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A Further Result on the Oscillation of Delay Difference Equations

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Abstract—In this paper, we are concerned with the delay difference equations of the form

$$y_{n+1} - y_n + p_n y_{n-k} = 0, \qquad n = 0, 1, 2, \dots, \tag{$*$}$$

where $p_n \geq 0$ and k is a positive integer. We prove by using a new technique that

$$\sum_{n=0}^{\infty} \left[\sum_{i=n}^{n+k} p_i \ln \left(\sum_{i=n}^{n+k} p_i + 1 - \operatorname{sign} \sum_{i=n}^{n+k} p_i \right) - \sum_{i=n+1}^{n+k} p_i \ln \left(\sum_{i=n+1}^{n+k} p_i + 1 - \operatorname{sign} \sum_{i=n+1}^{n+k} p_i \right) \right] = \infty$$

guarantees that all solutions of equation ($*$) oscillate, which improves many previous well-known results. In particular, our theorems also fit the case where $\sum_{i=n-k}^{n-1} p_i \leq k^{k+1}/(k+1)^{k+1}$. In addition, we present a nonoscillation sufficient condition for equation ($*$).

Keywords—Oscillation, Eventually positive solution, Difference equation.

1. INTRODUCTION

We are concerned with the delay difference equations of the form

$$y_{n+1} - y_n + p_n y_{n-k} = 0, \qquad n = 0, 1, 2, \dots, \tag{1}$$

where $\{p_n\}$ is a sequence of nonnegative real numbers and k is a positive integer.

As is customary, a solution $\{y_n\}$ of (1) is said to be oscillatory if the terms y_n of the sequence are not eventually positive or eventually negative. Otherwise, the solution is called nonoscillatory.

In 1989, Erbe and Zhang [1] first proved that if

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}}, \tag{2}$$

then every solution of (1) oscillates. If

$$p_n \leq \frac{k^k}{(k+1)^{k+1}}, \qquad \text{for large } n, \tag{3}$$

then (1) has a nonoscillatory solution.

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In the same year, Ladas, Philos and Sficas [2] proved that condition (2) can be replaced by the weaker condition

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > \left(\frac{k}{k+1}\right)^{k+1}. \tag{4}$$

We remark that condition (4) is a discrete analogue of a well-known oscillation criteria

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > \frac{1}{e} \tag{5}$$

for the following delay differential equation:

$$x'(t) + p(t)x(t-\tau) = 0, \tag{6}$$

where $p(t) \in C([0,\infty),[0,\infty))$ and τ is a positive constant. A nonoscillation criteria for (6) corresponding to (5) is

$$\int_{t-\tau}^{t} p(s)\,ds \le \frac{1}{e}, \qquad \text{for large } t. \tag{7}$$

As a discrete analogue of (7), Ladas [3] presented the following open problem in 1990.

OPEN PROBLEM A. Assume that

$$\sum_{i=n-k}^{n-1} p_i \le \left(\frac{k}{k+1}\right)^{k+1}, \qquad \text{for large } n. \tag{8}$$

Does (1) have an eventually positive solution?

In 1994, Yu, Zhang and Wang [4] obtained a better sufficient condition than (4) for the oscillation of all solutions of (1) and used the result to construct an example which shows that the answer to Open Problem A is negative. That is, if for some integer $N \ge 1$,

$$\sup_{\lambda\in E, n\ge N} \left[\lambda \prod_{i=n-k}^{n-1} (1-\lambda p_i)\right] < 1, \tag{9}$$

where $E = \{\lambda > 0 \mid 1 - \lambda p_n > 0,\ n = 0,1,2,\dots\}$, then every solution of (1) oscillates; if there exists a $\lambda_0 \in E$ such that

$$\lambda_0 \prod_{i=n-k}^{n-1} (1-\lambda_0 p_i) \ge 1, \qquad \text{for large } n, \tag{10}$$

then (1) has an eventually positive solution.

