The Poles and Zeros of a Linear Time-Varying System*

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ABSTRACT

For linear time-varying discrete-time and continuous-time systems, a notion of poles and zeros is developed in terms of factorizations of operator polynomials with time-varying coefficients. In the discrete-time case, it is shown that the poles can be computed by solving a nonlinear recursion with time-varying coefficients. In the continuous-time case, the poles can be calculated by solving a nonlinear differential equation with time-varying coefficients. The theory is applied to the study of the zero-input response and asymptotic stability. It is shown that if a time-varying analogue of the Vandermonde matrix is invertible, the zero-input response can be decomposed into a sum of modes associated with the poles. Stability is then studied in terms of the components of the modal decomposition.

1. INTRODUCTION

In this paper we develop a notion of poles and zeros for the class of linear time-varying systems. Our notion is defined in terms of the factorization of polynomials with time-varying coefficients. Although the idea of factoring polynomials with time-varying coefficients is not new [e.g., see Ore (1933), Amitsur (1954), Newcomb (1970)], we present a new approach to constructing factorizations given in terms of the solutions to a nonlinear equation.

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We begin in the next section with the study of a linear time-varying discrete-time system specified by a second-order input-output difference equation. Various properties of factorizations are studied in Section 3, and then, in Section 4, we consider a general class of linear time-varying discrete-time systems specified by an nth-order input-output difference equation. In Section 5 the theory is applied to the study of the zero-input response and asymptotic stability. The continuous-time case is considered in Section 6.

2. POLES OF A SECOND-ORDER SYSTEM

With $\mathbb{Z}$ equal to the set of integers and $\mathbb{R}$ equal to the set of real numbers, let A denote the set of all functions defined on $\mathbb{Z}$ with values in $\mathbb{R}$. With $\mathbb{C}$ equal to the set of complex numbers, let $A_{\mathbb{C}}$ denote the set of all complex-valued functions defined on $\mathbb{Z}$. Clearly, the set A can be viewed as a subset of $A_{\mathbb{C}}$. Both A and $A_{\mathbb{C}}$ are commutative rings with pointwise addition and multiplication defined by

$$(a+b)(k)=a(k)+b(k),$$

$$(ab)(k)=a(k)b(k),$$

where $a(k), b(k)$ are arbitrary elements of A (or $A_{\mathbb{C}}$) and k is the discrete-time index.

Now consider the linear discrete-time system given by the second-order input-output difference equation

$$y(k+2)+a_1(k)y(k+1)+a_0(k)y(k)=b(k)u(k), \tag{1}$$

where $y(k)$ is the output at time k, $u(k)$ is the input at time k, and $a_0(k), a_1(k), b(k)$ are elements of A. We shall write the left side of (1) in operator form as follows.

For any positive integer i, let z^i denote the i-step left shift operator on A defined by

$$z^i f(k)=f(k+i), \qquad f\in A.$$

For any $a(k)\in A$, let $a(k)z^i$ denote the operator on A defined by

$$\left[a(k)z^i\right]f(k)=a(k)f(k+i).$$

Then we can write (1) in the operator form

$$[z^2 + a_1(k)z + a_0(k)]y(k) = b(k)u(k). \tag{2}$$

Suppose that there exist functions $p_1(k), p_2(k)$ belonging to A_C such that

$$[z^2 + a_1(k)z + a_0(k)]y(k) = [z - p_1(k)]\{[z - p_2(k)]y(k)\}. \tag{3}$$

It follows from (3) that the given system can be viewed as a cascade connection of two first-order subsystems. To see this, let

$$v(k) = [z - p_2(k)]y(k), \tag{4}$$

so that

$$y(k+1) - p_2(k)y(k) = v(k). \tag{5}$$

Inserting the expression (4) for $v(k)$ into (3) and using (2), we have that

$$[z - p_1(k)]v(k) = b(k)u(k),$$

or

$$v(k+1) - p_1(k)v(k) = b(k)u(k). \tag{6}$$

From (5) and (6), we have the cascade realization of the system shown in Figure 1.

Given the decomposition of the system into the first-order subsystems shown in Figure 1, it is tempting to call $p_1(k)$ and $p_2(k)$ the poles of the system. We shall give a formal definition later. First, we want to characterize

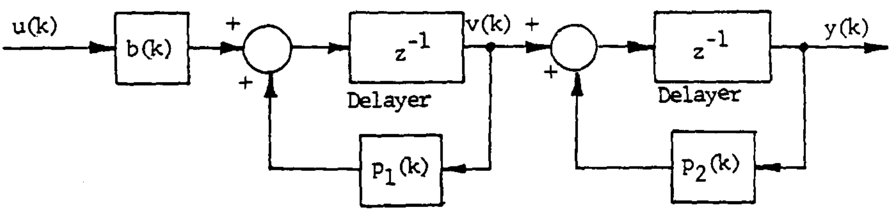

FIG. 1. Realization of system.

the cascade decomposition in terms of a polynomial factorization, and then we consider the existence and construction of factorizations.

Again suppose that there exist $p_1(k), p_2(k) \in A_C$ such that (3) is satisfied. We want to define a product

$$[z - p_1(k)] \circ [z - p_2(k)]$$

so that

$$\{[z - p_1(k)] \circ [z - p_2(k)]\} y(k) = [z - p_1(k)]\{[z - p_2(k)] y(k)\}. \quad (7)$$

Expanding the right side of (7), we obtain

$$\begin{aligned} &[z - p_1(k)]\{[z - p_2(k)] y(k)\} \\ &\quad = [z - p_1(k)][y(k+1) - p_2(k) y(k)] \\ &\quad = y(k+2) - [p_1(k) + p_2(k+1)] y(k+1) + p_1(k) p_2(k) y(k) \\ &\quad = \{z^2 - [p_1(k) + p_2(k+1)] z + p_1(k) p_2(k)\} y(k). \end{aligned}$$

Inserting this into the right side of (7), we have that

$$[z - p_1(k)] \circ [z - p_2(k)] = z^2 - [p_1(k) + p_2(k+1)] z + p_1(k) p_2(k). \quad (8)$$

Equation (8) shows that the multiplication $\circ$ is the usual polynomial multiplication except that

$$z \circ p_2(k) = p_2(k+1) z. \quad (9)$$

As a result of the property (9), the multiplication $\circ$ is called a skew polynomial multiplication. In Section 4 we define a ring of polynomials with the property (9).

