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Finitely additive representation of L^p spaces

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Abstract

Let $\bar{\lambda}$ be any atomless and countably additive probability measure on the product space $\{0,1\}^{\mathcal{N}}$ with the usual σ-algebra. Then there is a purely finitely additive probability measure λ on the power set of a countable subset $T \subset \bar{T}$ such that $L^p(\bar{\lambda})$ can be isometrically isomorphically embedded as a closed subspace of $L^p(\lambda)$. The embedding is strict. It is also 'canonical,' in the sense that it maps simple and continuous functions on $\bar{T}$ to their restrictions to T.

Keywords: Finitely additive probabilities; L^p spaces

1. Introduction

A central goal of classical analysis is to identify the classes of functions that can be meaningfully integrated. The standard theory endows a space with a σ-algebra and a countably additive probability measure, then studies integrals of measurable functions on that space. This gives rise to the classic L^p-spaces that have been the backbone of integration theory in many fields.

There are, however, many situations in probability where the classical approach is inadequate. Examples are too numerous to exhaustively list here, so I confine myself to just a few. In his *Foundation of Statistics*, Savage [14] advocated that acts should encompass *all* functions on a given state space. He argued that requiring acts to be measurable with respect to some restrictive σ-algebra (i.e., smaller than the power set) rests on questionable normative and behavioral

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grounds. A similar theme resurfaces in Dubins and Savage's classic book [9] on the theory of gambling and stochastic processes. Another context where standard measurability requirements are restrictive is in modeling functions that oscillate excessively. An example is the problem of "chattering" approximate solutions to optimal control problems (e.g., Kushner [12]). Section 4 explains how the analysis of this paper can be used to model chattering controls. Other examples where similar phenomena occur arise in modeling complexity (Al-Najjar, Anderlini, and Felli [3]), modeling randomness in large populations (Al-Najjar [1]), and in the study of large games (Al-Najjar [2]).

In these and other contexts a theory of integration that dispenses with measurability requirements may be preferable on both conceptual and technical grounds. The approach proposed here is integration with respect to finitely additive probability measures on the power set of a given space. Of course, the theory of integration with respect to finitely additive measures has been known for almost as long as its standard countably additive counterpart. What hampered its adoption is the lack of tractability: Key limit theorems fail, the Radon–Nikodym theorem is not valid, and L^p spaces are not complete, to name just a few difficulties. This paper provides a setting to address these issues.

For simplicity, I focus on the probability space $(\bar{T}, \bar{\mathcal{B}}, \bar{\lambda})$ where $\bar{T} = \{0,1\}^{\mathcal{N}}$,[1] $\bar{\mathcal{B}}$ is the σ-algebra generated by the product topology on $\bar{T}$, and $\bar{\lambda}$ an atomless, countably additive probability measure on $\bar{\mathcal{B}}$.

The main result of this paper is that there is a continuous linear operator $\Phi : L^p(\bar{\lambda}) \to L^p(\lambda)$, where λ is a purely finitely additive probability measure on the power set $2^{\bar{T}}$. In fact, λ has countable support $T \subset \bar{T}$ and the function space $L^p(\lambda)$ is complete. The operator Φ is an isometric isomorphism between $L^p(\bar{\lambda})$ and a closed linear subspace $H \subset L^p(\lambda)$. The embedding is 'canonical' in the sense that there is a unique such Φ mapping simple and continuous functions on $\bar{T}$ to their corresponding restrictions to T.

To the extent that L^p spaces are the objects of interest, the above shows that there is nothing to lose in working with $L^p(\lambda)$ rather than the standard $L^p(\bar{\lambda})$. On the other hand, $L^p(\lambda)$ is strictly richer $L^p(\bar{\lambda})$. In fact, when $p = 2$, $L^p(\lambda)$ has an uncountable orthogonal set, and hence non-separable. This may seem puzzling at first, since $\bar{T}$ contains T and thus, naively, should have richer spaces of functions. The puzzle is resolved by noting that $\bar{\mathcal{B}}$-measurability severely restricts the class of admissible functions on $\bar{T}$, while any function on T is measurable and every bounded function is integrable.

