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The Igusa Local Zeta Function Associated with the Singular Cases of the Determinant and the Pfaffian

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This paper describes the theory of the Igusa local zeta function associated with

quaternion division algebra over K.

1. INTRODUCTION

To an arbitrary polynomial $f(x)$ in n variables with coefficients in a local field K we associate a distribution $|f|^s$ on K, called the "complex power" of $f(x)$ as

$$|f|^s(\Phi) = \int_{K^n} |f(x)|_K^s \, \Phi(x) \, dx,$$

in which $|\cdot|_K$ is an absolute value in K, Φ is a Schwartz–Bruhat function, and dx is a Haar measure on K^n. The complex parameter s above is restricted to the right half plane and a fundamental theorem states that $|f|^s$ has a meromorphic continuation to the whole s-plane. Furthermore, if K is a p-adic field with q as the cardinality of its residue field, then $|f|^s(\Phi)$ is a rational function of $t = q^{-s}$. This theorem was proved by Atiyah, Bernstein, S. I. Gel'fand, and Igusa in several papers published between 1969 and 1975 [1, 3, 9]. In the p-adic case, these complex powers are called Igusa local zeta functions. Any discussion of developments in this field should also mention the earlier works of Gel'fand and Shilov [7] in which this theorem was proved for a quite general $f(x)$ and the works of Sato and others on prehomogeneous vector spaces [18, 19].

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In the p-adic case the theory of complex powers is not as well understood as it is in the Archimedean case. For example, in the Archimedean case the real poles of $|f|^s(\Phi)$ are known to be the zeros of the Bernstein polynomial [2] and hence by Malgrange [14] related to an eigenvalue of the local monodromy of f. Igusa has conjectured a similar relationship in the p-adic case [12]. For an excellent survey of the conjectures and results surrounding the Igusa local zeta function, please see Denef's report [4] and the work of Meuser [15, 16, 17]. Motivated by the need to have a better understanding of the p-adic case, Igusa has determined the local zeta function $Z(t)=|f|^s(\Phi)$ for a large number of group invariants $f(x)$, where $\Phi=\phi_{X^0}$ is the characteristic function of the lattice of integral points of K^n. In this paper, results are given where $f(x)$ is the determinant of a Hermitian matrix of degree m with coefficients in: (1) a ramified quadratic extension of K; and (2) the unique quaternion division algebra over K.

These two cases complete the determination of local zeta functions under the following classification. Let C be a composition algebra over a number field F, denote by X the vector space of Hermitian matrices of degree m with coefficients in C and by $f(x)$ the determinant (or the generic norm) of X. For any p-adic completion F_v of F with a residue field of q_v elements, denote the lattice of integral points of $X_v=X\otimes F_v$ by X_v^0. In this situation the local zeta function associated to $f(x)$ is

$$Z(t)=\int_{X_v^0}|f(x)|_v^s\,dx,$$

in which $|\cdot|_v$ is the absolute value on F_v, dx is a Haar measure on X_v and s is a complex variable in the right half plane. By the general theorem mentioned above, $Z(t)$ is always a rational function of $t=q_v^{-s}$. Under this classification $Z(t)$ has been determined for almost all v [10], i.e. excluding a finite number of singular v's. By a classical theorem [13] there are four types of C; they are F itself, a quadratic extension of F, a quaternion algebra over F, and an octonion algebra over F. Of these types (1) and (2) above (where $K=F_v$) are the singular cases and the determination of the rational function $Z(t)$ in these cases completes the determination of $Z(t)$ for all v.

In both cases, recursion formulae for $Z(t)=Z_m(t)$ are obtained and the $Z_m(t)$ are determined for all m. In so doing, the proofs use a classical identity of Gauss. In case (2), it is natural to consider a similarly defined $Z_m^*(t)$ in which the original Φ is replaced by its Fourier transform. And, indeed, two recursion formulae are obtained involving $Z_m(t)$ and $Z_m^*(t)$. In addition in case (2), we verify a functional equation which states that up to sign $Z_m^*(q^{2m-1}t)$ and $Z_m(t^{-1})$ differ by a product of m Tate local gamma factors. This functional equation is evidence for Igusa's sign conjecture for vector spaces over central division algebras [11].

