On Some Finite Difference Inequalities in Two Independent Variables

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In this article, we establish some new finite difference inequalities in two independent variables which can be used as tools in the study of certain classes of finite difference and sum-difference equations.

Key Words: finite difference inequalities; two independent variables; finite difference and sum-difference equations; sums and products; nonincreasing; nondecreasing; subadditive and submultiplicative functions.

1. INTRODUCTION

In [3], Mate and Nevai proved the following finite difference inequality.

\[ \text{Lemma MN. Let } u(n) > 0, \ p(n) > 0 \text{ be real-valued functions defined on integers and let } c > 0 \text{ be a constant. If } \]
\[ u(n) \leq c + \sum_{s = n + 1}^{\infty} p(s)u(s), \]
then,
\[ u(n) \leq c \exp \left( \sum_{s = n + 1}^{\infty} p(s) \right). \]

In analyzing the dynamics of a physical system governed by nonlinear finite difference equations involving many independent variables, one often needs some kinds of finite difference inequalities for proving various theorems or approximating various functions. The desire to widen the
scope of applications of such inequalities recently resulted in the necessity of discovering new finite difference inequalities which are applicable in the situations for which the earlier inequalities do not apply directly, see [1–8] and some of the references cited therein. In the present article, we offer some basic finite difference inequalities similar to those given in Lemma MN, involving functions of two independent variables. The inequalities given here can be used as tools in the analysis of problems in the theory of certain finite difference and sum-difference equations in two independent variables.

2. STATEMENT OF RESULTS

In what follows, we denote by $R$ the set of real numbers, $R_+ = [0, \infty)$ and $N_0 = \{0, 1, 2, \ldots \}$. We use the usual conventions that empty sums and products are taken to be 0 and 1, respectively. We also assume that all the sums and products involved throughout the discussion exist on the respective domains of their definitions.

Our main results are established in the following theorems.

**Theorem 1.** Let $u(m, n), a(m, n), b(m, n)$ be real-valued nonnegative functions defined for $m, n \in N_0$.

(a) Let $a(m, n)$ be nonincreasing in each variable $m, n \in N_0$. If

$$u(m, n) \leq a(m, n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s, t)u(s, t)$$  \hspace{0.5cm} (2.1)

for $m, n \in N_0$, then

$$u(m, n) \leq a(m, n) \prod_{s=m+1}^{\infty} \left[ 1 + \sum_{t=n+1}^{\infty} b(s, t) \right]$$  \hspace{0.5cm} (2.2)

for $m, n \in N_0$.

(b) Let $a(m, n)$ be nondecreasing in $m \in N_0$ and nonincreasing in $n \in N_0$. If

$$u(m, n) \leq a(m, n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} b(s, t)u(s, t)$$  \hspace{0.5cm} (2.3)
for \( m, n \in \mathbb{N}_0 \), then

\[
    u(m, n) \leq a(m, n) \prod_{s=0}^{m-1} \left[ 1 + \sum_{t=n+1}^{\infty} b(s, t) \right]
\]  

(2.4)

for \( m, n \in \mathbb{N}_0 \).

**Theorem 2.** Let \( u(m, n), f(m, n), g(m, n), h(m, n) \) be real-valued non-negative functions defined for \( m, n \in \mathbb{N}_0 \) and \( u(m, n) \geq u_0 > 0 \), \( u_0 \) is a real constant. Let \( W(r) \) be a real-valued continuous, positive, strictly nondecreasing, subadditive, and submultiplicative function on \( I = [u_0, \infty) \) and let \( H(r) \) be a real-valued, continuous, positive, and nondecreasing function on \( I \).

(b) If

\[
    u(m, n) \leq f(m, n) + g(m, n)H\left( \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} h(s, t)W(u(s, t)) \right)
\]

(2.5)

for \( m, n \in \mathbb{N}_0 \), then for \( 0 \leq m \leq m_1, 0 \leq n \leq n_1, m, m_1, n, n_1 \in \mathbb{N}_0 \),

\[
    u(m, n) \leq f(m, n) + g(m, n)H\left( G\left( \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} h(s, t)W(f(s, t)) \right) \right)
    + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} h(s, t)W(g(s, t)) \right),
\]

(2.6)

where

\[
    G(r) = \int_{r_0}^{r} \frac{ds}{W(H(s))}, \quad r \geq u_0 \text{ with } r_0 \geq u_0,
\]

(2.7)

\( G^{-1} \) is the inverse function of \( G \) and

\[
    G\left( \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} h(s, t)W(f(s, t)) \right) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} h(s, t)W(g(s, t)) \right) \in \text{Dom}(G^{-1})
\]
for $0 \leq m \leq m_1$, $0 \leq n \leq n_1$, $m, m_1, n, n_1 \in N_0$.