The analysis of this paper hinges on the completeness of $L^p(\lambda)$ which, for $1 \leqslant p < \infty$, is not guaranteed in the finitely additive setting. Many useful characterizations of the completeness of $L^p(\lambda)$ (such as λ having the Radon–Nikodym property) were reported by Gangopadhyay [10], Gangopadhyay and Rao [11] and Basile and Bhaskara Rao [4]. The results in this paper are made possible by the work of Blass, Frankiewicz, Plebanek, and Ryll-Nardzewski [8] who provide a simple sufficient condition for completeness. This paper subscribes to a broader methodology of using finitely additive spaces to overcome difficulties appearing in standard settings. An interesting recent illustration is Berti, Regazzini, and Rigo's study [5] of Brownian motion.

[1] The analysis is conducted here for the product space $\bar{T} = \{0,1\}^{\mathcal{N}}$ for simplicity. It can be readily generalized to other spaces, including [0, 1], at the expense of more cumbersome notation and arguments.

2. Embedding theorem

Let $\bar{\mathcal{B}}_k$ denote the algebra on $\bar{T}$ generated by the first k coordinates. Note that $\bigcup_{k=1}^{\infty} \bar{\mathcal{B}}_k$ is itself an algebra and that $\bar{\mathcal{B}}$ is the σ-algebra it generates.

For the remainder of this paper, fix a countable dense subset $T \subset \bar{T}$ and let Σ denote its power set. Define $\mathcal{B}_k = \{B \cap T\colon\ B \in \bar{\mathcal{B}}_k\}$. A function $\bar{f} : \bar{T} \to \mathcal{R}$ (respectively $f : T \to \mathcal{R}$) is *simple* if it is measurable with respect to $\bar{\mathcal{B}}_k$ (respectively $\mathcal{B}_k$) for some k. Let $\bar{\mathcal{S}}$ (respectively $\mathcal{S}$) denote the set of all simple functions on $\bar{T}$ (respectively T), and note that $\bar{\mathcal{S}}$ and $\mathcal{S}$ are linear spaces. Define $\phi : \bar{\mathcal{S}} \to \mathcal{R}^T$ by $\phi(\bar{f})(t) \equiv \bar{f}(t),\ t \in T$, i.e., $\phi(\bar{f})$ is the restriction of $\bar{f}$ to T. The next lemma is obvious:

Lemma 2.1. *ϕ is a linear isomorphism from $\bar{\mathcal{S}}$ onto $\mathcal{S}$.*

In light of this lemma, we shall often drop the '$^-$' in referring to $\bar{\mathcal{B}}_k$, $\bar{\mathcal{S}}$ and the sets and functions they contain. It is also worth noting that the restriction to T of a continuous function $\bar{c} : \bar{T} \to \mathcal{R}$ is *uniformly* continuous. Conversely, any uniformly continuous function on T has a unique continuous extension to $\bar{T}$.

A critical fact underlying the analysis of this paper is the completeness of $L^p(\lambda)$, $1 \leqslant p < \infty$. Gangopadhyay [10], Gangopadhyay and Rao [11] and Basile and Bhaskara Rao [4] established that completeness of L^p spaces on finitely additive measure spaces is equivalent to a property which, in our context, is:

(AP) For any increasing sequence of sets $A_1, A_2, \ldots$ there is a set A such that $\lambda(A) = \lim_{k\to\infty} \lambda(A_k)$ and for every k, $\lambda(A_k - A) = 0$.

Blass, Frankiewicz, Plebanek, and Ryll-Nardzewski [8] showed that, when the underlying space is the integers and λ is an extension of the density, (AP) holds when λ is defined in terms of a free ultrafilter on the integers containing a *thin* set; i.e., a set $X \subset \mathcal{N}$ which, when enumerated in an increasing manner $l_1 < l_2 < \cdots$, satisfies $\frac{l_n - l_{n-1}}{l_n} \to 1$. The following lemma therefore ensures that $L^p(\lambda)$ is complete:

Lemma 2.2. *For any atomless and countably additive probability $\bar{\lambda}$ on $(\bar{T}, \bar{\mathcal{B}})$, there is an enumeration $\{t_1, \ldots\}$ of T and a free ultrafilter $\mathcal{U}$ on the positive integers containing a thin set $\{l_1, \ldots\}$ such that*:

$$\bar{\lambda}(B) = \mathcal{U}\text{-}\lim_n \lambda_n(B), \quad \forall B \in \bigcup_{k=1}^{\infty} \mathcal{B}_k,{}^{2} \tag{1}$$

where λ_n is the uniform distribution on $\{t_1, \ldots, t_{l_n}\}$. The set function

$$\lambda(A) \equiv \mathcal{U}\text{-}\lim_n \lambda_n(A), \quad A \subset T, \tag{2}$$

is a finitely additive probability measure on 2^T.