We now consider the existence of $p_1(k), p_2(k) \in A_C$ such that (3) is satisfied. Combining (3), (7), and (8), we have that

$$z^2 - [p_1(k) + p_2(k+1)] z + p_1(k) p_2(k) = z^2 + a_1(k) z + a_0(k). \quad (10)$$

Equating coefficients of z in (10), we see that $p_1(k)$ and $p_2(k)$ must satisfy the equations

$$p_1(k)+p_2(k+1) = -a_1(k), \tag{11}$$

$$p_1(k)p_2(k) = a_0(k) \tag{12}$$

Multiplying both sides of (11) by $p_2(k)$ and using (12), we obtain

$$p_2(k+1)p_2(k)+a_1(k)p_2(k)+a_0(k)=0. \tag{13}$$

Note that (13) is a nonlinear first-order difference equation with time-varying coefficients. It is also interesting to note that the left side of (13) looks like the polynomial $z^2+a_1(k)z+a_0(k)$ evaluated at $z=p_2(k)$ with

$$z|_{z=p_2(k)} = p_2(k) \quad \text{and} \quad z^2|_{z=p_2(k)} = p_2(k+1)p_2(k).$$

Given the initial value $p_2(k_0)$ at initial time k_0, we can compute $p_2(k)$ for $k>k_0$ by solving (11) and (12) recursively, or by solving (13) recursively. If $p_2(k)\neq 0$ for all $k \geqslant k_0$, the solution is unique. Given $p_2(k_0)$ selected at random, the probability is zero that $p_2(k)$ will be zero for some value of $k \geqslant k_0$. In other words, for almost all initial values $p_2(k_0)\in \mathbf{C}$, (13) has a unique solution $p_2(k)$ with $p_2(k)\neq 0$ for all $k\geqslant k_0$. Furthermore, $p_1(k)$ can be computed from (11). We therefore have the following result.

PROPOSITION 1. *For almost all* $\gamma \in \mathbf{C}$, *the operator polynomial* $z^2 + a_1(k)z+a_0(k)$ *has a unique skew polynomial factorization* $[z-p_1(k)]\circ[z-p_2(k)]$ *for* $k>k_0$ *with* $p_2(k_0)=\gamma$.

It should be stressed that we are asserting the (generic) existence of a unique factorization over the time interval $k>k_0$ with the given initial value $p_2(k_0)$. If we allow the initial value $p_2(k_0)$ to range over all of $\mathbf{C}$, in general we will obtain an infinite collection of skew polynomial factorizations.

If the given system is time invariant for $k \leqslant k_0$ [i.e., $a_0(k)$ and $a_1(k)$ are constant for $k\leqslant k_0$], we can take the initial value $p_2(k_0)$ to be equal to one of the initial poles. If the system is not time invariant prior to $k=k_0$, we could take $p_2(k_0)$ to be one of the zeros of the polynomial $z^2+a_1(k_0)z+a_0(k_0)$.

Now suppose that $p_2(k_1)=0$ for some $k_1>k_0$. From (12), we have

$$p_1(k_1)p_2(k_1)=a_0(k_1).$$

Thus if $a_0(k_1) \neq 0$, there is no solution for $p_1(k_1)$. If $a_0(k_1)=0$, we can take $p_1(k_1)$ to be any desired value, so in this case there are an infinite number of solutions. If $p_2(k_1)$ is zero for some $k_1 > k_0$, we could reinitialize by taking $p_2(k_1) \neq 0$, but the resulting factorization would only be valid for $k > k_1$.

If $p_1(k), p_2(k)$ are the solutions to (11) and (12) with $p_2(k_0)=\gamma \in \mathbf{C}$, then the complex conjugates $\bar{p}_1(k), \bar{p}_2(k)$ are the solutions to (11) and (12) with initial value $\bar{\gamma}$. This can be seen by simply taking the complex conjugate of both sides of (11) and (12). Unlike the time-invariant case, in general $\bar{p}_1(k) \neq p_2(k)$ and $\bar{p}_2(k) \neq p_1(k)$. Thus, in general there are two different solutions to (11) and (12) with initial value $p_2(k_0)=\gamma$ or $p_2(k_0)=\bar{\gamma}$. We call the ordered sets $(p_1(k), p_2(k))$ and $(\bar{p}_1(k), \bar{p}_2(k))$ pole sets on $k > k_0$ with respect to the initial values $\gamma, \bar{\gamma}$. The elements $p_2(k)$ and $\bar{p}_2(k)$ are called the right poles on $k > k_0$, and $p_1(k), \bar{p}_1(k)$ are the left poles on $k > k_0$.

In the following two examples, the pole sets were computed by solving (11) and (12) recursively. In particular, given $p_2(k_0)$, we used (12) to compute $p_1(k_0)$, then (11) to compute $p_2(k_0+1)$, then (12) to compute $p_1(k_0+1)$, and so on.

Example 1. The coefficients of the input-output difference equation (1) are

$$a_0(k)=0.5 \qquad \text{for all} \quad k \in \mathbb{Z},$$

$$a_1(k)=\begin{cases} -1, & k \leqslant 0, \\ -1+(2.5k/200), & 0<k<200, \\ 1.5, & k \geqslant 200. \end{cases}$$

For $k<0$,

$$z^2+a_1(k)z+a_0(k)=z^2-z+0.5=(z-0.5-j0.5)(z-0.5+j0.5),$$

and as $k \to \infty$

$$z^2+a_1(k)z+a_0(k) \to z^2+1.5z+0.5=(z+1)(z+0.5).$$

The poles with $p_2(0)=0.5-j0.5$ are shown in Figure 2. Note that the poles are not complex conjugates.

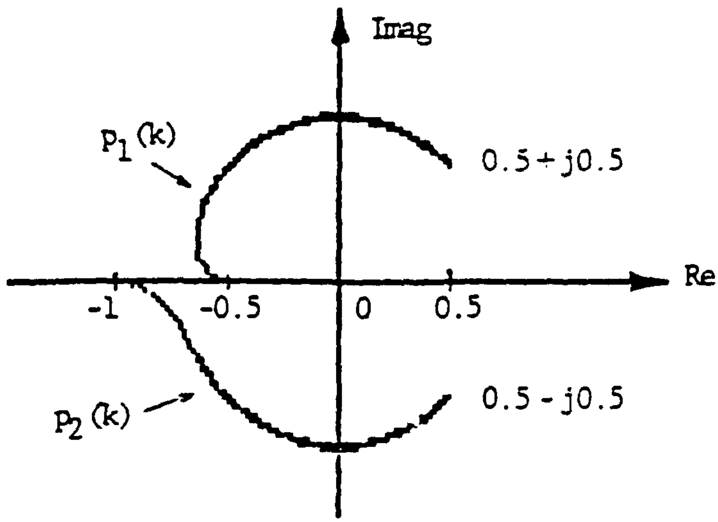

FIG. 2. Poles with $p_2(0) = 0.5 - j0.5$.

