Norm Approximation by Polynomials in Some Weighted Bergman Spaces

Ali Abkar

Department of Mathematics, Imam Khomeini International University, P.O. Box 288, Qazvin 34194, Iran; and Department of Mathematics, Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-1795, Tehran, Iran
E-mail: Abkar@ipm.ir

Communicated by D. Sarason
Received November 2, 2000; revised August 1, 2001; accepted August 1, 2001

The polynomials are shown to be dense in weighted Bergman spaces in the unit disk whose weights are superbiharmonic and vanish in an average sense at the boundary. This leads to an alternative proof of the Aleman–Richter–Sundberg Beurling-type theorem for zero-based invariant subspaces in the classical Bergman space. Additional consequences are deduced.

1. INTRODUCTION

We denote by \( \mathbb{D} \) the unit disk, and by \( \mathbb{T} \) its boundary in the complex plane. A weight function, or simply a weight, in \( \mathbb{D} \) is any continuous positive function \( \omega: \mathbb{D} \to [0, +\infty[ \). The weighted Bergman space \( L^p_a(\mathbb{D}, \omega) \), \( 0 < p < +\infty \), consists of all analytic functions \( f \) on \( \mathbb{D} \) such that

\[
\|f\|_{L^p_a(\mathbb{D}, \omega)} = \left( \int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) \right)^{1/p} < +\infty,
\]

where \( dA(z) = \pi^{-1} \, dx \, dy \) is the normalized area measure on \( \mathbb{D} \). If \( 1 \leq p < +\infty \), it follows that \( L^p_a(\mathbb{D}, \omega) \) is a Banach space with the above norm, and for \( 0 < p < 1 \), it is a quasi-Banach space. For \( p = 2 \), \( L^2_a(\mathbb{D}, \omega) \) is a Hilbert space of analytic functions with the inner product

\[
\langle f, g \rangle_{L^2_a(\mathbb{D}, \omega)} = \int_{\mathbb{D}} f(z) \overline{g(z)} \omega(z) \, dA(z), \quad f, g \in L^2_a(\mathbb{D}, \omega).
\]

This research was supported in part by a grant from Institute for Theoretical Physics and Mathematics (IPM), Tehran, Iran.
In case that the weight $\omega$ is identically $1$, the corresponding space is called the Bergman space, and we write $L^p_\omega(\mathbb{D})$ for it, dropping the indication of the weight.

A function $u$ defined on $\mathbb{D}$ is said to be superbiharmonic provided that $\Delta u \geq 0$, where $\Delta$ stands for the Laplace operator in the complex plane:

$$\Delta = \Delta_z = \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad (z = x + iy).$$

In this paper, we shall consider a class of (non-radial) weights $\omega$ satisfying the following conditions:

1. $\Delta^2 \omega \geq 0$, the weight function $\omega$ is superbiharmonic, and
2. $\lim_{r \to 1} \int_{\mathbb{T}} \omega(rz) \, d\sigma(z) = 0$, where $d\sigma$ is the normalized arc-length measure on $\mathbb{T}$.

The main result of this paper (Theorem 2.6) states that the polynomials are dense in the weighted Bergman space $L^p_\omega(\mathbb{D})$, $0 < p < +\infty$, for weights satisfying the conditions (i) and (ii). The result was motivated by a question about invariant subspaces in the classical Bergman space $L^2(\mathbb{D})$. Indeed, the question of weighted polynomial approximation for weights satisfying the condition (i) together with a stronger condition than (ii) was raised by Hedenmalm in [4, p. 114]. The main result of this paper answers Hedenmalm’s question in the affirmative.

**Background.** A closed subspace $M$ of the Bergman space $L^p_\omega(\mathbb{D})$ is said to be invariant provided that $zM \subset M$. Assume that $M$ is an invariant subspace of $L^2(\mathbb{D})$ with index one; this means that the quotient space $M/zM$ has dimension one. In [6], Hedenmalm considered the following extremal problem,

$$\sup \{ \Re f^{(j)}(0) : f \in M, \|f\|_{L^2(\mathbb{D})} \leq 1 \},$$

where $j$ is the multiplicity of the common zero at the origin of all the functions in $M$. The unique solution to this problem is called the Hedenmalm extremal function for $M$ and is denoted by $\varphi_M$. The extremal functions in the Bergman spaces play the same role as the inner functions play in the Hardy spaces. Hedenmalm further showed that there is a function $\Phi_{\varphi_M} \in C(\mathbb{D}) \cap C^\omega(\mathbb{D})$ such that

1. $\Phi_{\varphi_M} = 0$ on $\mathbb{T}$ and $\Delta \Phi_{\varphi_M} = (|\varphi_M|^2 - 1)$ inside $\mathbb{D},$
2. $0 \leq \Phi_{\varphi_M}(z) \leq 1 - |z|^2$, for $z \in \mathbb{D}$.

A particularly simple class of invariant subspaces is the one characterized by a zero sequence; a subset $A \subset \mathbb{D}$ is said to be a Bergman space zero
sequence provided that there is a function in the Bergman space which vanishes precisely on $L$. Let $M$ be the invariant subspace of $L^2_a$ consisting of all functions which vanish on $A$, and let $\varphi_M$ be the corresponding extremal function for $M$. Now, the following question is natural: Does $\varphi_M$ generate the invariant subspace $M$?