Proof. Let $\{t_1^*, \ldots\}$ be an arbitrary enumeration of T. Set $T_0 = \emptyset$. For $k = 1, \ldots$ find a finite set $T_k \subset T$ such that:

[2] For a sequence of real numbers $\{x_n\}$, $\mathcal{U}$-$\lim_n x_n = \alpha$ means: $\{n\colon\ |x_n - \alpha| < \epsilon\} \in \mathcal{U}$ for every $\epsilon > 0$.

1. $\{t_1^*, \dots, t_k^*\} \cup T_{k-1} \subset T_k$,
2. $\frac{\#T_k - \#T_{k-1}}{\#T_k} > 1 - \frac{1}{k}$,[3]
3. $|\frac{\#(B \cap T_k)}{\#T_k} - \bar{\lambda}(B)| < \frac{1}{k}, \forall B \in \mathcal{B}_k$.

This can be done since T is dense and $\mathcal{B}_k$ is finite.

Let $\{t_1, \dots\}$ be any sequence of distinct elements in T such that $T_k = \{t_1, \dots, t_{\#T_k}\}$. That is, $\{t_1, \dots\}$ is an enumeration of T that is consistent with the T_k's.

Let $\mathcal{U}$ be a free ultrafilter that contains the sequence of integers $\{\#T_1, \#T_2, \dots\}$ and set $l_n = \#T_n$. By the construction above, $\{l_n\}$ is thin and $\lambda_n(B) \to \bar{\lambda}(B)$, so $\lambda(B) = \bar{\lambda}(B)$. That λ is a finitely additive probability on 2^T is well known and easy to verify (see, e.g., Blass, Frankiewicz, Plebanek, and Ryll-Nardzewski [8]). □

Our use of integration on finitely additive measure spaces follows Bhaskara Rao and Bhaskara Rao [6]. The next lemma is obvious:

Lemma 2.3. *Suppose that $\bar{\lambda}$ and λ are as in Lemma 2.2. Then ϕ preserves integrals: for every $f \in \mathcal{S}$,*

$$\int_B f \, d\bar{\lambda} = \int_B \phi(f) \, d\lambda, \quad \forall B \in \bigcup_{k=1}^{\infty} \mathcal{B}_k.$$

For probability measures $\bar{\lambda}$ and λ on $(\bar{T}, \bar{\mathcal{B}})$ and (T, Σ), the spaces $L^p(\bar{\lambda})$ and $L^p(\lambda)$ are defined in the usual way.[4] We shall use '$\equiv_{p,\nu}$' and '$[\cdot]_{p,\nu}$', $\nu \in \{\lambda, \bar{\lambda}\}$, to denote equivalence relationships and equivalence classes. We typically drop p and ν when they are clear from the context. In fact, we will drop references to equivalence classes when there is no risk of ambiguity. Thus, for a function Φ defined on $L^p(\bar{\lambda})$, we will write $\Phi(\bar{f})$ instead of $\Phi([\bar{f}])$.

The following is the paper's main embedding theorem:

Theorem 1. *For $1 \leqslant p < \infty$ and any atomless and countably additive probability measure $\bar{\lambda}$ on $(\bar{T}, \bar{\mathcal{B}})$, there is a purely finitely additive probability measure λ on (T, Σ) and a unique bounded linear operator $\Phi : L^p(\bar{\lambda}) \to L^p(\lambda)$ such that*

$$\Phi(f) = [\phi(f)], \quad \forall f \in \mathcal{S}.$$

The operator Φ preserves integrals: for every $[\bar{f}] \in L^p(\bar{\lambda})$,

$$\int_B \bar{f} \, d\bar{\lambda} = \int_B \Phi(\bar{f}) \, d\lambda, \quad \forall B \in \bigcup_{k=1}^{\infty} \mathcal{B}_k. \tag{3}$$

In fact, Φ is an isometric isomorphism onto a closed linear subspace $H \subset L^p(\lambda)$; that is, Φ is one-to-one and $\|\Phi(\bar{f})\| = \|\bar{f}\|$ for every $[\bar{f}] \in L^p(\bar{\lambda})$.