EXAMPLE 2. Now suppose that

$$a_0(k) = 0.5 \qquad \text{for all} \quad k \in \mathbb{Z},$$

$$a_1(k) = \begin{cases} -1, & k \leqslant 0, \\ -1 + (2.5k/50), & 0 < k < 50, \\ 1.5, & k \geqslant 50. \end{cases}$$

As in Example 1, the initial poles are $0.5 + j0.5$, $0.5 - j0.5$, and as $k \to \infty$

$$z^2 + a_1(k)z + a_0(k) \to z^2 + 1.5z + 0.5 = (z+1)(z+0.5).$$

The main difference between this example and the previous one is that the rate of change of $a_1(k)$ is much faster in this example. The poles with $p_2(0) = 0.5 - j0.5$ are shown in Figure 3. Note the erratic behavior of the poles as k is increased from the initial value $k = 0$. This is a result of the rapid variation of the coefficient $a_1(k)$.

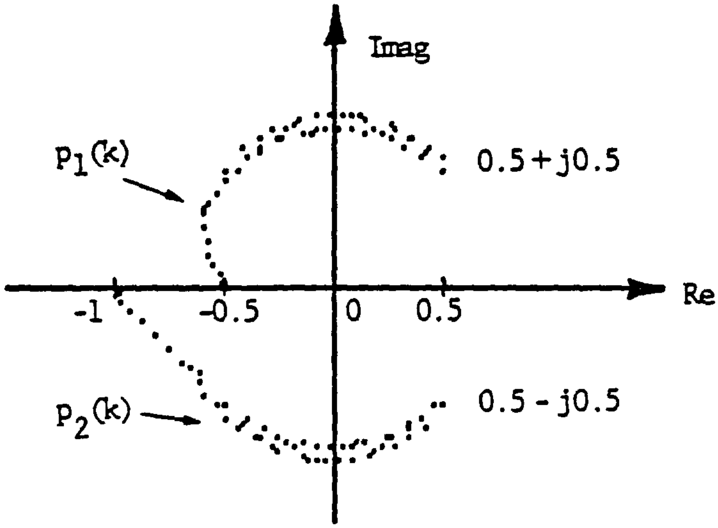

FIG. 3. Poles with $p_2(0) = 0.5 - j0.5$.

3. PROPERTIES OF FACTORIZATIONS

Again consider the discrete-time system given by the second-order input-output difference equation (1). Let $\hat{p}_1(k)$ and $\hat{p}_2(k)$ denote the zeros of the polynomial $z^2 + a_1(k)z + a_0(k)$; that is,

$$[z - \hat{p}_1(k)][z - \hat{p}_2(k)] = z^2 + a_1(k)z + a_0(k),$$

where $[z - \hat{p}_1(k)][z - \hat{p}_2(k)]$ is the ordinary product of two polynomials. We can write this product in the form

$$\begin{aligned}[z - \hat{p}_1(k)][z - \hat{p}_2(k)] \\ &= z^2 - [\hat{p}_1(k) + \hat{p}_2(k)]z + \hat{p}_1(k)\hat{p}_2(k) \\ &= z^2 - [\hat{p}_1(k) + \hat{p}_2(k+1)]z + \hat{p}_1(k)\hat{p}_2(k) + [\hat{p}_2(k+1) - \hat{p}_2(k)]z \\ &= [z - \hat{p}_1(k)] \circ [z - \hat{p}_2(k)] + [\hat{p}_2(k+1) - \hat{p}_2(k)]z. \qquad (14)\end{aligned}$$

If the system is slowly varying in the sense that $|\hat{p}_2(k+1) - \hat{p}_2(k)|$ is small for $k > k_0$, from (14) we have that

$$[z - \hat{p}_1(k)][z - \hat{p}_2(k)] \doteq [z - \hat{p}_1(k)] \circ [z - \hat{p}_2(k)]. \tag{15}$$

Here $\doteq$ means that the magnitude of the difference of the coefficients of the polynomials is small for $k > k_0$. The relationship (15) corresponds to the well-known result that slowly varying systems can be studied using the "frozen-time approach."

Now suppose that $a_0(k) \to c_0$ and $a_1(k) \to c_1$ as $k \to \infty$. Let r_1 and r_2 denote the zeros of $z^2 + c_1 z + c_0$. If $(p_1(k), p_2(k))$ is a pole set on $k > k_0$, it turns out that $p_2(k)$ will not converge to r_1 or r_2 in general. In particular, if r_1 and r_2 are complex numbers and $p_2(k_0)$ is a real number, by (13) $p_2(k)$ is real for all $k > k_0$, and thus $p_2(k)$ cannot converge to r_1 or r_2.

As a special case, suppose that

$$a_0(k) = c_0 \text{ and } a_1(k) = c_1 \qquad \text{for all} \quad k \geqslant k_1.$$

Given the pole set $(p_1(k), p_2(k))$, let $\Delta(k) = p_2(k) - r_2$, so that

$$\Delta(k+1) = p_2(k+1) - r_2. \tag{16}$$

By (13),

$$p_2(k+1) = -a_1(k) - \frac{a_0(k)}{p_2(k)},$$

and thus for $k \geqslant k_1$,

$$p_2(k+1) = -c_1 - \frac{c_0}{p_2(k)}. \tag{17}$$

Inserting (17) into (16) gives

$$\Delta(k+1) = -c_1 - \frac{c_0}{p_2(k)} - r_2$$

$$\Delta(k+1) = \frac{-(c_1 + r_2)p_2(k) - c_0}{p_2(k)}.$$

Now $p_2(k) = \Delta(k) + r_2$, and therefore

$$\Delta(k+1) = \frac{-\left(r_2^2 + c_1 r_2 + c_0\right) - (c_1 + r_2)\Delta(k)}{\Delta(k) + r_2}. \tag{18}$$

But since r_1 and r_2 are the zeros of $z^2 + c_1 z + c_0$, we have that

$$r_2^2 + c_1 r_2 + c_0 = 0 \quad \text{and} \quad -\left(c_1 + r_2\right) = r_1. \tag{19}$$

Using (19) in (18), we obtain

$$\Delta(k+1) = \frac{r_1 \Delta(k)}{\Delta(k) + r_2}, \qquad k \geqslant k_1. \tag{20}$$

We can convert (20) into a linear difference equation by using the transformation

$$g(k) = \frac{1}{\Delta(k)}. \tag{21}$$

Combining (20) and (21) gives

$$g(k+1) = \frac{1}{r_1}\left[r_2 g(k) + 1\right]. \tag{22}$$

Now (22) can be solved using standard techniques, and once we have determined $g(k)$, we can compute $\Delta(k)$ using (21). If $r_1 \neq r_2$, the result is

$$\Delta(k) = \frac{(r_1 - r_2)\Delta(k_1)}{\Delta(k_1) + \left[(r_1 - r_2) - \Delta(k_1)\right](r_2/r_1)^{k-k_1}}, \qquad k \geqslant k_1. \tag{23}$$

From (23), we see that if $\Delta(k_1) \neq r_1 - r_2$ [i.e., $p_2(k_1) \neq r_1$] and $|r_2/r_1| > 1$, then $\Delta(k)$ converges to zero as $k \to \infty$, and $p_2(k)$ converges to r_2 as $k \to \infty$. Clearly, the condition $|r_2/r_1| > 1$ requires that r_1 and r_2 be real (not complex conjugates).