Since for $\lambda \in E$,

$$\lambda \prod_{i=n-k}^{n-1} (1-\lambda p_i) \le \frac{(k/(k+1))^{k+1}}{\sum_{i=n-k}^{n-1} p_i},$$

which follows that if (4) holds then (9) holds naturally. However, the converse is not true. As one has seen, from the example given in [4], oscillation criteria (9) fit the case when (8) holds.

The negative answer to Open Problem A shows that the discrete analogue of the oscillation results for delay differential equations may be not true. Therefore, it is valuable to study the oscillation criteria for delay difference equations.

The aim of this note is to give some new explicit conditions for oscillation and nonoscillation for (1). Our oscillation criterion improves condition (4), in which condition (2) or (4) or even the condition

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 0 \tag{11}$$

is no longer necessary. In particular, our results are still effective when (8) holds.

2. LEMMAS

LEMMA 1. *Let $\{y_n\}$ be an eventually positive solution of (1). If*

$$\limsup_{n\to\infty} p_n > 0,$$

then

$$\liminf_{n\to\infty} \frac{y_{n-k}}{y_n} < \infty.$$

PROOF. From (1), we have eventually

$$p_n = \frac{y_n}{y_{n-k}} - \frac{y_{n+1}}{y_{n-k}} \leq \frac{1}{y_{n-k}/y_n}.$$

It follows that

$$\limsup_{n\to\infty} p_n \leq \frac{1}{\liminf_{n\to\infty}(y_{n-k}/y_n)},$$

which implies that the conclusion of Lemma 1 is true. The proof is complete.

LEMMA 2. *(See [1].) If (1) has an eventually positive solution, then*

$$\sum_{i=n-k}^{n} p_i \leq 1$$

eventually.

LEMMA 3. *Assume that $a, b > 0$, $x_i \geq 0$, $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} x_i = b$. Then*

$$\sum_{i=1}^{k}(a + x_i)\ln(a + x_i) \geq k\left(a + \frac{b}{k}\right)\ln\left(a + \frac{b}{k}\right).$$

PROOF. Let $L(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{k}(a + x_i)\ln(a + x_i) + \lambda(\sum_{i=1}^{k} x_i - b)$. Then

$$\frac{\partial L}{\partial x_i} = \ln(a + x_i) + 1 + \lambda, \qquad i = 1, 2, \ldots, k,$$

$$\frac{\partial^2 L}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{a + x_i}, \qquad i, j = 1, 2, \ldots, k,$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

By a simple calculation, it is easy to see that the unique stationary point of the function $L(x_1, x_2, \ldots, x_k)$ is $(b/k, b/k, \ldots, b/k)$. Since the matrix

$$M = \begin{pmatrix} \dfrac{\partial^2 L}{\partial x_1^2} & \dfrac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 L}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 L}{\partial x_2 \partial x_1} & \dfrac{\partial^2 L}{\partial x_2^2} & \cdots & \dfrac{\partial^2 L}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 L}{\partial x_n \partial x_1} & \dfrac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 L}{\partial x_n^2} \end{pmatrix}_{x_i = b/k} = \begin{pmatrix} \dfrac{k}{ak+b} & & & \\ & \dfrac{k}{ak+b} & & \\ & & \ddots & \\ & & & \dfrac{k}{ak+b} \end{pmatrix}$$

is positive definite. It is easy to know that the point $(b/k, b/k, \ldots, b/k)$ is the minimum point of the function L. Hence, in view of Lagrange's method of multipliers, we obtain

$$\sum_{i=1}^{k}(a + x_i)\ln(a + x_i) \geq k\left(a + \frac{b}{k}\right)\ln\left(a + \frac{b}{k}\right).$$

The proof is complete.

3. MAIN RESULTS

THEOREM 1. *If*

$$\sum_{n=0}^{\infty}\left[\sum_{i=n}^{n+k} p_i \ln\left(\sum_{i=n}^{n+k} p_i + 1 - \operatorname{sign}\sum_{i=n}^{n+k} p_i\right) - \sum_{i=n+1}^{n+k} p_i \ln\left(\sum_{i=n+1}^{n+k} p_i + 1 - \operatorname{sign}\sum_{i=n+1}^{n+k} p_i\right)\right] = \infty, \quad (12)$$

then every solution of (1) oscillates.