(b$_2$) If

$$u(m, n) \leq f(m, n) + g(m, n) H\left(\sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} h(s, t)W(u(s, t))\right)$$

for $m, n \in N_0$, then for $0 \leq m \leq m_2$, $0 \leq n \leq n_2$, $m, m_2, n, n_2 \in N_0$,

$$u(m, n) \leq f(m, n) + g(m, n) H\left(G^{-1}\left[G\left(\sum_{s=0}^{\infty} \sum_{t=1}^{\infty} h(s, t)W(f(s, t))\right)\right.ight.$$  

$$+ \left.\sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} h(s, t)W(g(s, t))\right]\right),$$

where $G, G^{-1}$ are as defined in part (b$_1$) and

$$G\left(\sum_{s=0}^{\infty} \sum_{t=1}^{\infty} h(s, t)W(f(s, t))\right) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} h(s, t)W(g(s, t))$$

$$\in \text{Dom}(G^{-1})$$

for $0 \leq m \leq m_2$, $0 \leq n \leq n_2$, $m, m_2, n, n_2 \in N_0$.

**Theorem 3.** Let $u(m, n), a(m, n), b(m, n)$ be real-valued nonnegative functions defined for $m, n \in N_0$ and let $L: N_0^2 \times R_+ \rightarrow R_+$ be a function which satisfies the condition,

$$0 \leq L(m, n, u) - L(m, n, v) \leq M(m, n, v)(u - v)$$

for $u \geq v \geq 0$, where $M(m, n, v)$ is a real-valued nonnegative function defined for $m, n \in N_0$, $v \in R_+$.

(c$_1$) If

$$u(m, n) \leq a(m, n) + b(m, n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} L(s, t, u(s, t))$$

for $m, n \in N_0$, then

$$u(m, n) \leq a(m, n) + b(m, n) e(m, n)$$  

$$\times \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} M(s, t, a(s, t))b(s, t)\right]$$

(2.11)
for \( m, n \in N_0 \), where
\[
e(m, n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} L(s, t, a(s, t))
\] (2.12)
for \( m, n \in N_0 \).

(c2) If
\[
u(m, n) \leq a(m, n) + b(m, n) \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} L(s, t, u(s, t))
\] (2.13)
for \( m, n \in N_0 \), then
\[
u(m, n) \leq a(m, n) + b(m, n) \bar{\nu}(m, n)
\times \prod_{s=0}^{m-1} \left[ 1 + \sum_{t=n+1}^{\infty} M(s, t, a(s, t))b(s, t) \right]
\] (2.14)
for \( m, n \in N_0 \), where
\[
\bar{\nu}(m, n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} L(s, t, a(s, t))
\] (2.15)
for \( m, n \in N_0 \).

THEOREM 4. Let \( u(m, n), a(m, n), b(m, n) \) be real-valued nonnegative functions defined for \( m, n \in N_0 \) and let \( L: N_0^2 \times R_+ \rightarrow R_+ \) be a function which satisfies the condition,
\[
0 \leq L(m, n, u) - L(m, n, v) \leq M(m, n, v) \phi^{-1}(u - v)
\]
for \( u \geq v \geq 0 \), where \( M(m, n, v) \) is defined as in Theorem 3, let \( \phi: R_+ \rightarrow R_+ \)
be a continuous and strictly increasing function with \( \phi(0) = 0 \), \( \phi^{-1} \) is the inverse function of \( \phi \)
and
\[
\phi^{-1}(uv) \leq \phi^{-1}(u) \phi^{-1}(v)
\]
for \( u, v \in R_+ \).