Summarizing the above constructions, we have the following result.

PROPOSITION 2. *Suppose that* $a_0(k) = c_0$ *and* $a_1(k) = c_1$ *for* $k \geqslant k_1$, *and let* r_1, r_2 *be the zeros of* $z^2 + c_1 z + c_0$. *If* $|r_2/r_1| > 1$, *then for any pole set* $(p_1(k), p_2(k))$ *with* $p_2(k_1) \neq r_1$, $p_2(k)$ *converges to* r_2 *as* $k \to \infty$.

The hypothesis of Proposition 2 is satisfied in Examples 1 and 2 given in the previous section, and thus in these cases $p_2(k)$ does converge to r_2 ($= -1$). In the following example, we modify Example 2 so that r_1 and r_2 are complex.

EXAMPLE 3. Suppose that

$$a_0(k) = 0.5 \qquad \text{for all} \quad k \in \mathbb{Z},$$

$$a_1(k) = \begin{cases} -1, & k \leqslant 0, \\ -1 + (2.4k/50), & 0 < k < 50, \\ 1.4, & k \geqslant 50. \end{cases}$$

As in Example 2, the initial poles are $0.5 + j0.5$, $0.5 - j0.5$, but now

$$z^2 + a_1(k)z + a_0(k) \to z^2 + 1.4z + 0.5 \qquad \text{as} \quad k \to \infty.$$

In this example, $r_1 = -0.7 + j0.1$ and $r_2 = \bar{r}_1$. The poles with $p_2(0) = 0.5 - j0.5$ are plotted in Figure 4. In this case, the poles $p_1(k)$ and $p_2(k)$ continuously encircle points in the complex plane as $k \to \infty$.

To avoid the type of behavior displayed in Example 3, we can reinitialize the recursion for computing the poles. For example, again letting $\hat{p}_1(k)$ and $\hat{p}_2(k)$ denote the zeros of $z^2 + a_1(k)z + a_0(k)$, if $|\hat{p}_2(k+1) - \hat{p}_2(k)|$ is small for some range of $k < k_1$, we could set $p_2(k_1) = \hat{p}_2(k_1)$ and continue with the recursion.

We conclude this section by considering the relationship between two different poles sets. Let $(p_1(k), p_2(k))$ and $(q_1(k), q_2(k))$ be two pole sets for the system given by (1). Then by (11) and (12), we have that

$$p_1(k) + p_2(k+1) = q_1(k) + q_2(k+1), \tag{24}$$

$$p_1(k)p_2(k) = q_1(k)q_2(k). \tag{25}$$

Multiplying both sides of (24) by $q_2(k)$ and using (25), we obtain

$$p_1(k)[q_2(k) - p_2(k)] = q_2(k)[q_2(k+1) - p_2(k+1)]. \tag{26}$$

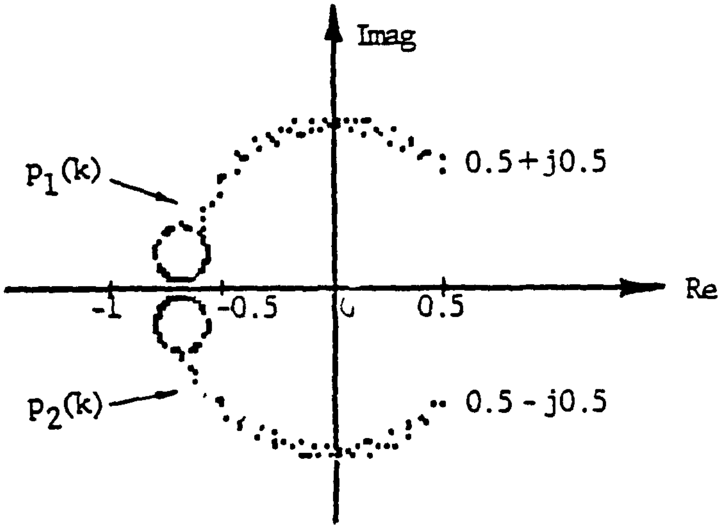

FIG. 4. Poles with $p_2(0) = 0.5 - j0.5$.

Defining

$$w(k) = q_2(k) - p_2(k),$$

we can write (26) in the form

$$p_1(k)w(k) = q_2(k)w(k+1). \tag{27}$$

If $w(k) \neq 0$ for all $k \geqslant k_0$, from (27) we have that

$$p_1(k) = \frac{w(k+1)}{w(k)} q_2(k), \qquad k \geqslant k_0. \tag{28}$$

The relationship (28) shows that it is possible to compute the left pole $p_1(k)$ of the pole set $(p_1(k), p_2(k))$ from the right pole $q_2(k)$ of the pole set $(q_1(k), q_2(k))$. Note that if $q_1(k) = \bar{p}_1(k)$ and $q_2(k) = \bar{p}_2(k)$, then by (28)

$$p_1(k) = \left[\frac{\bar{p}_2(k+1) - p_2(k+1)}{\bar{p}_2(k) - p_2(k)}\right]\bar{p}_2(k),$$

assuming that $p_2(k) \neq \bar{p}_2(k)$. This result shows that in general $p_1(k)$ is not the complex conjugate of $p_2(k)$.

4. THE nTH-ORDER CASE

Let $A[z]$ ($A_C[z]$) denote the set of all operator polynomials in the left shift operator z with coefficients in A (A_C). With the usual polynomial addition and with multiplication defined in terms of

$$z^i \circ z^j = z^{i+j}$$

$$z^i \circ a(k) = a(k+i)z^i, \qquad a(k) \in A \text{ (or } A_C),$$

$A[z]$ and $A_C[z]$ are skew (noncommutative) rings. These rings have appeared in the work of Hafez (1975), Kamen and Hafez (1975), and Kamen, Khargonekar, and Poolla (1985).

Now given

$$a(z,k) = z^n + \sum_{i=0}^{n-1} a_i(k)z^i \in A[z]$$

and

$$b(z,k) = \sum_{i=0}^{n} b_i(k)z^i \in A[z],$$

consider the linear discrete-time system defined by the input-output equation

$$a(z,k)y(k) = b(z,k)u(k),$$

where as before, $y(k)$ is the output and $u(k)$ is the input.

