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Multiple Symmetric Nonnegative Solutions of Second-Order Ordinary Differential Equations

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Abstract—The existence of multiple nonnegative solutions of the equations $-x'' = f(x, x')$ subject to $x(0) = x(1) = 0$ is studied. The result is obtained that there are at least three symmetric nonnegative solutions if certain conditions are imposed on f.

Keywords—Boundary value problem, Cone, Multiple solutions.

1. INTRODUCTION

In this paper, we shall consider the existence of multiple nonnegative solutions for the second-order boundary value problem

$$-x'' = f(x, x'), \qquad 0 \le t \le 1, \tag{1.1}$$

$$x(0) = x(1) = 0, \tag{1.2}$$

where $f : R \times R \to [0, +\infty)$ is continuous. Many authors [1–3] have studied the equations $-x'' = f(t, x)$ and $-x'' = g(t)f(x)$, but in all these equations the nonlinear term does not include x'. If the equations above include x', the relative boundary value problem will be more complicated. Motivated by the paper [4], we will impose certain conditions on f which ensure the existence of at least three symmetric nonnegative solutions of (1.1) and (1.2).

2. SOME DEFINITIONS AND RESULTS

In order to state and prove our results, we need some notation and conclusions about the theory of cones in Banach spaces. We denote the closure of a set D by $\bar{D}$. $C^1[0, 1]$ denotes the space of all continuously differentiable functions on $[0, 1]$ endowed with the maximum norm

$$\|x\| = \max\left\{\max_{0\le t\le 1} |x(t)|, \max_{0\le t\le 1} |x'(t)|\right\}, \qquad \forall\, x \in C^1[0, 1]. \tag{2.1}$$

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DEFINITION 2.1. *Let E be a real Banach space. A closed, convex set $P \subset E$ is called a cone if the following conditions are satisfied:*

(i) *if $x \in P$, then $\lambda x \in P$ for any $\lambda \geq 0$;*
(ii) *if $x \in P$ and $-x \in P$, then $x = 0$.*

A cone P induces a partial ordering $\leq$ in E by $x \leq y$ if and only if $y - x \in P$.

DEFINITION 2.2. *Let E be a Banach space, $P \subset E$ a cone in E. A map α is said to be a nonnegative continuous concave functional on P if $\alpha : P \to [0, +\infty)$ is continuous and*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y), \qquad \forall x, y \in P \quad \text{and} \quad \forall t \in [0,1]. \tag{2.2}$$

DEFINITION 2.3. *For positive numbers a, b, r with $a < b$ and α a nonnegative continuous concave functional on a cone P, define convex sets P_r and $P(\alpha, a, b)$, respectively, by*

$$P_r = \{x \in P : \|x\| < r\}$$

and

$$P(\alpha, a, b) = \{x \in P : a \leq \alpha(x),\ \|x\| \leq b\}.$$

The following well-known theorem is very crucial in our arguments; see [5] for a proof.

THEOREM 2.1. *(See [5].) Let $A : \bar{P}_c \to \bar{P}_c$ be a completely continuous operator and α be a nonnegative continuous concave functional on P such that $\alpha(x) \leq \|x\|$ for all $x \in \bar{P}_c$. Suppose there exist a, b, d with $0 < a < b < d \leq c$, such that*

(i) *$\{x \in P(\alpha, b, d) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Ax) > b$, for all $x \in P(\alpha, b, d)$;*
(ii) *$\|Ax\| < a$, for all $x \in \bar{P}_a$;*
(iii) *$\alpha(Ax) > b$, for all $x \in P(\alpha, b, c)$ with $\|Ax\| > d$.*

Then A has at least three fixed points, x_1, x_2, and x_3 satisfying

$$\|x_1\| < a, \quad b < \alpha(x_2) \quad \text{and,} \quad \|x_3\| > a, \qquad \text{with } \alpha(x_3) < b.$$

3. MULTIPLE SYMMETRIC NONNEGATIVE SOLUTIONS

It is well known that BVP (1.1),(1.2) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s) f(x(s), x'(s))\, ds, \qquad \forall t \in [0,1],$$

where $G(t,s)$ is Green's function as follows:

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $E = C^1[0,1]$, define $A : E \to E$ by

$$(Ax)(t) = \int_0^1 G(t,s) f(x(s), x'(s))\, ds, \qquad \forall x \in E. \tag{3.1}$$

Thus, BVP (1.1),(1.2) has a solution $x(t)$ if and only if x is a fixed point of the operator A, i.e., $x = Ax$.

LEMMA 3.1. *$D \subset E$ is relative compact if and only if both the functions $x \in D$ and x' are uniformly bounded and equicontinuous on $[0,1]$.*

PROOF. An application of the Arzela-Ascoli theorem completes the proof.