For $p = 2$, Φ also preserves inner products: $(\bar{f}|\bar{g}) = (\Phi(\bar{f})|\Phi(\bar{g}))$ for every $\bar{f}, \bar{g} \in L^2(\bar{\lambda})$.

[3] The notation # denotes the cardinality of a finite set.

[4] For the theory of L^p spaces on finitely additive measure spaces, see Bhaskara Rao and Bhaskara Rao [6].

The theorem establishes an embedding of equivalence classes of functions. It does not claim that one can map arbitrary measurable subsets of $\bar{T}$ to subsets of T in any sensible way. Notice also that two functions f and g such that $f - g$ is a null function[5] are L^p-equivalent and thus correspond to the same element of $L^p(\lambda)$. But it is possible that $|f(t) - g(t)| > 0$ for every t, since a null function need not be equal to 0 almost everywhere in the finitely additive setting. The point is that the difference between f and g can be made 'infinitely' small, and thus irrelevant from the perspective of L^p spaces.

Proof of Theorem 1. Define Φ on $\mathcal{S}$ by $\Phi(f) \equiv [\phi(f)]$. Lemma 2.3 and Eq. (3) readily imply that Φ is uniformly continuous. Since $\mathcal{S}$ is dense in $L^p(\bar{\lambda})$ and $L^p(\lambda)$ is complete, Φ has a unique uniformly continuous extension from $L^p(\bar{\lambda})$ into $L^p(\lambda)$ (Royden [13, Proposition 11, p. 149]), which we shall also denote by Φ.

To avoid repetition, we shall assume in the remainder of the proof that $\bar{f}, \bar{g} \in L^p(\bar{\lambda})$ and $f_n \to \bar{f}$ and $g_n \to \bar{g}$ in L^p; all functions indexed by n are simple.

To check that Φ is linear, take any pair of real numbers a and b, then

$$\begin{aligned}\Phi(a\bar{f} + b\bar{g}) &= \Phi\big(\lim(af_n + bg_n)\big)\\ &= \lim \Phi(af_n + bg_n)\\ &= a \lim \Phi(f_n) + b \lim \Phi(g_n)\\ &= a\Phi(\bar{f}) + b\Phi(\bar{g}).\end{aligned}$$

A similar argument shows that Φ preserves inner products when $p = 2$:

$$\begin{aligned}(\bar{f}|\bar{g}) &= \lim(f_n|g_n)\\ &= \lim\big(\Phi(f_n)|\Phi(g_n)\big)\\ &= \big(\Phi(\lim f_n)|\Phi(\lim g_n)\big)\\ &= \big(\Phi(\bar{f})|\Phi(\bar{g})\big).\end{aligned}$$

Φ is an isometry since:

$$\big\|\Phi(\bar{f})\big\| = \big\|\Phi(\lim f_n)\big\| = \lim\big\|\Phi(f_n)\big\| = \lim \|f_n\| = \|\bar{f}\|.$$

Φ is obviously one–one since for any $\bar{f}$ and $\bar{g}$, $\|\Phi(\bar{f}) - \Phi(\bar{g})\| = \|\bar{f} - \bar{g}\|$. □

The next lemma shows that Φ preserves continuous functions:

Lemma 2.4. *Let $\bar{c}: \bar{T} \to \mathcal{R}$ be any continuous function and c is its restriction to T. Then $c \in \Phi(\bar{c})$.*

Proof. In this proof we will explicitly distinguish between simple functions on $\bar{T}$ and T. Let $\{\bar{f}_n\}$ be a sequence of simple functions converging uniformly to $\bar{c}$ on $\bar{T}$ (such sequence exists using an argument similar to that in Bhaskara Rao and Bhaskara Rao [6, p. 137]). Since $\sup_{t\in T} |f_n(t) - c(t)| = \sup_{t\in\bar{T}} |\bar{f}_n(t) - \bar{c}(t)|$, the sequence $\{f_n\}$ converges uniformly on T, hence $\|f_n - c\| \to 0$. On the other hand, the continuity of Φ implies: $[f_n] = [\phi(\bar{f}_n)] = \Phi(\bar{f}_n) \to \Phi(\bar{c})$ in L^p. That is, for any $c' \in \Phi(\bar{c})$, we have $\|f_n - c'\| \to 0$. But then $\|c - c'\| = 0$, and $c \in \Phi(\bar{c})$ as required. □

[5] That is, $\lambda\{|f - g| > \epsilon\} = 0$ for every $\epsilon > 0$ (Bhaskara Rao and Bhaskara Rao [6, p. 88]).