We call $p_n(k) \in A_C$ a right pole of the system if there exists a $e(z,k) \in A_C[z]$ such that

$$a(z,k) = e(z,k) \circ [z - p_n(k)]. \tag{29}$$

We call $q(k) \in A_C$ a right zero of the system if there exists a $h(z,k) \in A_C[z]$ such that

$$b(z,k) = h(z,k) \circ [z - q(k)].$$

Suppose that there exist $p_n(k)$ and $e(z,k)$ such that (29) holds. Then writing

$$e(z,k) = z^{n-1} + \sum_{i=0}^{n-2} e_i(k) z^i,$$

we have that (29) is equivalent to the equations

$$e_{n-2}(k) = p_n(k+n-1) + a_{n-1}(k), \tag{30}$$

$$e_{i-1}(k) = e_i(k) p_n(k+i) + a_i(k), \qquad i = n-2, n-3, \dots, 1, \tag{31}$$

$$0 = e_0(k) p_n(k) + a_0(k). \tag{32}$$

Given the initial values $p_n(k_0 - n + 1), p_n(k_0 - n + 2), \dots, p_n(k_0)$, we can solve (30)–(32) recursively to compute $p_n(k)$ and the $e_i(k)$ for $k > k_0$. As in the case $n = 2$, a unique solution exists for almost all possible values of the initial data.

Once we have determined the factorization (29), we can then pull out a right factor from $e(z,k)$, and so on, until we obtain the pole set $(p_1(k), p_2(k), \dots, p_n(k))$. It is important to note that this is an ordered set. Due to the noncommutativity of multiplication in the ring $A_C[z]$, a permutation of the elements in a pole set would not result in another pole set [unless of course the $p_i(k)$ are constant].

As we now show, Equations (30)–(32) can be combined to yield a single equation for $p_n(k)$. First, multiplying both sides of (30) by $p_n(k+n-2)$ and using (31) with $i = n-2$, we obtain

$$e_{n-3}(k) = p_n(k+n-1) p_n(k+n-2) + a_{n-1}(k) p_n(k+n-2) + a_{n-2}(k). \tag{33}$$

Multiplying both sides of (33) by $p_n(k+n-3)$ and using (31) with $i = n-3$ gives

$$\begin{aligned} e_{n-4}(k) = {} & p_n(k+n-1) p_n(k+n-2) p_n(k+n-3) \\ & + a_{n-1}(k) p_n(k+n-2) p_n(k+n-3) \\ & + a_{n-2}(k) p_n(k+n-3) + a_{n-3}(k). \end{aligned}$$

Continuing, we obtain

$$0 = p_n(k+n-1)p_n(k+n-2)\cdots p_n(k) + \sum_{i=1}^{n-1} a_i(k)p_n(k+i-1)p_n(k+i-2)\cdots p_n(k) + a_0(k). \quad (34)$$

If $p_n(k) \neq 0$ for $k \geqslant k_0 - n + 1$, we can rewrite (34) in the form (assuming $n \geqslant 3$)

$$p_n(k+n-1) = -a_{n-1}(k) - \sum_{i=1}^{n-2} \frac{a_i(k)}{p_n(k+n-2)p_n(k+n-3)\cdots p_n(k+i)} - \frac{a_0(k)}{p_n(k+n-2)p_n(k+n-3)\cdots p_n(k)}. \quad (35)$$

Equation (35) can be solved recursively to compute the right pole $p_n(k)$.

We can compute a collection of right poles by solving (35) for different initial conditions. The calculation of right poles using (35) can be carried out in parallel. In the next section we will show that the computation of a set of right poles arises in the derivation of a modal decomposition of the zero-input response.

Instead of solving (35), we can compute right poles by solving the linear difference equation $a(z,k)y(k) = 0$. In particular, let $y(k)$ be the solution to this equation with the nonzero initial values $y(k_0 - n), y(k_0 - n + 1), \ldots, y(k_0 - 1)$, and suppose that $y(k) \neq 0$ for all $k \geqslant k_0$. Then from existing results [see Hafez (1975)], we have that $p_n(k) = y(k+1)/y(k)$ is a right pole on $k > k_0$. To show this, first observe that for any integer $i \geqslant 0$,

$$p_n(k+i)p_n(k+i-1)\cdots p_n(k) = \frac{y(k+i+1)}{y(k)}.$$

Then since $a(z,k)y(k) = 0$, we have that $p_n(k) = y(k+1)/y(k)$ satisfies (34), which shows that $p_n(k)$ is a right pole.

We conclude this section by showing that our notion of a zero can be interpreted to be a transmission blocking zero as in the time-invariant case. Let $q(k)$ be a right zero of the system defined above, so that

$$b(z,k) = h(z,k) \circ [z - q(k)].$$

Define

$$\phi_q(k,k_0)=\begin{cases}1, & k=k_0,\\ q(k-1)q(k-2)\cdots q(k_0), & k>k_0,\\ 0, & k<k_0.\end{cases}$$

The function $\phi_q(k,k_0)$ is called the mode associated with the right zero $q(k)$. Note that if $q(k)$ is constant, so that $q(k)=q$ for $k\geqslant k_0$, then

$$\phi_q(k,k_0)=q^{k-k_0}, \qquad k\geqslant k_0.$$

Let $q(k)$ be a right zero with associated mode $\phi_q(k,k_0)$. With the input $u(k)$ of the system equal to $\phi_q(k,k_0)$, we want to show that we can choose the initial output values $y(k_0-n), y(k_0-n+1),\dots,y(k_0-1)$ so that $y(k)=0$ for all $k\geqslant k_0$. First, we have that

$$\begin{aligned}b(z,k)u(k)&=h(z,k)\{[z-q(k)]u(k)\}\\ &=h(z,k)\big[\phi_q(k+1,k_0)-q(k)\phi_q(k,k_0)\big]\end{aligned}$$

By definition of $\phi_q(k,k_0)$,

$$\phi_q(k+1,k_0)=q(k)\phi_q(k,k_0), \qquad k\geqslant k_0,$$

and thus

$$b(z,k)u(k)=0 \qquad \text{for} \quad k\geqslant k_0.$$

Returning to the input-output difference equation, we then have that

$$y(k+n)=-\sum_{i=0}^{n-1}a_i(k)y(k+i), \qquad k\geqslant k_0. \tag{36}$$

If $y(k_0+i)=0$ for $i=0,1,\dots,n-1$, it follows from (36) that $y(k)=0$ for all $k\geqslant k_0$ as desired. If $a_0(k)\neq 0$ for $k=k_0-1,k_0-2,\dots,k_0-n$, it is a standard construction to show that the initial values $y(k_0-1)$, $y(k_0-2),\dots,y(k_0-n)$ can be chosen so that $y(k_0+i)=0$ for $i=0,1,\dots,n-1$. Thus we have the following result.

PROPOSITION 3. *Suppose that* $a_0(k) \neq 0$ *for* $k = k_0 - 1, k_0 - 2, \ldots, k_0 - n$. *Then with* $u(k) = \phi_q(k, k_0)$, *for some initial values* $y(k_0 - 1), y(k_0 - 2), \ldots, y(k_0 - n)$ *the output* $y(k)$ *is identically zero for all* $k \geqslant k_0$.