2. Igusa's Key Lemma

In this section Igusa's "Key Lemma" [10] is introduced. Let p be any arbitrary prime number and $\mathbf{Q}_p$ the Hensel p-adic field. In this paper K will denote a p-adic local field (a finite algebraic extension of $\mathbf{Q}_p$). The ring of integers of K will be denoted by O_K and the unique maximal ideal of O_K will be denoted by P_K. If we fix an element π_K in $P_K - P_K^2$ then π_K generates $P_K = \pi_K O_K$ in O_K. We let $U_K = O_K - \pi_K O_K$ be the group of units in O_K ("-" denotes set complement). Every element x of $K^\times = K - \{0\}$ can be uniquely expressed as $x = \pi_K^e u$ where e is an integer called the order of x and u is an element of U_K. The absolute value on K is the usual one: $|x|_K = |\pi_K^e u|_K = q^{-e}$ where q is the cardinality of the finite field O_K/P_K and $|0|_K = 0$. We take as dx the Haar measure on K^n normalized so that the measure of O_K is 1 and $d(\pi_K x) = q^{-1}\, dx$.

As explained in the introduction, we are concerned with the calculation of $Z(s)$; however, since $|x|_K^s = |\pi_K^e u|_K^s = s^{-se}$ for all $x \in K$ and s a complex number, we let $t = q^{-s}$ and think of $Z(s)$ as a function of t.

Key Lemma [10]. *Let $f(x) \in O_K[x_1, ..., x_n]$, $f(x)$ homogeneous of degree m, G a connected K-subgroup such that $f(g \cdot x) = \nu(g) f(x)$ for all $g \in G$ and ν a rational character of G. Let $G^0 = G(O_K) = G(K) \cap GL_n(O_K)$ and $\bar{G}^0 = G(\mathbf{F}_q)$ is the image of G under the cannonical map $GL_n(O_K) \to GL_n(\mathbf{F}_q)$. Let $R =$ a subset of points $x_o \in O_K$ such that $\mathbf{F}_q^n$ is a disjoint union of $\bar{G}^0 \cdot \bar{x}_0$ over all x_0 in R then*

$$Z(t) = \int_{O_K^n} |f(x)|_K^s\, dx = \frac{1}{1 - q^{-n} t^m} \sum_{x_0 \in R,\, \bar{x}_0 \notin 0} |\bar{G}^0 \cdot \bar{x}_0| \int_{x_0 + \pi_K O_K^n} |f(x)|_K^s\, dx$$

3. Ramified Case

Let $X = H_m(C)$ where $C = K'$ is a ramified quadratic extension of K. As K' is a quadratic extension of K, we have the natural involution on K' and can form Hermitian matricies over K', $H_m(K')$. $O_{K'} = \{a \in K' \mid |a|_{K'} \leqslant 1\}$ is the ring of integers in K'. $P_{K'} = \{a \in K' \mid |a|_{K'} < 1\}$ and $U_{K'} = \{a \in K' \mid |a|_{K'} = 1\}$ are the unique maximal ideal of $O_{K'}$ (the ideal of non-units) and the group of units in $O_{K'}$, respectively. If we choose and fix $\pi_{K'}$ in $P_{K'} - P_{K'}^2$ then $\pi_{K'}$ generates $P_{K'}$ in $O_{K'}$ and $\pi_{K'}^2$ and π_K differ at most by a unit. Hence, $O_K/P_K = O_{K'}/P_{K'}$ and if q is the cardinality of both residue fields then $|\pi_{K'}|_{K'} = |\pi_K|_K^{1/2}$. K' is complete with respect to the absolute value $|\cdot|_{K'}$. In this case, we will assume that 2 does not divide q to get the simpler orbital decomposition in equation (3.1). Under these conditions, we will calculate

$$Z(t) = \int_{X^0} |f(x)|_K^s \, dx,$$

where $C^0 = O_{K'}$, $X^0 = H_m(C^0)$, $f(x) = \det(x)$, $\mu(O_{K'}) = \mu(O_K) = 1$, $d(\pi_{K'} x) = q^{-1/2}\, dx$, and G^0 is the image of $GL_m(C^0)$ in $GL(X^0)$ under the map $g \to$ "$x \to g \cdot x = gx{}^tg'$", in which ${}^tg'$ is the Hermitian adjoint of g.