LEMMA 3.2. *A defined by (3.1) is a completely continuous operator.*

PROOF. Notice that (3.1) and

$$\begin{aligned}(Ax)'(t) &= -\int_0^t sf(x(s),x'(s))\,ds + \int_t^1 (1-s)f(x(s),x'(s))\,ds \\ &= \int_t^1 f(x(s),x'(s))\,ds - \int_0^1 sf(x(s),x'(s))\,ds, \qquad \forall t\in[0,1], \quad \forall x\in E,\end{aligned}$$

we have that if $D \subset E$ is bounded, then all functions Ax and $(Ax)'$ for all $x \in D$ are uniformly bounded and equicontinuous on $[0,1]$, it follows from Lemma 3.1 that A is completely continuous. And this completes the proof of the lemma.

Next we define a cone P in E by

$$P = \left\{x \in E : x \text{ is concave, symmetric for } \frac{1}{2} \text{ and nonnegative on } [0,1]\right\},$$

and define the nonnegative continuous concave functional $\alpha : P \to [0,+\infty)$ by

$$\alpha(x) = \min_{\eta\le t\le 1-\eta} x(t), \qquad \forall x \in P,$$

where the constant $\eta \in (0,1/2)$.

LEMMA 3.3. *If the function $x \in P$, then x is increasing on $[0,1/2]$ but x' is decreasing on $[0,1/2]$.*

With the use of Lemma 3.3 we have that

$$\alpha(x) = x(\eta) \le \|x\|, \qquad \forall x \in P, \tag{3.2}$$

and

$$\|x\| = \max\left\{x\left(\frac{1}{2}\right), x'(0)\right\}, \qquad \forall x \in P. \tag{3.3}$$

We now present our result of this paper.

THEOREM 3.1. *Let $0 < a < b < c/M$, $M = 1/\eta(1-2\eta)$, and f satisfies*

(H_1) *$f(u,v) < 2a$, for all $0 \le u \le a$, $|v| \le a$;*

(H_2) *$f(u,v) \ge [\eta(1/2-\eta)]^{-1}b$, for all $b \le u \le Mb$, $|v| \le Mb$;*

(H_3) *$f(u,v) \le 2c$, for all $0 \le u \le c$, $|v| \le c$;*

(H_4) *$f(u,v) = f(u,-v)$, for all $0 \le u < +\infty$, $-\infty < v < +\infty$;*

(H_5) *$f(u_1,v_1) \le f(u_2,v_2)$, for all $0 \le u_1 \le u_2 \le c$, $0 \le v_2 \le v_1 \le c$.*

Then, BVP (1.1),(1.2) has at least three symmetric nonnegative solutions x_1, x_2, and x_3 satisfying $\|x_1\| \le a$, $b < \alpha(x_2)$, and $\|x_3\| > a$, with $\alpha(x_3) < b$.

PROOF. Lemma 3.2 implies that the operator A defined by (3.1) is completely continuous from E into E. We first note that for any $x \in P$, $(Ax)(t) \ge 0$ and $(Ax)''(t) = -f(x(t), x'(t)) \le 0$ for $0 \le t \le 1$, and second note that Assumption (H_4) yields for $0 \le t \le 1$,

$$\begin{aligned}(Ax)(t) &= \int_0^1 G(t,s)f(x(s),x'(s))\,ds \\ &= (1-t)\int_0^t sf(x(s),x'(s))\,ds + t\int_t^1 (1-s)f(x(s),x'(s))\,ds \\ &= (1-t)\int_{1-t}^1 (1-w)f(x(1-w),x'(1-w))\,dw \\ &\quad + t\int_0^{1-t} wf(x(1-w),x'(1-w))\,dw\end{aligned}$$

$$
\begin{aligned}
&= (1-t)\int_{1-t}^{1}(1-w)f(x(w),-x'(w))\,dw + t\int_0^{1-t} wf(x(w),-x'(w))\,dw\\
&= (1-t)\int_{1-t}^{1}(1-w)f(x(w),x'(w))\,dw + t\int_0^{1-t} wf(x(w),x'(w))\,dw\\
&= (Ax)(1-t).
\end{aligned}
$$

Consequently $Ax \in P$, that is, $A : P \to P$.

Now, we divide our proof into three steps.