PROOF. Assume the contrary. Without loss of generality, we may assume that (1) has an eventually positive solution $\{y_n\}$. Choose a positive integer n_1 so large that

$$y_n > 0 \quad \text{and} \quad y_{n+1} - y_n \le 0, \qquad n \ge n_1. \quad (13)$$

Define two functions $p(t)$ and $x(t)$ as follows:

$$\begin{aligned} p(t) &= p_n, & n \le t < n+1, \quad n = 0,1,2,\dots, \\ x(t) &= y_n + (y_{n+1} - y_n)(t-n), & n \le t < n+1, \quad n = 0,1,2,\dots. \end{aligned}$$

Let $x'(t)$ denote derivative on the right. Then $x'(t) = y_{n+1} - y_n$ for $n \le t < n+1$, $n = 0,1,2,\dots$. Hence, we may rewrite (1) as

$$x'(t) + p(t)x([t-k]) = 0, \qquad n \ge 0, \quad (14)$$

where and in the sequel, $[\cdot]$ denotes the greatest integer function. In view of (13), then

$$x(t) > 0 \quad \text{and} \quad x'(t) \le 0, \qquad t \ge n_1. \quad (15)$$

Set $\lambda(t) = -x'(t)/x(t)$ for $t \ge n_1$. Then $\lambda(t) \ge 0$ for $t \ge n_1$. It follows from (14) that

$$\lambda(t) = p(t)\exp\left(\int_{[t-k]}^{t} \lambda(s)\,ds\right), \qquad t \ge n_1 + k$$

or

$$\lambda(t)\int_t^{[t+k+1]} p(s)\,ds = p(t)\int_t^{[t+k+1]} p(s)\,ds \cdot \exp\left(\int_{[t-k]}^{t} \lambda(s)\,ds\right), \qquad t \ge n_1 + k. \quad (16)$$

One can easily show that

$$\varphi(r)re^x \ge \varphi(r)x + \varphi(r)\ln(er + 1 - \operatorname{sign} r), \qquad \text{for } r \ge 0 \quad \text{and} \quad x \in \mathbf{R}, \quad (17)$$

where $\varphi(0) = 0$ and $\varphi(r) \ge 0$ for $r > 0$.

By the definition of $p(t)$, we see that $p(t)$ is right-continuous. Therefore, $\int_t^{[t+k+1]} p(s)\,ds = 0$ implies that $p(t) = 0$. Employing inequality (17) on the right-hand side of (16), we get

$$\begin{aligned} \lambda(t)\int_t^{[t+k+1]} p(s)\,ds &= p(t)\int_t^{[t+k+1]} p(s)\,ds \exp\left(\int_{[t-k]}^{t} \lambda(s)\,ds\right) \\ &\ge p(t)\int_{[t-k]}^{t} \lambda(s)\,ds + p(t)\ln\left[e\int_t^{[t+k+1]} p(s)\,ds \right. \\ &\quad \left. + 1 - \operatorname{sign}\left(\int_t^{[t+k+1]} p(s)\,ds\right)\right], \qquad t \ge n_1 + k. \end{aligned} \quad (18)$$

Set $A(t) = e\int_t^{[t+k+1]} p(s)\,ds + 1 - \operatorname{sign}\left(\int_t^{[t+k+1]} p(s)\,ds\right)$. Then (18) can be rewritten as

$$\lambda(t)\int_t^{[t+k+1]} p(s)\,ds - p(t)\int_{[t-k]}^{t} \lambda(s)\,ds \geq p(t)\ln A(t), \qquad t \geq n_1 + k. \tag{19}$$