3.1. *Orbital Decomposition*

Since $O_K/\pi_K O_K$ and $O_{K'}/\pi_{K'} O_{K'}$ are both isomorphic to the finite field with q elements, $\mathbf{F}_q$, if we let $H_m(\pi_{K'}^{-1} O_{K'})$ denote the set of Hermitian matrices of X with diagonal entries in O_K and off-diagonal entries in $\pi_{K'}^{-1} O_{K'}$, there is the isomorphism

$$H_m(O_{K'})/\pi_K H_m(\pi_{K'}^{-1} O_{K'}) \cong H_m(\mathbf{F}_q).$$

Before applying the Key Lemma, we need to determine the orbital structure of $H_m(\mathbf{F}_q)$ and $\pi_K H_m(\pi_{K'}^{-1} O_{K'})$ under the action of $\bar{G} = GL_m(\mathbf{F}_q)$ where $\bar{G} = G^0 \bmod \pi_{K'}$. By the diagonalization of quadratic forms [5, p. 156], we have the following decomposition of $H_m(\mathbf{F}_q)$ into disjoint orbits when 2 does not divide q:

$$H_m(\mathbf{F}_q) = \{0\} \cup \left[\bigcup_{k=1}^{m} \left\{ \bar{G} \cdot \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix} \cup \bar{G} \cdot \begin{pmatrix} 1_{k-1} & 0 & 0 \\ 0 & \bar{\varepsilon} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \right]; \quad (3.1)$$

here $\bar{\varepsilon}$ is in $\mathbf{F}_q^\times$ and is not a square.

To decompose $\pi_K H_m(\pi_{K'}^{-1} O_{K'})$ into its orbits, write any $x \in \pi_K H_m(\pi_{K'}^{-1} O_{K'})$ as $x = \pi_{K'} A + \pi_K B$ where $B \in H_m(O_{K'})$ and $A \in \mathrm{Alt}_m(O_K) - \pi_K \mathrm{Alt}_m(O_K)$, the alternating or skew-symmetric matrices. Clearly,

$$\pi_K H_m(\pi_{K'}^{-1} O_{K'})/\pi_K H_m(O_{K'}) \cong \pi_{K'} \mathrm{Alt}_m(\mathbf{F}_q).$$

The orbital decomposition of $Alt_m(\mathbf{F}_q)$ into disjoint orbits is known [8] to be

$$\mathrm{Alt}_m(\mathbf{F}_q) = \{0\} \cup \left\{ \bigcup_{k=1}^{[m/2]} \bar{G} \cdot \begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad (3.2)$$

where $[\cdot]$ is the Gauss symbol or the greatest integer function and E_k is the $(2k \times 2k)$ block matrix with k copies of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ down the main diagonal and zeros elsewhere.

3.2. *Cardinality of the orbits*, $|\bar{G} \cdot \overline{x_0}|$.

The cardinality of each orbit $|\bar{G} \cdot \overline{x_0}| = |\bar{G}|/|\bar{H}|$ where $\overline{x_0}$ is the orbit representative and $\bar{H}$ is the stabilizer of $\overline{x_0}$ in $\bar{G}$. Letting $(i) = (1 - q^{-i})$, we have the following formulae of Dickson [5, pages 78, 160, 94]:

$$|GL_m(\mathbf{F}_q)| = q^{m^2} \prod_{i=1}^{m} (i), \qquad |Sp_{2r}(\mathbf{F}_q)| = q^{r(2r+1)} \prod_{i=1}^{r} (2i),$$

$$|SO_m(\mathbf{F}_q)| = q^{m(m-1)/2} \begin{cases} \displaystyle\prod_{i=1}^{(m-1)/2} (2i) & m \text{ odd} \\ \displaystyle(1 - \chi(d)\, q^{-m/2}) \prod_{i=1}^{m/2-1} (2i) & m \text{ even} \end{cases};$$

here $d = (-1)^{m(m-1)/2} \det(\textit{coefficient matrix})$ and χ is the unique non-trivial quadratic character on $\mathbf{F}_q$.