3. How rich is $L^p(\lambda)$?

$L^p(\lambda)$ strictly includes $L^p(\bar{\lambda})$. To see this, let M denote the set of all functions $s: T \to \{0, 1\}$ endowed with the σ-algebra $\mathcal{M}$ generated by all events of the form $\{s\colon\ s(t) = 0\}$, $t \in T$. Let P denote the (countably additive) probability measure on $(M, \mathcal{M})$ generated by i.i.d. flips of a balanced coin for each $t \in T$. By the law of large numbers, there is $A \subset M$ with $P(A) = 1$ such that any $s \in A$ satisfies

$$\int_B s\, d\lambda = 0.5\lambda(B), \quad \text{for all } B \in \bigcup_{k=1}^{\infty} \mathcal{B}_k. \tag{4}$$

But for any such s there can be no measurable function $\bar{s}$ on $\bar{T}$ such that $s \in \Phi(\bar{s})$. For if this were true, then $\bar{s}$ must satisfy:

$$\int_B \bar{s}\, d\bar{\lambda} = 0.5\bar{\lambda}(B), \quad \text{for all } B \in \bigcup_{k=1}^{\infty} \mathcal{B}_k.$$

The indefinite integral on the LHS is a finitely additive set function on $\bigcup_{k=1}^{\infty} \mathcal{B}_k \subset \bar{\mathcal{B}}$. But then it must be countably additive on $\bigcup_{k=1}^{\infty} \mathcal{B}_k$, and thus admits a unique countably additive extension $\bar{\nu}$ to $\bar{\mathcal{B}}$ (Billingsley [7, Theorems 2.3 and 3.1, respectively]). This implies:

$$\bar{\nu}(A) \equiv \int_A \bar{s}\, d\bar{\lambda} = 0.5\bar{\lambda}(A), \quad \forall A \in \bar{\mathcal{B}},$$

So $\bar{s}$ may be taken to be the constant function 0.5. Since s takes values in $\{0, 1\}$, Lemma 2.4 implies that $s \notin \Phi(\bar{s})$.

We have thus shown:

Theorem 2. *For every* $1 \leqslant p < \infty$, $L^p(\lambda) - \Phi(L^p(\bar{\lambda})) \neq \emptyset$.

For $p = 2$ we can get a sharper result:

Theorem 3. $L^2(\lambda)$ *has an uncountable number of orthogonal elements and is thus not separable.*

Proof. Let $(M, \mathcal{M})$, P and A be as above. For $s \in A$ let $\hat{s} = s = 0.5$, and define $\hat{A} \equiv \{s - 0.5, s \in A\}$. Thus, $\hat{A}$ is the same as A except that each of its members has mean zero.

Choose $\hat{s}^1 \in \hat{A}$. Let '$\perp$' denote orthogonality in either $L^p(\bar{\lambda})$ or $L^p(\lambda)$. We show that for each ordinal $\alpha > 1$ but strictly smaller than the first uncountable ordinal $\aleph_1$ we can choose $\hat{s}^\alpha \in \hat{A}$ such that $\hat{s}^\alpha \perp \hat{s}^\beta$ whenever $\beta < \alpha$. In this case, $\{\hat{s}^\alpha\colon\ 1 \leqslant \alpha < c\}$ is an uncountable collection of orthogonal elements of $L^2(\lambda)$, as required.

Suppose that $\hat{s}^\beta$ has been defined for every $\beta < \alpha < c$. Since $\alpha < \aleph_1$, the set of such β's is countable. For a given such β, define $Z^\beta \equiv \{s \in \{0, 1\}^T\colon\ \hat{s} \perp \hat{s}^\beta\}$. The law of large numbers implies $P(Z^\beta) = 1$. Since $\{Z^\beta\colon\ \beta < \alpha\}$ is a countable collections of P-measure 1 sets and P is countably additive, it follows that $P(\bigcap_{\beta<\alpha} Z^\beta) = 1$. Since P is atomless, the set

$$\left(\bigcap_{\beta<\alpha} Z^\beta\right) \cap \hat{A}$$

is uncountable (hence non-empty!). Any choice of $\hat{s}^\alpha$ in this set has the desired properties. $\square$

4. Representation of $L^p(\lambda) - \Phi(L^p(\bar{\lambda}))$

What do points in $L^p(\lambda) - \Phi(L^p(\bar{\lambda}))$ correspond to? The next theorem answers this question.