Thus for any integer $N > n_2 + 2k$ $(n_2 = n_1 + k)$,

$$\int_{n_2}^{N} \lambda(t)\int_t^{[t+k+1]} p(s)\,ds\,dt - \int_{n_2}^{N} p(t)\int_{[t-k]}^{t} \lambda(s)\,ds\,dt \geq \int_{n_2}^{N} p(t)\ln A(t)\,dt. \tag{20}$$

Let $D_1 = \{(t,s) \mid n_2 \leq t < N, [t-k] \leq s \leq t\}$ and $D_2 = \{(s,t) \mid n_2 \leq s < N-k, s \leq t < [s+k+1]\}$. Clearly, $D_2 \subset D_1$. Hence

$$\int_{n_2}^{N} p(t)\int_{[t-k]}^{t} \lambda(s)\,ds\,dt = \iint_{D_1} p(t)\lambda(s)\,ds\,dt \geq \iint_{D_2} p(t)\lambda(s)\,ds\,dt. \tag{21}$$

Since

$$\begin{aligned}\iint_{D_2} p(t)\lambda(s)\,ds\,dt &= \sum_{i=n_2}^{N-k-1} \int_i^{i+1} \lambda(s)\int_s^{i+k+1} p(t)\,dt\,ds \\ &= \int_{n_2}^{N-k} \lambda(s)\int_s^{[s+k+1]} p(t)\,dt\,ds \\ &= \int_{n_2}^{N-k} \lambda(t)\int_t^{[t+k+1]} p(s)\,ds\,dt.\end{aligned} \tag{22}$$

It follows from (20), (21), and (22) that

$$\int_{N-k}^{N} \lambda(t)\int_t^{[t+k+1]} p(s)\,ds\,dt \geq \int_{n_2}^{N} p(t)\ln A(t)\,dt. \tag{23}$$

Note that for $n \leq t < n+1$,

$$\int_t^{[t+k+1]} p(s)\,ds \leq \int_n^{n+k+1} p(s)\,ds = \sum_{i=n}^{n+k} p_i.$$

By Lemma 2, we have eventually

$$\int_t^{[t+k+1]} p(s)\,ds \leq 1. \tag{24}$$

From (23) and (24), we obtain

$$\lim_{N\to\infty} \int_{N-k}^{N} \lambda(t)\,dt \geq \int_{n_2}^{\infty} p(t)\ln A(t)\,dt$$

or

$$\lim_{N\to\infty} \frac{y_{N-k}}{y_N} \geq \int_{n_2}^{\infty} p(t)\ln A(t)\,dt. \tag{25}$$

Set $E = \{n \geq n_2 \mid p_n > 0\}$. Then

$$\begin{aligned}
\int_{n_2}^{\infty} p(t) \ln A(t)\, dt &= \sum_{n=n_2}^{\infty} \int_n^{n+1} p(t) \ln \left[e \int_t^{[t+k+1]} p(s)\, ds + 1 - \operatorname{sign}\left(\int_t^{[t+k+1]} p(s)\, ds \right) \right] dt \\
&= \sum_{n=n_2}^{\infty} p_n \int_n^{n+1} \ln \left[e \int_t^{n+k+1} p(s)\, ds + 1 - \operatorname{sign}\left(\int_t^{n+k+1} p(s)\, ds \right) \right] dt \\
&= \sum_{n\in E} p_n \int_n^{n+1} \ln \left[e \sum_{i=n+1}^{n+k} p_i + e p_n (n+1-t) \right] dt \\
&= \sum_{n\in E} \left[\sum_{i=n}^{n+k} p_i \ln \left(\sum_{i=n}^{n+k} p_i \right) - \sum_{i=n+1}^{n+k} p_i \ln \left(\sum_{i=n+1}^{n+k} p_i + 1 - \operatorname{sign} \sum_{i=n+1}^{n+k} p_i \right) \right] \\
&= \sum_{n=n_2}^{\infty} \left[\sum_{i=n}^{n+k} p_i \ln \left(\sum_{i=n}^{n+k} p_i + 1 - \operatorname{sign} \sum_{i=n}^{n+1} p_i \right) \right. \\
&\qquad \left. - \sum_{i=n+1}^{n+k} p_i \ln \left(\sum_{i=n+1}^{n+k} p_i + 1 - \operatorname{sign} \sum_{i=n+1}^{n+k} p_i \right) \right].
\end{aligned}$$