5. THE ZERO-INPUT RESPONSE AND STABILITY

In this section we assume that the input $u(k)$ to the system is zero for all $k \geqslant k_0$, so that the input-output difference equation reduces to

$$y(k+n) + \sum_{i=0}^{n-1} a_i(k) y(k+i) = 0. \tag{37}$$

Using our operator notation, (37) can be written in the form

$$a(z,k) y(k) = 0. \tag{38}$$

The initial conditions for (37) [or (38)] are the values $y(k_0)$, $y(k_0+1), \ldots, y(k_0+n-1)$.

Let $p_{n1}(k), p_{n2}(k), \ldots, p_{nn}(k)$ be n right poles; that is, suppose that there exist polynomials $e_i(z,k) \in A_{\mathbb{C}}[z]$ such that

$$a(z,k) = e_i(z,k) \circ [z - p_{ni}(k)] \qquad \text{for} \quad i = 1, 2, \ldots, n.$$

Let $V(k)$ denote the time-varying version of the $n \times n$ Vandermonde matrix defined by

$$V(k) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ p_{n1}(k) & p_{n2}(k) & \cdots & p_{nn}(k) \\ p_{n1}(k+1)p_{n1}(k) & p_{n2}(k+1)p_{n2}(k) & \cdots & p_{nn}(k+1)p_{nn}(k) \\ \vdots & \vdots & & \vdots \\ p_{n1}(k+n-2)\cdots p_{n1}(k) & p_{n2}(k+n-2)\cdots p_{n2}(k) & \cdots & p_{nn}(k+n-2)\cdots p_{nn}(k) \end{bmatrix}.$$

Finally, let $\phi_{p_{ni}}(k, k_0)$ denote the mode associated with the pole $p_{ni}(k)$. Recall that

$$\phi_{p_{ni}}(k, k_0) = \begin{cases} 1, & k = k_0, \\ p_{ni}(k-1) p_{ni}(k-2) \cdots p_{ni}(k_0), & k > k_0, \\ 0, & k < k_0. \end{cases}$$

We then have the following result.

THEOREM 1. *If the determinant of* $V(k_0)$ *is nonzero, then for any initial values* $y(k_0), y(k_0+1), \dots, y(k_0+n-1)$ *there exist constants* $c_1, c_2, \dots, c_n$ *such that the solution to* $a(z,k)y(k)=0$ *can be written in the form*

$$y(k) = \sum_{i=1}^{n} c_i \phi_{p_{ni}}(k, k_0) \qquad \textit{for} \quad k \geqslant k_0. \tag{39}$$

Proof. We will first show that $y(k) = c\phi_{p_{ni}}(k, k_0)$ is a solution to $a(z,k)y(k)=0$, where c is a constant. We have that

$$a(z,k)c\phi_{p_{ni}}(k,k_0) = ce_i(z,k)\big[(z - p_{ni}(k))\phi_{p_{ni}}(k,k_0)\big]$$

$$= ce_i(z,k)\big[\phi_{p_{ni}}(k+1,k_0) - p_{ni}(k)\phi_{p_{ni}}(k,k_0)\big].$$

By definition of the mode $\phi_{p_{ni}}(k, k_0)$,

$$\phi_{p_{ni}}(k+1,k_0) = p_{ni}(k)\phi_{p_{ni}}(k,k_0), \qquad k \geqslant k_0,$$

and thus

$$a(z,k)c\phi_{p_{ni}}(k,k_0) = 0 \qquad \text{for} \quad k \geqslant k_0.$$

Using linearity, we have that the right side of (39) is a solution to $a(z,k)y(k)=0$. Again using the definition of the modes $\phi_{p_{ni}}(k,k_0)$, from (39) we also have

$$\begin{bmatrix} y(k_0) \\ y(k_0+1) \\ \vdots \\ y(k_0+n-1) \end{bmatrix} = V(k_0)\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Since $V(k_0)$ is invertible, we can solve the above matrix equation for the constants $c_1, c_2, \dots, c_n$. We therefore have the desired result. ∎

Equation (39) is the time-varying version of the well-known modal decomposition of the zero-input response in the time-invariant case. In fact, if the right poles $p_{ni}(k)$ are constant, so that $p_{ni}(k) = p_i$ for $k \geqslant k_0$, then (39) simplifies to the modal decomposition in the time-invariant case given by

$$y(k) = \sum_{i=1}^{n} c_i p_i^{k-k_0}, \qquad k \geqslant k_0.$$

The repeated root case is left to another paper [see Kamen (in preparation)].

We conclude this section by applying the modal decomposition (39) to the study of asymptotic stability. Recall that the given system is asymptotically stable (a.s.) on $k > k_0$ if the zero-input response $y(k)$ converges to zero as $k \to \infty$ for any initial values $y(k_0), y(k_0+1), \ldots, y(k_0+n-1)$. The system is uniformly asymptotically stable (u.a.s.) on $k > k_0$ if for any real number $\varepsilon > 0$, there is a positive integer N_ε such that

$$|y(k+N_\varepsilon)| < \varepsilon |y(k)| \qquad \text{for all} \quad k > k_0.$$

THEOREM 2. *Let $p_{n1}(k), p_{n2}(k), \ldots, p_{nn}(k)$ be n right poles with associated Vandermonde matrix $V(k)$, and suppose that $V(k_0)$ is invertible. Then the system is a.s. on $k > k_0$ if and only if*

$$|p_{ni}(k-1)p_{ni}(k-2)\cdots p_{ni}(k_0)| \to 0 \quad \text{as } k \to \infty \qquad \text{for} \quad i = 1,2,\ldots,n. \tag{40}$$

The system is u.a.s. on $k > k_0$ if and only if for any real number $\varepsilon > 0$, there is a positive integer N_ε such that

$$|p_{ni}(k+N_\varepsilon-1)p_{ni}(k+N_\varepsilon-2)\cdots p_{ni}(k)| < \varepsilon$$

$$\text{for all} \quad k > k_0 \text{ and } i = 1,2,\ldots,n.$$

The proof of Theorem 2 follows easily from the modal decomposition (39). The details are omitted.

By Theorem 2, we see that testing for a.s. or u.a.s. can be reduced to testing the stability of the first-order systems corresponding to the components of the modal decomposition (39).

It follows from (40) that a sufficient condition for asymptotic stability on $k > k_0$ (assuming $p_{ni}(k) \not\to 0$ as $k \to \infty$) is

$$|p_{ni}(k)| < 1, \qquad k > k_0, \quad i = 1,2,\ldots,n. \tag{41}$$

It is interesting to note that if the $p_{ni}(k)$ in (41) are replaced by the ordinary zeros of $a(z, k)$, then the condition is no longer sufficient (or necessary) for stability.