Hedenmalm [6, Corollary 2.4] obtained the following reduction of the problem. Denoting by $[\varphi_M]$ the invariant subspace generated by $\varphi_M$, he found that

$$[\varphi_M] = \varphi_M \cdot \mathcal{A}^2(\varphi_M),$$

where $\mathcal{A}^2(\varphi_M)$ is the closure of the polynomials in the Hilbert space

$$\mathcal{A}^2(\varphi_M) = \left\{ f \in L^2_a : \int_D \Phi \varphi_M(z) A |f(z)|^2 dA(z) < +\infty \right\},$$

with the following norm

$$\|f\|_{\mathcal{A}^2(\varphi_M)}^2 = \|f\|_{L^2_a(D)}^2 + \int_D \Phi \varphi_M(z) A |f(z)|^2 dA(z).$$

By [6, Theorem 4.2], we know that $M \subset \varphi_M \cdot \mathcal{A}^2(\varphi_M)$. Now, the question if $\varphi_M$ generates $M$ becomes the question of whether the polynomials are dense in the space $\mathcal{A}^2(\varphi_M)$. We will show that the norm in $\mathcal{A}^2(\varphi_M)$ is comparable to

$$\|f\|_{\mathcal{A}^2(\varphi_M)}^2 = |f(0)|^2 + \int_D \omega \varphi_M(z) A |f(z)|^2 dA(z),$$

where $\omega \varphi_M(z) = (1 - |z|^2)^2 + \Phi \varphi_M(z)$. Hence the polynomials are dense in $\mathcal{A}^2(\varphi_M)$ if and only if they are dense in the weighted Bergman space $L^2_a(D, \omega \varphi_M)$. It follows that $\varphi_M$ generates $M$ if the polynomials are dense in the weighted Bergman space $L^2_a(D, \omega \varphi_M)$, where the weight function $\omega \varphi_M$ has the following properties:

1. $\Delta^2 \omega \varphi_M > 0$, and
2. $0 < \omega \varphi_M(z) \lesssim 2(1 - |z|^2)$.

As an application of Theorem 2.6, we prove that $\varphi_M$ generates the invariant subspace $M$ (see Theorem 3.1), a result which was proved in a similar fashion by Aleman, Richter, and Sundberg (see [2, Propositions 4.4, 4.5]). Indeed, they proved that the polynomials are dense in $\mathcal{A}^2(\varphi_M)$, so that $\mathcal{A}^2(\varphi_M) = \mathcal{A}^2(\varphi_M)$ and hence $[\varphi_M] = \varphi_M \cdot \mathcal{A}^2(\varphi_M)$. This answers the question of polynomial approximation for the weights $\omega \varphi_M$ considered above. They showed that

$$\int_D \Phi \varphi_M(z) A |f(rz)|^2 dA(z) \to \int_D \Phi \varphi_M(z) A |f(z)|^2 dA(z), \quad \text{as} \quad r \to 1,$$
the proof of which uses the representation formula

$$\Phi_{\varphi_M}(z) = \int_D \Gamma(z, \zeta) \, \partial \varphi_M(\zeta) \, dA(\zeta), \quad z \in \mathbb{D},$$

whereby $\Gamma(z, \zeta)$ is the \textit{biharmonic Green function} for the operator $\mathcal{A}^2$ in the unit disk:

$$\Gamma(z, \zeta) = \frac{|z - \zeta|^2}{|1 - \overline{z}\zeta|}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.$$  

Weights $\omega$ satisfying the conditions (i) and (ii) need not have as simple a representation as the function $\Phi_{\varphi_M}$ above. Nevertheless, according to a result due to the author and Hedenmalm [1], the weight function $\omega$ can be represented by

(1.1) \hspace{1cm} \omega(\zeta) = \int_D \Gamma(z, \zeta) \, \mathcal{A}^2 \omega(z) \, dA(z) + \int_T H(z, \zeta) \, d\mu(z), \quad \zeta \in \mathbb{D},

where $d\mu$ is a positive Borel measure on $T$ and

$$H(z, \zeta) = \frac{(1 - |\zeta|^2)^2}{|1 - \overline{z}\zeta|^2}, \quad (z, \zeta) \in \mathbb{T} \times \mathbb{D},$$

is called the \textit{harmonic compensator} (indeed, here we are using a special case of a representation formula established in [1]).

To prove the main approximation theorem (Theorem 2.6), we consider $\tilde{f}$—the dilation of $f$ by $r$—defined by $f_r(z) = f(rz)$ for $0 \leq r < 1$ and $z \in \mathbb{D}$. We then prove that $f_r \to f$ in norm, as $r \to 1$, that is,

$$\lim_{r \to 1} \int_D |f_r(z) - f(z)| \, \omega(z) \, dA(z) = 0,$$

from which the result follows.