In view of (12),

$$\int_{n_2}^{\infty} p(t) \ln A(t)\, dt = \infty. \tag{26}$$

Substituting this into (25), we have

$$\lim_{n\to\infty} \frac{y_{n-k}}{y_n} = \infty. \tag{27}$$

On the other hand, from (12), we can show that

$$\limsup_{n\to\infty} p_n > 0. \tag{28}$$

In fact, if (28) is not true, then $\lim_{n\to\infty} p_n = 0$. Hence, $\sum_{i=n}^{n+k} p_i < 1/e$ for large n. Since the function $x \ln x$ is decreasing in $(0, 1/e)$, it follows that for large n,

$$\sum_{i=n}^{n+k} p_i \ln \left(\sum_{i=n}^{n+k} p_i + 1 - \operatorname{sign} \sum_{i=n}^{n+k} p_i \right) - \sum_{i=n+1}^{n+k} p_i \ln \left(\sum_{i=n+1}^{n+k} p_i + 1 - \operatorname{sign} \sum_{i=n+1}^{n+k} p_i \right) \leq 0,$$

and so

$$\sum_{n=0}^{\infty} \left[\sum_{i=n}^{n+k} p_i \ln \left(\sum_{i=n}^{n+k} p_i + 1 - \operatorname{sign} \sum_{i=n}^{n+k} p_i \right) - \sum_{i=n+1}^{n+k} p_i \ln \left(\sum_{i=n+1}^{n+k} p_i + 1 - \operatorname{sign} \sum_{i=n+1}^{n+k} p_i \right) \right] < \infty,$$

which contradicts (12). Therefore, (28) holds. Hence, by Lemma 1, we have

$$\liminf_{n\to\infty} \frac{y_{n-k}}{y_n} < \infty.$$

This contradicts (27) and complete the proof.

From Theorem 1, we obtain immediately the following.

COROLLARY 1. *If there exists an integer $N > 0$ such that $\sum_{i=n+1}^{n+k} p_i > 0$ for $n \geq N$, and*

$$\sum_{n=N}^{\infty} \left[\sum_{i=n}^{n+k} p_i \ln \left(\sum_{i=n}^{n+k} p_i \right) - \sum_{i=n+1}^{n+k} p_i \ln \left(\sum_{i=n+1}^{n+k} p_i \right) \right] = \infty, \tag{29}$$

then every solution of (1) oscillates.

REMARK 1. Corollary 1 substantially improves condition (4). In fact, if (4) holds, then there exist $\theta > 1$ and an integer $s > 0$ such that

$$\sum_{i=n+1}^{n+k} p_i \geq \theta \left(\frac{k}{k+1}\right)^{k+1} = \theta_1, \qquad \text{for } n \geq sk. \tag{30}$$

For $d \geq 0$, the function $(d+x)\ln(d+x) - x\ln x$ is nondecreasing in $(0, +\infty)$, it follows that for $n \geq sk$,

$$\sum_{i=n}^{n+k} p_i \ln\left(\sum_{i=n}^{n+k} p_i\right) - \sum_{i=n+1}^{n+k} p_i \ln\left(\sum_{i=n+1}^{n+k} p_i\right) \geq (p_n + \theta_1)\ln(p_n + \theta_1) - \theta_1 \ln \theta_1,$$

and so for $j = s, s+1, s+2, \dots$,

$$\begin{aligned}
&\sum_{n=jk}^{(j+1)k-1} \left[\sum_{i=n}^{n+k} p_i \ln\left(\sum_{i=n}^{n+k} p_i\right) - \sum_{i=n+1}^{n+k} p_i \ln\left(\sum_{i=n+1}^{n+k} p_i\right)\right] \\
&\quad \geq \sum_{n=jk}^{(j+1)k-1} \left[(p_n + \theta_1)\ln(p_n + \theta_1) - \theta_1 \ln \theta_1\right] \\
&\quad \geq k \left[\left(\theta_1 + \frac{1}{k}\sum_{n=jk}^{(j+1)k-1} p_n\right) \ln\left(\theta_1 + \frac{1}{k}\sum_{n=jk}^{(j+1)k-1} p_n\right) - \theta_1 \ln \theta_1\right].
\end{aligned} \tag{31}$$