(d1) If
\[
u(m, n) \leq a(m, n) + b(m, n) \phi \left( \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} L(s, t, u(s, t)) \right)
\] (2.16)
for \( m, n \in N_0 \), then
\[
u(m, n) \leq a(m, n) + b(m, n) \phi \left( e(m, n)
\times \prod_{s=m+1}^{\infty} \left[ 1 + \sum_{t=n+1}^{\infty} M(s, t, a(s, t)) \phi^{-1}(b(s, t)) \right] \right)
\] (2.17)
for \( m, n \in \mathbb{N}_0 \), where \( e(m, n) \) is defined by (2.12).

\((d_2)\) If

\[
 u(m, n) \leq a(m, n) + b(m, n) \phi \left( \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} L(s, t, u(s, t)) \right)
\]  
(2.18)

for \( m, n \in \mathbb{N}_0 \), then

\[
 u(m, n) \leq a(m, n) + b(m, n) \phi \left( \tilde{e}(m, n) \right.
\]

\[
 \times \prod_{s=0}^{m-1} \left[ 1 + \sum_{t=n+1}^{\infty} M(s, t, a(s, t)) \phi^{-1}(b(s, t)) \right] \]  
(2.19)

for \( m, n \in \mathbb{N}_0 \), where \( \tilde{e}(m, n) \) is defined by (2.15).

3. PROOFS OF THEOREMS 1–4

We give the details of the proofs of \((a_1), (b_1), (c_1), (d_1)\) only. The proofs of \((a_2), (b_2), (c_2), (d_2)\) can be completed similarly with suitable modifications.

\((a_1)\) First, we assume that \( a(m, n) > 0 \) for \( m, n \in \mathbb{N}_0 \). From (2.1), it is easy to observe that

\[
 \frac{u(m, n)}{a(m, n)} \leq 1 + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s, t) \frac{u(s, t)}{a(s, t)}.
\]  
(3.1)

Define a function \( z(m, n) \) by the right side of (3.1), then

\[
 \frac{u(m, n)}{a(m, n)} \leq z(m, n)
\]

and

\[
 \left[ z(m, n) - z(m + 1, n) \right] - \left[ z(m, n + 1) - z(m + 1, n + 1) \right] = b(m + 1, n + 1) \frac{u(m + 1, n + 1)}{a(m + 1, n + 1)}
\]

\[
 \leq b(m + 1, n + 1) z(m + 1, n + 1).
\]  
(3.2)
From (3.2) and using the facts that $z(m,n) > 0$, $z(m + 1, n + 1) \leq z(m + 1, n)$ for $m, n \in N_0$, we observe that

$$\frac{[z(m,n) - z(m+1,n)]}{z(m + 1,n)} - \frac{[z(m,n+1) - z(m+1,n+1)]}{z(m + 1,n + 1)} \leq b(m + 1, n + 1).$$

(3.3)

Keeping $m$ fixed in (3.3) set $n = t$ and sum over $t = n, n + 1, \ldots, q - 1$ ($q \geq n + 1$ is arbitrary in $N_0$) to obtain

$$\frac{[z(m,n) - z(m+1,n)]}{z(m + 1,n)} - \frac{[z(m,q) - z(m+1,q)]}{z(m + 1,q)} \leq \sum_{t=n+1}^{q} b(m + 1,t).$$

(3.4)

Noting that $\lim_{q \to \infty} z(m,q) = \lim_{q \to \infty} z(m + 1,q) = 1$ and by letting $q \to \infty$ in (3.4) we get

$$\frac{[z(m,n) - z(m+1,n)]}{z(m + 1,n)} \leq \sum_{t=n+1}^{\infty} b(m + 1,t),$$

i.e.,

$$z(m,n) \leq \left[ 1 + \sum_{t=n+1}^{\infty} b(m + 1,t) \right] z(m + 1,n).$$

(3.5)

Now, by keeping $n$ fixed in (3.5) and by setting $m = s$ and by substituting $s = m, m + 1, \ldots, p - 1$ ($p \geq m + 1$ is arbitrary in $N_0$) successively, we obtain

$$z(m,n) \leq z(p,n) \prod_{s=m+1}^{p} \left[ 1 + \sum_{t=n+1}^{\infty} b(s,t) \right].$$

(3.6)

Noting that $\lim_{p \to \infty} z(p,n) = 1$, and letting $p \to \infty$ in (3.6), we get

$$z(m,n) \leq \prod_{s=m+1}^{\infty} \left[ 1 + \sum_{t=n+1}^{\infty} b(s,t) \right].$$

(3.7)

Using (3.7) in (3.1), we get the required inequality in (2.2).
If \( a(m, n) \) is nonnegative, we carry out the above procedure with \( a(m, n) + \epsilon \) instead of \( a(m, n) \), where \( \epsilon > 0 \) is an arbitrary small constant, and subsequently pass to the limit as \( \epsilon \to 0 \) to obtain (2.2).