2. AN APPROXIMATION THEOREM

In [2], the authors studied properties of a function which is closely related to the biharmonic Green function $\Gamma(z, \zeta)$, namely,

$$\tilde{F}(z, \zeta) = \frac{(1 - |z|^2)^2 (1 - |\zeta|^2)^2}{|1 - \overline{z}\zeta|^2}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.$$
Among other things, they proved that $\tilde{F}(z, \zeta)$ has some kind of monotonicity property. More precisely, for fixed $z \in \mathbb{D}$, the function $r \mapsto r\tilde{F}(z, \zeta/r)$ is increasing on the interval $([\zeta], 1)$ (see [2, p. 285]). In the following lemma, we shall see that for fixed $z \in \mathbb{T}$, the harmonic compensator

$$H(z, \zeta) = \frac{(1-|\zeta|^2)^2}{|1-\bar{z}\zeta|^2}, \quad (z, \zeta) \in \mathbb{T} \times \mathbb{D},$$

enjoys the same monotonicity property; the function $rH(z, \zeta/r)$ is an increasing function of $r$ for $|\zeta| < r < 1$. Here, it is important to note that the first argument in $\tilde{F}(z, \zeta)$ belongs to the unit disk, while the first argument in $H(z, \zeta)$ belongs to the unit circle. As a result, $\tilde{F}(z, \zeta)$ vanishes for $z \in \mathbb{T}$, while $H(z, \zeta)$ does not, meaning that these functions are quite different, that is, if we look at them as functions of $\zeta$, then $H(z, \zeta)$ is not a constant multiple of $\tilde{F}(z, \zeta)$. Therefore, the monotonicity property of the harmonic compensator does not follow from that of $\tilde{F}(z, \zeta)$. However, the argument made in [2] to establish the monotonicity property of $\tilde{F}(z, \zeta)$ works well in this situation; it is enough to verify that the derivative with respect to $r$ of the function $rH(z, \zeta/r)$ is positive, for $|\zeta| < r < 1$. For a proof based on this idea (attributed to the current author) see ([7, Lemma 3.13]).

**Lemma 2.1.** Let $z \in \mathbb{T}$ be fixed and consider

$$H(z, \zeta) = \frac{(1-|\zeta|^2)^2}{|1-\bar{z}\zeta|^2}, \quad \zeta \in \mathbb{D}.$$

Then for $0 < r < 1$ and $|\zeta| < r$, the function $r \mapsto rH(z, \zeta)$ is increasing in $r$.

**Proof.** We first make the following simple observation: if $Re \ a \leq b < r$, then

$$\left|\frac{r-a}{r-b}\right|^2 = \left|1 + \frac{b-a}{r-b}\right|^2 = 1 + \frac{|b-a|^2}{(r-b)^2} + \frac{2 \text{Re}(b-a)}{r-b}$$

is a decreasing function of $r$.

To prove the lemma, we note that

$$rH\left(z, \frac{\zeta}{r}\right) = \frac{(r^2-|\zeta|^2)^2}{r^2|r-\bar{\zeta}z|^2} = \frac{(r^2-|\zeta|^2)^2}{r^2\left|\frac{r-\bar{\zeta}z}{r-|\zeta|}\right|^2 + 4 |\zeta|\left|\frac{r-\bar{\zeta}z}{r-|\zeta|}\right|^2}.$$
According to the above observation, the denominator is a decreasing function of \( r \) (with \( a = \overline{z} \) and \( b = |z| < r \)), on the other hand, the numerator is an increasing function of \( r \), hence the lemma follows.

In the following lemma we collect some known facts on the biharmonic Green function. We then use this lemma to supply an alternative proof for the main result of this paper. We mention that parts (c) and (d) of Lemma 2.2 coincide with parts (a) and (b) of Lemma 2.3 in [2]; part (b) is rather well known. The merit here is the deduction of parts (b), (c), and (d) from a power series representation of \( I(z, \zeta) \) which is established by a direct computation.

**Lemma 2.2.** Let \( I(z, \zeta) \) denote the biharmonic Green function for the unit disk. Then:

(a) \( I(z, \zeta) \) has the power series representation

\[
I(z, \zeta) = \sum_{n=0}^{\infty} \frac{[(1-|z|)^2(1-|\zeta|^2)]^{n+2}}{(n+1)(n+2)|1-\overline{z}\zeta|^{2(n+1)}}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.
\]

(b) \( I(z, \zeta) > 0 \), for every \( (z, \zeta) \in \mathbb{D} \times \mathbb{D} \).

(c) For every \( (z, \zeta) \in \mathbb{D} \times \mathbb{D} \) we have

\[
\frac{1}{2} \frac{(1-|z|^2)^2(1-|\zeta|^2)^2}{|1-\overline{z}\zeta|^2} \leq I(z, \zeta) \leq \frac{(1-|z|^2)^2(1-|\zeta|^2)^2}{|1-\zeta|^2}.
\]

(d) For every \( (z, \zeta) \in \mathbb{D} \times \mathbb{D} \) we have

\[
\frac{1}{2} (1-|z|)^2(1-|\zeta|^2)^2 \leq I(z, \zeta) \leq (1+|z|)^2(1-|\zeta|^2)^2.
\]

**Proof.** First of all we recall the elementary identity

\[
1 - \left| \frac{z - \zeta}{1 - \overline{z}\zeta} \right|^2 = \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{z}\zeta|^2}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.
\]