To compute the cardinality of the orbits in equation (3.1), let

$$\overline{x_0} = \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1_{k-1} & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $|\bar{G} \cdot \overline{x_0}|$ be the sum of the cardinalities of the orbits for these two rank k representatives. Then take $\bar{g} = \left(\begin{smallmatrix} g_1 & g_{12} \\ g_{21} & g_2 \end{smallmatrix}\right) \in \bar{H}$ then $g_1 x_0^t g_1 = x_0$ and $g_{21} = (0)$. Therefore, $g_1 \in O_k(x_0)(\mathbf{F}_q)$, $g_{12} \in \mathrm{Mat}_{k, m-k}(\mathbf{F}_q)$ and $g_2 \in GL_{m-k}(\mathbf{F}_q)$. Thus, $|\bar{H}| = |GL_{m-k}(\mathbf{F}_q)|\, q^{k(m-k)}\, |O_k(x_0)(\mathbf{F}_q)|$ and the sum of the cardinalities of the orbits for the two rank k representatives is

$$\frac{1}{2}\left[\frac{|GL_m(\mathbf{F}_q)|}{|GL_{m-k}(\mathbf{F}_q)|\, q^{k(m-k)}\, |SO_k^{\chi(d)=1}(\mathbf{F}_q)|} + \frac{|GL_m(\mathbf{F}_q)|}{|GL_{m-k}(\mathbf{F}_q)|\, q^{k(m-k)}\, |SO_k^{\chi(d)=-1}(\mathbf{F}_q)|}\right],$$

since $SO_k(\mathbf{F}_q)$ is a subgroup of index 2 in $O_k(x_0)(\mathbf{F}_q)$. By Dickson's formulae,

$$|\bar{G} \cdot \overline{x_0}| = q^{-k(k-2m-1)/2} \frac{\prod_{i=1}^{k} (m-k+i)}{\prod_{j=1}^{[k/2]} (2j)}. \tag{3.3}$$

To compute the orbits of equation (3.2), let $\overline{x_o} = \pi_{K'} \cdot \left(\begin{smallmatrix} E_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Take $\bar{g} = \left(\begin{smallmatrix} g_1 & g_{12} \\ g_{21} & g_2 \end{smallmatrix}\right) \in \bar{H}$ then $g_1 E_r^t g_1 = E_r$ and $g_{21} = (0)$. Therefore, $g_1 \in Sp_{2r}(\mathbf{F}_q)$, $g_{12} \in Mat_{2r, m-2r}(\mathbf{F}_q)$ and $g_2 \in GL_{m-2r}(\mathbf{F}_q)$. Thus, $|\bar{H}| = |GL_{m-2r}(\mathbf{F}_q)|$ $q^{2r(m-2r)}$ $|SP_{2r}(\mathbf{F}_q)|$ and

$$|\bar{G} \cdot \overline{x_0}| = \frac{|GL_m(\mathbf{F}_q)|}{|GL_{m-2r}(\mathbf{F}_q)|\, q^{2r(m-2r)}\, |SP_{2r}(\mathbf{F}_q)|}.$$

By Dickson's formulae,

$$|\bar{G} \cdot \overline{x_0}| = q^{r(2m-2r-1)} \frac{\prod_{i=1}^{2r} (m-2r+i)}{\prod_{l=1}^{r} (2l)}. \tag{3.4}$$

3.3. *Two Partial Integrals and a formula for* $Z_m(t)$

LEMMA 1 (First Partial Integral). *For* $\alpha_1, \alpha_2, ..., \alpha_k \in O_K - \pi_K O_K$ $(0 \leqslant k \leqslant m)$

$$I_{m,k} = \int_{\mathrm{diag}(\alpha_1, ..., \alpha_k, 0) + \pi_K H_m(\pi_{K'}^{-1} O_{K'})} |det(x)|_K^s \, dx$$

then $I_{m,k} = q^{-m} I_{m-1,k-1}$.