It should be mentioned that the condition (41) is not necessary for asymptotic stability in general. For example, if the $p_{ni}(k)$ are periodic for $k > k_0$ with period two, the system is a.s. if and only if

$$|p_{ni}(k_0+1)p_{ni}(k_0)| < 1 \qquad \text{for} \quad i = 1,2,\dots,n.$$

6. CONTINUOUS-TIME CASE

All of the constructions given in the previous sections have an analogue in the continuous-time case. A sketch of this case is given below.

With D equal to the derivative operator, consider the linear continuous-time system given by the input-output differential equation

$$a(D,t)y(t) = b(D,t)u(t), \tag{42}$$

where

$$a(D,t) = D^n + \sum_{i=0}^{n-1} a_i(t)D^i$$

$$b(D,t) = \sum_{i=0}^{n} b_i(t)D^i.$$

We assume that the coefficients $a_i(t)$ and $b_i(t)$ can be differentiated a suitable number of times.

We begin by considering the second-order case ($n = 2$), so that

$$a(D,t) = D^2 + a_1(t)D + a_0(t).$$

Suppose that there exist functions $p_1(t)$ and $p_2(t)$ (both of which may be complex valued) such that

$$(D^2 + a_1(t)D + a_0(t))y(t) = (D - p_1(t))[(D - p_2(t))y(t)]. \tag{43}$$

Expanding the right side of (43), we obtain

$$[D - p_1(t)]\{[D - p_2(t)]y(t)\}$$
$$= \{D^2 - [p_1(t) + p_2(t)]D + p_1(t)p_2(t) - \dot{p}_2(t)\}y(t), \quad (44)$$

where $\dot{p}_2(t)$ is the derivative of $p_2(t)$.

Now we want to define a polynomial multiplication $\circ$ such that

$$\{[D - p_1(t)] \circ [D - p_2(t)]\}y(t) = [D - p_1(t)]\{[D - p_2(t)]y(t)\}. \quad (45)$$

Comparing (44) and (45), we see that

$$[D - p_1(t)] \circ [D - p_2(t)]$$
$$= D^2 - [p_1(t) + p_2(t)]D + p_1(t)p_2(t) - \dot{p}_2(t). \quad (46)$$

Note that

$$D \circ p_2(t) = p_2(t)D + \dot{p}_2(t),$$

and thus the multiplication $\circ$ is noncommutative. Also note that the noncommutativity is different from that in the discrete-time case [see (9)].

If there exist $p_1(t)$ and $p_2(t)$ such that (43) holds, we call the ordered set $(p_1(t), p_2(t))$ a pole set, and we call $p_2(t)$ a right pole. It follows from (43)–(46) that $(p_1(t), p_2(t))$ is a pole set if and only if

$$p_1(t) + p_2(t) = -a_1(t), \quad (47)$$

$$p_1(t)p_2(t) - \dot{p}_2(t) = a_0(t). \quad (48)$$

Multiplying both sides of (47) by $p_2(t)$ and using (48), we have that

$$p_2^2(t) + \dot{p}_2(t) + a_1(t)p_2(t) + a_0(t) = 0. \quad (49)$$

Equation (49) is a nonlinear first-order differential equation with time-varying coefficients. In fact, (49) is a special case of the Riccati equation. Also note that the left side of (49) looks like the polynomial $D^2 + a_1(t)D + a_0(t)$

evaluated at $D = p_2(t)$ with

$$D|_{D=p_2(t)} = p_2(t) \quad \text{and} \quad D^2|_{D=p_2(t)} = p_2^2(t) + \dot{p}_2(t).$$

When $a_0(t) = c_0$ and $a_1(t) = c_1$ for all $t > t_1$, we shall show that (49) has a unique solution $p_2(t)$ for almost all initial conditions $p_2(t_1) \in \mathbf{C}$.

Let r_1 and r_2 denote the zeros of $D^2 + c_1 D + c_0$, and define

$$\Delta(t) = p_2(t) - r_2.$$

Then

$$\dot{\Delta}(t) = \dot{p}_2(t) = -p_2^2(t) - c_1 p_2(t) - c_0 \qquad \text{for} \quad t > t_1.$$

Inserting $p_2(t) = \Delta(t) + r_2$ into this expression gives

$$\dot{\Delta}(t) = -\Delta^2(t) - (r_2 - r_1)\Delta(t) \qquad \text{for} \quad t > t_1. \tag{50}$$

Defining $g(t) = 1/\Delta(t)$, we can convert (50) into the linear differential equation

$$\dot{g}(t) = (r_2 - r_1) g(t) + 1. \tag{51}$$

Solving (51) and using the relationship $\Delta(t) = 1/g(t)$, we obtain

$$\Delta(t) = \frac{(r_1 - r_2)\Delta(t_1)}{\Delta(t_1) + [(r_1 - r_2) - \Delta(t_1)] \exp[-(r_1 - r_2)(t - t_1)]}, \qquad t > t_1. \tag{52}$$

In deriving (52), we have assumed that $r_1 \neq r_2$.

It is interesting that (52) has exactly the same form as in the discrete-time case [see (23)], except that the exponential term is replaced by $(r_2/r_1)^{k-k_1}$ in the discrete-time case. From (52), we see that there is a unique solution $p_2(t) = \Delta(t) + r_2$ for any initial condition $p_2(t_1) \neq r_1$.

Returning to the nth-order case defined by (42), we call $p_n(t)$ a right pole of the system if there is a polynomial $e(D, t)$ such that

$$a(D, t) = e(D, t) \circ [D - p_n(t)],$$

and we call $q(t)$ a right zero of the system if there exists a polynomial $h(D,t)$ such that

$$b(D,t) = h(D,t) \circ [D - q(t)].$$

It is assumed that the coefficients of $e(D,t)$ and $h(D,t)$ belong to some space of functions that can be differentiated an appropriate number of times. The precise specification of the space of coefficients will not be considered.

The right poles can be computed by solving a $(n-1)$th-order nonlinear differential equation with time-varying coefficients. The equation is defined as follows. First, given a $n-1$ times differentiable function $p(t)$, for $i = 1, 2, \ldots, n-1$ define $(S^i p)(t)$ recursively by

$$(Sp)(t) = p^2(t) + \dot{p}(t),$$

$$(S^2 p)(t) = (S(Sp))(t) = p(t)(Sp)(t) + \frac{d}{dt}(Sp)(t)$$

$$= p^3(t) + 3p(t)\dot{p}(t) + \ddot{p}(t),$$

$$(S^i p)(t) = (S(S^{i-1}p))(t) = p(t)(S^{i-1}p)(t) + \frac{d}{dt}(S^{i-1}p)(t).$$

Then $p_n(t)$ is a right pole of the system (42) if and only if $p_n(t)$ satisfies the nonlinear differential equation (assuming $n \geqslant 3$)

$$(S^{n-1}p_n)(t) + \sum_{i=2}^{n-1} a_i(t)(S^{i-1}p_n)(t) + a_1(t)p_n(t) + a_0(t) = 0.$$

This equation is the continuous-time counterpart to the nonlinear difference equation (34) in the discrete-time case. The proof that $p_n(t)$ is a right pole if and only if it satisfies the above equation is omitted.