**Proof of (a).** Putting

\[
x = \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\overline{z}\zeta|^2}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D},
\]

we get \( |z-\zeta|^2 = |1-\overline{z}\zeta|^2 (1-x) \). We now use the defining formula for the biharmonic Green function to obtain

\[
I(z, \zeta) = |1-\overline{z}\zeta|^2 (1-x) \log(1-x) + (1-|z|^2)(1-|\zeta|^2)
\]

\[
= |1-\overline{z}\zeta|^2 [(1-x) \log(1-x) + x].
\]
Now, a manipulation with the power series expansion of \( \log(1-x) \) on the right-hand side of (2.3) yields

\[
(1-x) \log(1-x) + x = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n} - \sum_{n=2}^{\infty} \frac{x^n}{n} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+1)(n+2)}.
\]

It then follows from (2.2), (2.3), and (2.4) that

\[
\Gamma(z, \zeta) = |1 - \zeta|^2 \sum_{n=0}^{\infty} \frac{(1 - |z|^2)^{n+2} (1 - |\zeta|^2)^{n+2}}{(n+1)(n+2) |1 - \zeta z|^{2(n+2)}}
\]

Hence (a) is proved. Part (b) follows immediately from (a). To verify (c), we note that the representing series of \( \Gamma(z, \zeta) \) in part (a) above consists of nonnegative terms, so that the first term for \( n = 0 \) can be served as a lower bound for the biharmonic Green function. Hence the first inequality in (c) follows. To obtain an upper bound for \( \Gamma(z, \zeta) \), we use part (a) to write

\[
\Gamma(z, \zeta) = \sum_{n=0}^{\infty} \frac{(1 - |z|^2)^{n+2} (1 - |\zeta|^2)^{n+2}}{(n+1)(n+2) |1 - \zeta z|^{2(n+1)}}.
\]

where \( x \) is given by (2.2). Since \( 0 \leq x \leq 1 \), it follows that the sum of the last series in (2.5) is no bigger than 1, so that the second inequality in (c) follows. As in [2], part (d) follows from part (c) and the inequalities

\[
\frac{1}{(1 + |z|)^2} \leq \frac{1}{|1 - \zeta z|^2} \leq \frac{1}{(1 - |z|)^2}.
\]

The proof is complete.

We now proceed to prove that the biharmonic Green function \( \Gamma(z, \zeta) \) has the same monotonicity property as the harmonic compensator \( H(z, \zeta) \), refuting a statement made in [2, p. 307].

**Lemma 2.3.** For \( 0 < r < 1, z \in \mathbb{D}, \) and \( |\zeta| < r \), the function \( r \Gamma(z, \zeta) \) is an increasing function of \( r \).
Proof. We use the following formula, due to Hadamard, to represent the biharmonic Green function $\Gamma(z, \zeta)$ in terms of the harmonic compensator $H(z, \zeta)$ (see [5, p. 74]):

$$
\Gamma(z, \zeta) = \frac{1}{\pi} \int_{\max \{|z|, |\zeta|\}}^{\infty} H \left( e^{\omega}, \frac{z}{s} \right) H \left( e^{\omega}, \frac{\zeta}{s} \right) s \, d\theta \, ds, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.
$$

Hence for $|\zeta| < r$, we have

$$
r \Gamma \left( \frac{z - \zeta}{r} \right) = \frac{1}{\pi} \int_{\max \{|z|, |\zeta|/r\}}^{\infty} H \left( e^{\omega}, \frac{z}{s} \right) r s H \left( e^{\omega}, \frac{\zeta}{rs} \right) d\theta \, ds.
$$

Note that the interval

$$\left[ \max \left\{ |z|, \frac{|\zeta|}{r} \right\}, 1 \right]
$$

gets bigger as $r$ increases. On the other hand, since $\frac{|\zeta|}{r} < s$, we have $|\zeta| < rs$, and hence

$$rs H \left( e^{\omega}, \frac{\zeta}{rs} \right)
$$
is an increasing function of $r$, according to Lemma 2.1. The proof is complete.

**Proposition 2.4.** Let $\omega$ be a superbiharmonic weight function satisfying the condition (ii). Then the function $r \omega(z/r)$, for $0 < r < 1$ and $|z| < r$, is an increasing function of $r$.

**Proof.** This is a direct consequence of the representation formula (1.1), Lemma 2.1, and Lemma 2.3.

We need the following well-known result [3, p. 21; 7, p. 66]:

**Lemma 2.5.** Suppose that $\mu$ is a finite positive measure on a measure space $X$, $0 < p < +\infty$, and $f_n, f$ are measurable functions such that

$$
\limsup_{n \to +\infty} \int_X |f_n|^p \, d\mu \leq \int_X |f|^p \, d\mu < +\infty,
$$

and $f_n \to f$ almost everywhere with respect to the measure $\mu$. Then

$$
\lim_{n \to +\infty} \int_X |f - f_n|^p \, d\mu = 0.
$$
We can now state the main result of this paper.

**Theorem 2.6 (0 < p < +∞).** Let \( \omega \) be a superbiharmonic weight function satisfying the condition (ii). Let \( f \) be a function in the weighted Bergman space \( L_p^*(D, \omega) \), and let \( f_r \) denote the dilation of \( f \) by \( r, 0 < r < 1 \). Then we have

\[
\|f_r - f\|_{L_p^*(D, \omega)} \to 0, \quad \text{as} \quad r \to 1.
\]

In particular, the polynomials are dense in the weighted Bergman space \( L_p^*(D, \omega) \).