Remark. By repeated application of this lemma

$$I_{m,k} = q^{k(k-2m-1)/2} \int_{\pi_K H_{m-k}(\pi_{K'}^{-1} O_{K'})} |\det(x)|_K^s \, dx.$$

Proof. For any $x \in \mathrm{diag}(\alpha_1, ..., \alpha_k, 0) + \pi_K H_m(\pi_{K'}^{-1} O_{K'})$, we can write $x = \left(\begin{smallmatrix} \alpha & {}^t y' \\ y & z \end{smallmatrix}\right)$ where $\alpha \in O_K - \pi_K O_K$, $y \in \pi_{K'} O_{K'}^{m-1}$ and $z \in \mathrm{diag}(\alpha_2, ..., \alpha_k, 0) + \pi_K H_{m-1}(\pi_{K'}^{-1} O_{K'})$. Diagonalize x as

$$x = \begin{pmatrix} 1 & 0 \\ \alpha^{-1} y & 1_{m-1} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & z^* \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\, {}^t y' \\ 0 & 1_{m-1} \end{pmatrix},$$

where $z^* = z - \alpha^{-1} y\, {}^t y' \equiv z \bmod \pi_K$. Returning to the partial integral,

$$I_{m,k} = \int_{(\alpha + \pi_K O_K) \times \pi_{K'} O_{K'}^{m-1}} \left\{ \int_{\mathrm{diag}(\alpha_2, ..., \alpha_k, 0) + \pi_K H_{m-1}(\pi_{K'}^{-1} O_{K'})} \left| \det \begin{pmatrix} \alpha & {}^t y' \\ y & z \end{pmatrix} \right|_K^s dz \right\} d\alpha\, dy.$$

By the diagonalization, the fact that $dz = dz^*$ and the fact that α is a unit, the expression above in curly brackets is exactly $I_{m-1,k-1}$. And we see that $I_{m,k} = I_{m-1,k-1} \mathrm{vol}(\alpha + \pi_K O_K) \mathrm{vol}(\pi_{K'} O_{K'}^{m-1}) = q^{-m} I_{m-1,k-1}$. ∎

LEMMA 2 (Second Partial Integral). *If* $0 \leqslant r \leqslant [m/2]$ *and*

$$J_{m,2r} = \int_{\pi_{K'}\left(\begin{smallmatrix} E_r & 0 \\ 0 & 0 \end{smallmatrix}\right) + \pi_K H_m(O_{K'})} |\det(x)|_K^s \, dx,$$

where E_r *is the* $(2r \times 2r)$ *alternating matrix defined in section* 3.1 *then*

$$J_{m,2r} = q^{-m^2} t^{m-r} Z_{m-2r}(t).$$

Proof. By induction. If $r=0$, make the change of variables $x = \pi_K z$ then $dx = q^{-m^2}\, dz$, then $|\det(x)|_K^s = t^m |\det(z)|_K^s$ and $J_{m,0} = q^{-m^2} t^m \times \int_{H_m(O_{K'})} |\det(z)|_K^s \, dz = q^{-m^2} t^m Z_m(t)$.

Assume Lemma 2 holds for $J_{m-2,2r-2}$. For any $x \in \pi_{K'}\left(\begin{smallmatrix} E_r & 0 \\ 0 & 0 \end{smallmatrix}\right) + \pi_K H_m(O_{K'})$, we can write $x = \left(\begin{smallmatrix} \beta & -{}^t y' \\ y & z \end{smallmatrix}\right)$ where $\beta \in \pi_K O_K$, $y \in \pi_{K'}(-e_1 + \pi_{K'} O_{K'}^{m-1})$, ${}^t e_1 = (1, 0, ..., 0)$, and $z \in \mathrm{Mat}_{m-1,m-1}$. Make the change of variables $(\beta, y) \to (\pi_K \beta, \pi_{K'} y)$ then $d\beta\, dy \to q^{-m}\, d\beta\, dy$. Since