(b) Define a function \( z(m, n) \) by

\[
z(m, n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} h(s, t)W(u(s, t)), \tag{3.8}
\]

then from (2.5) we have

\[
u(m, n) \leq f(m, n) + g(m, n)H(z(m, n)). \tag{3.9}
\]

From (3.8) and (3.9), we observe that

\[
z(m, n) \leq \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} h(s, t)W(f(s, t) + g(s, t)H(z(s, t)))
\]

\[
\leq \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} h(s, t)[W(f(s, t)) + W(g(s, t))W(H(z(s, t)))]
\]

\[
\leq \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} h(s, t)W(f(s, t)) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} h(s, t)W(g(s, t))W(H(z(s, t))). \tag{3.10}
\]

Define a function \( v(m, n) \) by the right side of (3.10). Then, \( z(m, n) \leq v(m, n) \) and

\[
[v(m, n) - v(m + 1, n)] - [v(m, n + 1) - v(m + 1, n + 1)]
\]

\[
= h(m + 1, n + 1)W(g(m + 1, n + 1))W(H(z(m + 1, n + 1)))
\]

\[
\leq h(m + 1, n + 1)W(g(m + 1, n + 1))W(H(v(m + 1, n + 1))). \tag{3.11}
\]

From (3.11) and the fact that \( v(m + 1, n + 1) \leq v(m + 1, n) \), we observe that

\[
\frac{[v(m, n) - v(m + 1, n)]}{W(H(v(m + 1, n)))} - \frac{[v(m, n + 1) - v(m + 1, n + 1)]}{W(H(v(m + 1, n + 1)))}
\]

\[
\leq h(m + 1, n + 1)W(g(m + 1, n + 1)). \tag{3.12}
\]
Keeping \( m \) fixed in (3.12), substituting \( n = t \), and taking the sum over \( t = n, n + 1, \ldots, q - 1, (q \geq n + 1 \) is arbitrary in \( N_0 \), we obtain

\[
\frac{[v(m, n) - v(m + 1, n)]}{W(H(v(m + 1, n)))} \leq \frac{[v(m, q) - v(m + 1, q)]}{W(H(v(m + 1, q)))} \leq \sum_{t=n+1}^{q} h(m + 1, t)W(g(m + 1, t)). \tag{3.13}
\]

Noting that \( \lim_{q \to \infty} v(m, q) = \lim_{q \to \infty} v(m + 1, q) = \sum_{t=1}^{\infty} \sum_{t=1}^{\infty} h(s, t)W(f(s, t)) \) and by letting \( q \to \infty \) in (3.13), we have

\[
\frac{[v(m, n) - v(m + 1, n)]}{W(H(v(m + 1, n)))} \leq \sum_{t=n+1}^{\infty} h(m + 1, t)W(g(m + 1, t)). \tag{3.14}
\]

From (2.7) and (3.14), we have

\[
G(v(m, n)) - G(v(m + 1, n)) = \int_{v(m + 1, n)}^{v(m, n)} ds \frac{ds}{W(H(s))} \leq \frac{[v(m, n) - v(m + 1, n)]}{W(H(v(m + 1, n)))} \leq \sum_{t=n+1}^{\infty} h(m + 1, t)W(g(m + 1, t)). \tag{3.15}
\]

Now, keeping \( n \) fixed in (3.15), substituting \( m = s \), and taking the sum over \( s = m, m + 1, \ldots, p - 1, (p \geq m + 1 \) is arbitrary in \( N_0 \), we obtain

\[
G(v(m, n)) - G(v(p, n)) \leq \sum_{s=m+1}^{p} \sum_{t=n+1}^{\infty} h(s, t)W(g(s, t)). \tag{3.16}
\]