Right poles can also be computed by solving the nth-order linear differential equation $a(D,t)y(t) = 0$. In particular, let $y(t)$ be the solution to this equation with the initial values $y^{(i)}(t_0)$, $i = 0, 1, \ldots, n-1$, where $y^{(i)}(t)$ is the ith derivative of $y(t)$. If $y(t) \neq 0$ for $t > t_0$, then $p_n(t) = \dot{y}(t)/y(t)$ is a right pole on $t > t_0$. This result follows directly from known results [see Amitsur (1954)].

If $p(t)$ is a right pole or a right zero, we define the mode associated with $p(t)$ by

$$\phi_p(t,t_0)=\begin{cases}0, & t<t_0,\\ \exp\left[\int_{t_0}^{t}p(\lambda)\,d\lambda\right], & t\geqslant t_0.\end{cases}$$

We then have the following continuous-time counterpart to Proposition 3.

PROPOSITION 4. *Let $q(t)$ be a right zero of the system. Then with the input $u(t)=\phi_q(t,t_0)$, there exist initial conditions $y^{(i)}(t_0)$, $i=0,1,\ldots,n-1$, such that the output response is identically zero for $t>t_0$.*

The proof of this proposition is very similar to the proof of Proposition 3 and is therefore omitted.

Now suppose that $p_{n1}(t), p_{n2}(t),\ldots,p_{nn}(t)$ are n right poles of the system, and define $(S^i p_{ni})(t)$ recursively as given above. Let $V(t)$ denote the generalized Vandermonde matrix defined by

$$V(t)=\begin{bmatrix}1 & 1 & \cdots & 1\\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t)\\ (Sp_{n1})(t) & (Sp_{n2})(t) & \cdots & (Sp_{nn})(t)\\ \vdots & \vdots & & \vdots\\ (S^{n-2}p_{n1})(t) & (S^{n-2}p_{n2})(t) & \cdots & (S^{n-2}p_{nn})(t)\end{bmatrix}$$

We then have the following counterpart to Theorem 1:

THEOREM 3. *Suppose that the determinant of $V(t_0)$ is nonzero. Then for any initial conditions $y^{(i)}(t_0)$, $i=1,2,\ldots,n-1$, there exist constants $c_1,c_2,\ldots,c_n$ such that the solution to $a(D,t)y(t)=0$ can be written in the form*

$$y(t)=\sum_{i=1}^{n}c_i\phi_{p_{ni}}(t,t_0)\qquad \textit{for}\quad t>t_0. \tag{53}$$

Again the proof is very similar to the one given in the discrete-time case, so we shall omit the details.

If the right poles $p_{ni}(t)$ are constant, so that $p_{ni}(t)=p_i$ for $t>t_0$, the modal decomposition (53) simplifies to the well-known form in the time-invariant case given by

$$y(t)=\sum_{i=1}^{n} c_i \exp[p_i(t-t_0)], \qquad t>t_0.$$

It follows directly from (53) that the system is asymptotically stable (a.s.) on $t>t_0$ if and only if

$$|\phi_{p_{ni}}(t,t_0)| \to 0 \quad \text{as } t\to\infty \qquad \text{for} \quad i=1,2,\ldots,n. \tag{54}$$

A condition which implies (54) is (assuming $\operatorname{Re} p_{ni}(t) \not\to 0$ as $t\to\infty$)

$$\operatorname{Re} p_{ni}(t)<0 \qquad \text{for} \quad t>t_0 \text{ and } i=1,2,\ldots,n. \tag{55}$$

If the $p_{ni}(t)$ in (55) are replaced by the ordinary zeros of $a(D,t)$, then the condition is no longer sufficient for stability.

7. DISCUSSION OF RESULTS AND OPEN QUESTIONS

We have developed a notion of poles and zeros in terms of a noncommutative factorization of operator polynomials with time-varying coefficients. It should be stressed that due in part to the noncommutativity, factorizations are not unique up to a permutation of the factors. This nonuniqueness is well known in the existing mathematics literature [Ore (1933) and Amitsur (1954)].

A key result in this work is that uniqueness of factorizations may be obtained by specifying the initial values of the factors. The initial values correspond to a "frame of reference," such as an initial time interval over which the system is time invariant (i.e., the coefficients of the operator polynomial are constant). As noted in Sections 2 and 3, it may be desirable to reinitialize factorizations over time intervals for which the system is time invariant or slowly varying.

A fundamental concept developed in this paper is the notion of the mode associated with a pole or zero. Our notion of mode is a natural generalization of the concept of mode in the time-invariant case. As seen from the results derived in this paper, the modal theory in the time-varying case is very similar to the well-known modal theory of time-invariant systems. A good example of this correspondence is the decomposition of the zero-input response into a sum of modes as derived in Sections 5 and 6.

It is an interesting open question how much of the theory of time-invariant systems can be extended to the time-varying case by using the concepts of poles, zeros, and modes defined in this paper. In Sections 5 and 6 we showed that the stability of a linear time-varying system can be studied in terms of the right poles and associated modes, and that the resulting stability criteria are natural generalizations of the pole criteria for stability in the time-invariant case. It appears that in addition to stability, other components of the theory of time-invariant systems have counterparts in the time-varying case. One example is system reduction, which can be studied in terms of the existence of common left poles and left zeros. The existence of common left factors has been studied by Hwang (1986) using a time-varying version of the resultant matrix. It should be possible to interrelate Hwang's work with the approach developed in this paper.

System properties and feedback control are additional topics that one may be able to study in terms of the notions of poles, zeros, and modes developed in this work. In the existing algebraic theory of linear time-varying systems [see Kamen, Khargonekar, and Poolla (1985), Poolla, Kamen, and Khargonekar (1986), Ilchmann, Nurnberger, and Schmale (1984)], it is known that Bezout identities (in a ring of noncommutative polynomials) are important in the control of time-varying systems. These identities are natural generalizations of the well-known Bezout-identity setup in the control of time-invariant systems. An interesting open question is whether or not the existence of Bezout identities in the time-varying case can be characterized in terms of the notion of the poles and zeros (or the associated modes) defined in this work. For example, if we define an unstable mode as one that grows without bound, is it possible to characterize stabilizability by feedback in terms of some notion of controllability of the unstable modes? We leave this as an open question.

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