$$\begin{pmatrix} \pi_K \beta & -\pi_{K'}\, {}^t y' \\ \pi_{K'} y & z \end{pmatrix} = \begin{pmatrix} \pi_{K'} & 0 \\ 0 & 1_{m-1} \end{pmatrix} \begin{pmatrix} \beta & -{}^t y' \\ y & z \end{pmatrix} \begin{pmatrix} \pi_{K'} & 0 \\ 0 & 1_{m-1} \end{pmatrix},$$

$$J_{m,2r} = q^{-m} t \int_{O_K \times Y \times Z} \left| \det \begin{pmatrix} \beta & -{}^t y' \\ y & z \end{pmatrix} \right|_K^s d\beta\, dy\, dz, \qquad (3.5)$$

where from now on $\beta \in O_K$, $y \in (-e_1 + \pi_{K'} O_{K'}^{m-1}) = Y$, and $z \in \mathrm{Mat}_{m-1,m-1} = Z$. Since y is a primitive vector, there exists a matrix $g \in GL_{m-1}(O_{K'})$ such that $y = -g \cdot e_1$. Since

$$\begin{pmatrix} \beta & -{}^t y' \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \beta & {}^t e_1 \\ -e_1 & z^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & {}^t g' \end{pmatrix},$$

where $z = g z^* {}^t g'$ and hence $dz = dz^*$, equation (3.5) above becomes

$$J_{m,2r} = q^{-m} t \int_{O_K \times Y \times Z^*} \left| \det \begin{pmatrix} \beta & {}^t e_1 \\ -e_1 & z^* \end{pmatrix} \right|_K^s d\beta\, dy\, dz^*, \qquad (3.6)$$

where $z^* \in \mathrm{Mat}_{m-1,m-1} = Z^*$. More precisely,

$$\begin{pmatrix} \beta & {}^t e_1 \\ -e_1 & z^* \end{pmatrix} = \begin{pmatrix} \beta & 1 & 0 & \cdot & 0 \\ -1 & \gamma & w_1' & \cdot & w_{m-2}' \\ 0 & w_1 & & & \\ \cdot & \cdot & & x^* & \\ 0 & w_{m-2} & & & \end{pmatrix},$$

where $x^* \in X^* = \pi_{K'}\left(\begin{smallmatrix} E_{r-1} & 0 \\ 0 & 0 \end{smallmatrix}\right) + \pi_K H_{m-2}(O_{K'})$, $\gamma \in \pi_K O_K$, and $w_1 \in \pi_K O_{K'}$ for $1 \leqslant i \leqslant m-2$. Denote by $w \in \pi_K O_{K'}^{m-2} = W$ the $(m-2) \times 1$ column vector formed by the w_i, $1 \leqslant i \leqslant m-2$. Now, $\det\left(\begin{smallmatrix} \beta & {}^t e_1 \\ -e_1 & z^* \end{smallmatrix}\right) \equiv \det(x^*) \bmod \pi_K$, since $\beta \det(z^*) = \beta(\gamma \det(x^*) - {}^t w' \operatorname{adj}(x^*)\, w) \equiv 0 \bmod \pi_K$ where $x^* \operatorname{adj}(x^*) = \det(x^*) \cdot 1_{m-2}$ and equation (3.6) becomes

$$J_{m,\,2r} = q^{-m} t \int_{O_K \times Y \times \Gamma \times W} \left\{ \int_{X^*} |\det(x^*)|_K^s \, dx^* \right\} d\beta \, dy \, d\gamma \, dw.$$

The expression in curly brackets above is precisely $J_{m-2,\,2r-2}$ and we have that

$$J_{m,\,2r} = q^{-m} t \operatorname{vol}(Y) \operatorname{vol}(\Gamma) \operatorname{vol}(W)\, J_{m-2,\,2r-2} = q^{-(4m-4)} t J_{m-2,\,2r-2}. \tag{3.7}$$

Finally by the inductive hypothesis, $J_{m,\,2r} = q^{-m^2} t^{m-r} Z_{m-2r}(t)$. ∎

