u_{i+1}, ..., u_n, v_0, u_0, v_1, v_2 ..., v_j$

$$rc(Pc_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1. \tag{4}$$

From equation (3) and (4), we have $rc(Pc_n) = \left\lceil \frac{n}{2} \right\rceil + 1$. □

Theorem 3. *Let n be a positive integer at least 2, then*

$$rvc(Pc_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \leq 7, \\ \left\lceil \frac{n}{2} \right\rceil + 1 & \text{otherwise.} \end{cases}$$

Proof. We devide a proof into two cases.

Case 1. $n \leq 7$

Based on equation (2), we have $rvc(Pc_n) \geq (d-1)$. We may define a rainbow vertex $(d-1)$-coloring on Pc_n as shown in Fig 2. It is not difficult to verify that all graphs are rainbow-vertex connected.

Case2. $n \geq 8$

By using (2), we obtain

$$rvc(Pc_n) \geq \left\lceil \frac{n}{2} \right\rceil.$$

Suppose that b is an r-vertex coloring, where $r = \left\lceil \frac{n}{2} \right\rceil$. In what follows, we describe a coloring on Pc_n by b. First, color the inner vertices of the $v_{r-2} - u_n$ path. Without loss of generality, color the vertices as follows :

$$e(u_0) = 1$$
$$e(v_0) = 2$$
$$e(v_i) = i + 2, i \in [1, r-2].$$

Let P a subgraph of Pc_n whose the vertex set is $V(P) = \{v_i | i \in [1, n]\}$ and the edge set $E(P) = \{v_i v_{i+1} | i \in [1, n-1]\}$. In fact, for every two vertices with distance less than $d-2$ in P may not be colored with a same color. Consequently, v_{r-1} and v_r must be colored with 1, 2 or 3. Secondly, color v_{r-1} by one of the three colors, so that the color can not be used to color vertices $v_r, v_{r+1}, ..., v_{n-2}$. Since the $v_{r-1} - v_{n-1}$ path has length $d-2$, v_{n-1} must be colored with 1,2 or 3. Therefore, the $v_2 - v_{r+4}$ path or the $v_1 - v_{r+3}$ path are not rainbow vertex-path. This is due to a path which connect v_2 and v_{r+4} or v_1 and v_{r+3}, should have u_0, v_0, v_1 and v_n as its inner vertices, whilst v_{n-1} should have a same color with u_0, v_0 or v_1. Since Pc_n is not rainbow vertex-connected under b, we obtain

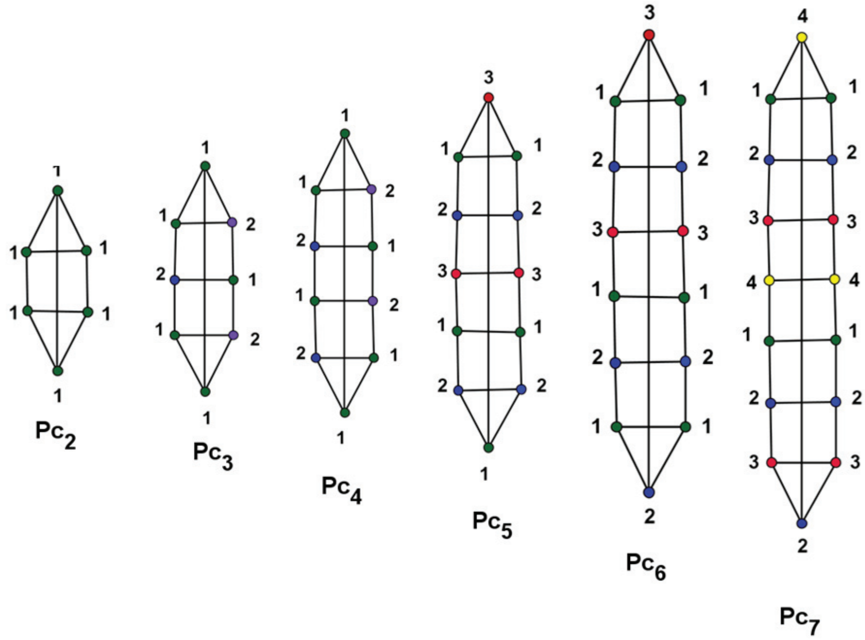

Fig. 1. A rainbow vertex $\left\lceil \frac{n}{2} \right\rceil$-coloring on Pc_n, for $n \in [2, 7]$

$$rvc(Pc_n) \geq \left\lceil \frac{n}{2} \right\rceil + 1. \quad (5)$$

Now, we need to prove that $rvc(Pc_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1$. We construct a vertex coloring $c' : V(Pc_n) \to [1, d]$ as follows :

$$\begin{aligned} c'(u_0) &= d - 1 \\ c'(u_i) &= i \bmod d, i \in [1, n] \\ c'(v_0) &= d \\ c'(v_i) &= i \bmod d, i \in [1, n]. \end{aligned}$$

In order to prove that Pc_n is rainbow-vertex connected under c', we devide the proof into 14 subcases. The subcases almost similar with cases in the proof of Theorem 2.2. So, we obtain

$$rc(Pc_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1. \quad (6)$$

From equation (5) and (6), we get $rvc(Pc_n) = \left\lceil \frac{n}{2} \right\rceil + 1$.
We conclude,

$$rvc(Pc_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \leq 7, \\ \left\lceil \frac{n}{2} \right\rceil + 1 & \text{otherwise.} \end{cases}$$

□

For illustration, we give a rainbow 6-coloring on Pc_{10} and a rainbow vertex 6-coloring on Pc_{10} in Fig 2 (a) and Fig 2 (b), respectively.

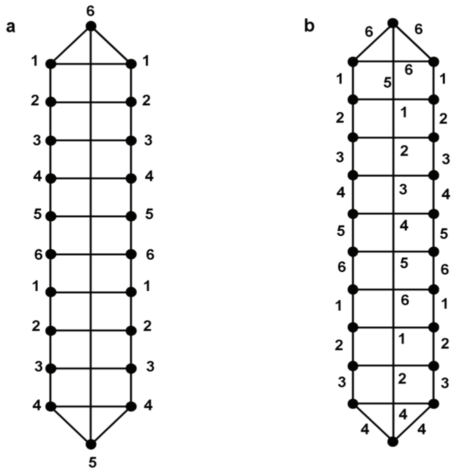

Fig. 2. (a) A rainbow 6-coloring on Pc_{10}; (b) A rainbow vertex 6-coloring on Pc_{10}.

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