STEP 1. We show

$$A\bar{P}_c \subset \bar{P}_c \tag{3.4}$$

and

$$A\bar{P}_a \subset P_a. \tag{3.5}$$

It is obvious that for any $x \in P$,

$$
\begin{aligned}
(Ax)(t) &= \int_0^1 G(t,s)f(x(s),x'(s))\,ds\\
&= (1-t)\int_0^t sf(x(s),x'(s))\,ds + t\int_t^1 (1-s)f(x(s),x'(s))\,ds\\
&= \int_0^t sf(x(s),x'(s))\,ds - t\int_0^1 sf(x(s),x'(s))\,ds + t\int_t^1 f(x(s),x'(s))\,ds\\
&= \int_0^t sf(x(s),x'(s))\,ds - t\left[\int_0^{1/2} sf(x(s),x'(s))\,ds + \int_{1/2}^1 sf(x(s),x'(s))\,ds\right]\\
&\quad + t\int_t^1 f(x(s),x'(s))\,ds\\
&= \int_0^t sf(x(s),x'(s))\,ds - t\left[\int_0^{1/2} sf(x(s),x'(s))\,ds + \int_0^{1/2} (1-w)f(x(w),x'(w))\,dw\right]\\
&\quad + t\int_t^1 f(x(s),x'(s))\,ds\\
&= \int_0^t sf(x(s),x'(s))\,ds - t\int_0^{1/2} f(x(s),x'(s))\,ds + t\int_t^1 f(x(s),x'(s))\,ds\\
&= \int_0^t sf(x(s),x'(s))\,ds - t\int_{1/2}^1 f(x(s),x'(s))\,ds + t\int_t^1 f(x(s),x'(s))\,ds,
\end{aligned}
$$

so we obtain that

$$(Ax)(t) = \int_0^t sf(x(s),x'(s))\,ds + t\int_t^{1/2} f(x(s),x'(s))\,ds, \qquad \forall\, t \in [0,1], \tag{3.6}$$

and

$$(Ax)'(t) = \int_t^{1/2} f(x(s),x'(s))\,ds, \qquad \forall\, t \in [0,1]. \tag{3.7}$$

For any given $x \in \bar{P}_c$, it follows from Assumption (H_3) that $f(x(t),x'(t)) \le 2c$, $0 \le t \le 1$. Thus, from (3.3),

$$\begin{aligned}\|Ax\| &= \max\left\{\max_{0\le t\le 1}|(Ax)(t)|, \max_{0\le t\le 1}|(Ax)'(t)|\right\},\\ &= \max\left\{(Ax)\left(\frac{1}{2}\right), (Ax)'(0)\right\}\\ &= \max\left\{\int_0^{1/2} sf(x(s),x'(s))\,ds, \int_0^{1/2} f(x(s),x'(s))\,ds\right\}\\ &\le \int_0^{1/2} 2c\,ds\\ &= c.\end{aligned}$$

Hence, (3.4) holds. In a similar argument, if $x \in \bar{P}_a$, then $Ax \in P_a$.

STEP 2. We show

$$\{x \in P(\alpha, b, Mb) : \alpha(x) > b\} \neq \emptyset \tag{3.8}$$

and

$$\alpha(Ax) > b, \qquad \text{for } x \in P(\alpha, b, Mb). \tag{3.9}$$

We note that $y(t) = Mb$, $0 \le t \le 1$, is a member of $P(\alpha, b, Mb)$ and $\alpha(y) = Mb > b$, thus (3.8) holds.

In addition, if $x \in P(\alpha, b, Mb)$, then $\alpha(x) \ge b$, $\|x\| \le Mb$, and so $b \le x(t) \le Mb$, $|x'(t)| \le Mb$, for all $\eta \le t \le 1/2$. Thus, for any $x \in P(\alpha, b, Mb)$, Assumption (H_2) yields that $f(x(t), x'(t)) \ge [\eta(1/2-\eta)]^{-1}b$, $\eta \le t \le 1/2$, and consequently, from (3.6),

$$\begin{aligned}\alpha(Ax) &= (Ax)(\eta)\\ &= \int_0^{\eta} sf(x(s),x'(s))\,ds + \eta\int_{\eta}^{1/2} f(x(s),x'(s))\,ds\\ &> \eta\int_{\eta}^{1/2} f(x(s),x'(s))\,ds\\ &\ge \eta\int_{\eta}^{1/2}\left[\eta\left(\frac{1}{2}-\eta\right)\right]^{-1} b\,ds\\ &= b.\end{aligned}$$

Hence, (3.9) holds.