Theorem 4. *For every bounded $f : T \to \mathcal{R}$ there is a $\bar{\mathcal{B}}$-measurable function $\bar{f} : \bar{T} \to \mathcal{R}$ such that*

$$\int_B \bar{f}\, d\bar{\lambda} = \int_B f\, d\lambda, \quad \forall B \in \bigcup_{k=1}^{\infty} \mathcal{B}_k. \tag{5}$$

$\bar{f}$ is unique (up to equivalence). If $p = 2$, then $\Phi(\bar{f})$ is the orthogonal projection of f on $\Phi(L^2(\bar{\lambda}))$:

$$\big\| f - \Phi(\bar{f}) \big\| \leqslant \| f - f' \|, \quad \forall f' \in \Phi\big(L^2(\bar{\lambda})\big). \tag{6}$$

The following commutative diagram illustrates the theorem for the case $p = 2$:

$$\begin{tikzcd}
f \arrow[r, "\text{Theorem 4}"] \arrow[dr, "\text{Projection on } \Phi(L^2(\bar{\lambda}))"'] & \bar{f} \arrow[d, equal, "\Phi"] \\
 & \Phi(\bar{f})
\end{tikzcd}$$

For a concrete example, take any function $s : T \to \{0, 1\}$ satisfying Eq. (4). How should s be represented in $L^p(\bar{\lambda})$? Theorem 4 represents it (up to equivalence) as the constant function 0.5, which is the best one can do to 'smooth out' the wildly fluctuating s under the constraint of $\bar{\mathcal{B}}$-measurability. The intuition is particularly sharp in the case $p = 2$: Theorem 4 may be interpreted as extracting the 'measurable part' of a function, in the sense of projecting it on the linear space spanned by $\cup_{k=1}^{\infty} \mathcal{B}_k$-measurable functions.

Proof of Theorem 4. Define

$$\bar{\nu}(B) \equiv \int_B f\, d\lambda, \quad B \in \bigcup_{k=1}^{\infty} \mathcal{B}_k. \tag{7}$$

Then $\bar{\nu}$ is a finitely additive set function on $\bigcup_{k=1}^{\infty} \mathcal{B}_k \subset \bar{\mathcal{B}}$. Using similar argument as above, $\bar{\nu}$ is countably additive and so must have a unique countably additive extension to $\bar{\mathcal{B}}$, which we shall also denote by $\bar{\nu}$.

Next we show that $\bar{\nu}$ is absolutely continuous with respect to $\bar{\lambda}$.[6] Define

$$f_k(t) \equiv \begin{cases} \bar{\nu}(B_k(t))/\bar{\lambda}(B_k(t)) & \text{if } \bar{\lambda}(B_k(t)) > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

where $\bar{B}_k(t)$ is the smallest (by set inclusion) set in $\mathcal{B}_k$ containing t. Then the sequence of functions $\{f_k\}_{N=1}^{\infty}$ is a martingale under $\bar{\lambda}$ and thus converges $\bar{\lambda}$-a.e. to a $\bar{\mathcal{B}}$-measurable function $\bar{f} : \bar{T} \to \mathcal{R} \cup \infty$ (Shiryayev [15, pp. 492–493]). Theorem 1, p. 493, in the same reference states

[6] Unlike the earlier argument, this requires proof since $\bar{\nu}$ is defined in terms of λ and not $\bar{\lambda}$. In the latter case absolute continuity of $\bar{\nu}$ is guaranteed by definition.

that $\bar{\nu}$ is absolutely continuous with respect to $\bar{\lambda}$ if $\bar{\nu}(\{t\colon \bar{f}(t)=\infty\}=0$, in which case $\bar{f}$ is its Radon–Nikodym derivative with respect to $\bar{\lambda}$. But whenever $\bar{\lambda}(B_k(t))>0$ we have

$$f_k(t)\equiv\frac{\bar{\nu}(B_k(t))}{\bar{\lambda}(B_k(t))}\equiv\frac{\int_{B_k(t)} f\,d\lambda}{\bar{\lambda}(B_k(t))}\leqslant\frac{(\sup_T|f|)\,\bar{\lambda}(B_k(t))}{\bar{\lambda}(B_k(t))}=\sup_T|f|<\infty.$$

Thus, $\bar{\nu}(\{t\colon \bar{f}(t)=\infty\}=0$, as required. The uniqueness of $\bar{f}$ follows from the uniqueness of the Radon–Nikodym derivative.