Noting that \( \lim_{p \to \infty} v(p, n) = \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} h(s, t)W(f(s, t)) \) and by taking \( p \to \infty \) in (3.16), we get

\[
v(m, n) \leq G^{-1} \left[ G \left( \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} h(s, t)W(f(s, t)) \right) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} h(s, t)W(g(s, t)) \right]. \tag{3.17}
\]
The required inequality in (2.6) follows from the fact that \( z(m, n) \leq v(m, n) \), (3.17) and (3.9). The subdomain \( 0 \leq m \leq m_1, 0 \leq n \leq n_1 \) is obvious.

(c) Define a function \( z(m, n) \) by

\[
z(m, n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} L(s, t, u(s, t)).
\]  \hspace{1cm} (3.18)

Then, from (2.10), we have

\[
u(m, n) \leq a(m, n) + b(m, n)z(m, n).
\]  \hspace{1cm} (3.19)

From (3.18), (3.19) and the hypotheses on \( L \), we observe that

\[
z(m, n) \leq \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[ L(s, t, a(s, t)) + b(s, t)z(s, t) \right. \\
- L(s, t, a(s, t)) + L(s, t, a(s, t)) \\
\leq e(m, n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} M(s, t, a(s, t))b(s, t)z(s, t),
\]  \hspace{1cm} (3.20)

where \( e(m, n) \) is defined by (2.12). Clearly \( e(m, n) \) is real-valued nonnegative and nonincreasing in each variable \( m, n \in \mathbb{N}_0 \). Now, an application of Theorem 1 part (a) to (3.20) yields

\[
z(m, n) \leq e(m, n) \prod_{s=m+1}^{\infty} \left[ 1 + \sum_{t=n+1}^{\infty} M(s, t, a(s, t))b(s, t) \right].
\]  \hspace{1cm} (3.21)

The desired inequality in (2.11) follows from (3.19) and (3.21).

(d) Define a function \( z(m, n) \) by (3.18), then from (2.16) we have

\[
u(m, n) \leq a(m, n) + b(m, n)\phi(z(m, n)).
\]  \hspace{1cm} (3.22)

From (3.18), (3.22) and the hypotheses on \( L \) and \( \phi \), it is easy to observe that

\[
z(m, n) \leq \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[ L(s, t, a(s, t)) + b(s, t)\phi(z(s, t)) \right. \\
- L(s, t, a(s, t)) + L(s, t, a(s, t)) \\
\leq e(m, n) \\
+ \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} M(s, t, a(s, t))\phi^{-1}(b(s, t)\phi(z(s, t))) \\
\leq e(m, n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} M(s, t, a(s, t))\phi^{-1}(b(s, t))z(s, t),
\]
where \( e(m, n) \) is defined by (2.12). Now, by following the last arguments as in the proof of \((c_1)\) we get the desired inequality in (2.17).

4. AN APPLICATION

In this section, we present an immediate application of Theorem 1 part \((a_1)\) to obtain the bound on the solution of a nonlinear sum-difference equation of the form,

\[
    u(m, n) = F(m, n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} B(m, n, s, t, u(s, t)),
\]

(4.1)

where \( u, F: N_0^2 \to R \), \( B: N_0^2 \times N_0^2 \times R \to R \) and

\[
    |F(m, n)| \leq a(m, n),
\]

(4.2)

\[
    |B(m, n, s, t, u)| \leq b(s, t)|u|,
\]

(4.3)

where \( a(m, n) \) and \( b(s, t) \) are as in Theorem 1 part \((a_1)\). Let \( u(m, n) \) be a solution of (4.1). From (4.1)–(4.3), we have

\[
    |u(m, n)| \leq a(m, n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s, t)|u(s, t)|.
\]

(4.4)

Now, an application of Theorem 1 part \((a_1)\) yields

\[
    |u(m, n)| \leq a(m, n) \prod_{s=m+1}^{\infty} \left[ 1 + \sum_{t=n+1}^{\infty} b(s, t) \right].
\]

(4.5)

The right-hand side of (4.5) gives an upper bound on the solution \( u(m, n) \) of (4.1) in terms of the known functions.

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