STEP 3. Let $x \in P(\alpha, b, c)$ with $\|Ax\| > Mb$. Then Assumption (H_5) yields $f(x(t), x'(t)) \ge f(x(\eta), x'(\eta))$, $\eta \le t \le 1/2$, and $f(x(t), x'(t)) \le f(x(\eta), x'(\eta))$, $0 \le t \le \eta$. And consequently,

$$\begin{aligned}\alpha(Ax) &= (Ax)(\eta)\\ &= \int_0^{\eta} sf(x(s),x'(s))\,ds + \eta\int_{\eta}^{1/2} f(x(s),x'(s))\,ds\\ &\ge \eta\int_{\eta}^{1/2} f(x(s),x'(s))\,ds\\ &= \frac{1}{M}[\eta(1-2\eta)]^{-1}\eta\int_{\eta}^{1/2} f(x(s),x'(s))\,ds\\ &= \frac{1}{M}\left(\int_{\eta}^{1/2} f(x(s),x'(s))\,ds + \eta\left(\frac{1}{2}-\eta\right)^{-1}\int_{\eta}^{1/2} f(x(s),x'(s))\,ds\right)\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{M}\left(\int_\eta^{1/2} f(x(s),x'(s))\,ds + \eta f(x(\eta),x'(\eta))\,ds\right)\\
&\geq \frac{1}{M}\left(\int_\eta^{1/2} f(x(s),x'(s))\,ds + \int_0^\eta f(x(s),x'(s))\,ds\right)\\
&= \frac{1}{M}\max\left\{\int_0^{1/2} sf(x(s),x'(s))\,ds, \int_0^{1/2} f(x(s),x'(s))\,ds\right\}\\
&= \frac{1}{M}\max\left\{\left|(Ax)\left(\frac{1}{2}\right)\right|, |(Ax)'(0)|\right\}\\
&= \frac{1}{M}\|Ax\|\\
&> b.
\end{aligned}
$$

Hence, an application of Theorem 2.1 completes the proof.

In the following, we present a few examples to which Theorem 3.1 may be applied.

EXAMPLE 3.1. Let $a = 1/17$, $b = 1$, $c = 525$, $\eta = 1/4$, $M = 8$, and

$$
f(u,v) = \begin{cases} 16u^2 + \dfrac{1}{17(1+v^2)}, & 0 \leq u \leq 8, \quad -\infty < v < +\infty,\\ \sqrt{u-8} + \dfrac{1}{17(1+v^2)} + 1024, & u > 8, \quad -\infty < v < +\infty. \end{cases}
$$

It is easy to check that the conditions in Theorem 3.1 hold. So it follows from Theorem 3.1 that BVP (1.1),(1.2) has at least three symmetric nonnegative solutions x_1, x_2, and x_3 satisfying $\|x_1\| \leq 1/17$, $1 < \alpha(x_2)$, and $\|x_3\| > 1/17$ with $\alpha(x_3) < 1$.

EXAMPLE 3.2. Let $a = 1/521$, $b = 1$, $c = 33372$, $\eta = 1/4$, $M = 8$, and

$$
f(u,v) = \begin{cases} \dfrac{1040u^2}{1+v^2}, & 0 \leq u \leq 8, \quad -\infty < v < +\infty,\\ \dfrac{66560 + \sqrt{u-8}}{1+v^2}, & u > 8, \quad -\infty < v < +\infty. \end{cases}
$$

It is easy to check that the conditions in Theorem 3.1 hold. So it follows from Theorem 3.1 that BVP (1.1),(1.2) has at least three symmetric nonnegative solutions x_1, x_2, and x_3 satisfying $\|x_1\| \leq 1/521$, $1 < \alpha(x_2)$, and $\|x_3\| > 1/521$ with $\alpha(x_3) < 1$.

If the norm is defined by

$$
\|x\| = \max\left\{\max_{0\leq t\leq 1} |x(t)|, \frac{1}{4}\max_{0\leq t\leq 1} |x'(t)|\right\}, \qquad \forall x \in C^1[0,1],
$$

as in the argument above, we shall have the following result similar to Theorem 3.1.

THEOREM 3.2. *Let $0 < a < b < c/M$, $M = 1/\eta(1-2\eta)$, and f satisfies*

- (H$_1$) *$f(u,v) < 8a$, for all $0 \leq u \leq a$, $|v| \leq 4a$;*
- (H$_2$) *$f(u,v) \geq [\eta(1/2-\eta)]^{-1}b$, for all $b \leq u \leq Mb$, $|v| \leq 4Mb$;*
- (H$_3$) *$f(u,v) \leq 8c$, for all $0 \leq u \leq c$, $|v| \leq 4c$;*
- (H$_4$) *$f(u,v) = f(u,-v)$, for all $0 \leq u < +\infty$, $-\infty < v < +\infty$;*
- (H$_5$) *$f(u_1,v_1) \leq f(u_2,v_2)$, for all $0 \leq u_1 \leq u_2 \leq c$, $0 \leq v_2 \leq v_1 \leq 4c$.*

Then, BVP (1.1),(1.2) has at least three symmetric nonnegative solutions x_1, x_2, and x_3 satisfying $\|x_1\| \leq a$, $b < \alpha(x_2)$, and $\|x_3\| > a$, with $\alpha(x_3) < b$.

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