In the last inequality, we have used Lemma 3. Since the function $x \ln x$ is increasing on $[1/e, \infty)$, and by (30)

$$\theta_1 + \frac{1}{k}\sum_{i=jk}^{(j+1)k-1} p_i \geq \frac{k+1}{k}\theta_1 = \theta\left(\frac{k}{k+1}\right)^k > \frac{1}{e},$$

hence, from (31), we have

$$\begin{aligned}
&\sum_{n=jk}^{(j+1)k-1} \left[\sum_{i=n}^{n+k} p_i \ln\left(\sum_{i=n}^{n+k} p_i\right) - \sum_{i=n+1}^{n+k} p_i \ln\left(\sum_{i=n+1}^{n+k} p_i\right)\right] \\
&\quad \geq k\left[\theta\left(\frac{k}{k+1}\right)^k \ln\left(\theta\left(\frac{k}{k+1}\right)^k\right) - \theta\left(\frac{k}{k+1}\right)^{k+1} \ln\left(\theta\left(\frac{k}{k+1}\right)^{k+1}\right)\right] \\
&\quad = \left(\frac{k}{k+1}\right)^{k+1} \theta \ln \theta > 0.
\end{aligned}$$

It follows that (29) holds with $N = sk$.

EXAMPLE 1. Consider the delay difference equation

$$y_{n+1} - y_n + p_n y_{n-3} = 0, \qquad n = 0, 1, 2, \dots, \tag{32}$$

where $p_{3n} = 0$, $p_{3n+1} = p_{3n+2} = d$, $n = 0, 1, 2, \dots$, $d \in (4/27, 81/512]$. Observe that

$$\sum_{i=n-3}^{n-1} p_i = 2d \leq \left(\frac{3}{4}\right)^4,$$

which implies that condition (8) is satisfied. But, on the other hand, for $j = 0, 1, 2, \ldots,$

$$\begin{aligned}
&\sum_{n=3j}^{3(j+1)-1}\left[\sum_{i=n}^{n+3} p_i \ln\left(\sum_{i=n}^{n+3} p_i\right) - \sum_{i=n+1}^{n+3} p_i \ln\left(\sum_{i=n+1}^{n+3} p_i\right)\right] \\
&\quad = \sum_{n=0}^{2}\left[\sum_{i=n}^{n+3} p_i \ln\left(\sum_{i=n}^{n+3} p_i\right) - \sum_{i=n+1}^{n+3} p_i \ln\left(\sum_{i=n+1}^{n+3} p_i\right)\right] \\
&\quad = 2(3d\ln 3d - 2d\ln 2d) \\
&\quad = 2d\ln\frac{27d}{4} > 0.
\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty}\left[\sum_{i=n}^{n+3} p_i \ln\left(\sum_{i=n}^{n+3} p_i\right) - \sum_{i=n+1}^{n+3} p_i \ln\left(\sum_{i=n+1}^{n+3} p_i\right)\right] = \infty.$$

In view of Corollary 1, every solution of (32) oscillates.

The above example shows that Theorem 1 and Corollary 1 fits the case when (8) holds. Next, we give another example for the comparison with condition (9) and obtain the oscillation of all solutions which cannot be obtained by condition (9).

EXAMPLE 2. Consider the delay difference equation

$$y_{n+1} - y_n + p_n y_{n-2} = 0, \qquad n = 0, 1, 2, \ldots, \tag{33}$$

where $p_{8n} = p_{8n+1} = 0$, $p_{8n+2} = p_{8n+3} = \cdots = p_{8n+7} = d$, $d \in (2/9, 1/3)$.