**Proof.** Let \( f \) be a function in the weighted Bergman space \( L_p^*(D, \omega) \), \( 0 < p < +\infty \). Since for every \( 0 < r < 1 \) the dilated function \( f_r \) can be approximated in norm by the polynomials, we only prove that \( f_r \to f \) in norm as \( r \to 1 \). It is easy to see that \( f_r \to f \) pointwise, hence by Lemma 2.5, what we need to verify is that

\[
\|f_r\|_{L_p^*(D, \omega)} \to \|f\|_{L_p^*(D, \omega)}, \quad \text{as} \quad r \to 1.
\]

We start by writing

\[
\|f_r\|_{L_p^*(D, \omega)}^p = \int_D |f_r(z)|^p \omega(z) \, dA(z),
\]

and then replace \( z \) by \( z/r \) on the right side to obtain

\[
\|f_r\|_{L_p^*(D, \omega)}^p = \frac{1}{r^3} \int_D |f(z)|^p \frac{r \omega(\frac{z}{r})}{r} \, dA(z).
\]

By Proposition 2.4, the integrand is an increasing function of \( r \), and therefore the monotone convergence theorem applies:

\[
\|f_r\|_{L_p^*(D, \omega)} \to \|f\|_{L_p^*(D, \omega)}, \quad \text{as} \quad r \to 1.
\]

The proof is complete.

**An alternative proof for Theorem 2.6.** It is possible to prove Theorem 2.6 without any resort to Lemma 2.3. To do this, we use Fubini’s theorem and formula (1.1) to write

\[
(2.6) \quad \|f_r\|_{L_p^*(D, \omega)}^p = \int_D \left( \int_D |f_r(z)|^p \Gamma(z, \zeta) \, dA(z) \right) \, d\omega(\zeta) \, dA(\zeta)
\]

\[
+ \int_D \left( \int_D |f_r(z)|^p H(\zeta, z) \, dA(z) \right) \, d\mu(\zeta).
\]
Replacing $z$ by $z/r$, we see that

$$
\int_{\mathbb{T}} \int_{\mathbb{D}} |f(z)|^p H(\zeta, z) \, dA(z) \, d\mu(\zeta) = \frac{1}{r^3} \int_{\mathbb{T}} \int_{\mathbb{D}} |f(z)|^p H\left(\frac{z}{r}, z\right) \, dA(z) \, d\mu(\zeta).
$$

Together with Lemma 2.1 and the monotone convergence theorem, this implies that

$$
(2.7) \quad \lim_{r \to 1} \int_{\mathbb{T}} \int_{\mathbb{D}} |f(z)|^p H(\zeta, z) \, dA(z) \, d\mu(\zeta) = \int_{\mathbb{T}} \int_{\mathbb{D}} |f(z)|^p H(\zeta, z) \, dA(z) \, d\mu(\zeta).
$$

As in the proof of Theorem 2.6, we want to show that

$$
\|f\|_{L^p(D, \omega)} \to \|f\|_{L^p(D, \omega)}, \quad \text{as} \quad r \to 1.
$$

To this end, by (2.6) and (2.7), it suffices to prove that

$$
(2.8) \quad \lim_{r \to 1} \int_{\mathbb{T}} \int_{\mathbb{D}} |f(z)|^p \Gamma(z, \zeta) \, dA(z) \, \Delta^2 \omega(\zeta) \, dA(\zeta)
$$

$$
= \int_{\mathbb{T}} \int_{\mathbb{D}} |f(z)|^p \Gamma(z, \zeta) \, dA(z) \, \Delta^2 \omega(\zeta).
$$

Recall that by Lemma 2.2(c)

$$
(2.9) \quad \frac{1}{2} \tilde{F}(z, \zeta) \leq \Gamma(z, \zeta) \leq \tilde{F}(z, \zeta), \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D},
$$

where $\tilde{F}$ is given by

$$
\tilde{F}(z, \zeta) = \frac{(1-|z|^2)^2}{|1-\zeta|^2}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.
$$

Moreover, according to [2, p. 285], for $|z| < r$, the function $r \tilde{F}(\zeta, z)$ is an increasing function of $r$. It follows from (2.9) and a change of variables that