Now assume that $p=2$. Clearly, it is sufficient to verify Eq. (6) for $f'\in\mathcal{S}$. Specifically, assume that f' is measurable with respect to $\mathcal{B}_k$ for some k. Let f_k be the function given in Eq. (8). Then, f_k is the conditional expectation (hence the orthogonal projection) of $\bar{f}$ on $\mathcal{B}_k$, so

$$E(f|\mathcal{B}_k)=f_k=E(\bar{f}|\mathcal{B}_k).$$

Hence, for every $l\geqslant k$,

$$\|f-f'\|\geqslant\big\|f-E(f|\mathcal{B}_k)\big\|=\big\|f-\Phi\big(E(\bar{f}|\mathcal{B}_k)\big)\big\|\geqslant\big\|f-\Phi\big(E(\bar{f}|\mathcal{B}_l)\big)\big\|.$$

By Levy's theorem (Shiryayev [15, p. 478]) $E(\bar{f}|\mathcal{B}_l)\to E(\bar{f}|\mathcal{B})=\bar{f}$, $\bar{\lambda}$-a.e., hence in L^2-norm since f is bounded. Since Φ is continuous, $\Phi(E(\bar{f}|\mathcal{B}_l))\to\Phi(\bar{f})$ and the result follows. □

As a further illustration, consider the following example, motivated by Kushner [12]. Let $\bar{\lambda}$ be the uniform distribution on $\bar{T}$ and let $f_n:\bar{T}\to\{0,1\}$, $n=1,2,\dots,$ be the sequence of functions $f_n(t)=\operatorname{sign}(\sin nt)$, where t is viewed as the binary expansion of a real number. Clearly, $\{f_n\}$ does not converge in L^p. On the other hand, for any continuous function $\bar{c}:\bar{T}\times\{0,1\}\to\mathcal{R}$,

$$\int_{\bar{T}}\bar{c}\big(t,f_n(t)\big)\,d\bar{\lambda}\to\int_{\bar{T}}\big[\hat{f}(t)\,\bar{c}(t,1)+\big(1-\hat{f}(t)\big)\,\bar{c}(t,0)\big]\,d\bar{\lambda},\tag{9}$$

where $\hat{f}$ is the constant function 0.5 on $\bar{T}$. That is, the sequence $\{f_n\}$ converges to a *distribution* $\hat{f}$ that puts equal weight on $\bar{c}(t,1)$ and $\bar{c}(t,0)$ at each t. The mode of convergence in Eq. (9) is known as *compact-weak convergence* (see Kushner [12, p. 48]). If we interpret $\bar{T}$ to be the interval $[0,1]$, then $\{f_n\}$ may be viewed as a sequence of ordinary controls and $\hat{f}$ is a *relaxed* control. Relaxed controls are artifacts introduced to ensure the closure of the space of ordinary controls.

Using the space T introduced in this paper, in this example any function s satisfying Eq. (4) is an ordinary control such that

$$\int_T c\big(t,f_n(t)\big)\,d\lambda\to\int_T c\big(t,s(t)\big)\,d\lambda\tag{10}$$

for every uniformly continuous c. That is, in this example, it is not necessary to use relaxed controls. To establish Eq. (10), we need to show that

$$\int_T c\big(t,s(t)\big)\,d\lambda=\int_T\big[0.5\,c(t,1)+0.5\,c(t,0)\big]\,d\lambda,$$

and so it suffices to show that

$$\int_T s(t)c(t,1)\,d\lambda=0.5\int_T c(t,1)\,d\lambda.\tag{11}$$

Suppose that $\{c_n\}$ is a sequence of simple functions that converge uniformly to c. Then the sequence of functions $t \mapsto s(t)c_n(t)$ also converges uniformly to $t \mapsto s(t)c(t)$. The result now follows from that facts that Eq. (11) holds for each c_n and that uniform convergence preserves integration (e.g., Bhaskara Rao and Bhaskara Rao [6, Theorem 4.4.20]).

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