By a simple calculation, one can obtain for $j = 0, 1, 2, \ldots,$

$$\begin{aligned}
&\sum_{n=8j}^{8j+7}\left[\sum_{i=n}^{n+2} p_i \ln\left(\sum_{i=n}^{n+2} p_i + 1 - \operatorname{sign}\sum_{i=n}^{n+2} p_i\right) - \sum_{i=n+1}^{n+2} p_i \ln\left(\sum_{i=n+1}^{n+2} p_i + 1 - \operatorname{sign}\sum_{i=n+1}^{n+2} p_i\right)\right] \\
&\quad = \sum_{n=0}^{7}\left[\sum_{i=n}^{n+2} p_i \ln\left(\sum_{i=n}^{n+2} p_i + 1 - \operatorname{sign}\sum_{i=n}^{n+2} p_i\right) - \sum_{i=n+1}^{n+2} p_i \ln\left(\sum_{i=n+1}^{n+2} p_i + 1 - \operatorname{sign}\sum_{i=n+1}^{n+2} p_i\right)\right] \\
&\quad = 4(3d\ln 3d - 2d\ln 2d) + (2d\ln 2d - d\ln d) + d\ln d \\
&\quad = 6d\ln\left(\frac{9d}{2}\right) > 0.
\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty}\left[\sum_{i=n}^{n+2} p_i \ln\left(\sum_{i=n}^{n+2} p_i + 1 - \operatorname{sign}\sum_{i=n}^{n+2} p_i\right) - \sum_{i=n+1}^{n+2} p_i \ln\left(\sum_{i=n+1}^{n+2} +1 - \operatorname{sign}\sum_{i=n+1}^{n+2} p_i\right)\right] = \infty.$$

In view of Theorem 1, every solution of (33) oscillates. The same conclusion, however, cannot be inferred from the aforementioned results. Since

$$\liminf_{n\to\infty}\sum_{i=n-2}^{n-1} p_i = 0 \quad \text{and} \quad \limsup_{n\to\infty}\sum_{i=n-2}^{n} p_i = 3d < 1,$$

and for $\lambda > 1$

$$\lambda\prod_{i=8n}^{8n+1}(1 - \lambda p_i) = \lambda > 1, \qquad n = 0, 1, 2, \ldots.$$

These show that conditions (4) and (9) do not hold. In fact, to the best of our knowledge, Example 2 does not satisfy the known oscillation conditions in the literature.

We now present a nonoscillation result for (1).

THEOREM 2. *If for large n,*

$$\sum_{i=n-k}^{n} p_i \le \frac{1}{e}, \tag{34}$$

then (1) has a nonoscillatory solution.

PROOF. By [5, Theorem 8.3.3], (1) has an eventually positive solution is equivalent to that the following delay differential equation:

$$x'(t) + p(t)x([t-k]) = 0 \tag{35}$$

has an eventually positive solution, where $p(t) = p_n, n \le t < n+1$, $n = 0, 1, 2, \dots$. By a similar method of proof of Corollary 2.1.1 in [6], we can prove that condition

$$\int_{[t-k]}^{t} p(s)\,ds \le \frac{1}{e}, \qquad \text{for large } t \tag{36}$$

guarantees that (35) has an eventually positive solution. Note that for $n \le t < n+1$,

$$\int_{[t-k]}^{t} p(s)\,ds \le \int_{n-k}^{n+1} p(s)\,ds = \sum_{i=n-k}^{n} p_i.$$

Hence, (34) implies that (36) holds, and so condition (34) guarantees that (1) has an eventually positive solution. The proof is complete.

Using the fact $k^{k+1}/(k+1)^{k+1} < 1/e$, we have the following nonoscillation condition from Theorem 2, which may be looked upon as a correction for (8).

COROLLARY 2. *If for large n,*

$$\sum_{i=n-k}^{n} p_i \le \left(\frac{k}{k+1}\right)^{k+1}, \tag{37}$$

then (1) has a nonoscillatory solution.

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