$$
(2.10) \quad \int_{\mathbb{D}} |f(z)|^p \Gamma(z, \zeta) \, dA(z) \, \Delta^2 \omega(\zeta) \leq \frac{1}{r^2} \int_{\mathbb{D}} |f(z)|^p \Gamma\left(\frac{z}{r}, \zeta\right) \, dA(z) \, \Delta^2 \omega(\zeta).
$$
We may assume that \( \frac{1}{2} \leq r < 1 \). It follows from (2.9), (2.10), the monotone convergence theorem, and Lemma 2.2 that
\[
\int_D |f_r(z)|^p \Gamma(z, \zeta) \, dA(z) A^2 \omega(\zeta) \leq 16 \int_D |f(z)|^p \Gamma(z, \zeta) \, dA(z) A^2 \omega(\zeta).
\]
This shows that the dominated convergence theorem can be applied to (2.8), so that the equality (2.8) follows if we prove that
\[
(2.11) \quad \lim_{r \to 1} \int_D |f_r(z)|^p \Gamma(z, \zeta) \, dA(z) = \int_D |f(z)|^p \Gamma(z, \zeta) \, dA(z).
\]
As for (2.11), we observe that by a change of variables,
\[
\int_D |f_r(z)|^p (1-|z|^2)^2 \, dA(z) = \frac{1}{r^2} \int_D |f(z)|^p (r^2-|z|^2)^2 \, dA(z),
\]
so that the monotone convergence theorem applies:
\[
\lim_{r \to 1} \int_D |f_r(z)|^p (1-|z|^2)^2 \, dA(z) = \int_D |f(z)|^p (1-|z|^2)^2 \, dA(z) < +\infty.
\]
Since
\[
F_r(z) = |f_r(z)|^p (1-|z|^2)^2 \to F(z) = |f(z)|^p (1-|z|^2)^2, \quad \text{as} \quad r \to 1,
\]
it follows from Lemma 2.5 (with \( p = 1 \)) that \( F_r \to F \) in \( L^1(\mathbb{D}, dA) \), as \( r \to 1 \); or
\[
\lim_{r \to 1} \int_D |f_r(z)|^p - |f(z)|^p \, (1-|z|^2)^2 \, dA(z) = 0.
\]
This, together with Lemma 2.2(d), implies that
\[
0 \leq \int_D |f_r(z)|^p - |f(z)|^p \, \Gamma(z, \zeta) \, dA(z)
\]
\[
\leq (1+|\zeta|)^2 \int_D |f_r(z)|^p - |f(z)|^p \, (1-|z|^2)^2 \, dA(z) \to 0, \quad \text{as} \quad r \to 1.
\]
It now follows that
\[
\left| \int_D (|f_r(z)|^p - |f(z)|^p) \Gamma(z, \zeta) \, dA(z) \right|
\]
\[
\leq \int_D |f_r(z)|^p - |f(z)|^p \, \Gamma(z, \zeta) \, dA(z) \to 0, \quad \text{as} \quad r \to 1.
\]
Hence (2.11) is proved. Combining (2.11), (2.8), (2.7) and (2.6) we conclude that
\[ \| f_r \|_{L^p(D, \omega)} \to \| f \|_{L^p(D, \omega)}, \quad \text{as } r \to 1, \]
which completes the alternative proof of Theorem 2.6.

Remark 2.7. In case that the weight function \( \omega \) is radial, that is, \( \omega \) depends only on \( |z| \), the statement that the polynomials are dense in such weighted Bergman spaces is well-known; for a proof see [8, p. 343; 9, p. 131].

3. SOME APPLICATIONS

As a consequence of Theorem 2.6, we have:

**Theorem 3.1** (Aleman–Richter–Sundberg). Let \( A \) be a zero sequence in the Bergman space \( L^2_a(D) \), and let \( M \) be the invariant subspace of \( L^2_a(D) \) consisting of all functions vanishing on \( A \). Then \( M \) is generated by its extremal function \( \varphi_M \).

**Proof.** Let \( [\varphi_M] \) denote the invariant subspace of \( L^2_a(D) \) generated by \( \varphi_M \). According to [6, Corollary 2.4] we know that \( [\varphi_M] = \varphi_M \cdot \mathcal{A}(\varphi_M) \), where \( \mathcal{A}(\varphi_M) \) is the closure of the polynomials in the space
\[ \mathcal{A}(\varphi_M) = \left\{ f \in L^2_a(D) : \int_D \Phi_{\varphi_M}(z) dA(z) < \infty \right\}, \]
with the norm
\[ \| f \|_{\mathcal{A}(\varphi_M)} = \| f \|_{L^2_a(D)} + \int_D \Phi_{\varphi_M}(z) dA(z). \]
Recall that \( \Phi_{\varphi_M} \) was defined in Section 1. It is clear that \( [\varphi_M] \subset M \), so what we need to know is the inclusion
\[ M \subset [\varphi_M] = \varphi_M \cdot \mathcal{A}(\varphi_M). \]
According to [6, Theorem 4.2], we have \( M \subset \varphi_M \cdot \mathcal{A}(\varphi_M) \), therefore it suffices to show that \( \mathcal{A}(\varphi_M) = \mathcal{A}(\varphi_M) \), that is the polynomials are dense in the space \( \mathcal{A}(\varphi_M) \). To this end, we shall verify that the norm in \( \mathcal{A}(\varphi_M) \) is comparable to
\[ \| f \|_{\mathcal{A}(\varphi_M)}^2 = |f(0)|^2 + \int_D \omega_{\varphi_M}(z) dA(z), \quad (3.1) \]
where 
\[ \omega_{\phi_M}(z) = (1 - |z|^2)^2 + \Phi_{\phi_M}(z). \]

It then follows from Theorem 2.6 that the polynomials are dense in the weighted Bergman space \( L^2_{\alpha}(D, \omega_{\phi_M}) \), because the weight \( \omega_{\phi_M} \) is super-biharmonic and \( 0 \leq \omega_{\phi_M} \leq 2(1 - |z|^2) \). We start by verifying that for \( f \in L^2_{\alpha}(D) \) with \( f(0) = 0 \) the following estimates hold:

\[
\frac{3}{2} \left\| f \right\|_{L^2_{\alpha}(D)}^2 \leq \int_D (1 - |z|^2)^2 A |f(z)|^2 dA(z) \leq 2 \left\| f \right\|_{L^2_{\alpha}(D)}^2.
\]

Indeed, an application of Green’s formula yields

\[
\int_D \frac{(1 - |z|^2)^2}{2} A |f(z)|^2 dA(z)
= \int_D (2 |z|^2 - 1) |f(z)|^2 dA(z)
= 2 \left\| zf \right\|_{L^2_{\alpha}(D)}^2 - \left\| f \right\|_{L^2_{\alpha}(D)}^2 \leq 2 \left\| f \right\|_{L^2_{\alpha}(D)}^2 - \left\| f \right\|_{L^2_{\alpha}(D)}^2 = \left\| f \right\|_{L^2_{\alpha}(D)}^2.
\]

Hence the right-hand side inequality in (3.2) follows. As for the left-hand side inequality in (3.2), we note that for \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) (here we use the fact that \( f(0) = 0 \)) we have

\[
\frac{|a_n|^2}{n+1} \leq \frac{3}{2} \frac{|a_n|^2}{n+2}, \quad \text{for} \quad n \geq 1.
\]

This implies that

\[
\frac{4}{3} \left\| f \right\|_{L^2_{\alpha}(D)}^2 \leq 2 \left\| zf \right\|_{L^2_{\alpha}(D)}^2,
\]

and finally,

\[
\int_D \frac{(1 - |z|^2)^2}{2} A |f(z)|^2 dA(z) = 2 \left\| zf \right\|_{L^2_{\alpha}(D)}^2 - \left\| f \right\|_{L^2_{\alpha}(D)}^2 \geq \frac{4}{3} \left\| f \right\|_{L^2_{\alpha}(D)}^2 - \left\| f \right\|_{L^2_{\alpha}(D)}^2 = \frac{1}{3} \left\| f \right\|_{L^2_{\alpha}(D)}^2.
\]
Hence (3.2) is proved. We now consider the function
\[ U(z) = -\log |z|^2 + |z|^2 - 1, \quad z \in \mathbb{D}. \]
Note that \( U(z) > 0 \) for every \( z \in \mathbb{D} \). It follows from Green’s formula that for \( f \in L^2_+(\mathbb{D}) \) we have
\[ \|f\|^2_{L^2_+(\mathbb{D})} = \|f(0)\|^2 + \int_{\mathbb{D}} U(z) \Delta |f(z)|^2 \, dA(z). \]  
(3.3)
It now follows from (3.2) and (3.3) that for every \( f \in L^2_+(\mathbb{D}) \) with \( f(0) = 0 \) the following holds:
\[ \frac{3}{4} \int_{\mathbb{D}} U(z) \Delta |f(z)|^2 \, dA(z) \leq \int_{\mathbb{D}} (1 - |z|^2)^2 \Delta |f(z)|^2 \, dA(z) \]
\[ \leq 2 \int_{\mathbb{D}} U(z) \Delta |f(z)|^2 \, dA(z). \]  
(3.4)
Replacing \( f \) by \( f - f(0) \) in (3.4), we see that (3.4) is valid for every \( f \in L^2_+(\mathbb{D}) \). We now use (3.3) and (3.4) to obtain
\[ \|f\|^2_{L^2_+(\mathbb{D})} = \|f(0)\|^2 + \int_{\mathbb{D}} U(z) \Delta |f(z)|^2 \, dA(z) \]
\[ = \|f(0)\|^2 + \int_{\mathbb{D}} U(z) \Delta |f(z)|^2 \, dA(z) + \int_{\mathbb{D}} \Phi_{\phi v}(z) \Delta |f(z)|^2 \, dA(z) \]
\[ \leq \frac{3}{2} \left\{ |f(0)|^2 + \int_{\mathbb{D}} ((1 - |z|^2)^2 + \Phi_{\phi v}(z)) \Delta |f(z)|^2 \, dA(z) \right\} \]
\[ = \frac{3}{2} \|f\|^2_{L^2_+(\mathbb{D})}. \]
(3.5)
Similarly, we prove that
\[ \|f\|^2_{L^2_+(\mathbb{D})} \geq \frac{1}{2} \|f\|^2_{L^2_+(\mathbb{D})}, \]
proving that the norm in \( L^2_+(\mathbb{D}) \) is equivalent to \( \|\cdot\| \) given by (3.1). This completes the proof.

Lemma 2.3 has further applications. It is shown in [2, Proposition 2.6] that for a given positive function \( v \) in the unit disk, and \( 0 \leq r \leq 1 \), the inequality
\[ I^r [v](z) \leq 2 I [v](z) \]
holds throughout \( \mathbb{D} \), where \( v \) denotes the dilation of \( v \) by \( r \), and \( I [v] \) is defined as
\[ I [v](z) = \int_{\mathbb{D}} I(z, \zeta) v(\zeta) \, dA(\zeta), \quad z \in \mathbb{D}. \]
We sharpen this estimate by dropping the constant 2 in the following proposition:

**Proposition 3.2.** Let \( v \) be a nonnegative function in \( D \). Then for \( 0 \leq r \leq 1 \) and \( z \in D \), we have

\[
\Gamma[r^3v](z) \leq \Gamma[v](z).
\]

**Proof.** Together with Lemma 2.3, a change of variables gives

\[
\Gamma[r^3v](z) = \int_D \Gamma(z, \zeta) r^3v(\zeta) \, dA(\zeta) = \int_D r\Gamma(z, \zeta) v(\zeta) \, dA(\zeta)
\]

\[
\leq \int_D \Gamma(z, \zeta) v(\zeta) \, dA(\zeta) = \Gamma[v](z),
\]

which is the desired result.

The following definition of a Bergman space inner function is now standard (see [2, 6]).

**Definition 3.3.** An analytic function \( \varphi \) in the unit disk is said to be an \( L^p(\mathbb{D}) \)-inner function provided that

\[
\int_0^1 |\varphi(z)|^p \, h(z) \, dA(z) = h(0),
\]

for every bounded harmonic function \( h \) in \( \mathbb{D} \).

In a Banach space of analytic functions \( X \), the dilation operator \( R_s : X \rightarrow X, 0 \leq s \leq 1 \), is defined by \( R_s f = f_s \), where \( f_s \) is the dilation of \( f \) by \( s \).

Let \( \varphi \) be an \( L^p(\mathbb{D}) \)-inner function, \( 0 < p < +\infty \). The closure of the polynomials in the weighted space \( L^p(\mathbb{D}, |\varphi|^p) \) is denoted by \( \mathcal{P}^p(|\varphi|^p) \). The authors in [2, Section 6] proved that for every even integer \( 2 \leq p < +\infty \), the operator \( R_s \) in the space \( \mathcal{P}^p(|\varphi|^p) \) is a contraction. If we define \( \tilde{R}_s(f) = sf_s \), for \( f \in \mathcal{P}^p(|\varphi|^p) \), it then follows from Lemma 2.3 that \( \tilde{R}_s \) is a contraction for all \( 1 \leq p < +\infty \).

**Theorem 3.4** (\( 1 \leq p < +\infty \)). Let \( \varphi \) be an \( L^p(\mathbb{D}) \)-inner function. Then the operator \( \tilde{R}_s(f) = sf_s \) is a contraction in \( \mathcal{P}^p(|\varphi|^p) \), the closure of the polynomials in the weighted space \( L^p(\mathbb{D}, |\varphi|^p) \).
Proof. Suppose \( f \in \mathcal{P}(|\varphi|^p) \). It follows from the definition that there exists a sequence of polynomials \( \{p_n\}_{n=1}^{\infty} \) such that

\[
\int_{\mathbb{D}} |p_n(z)|^p |\varphi(z)|^p \, dA(z) \to \int_{\mathbb{D}} |f(z)|^p |\varphi(z)|^p \, dA(z), \quad \text{as} \quad n \to +\infty.
\]

This means that \( f \varphi \in [\varphi] \), the invariant subspace of \( L_p^s(\mathbb{D}) \) generated by the inner function \( \varphi \). It is known \([2]\) that

\[
\|f\varphi\|_{L_p^s(\mathbb{D})} = \|f\|_{L_p^s(\mathbb{D})} + \int_{\mathbb{D}} \Gamma(z, \zeta) A_t |\varphi(z)|^p A_t |f(\zeta)|^p \, dA(z) \, dA(\zeta). \tag{3.5}
\]

Replacing \( f \) by \( f_s \) in (3.5), we obtain

\[
\|f_s\varphi\|_{L_p^s(\mathbb{D})} = \|f_s\|_{L_p^s(\mathbb{D})} + \int_{\mathbb{D}} \Gamma(z, \zeta) A_t |\varphi(z)|^p A_t |f_s(\zeta)|^p \, dA(z) \, dA(\zeta). \tag{3.6}
\]

We now replace \( \zeta \) by \( \frac{z}{s} \) in the last integral, and then multiply both sides by \( s \) to get

\[
s \|f_s\varphi\|_{L_p^s(\mathbb{D})} = s \|f_s\|_{L_p^s(\mathbb{D})} + \int_{\mathbb{D}} s \Gamma \left( z, \frac{\zeta}{s} \right) A_t |\varphi(z)|^p A_t |f(\zeta)|^p \, dA(z) \, dA(\zeta). \tag{3.7}
\]

It follows from Lemma 2.3 and the monotone convergence theorem that the second term on the right side of (3.7) is no bigger than the second term on the right side of (3.5). This together with the fact that for \( 0 < s < 1 \) we have \( s \|f_s\|_{L_p^s(\mathbb{D})} \leq \|f\|_{L_p^s(\mathbb{D})} \), implies that

\[
\|sf_s\varphi\|_{L_p^s(\mathbb{D})} \leq \|f\varphi\|_{L_p^s(\mathbb{D})}, \quad \text{for} \quad 0 < s < 1.
\]

This completes the proof.

ACKNOWLEDGMENTS

First of all, I express my sincere thanks to professor Håkan Hedenmalm (Lund). I am also grateful to the editor, Professor Donald Sarason (Berekely), whose invaluable suggestions improved the presentation of this paper. During the preparation of this paper I have had some oral/written communications with the following people: Alexander Borichev (Bordeaux), Stefan Richter (Tennessee), Stefan Jakobsson (KTH, Stockholm), and Alexandru Aleman (Lund). Here, I record my thanks to all of them.
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