THEOREM 1 (Recursion Formula for $Z_m(t)$). *For any non-negative integer m,*

$$Z_m(t) = \sum_{k=0}^{m} \left[\frac{\prod_{i=1}^{k} (m-k+i)}{\prod_{i=1}^{[k/2]} (2i)} \right] \sum_{r=0}^{[m-k)/2]} q^{r(2(m-k)-2r-1)} \times \left[\frac{\prod_{i=1}^{2r} (m-k-2r+i)}{\prod_{i=1}^{r} (2i)} \right] q^{-(m-k)^2} t^{m-k-r} Z_{m-k-2r}(t).$$

Proof. Applying the Key Lemma,

$$Z_m(t) = \sum_{k=0}^{m} |\bar{G} \cdot \overline{x_o}| \int_{x_0 + \pi_K H_m(\pi_{K'}^{-1} O_{K'})} |\det(x)|_K^s \, dx.$$

Using formula (3.3) for the cardinality of the orbits, in this case, and the first partial integral, we get

$$Z_m(t) = \sum_{k=0}^{m} \left[\frac{\prod_{i=1}^{k} (m-k+i)}{\prod_{i=1}^{[k/2]} (2i)} \right] q^{-k(k-2m-1)/2} I_{m,\,k}.$$

By the remark following Lemma 1 and a second application of the Key Lemma, $Z_m(t)$ becomes

$$Z_m(t) = \sum_{k=0}^{m} \left[\frac{\prod_{i=1}^{k} (m-k+i)}{\prod_{i=1}^{[k/2]} (2i)} \right] \sum_{r=0}^{[(m-k)/2]} \left| \bar{G} \cdot \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} \right| \times \int_{\pi_{K'}\left(\begin{smallmatrix} E_r & 0 \\ 0 & 0 \end{smallmatrix}\right) + \pi_K H_{m-k}(O_{K'})} |\det(x)|_K^s \, dx.$$

Due to formula (3.4) and the second partial integral, we get the following expression for $Z_m(t)$:

$$Z_m(t) = \sum_{k=0}^{m} \left[\frac{\prod_{i=1}^{k} (m-k+i)}{\prod_{i=1}^{[k/2]} (2i)} \right] \prod_{r=0}^{[(m-k)/2]} q^{r(2(m-k)-2r-1)} \times \left[\frac{\prod_{i=1}^{2r} (m-k-2r+i)}{\prod_{i=1}^{r} (2i)} \right] J_{m-k,\, 2r}.$$

By Lemma 2, the recursion formula is proved. ∎

3.4. *Closed Form Expression for* $Z_m(t)$.

The following identity of Gauss [6] will be used. If

$$F_{m,n}(x) = \prod_{i=1}^{n} \frac{(1-x^{m+i})}{(1-x^i)}$$

for m, n non-integers and $F_{m,0}(x) = 1$, then

(i) $F_{m,n}(x) = F_{n,m}(x)$,

(ii) $F_{i,j}(x) \cdot F_{i+j,k}(x) = F_{i,j+k}(x) \cdot F_{j,k}(x)$,

(iii) $F_{m,n}(x) = F_{m,n-1}(x) + x^n F_{m-1,n}(x)$, if $m, n \geqslant 1$.

The Gauss identity below follows from (iii):

$$\sum_{i+j=k} F_{i,j}(x)\, x^{i(i-1)/2} t^i = \prod_{i=1}^{k} (1 + x^{i-1} t).$$

Define the right hand side of the identity above to be R.H.S. $= G_k(x, t)$ where for $k = 0$, we let $G_k(x, t) = 1$.

LEMMA 3. *For any non-negative integer k, the following identity holds*:

$$1 = \sum_{j=0}^{k} x^{j^2} t^j F_{j,k-j}(x)\, G_{k-j}(x, -x^{j+1} t).$$

Proof. Apply the Gauss identity and the expression to be proved becomes

$$1 = \sum_{j=0}^{k} \sum_{p=0}^{k-j} F_{j,k-j}(x)\, F_{p,k-j-p}(x)(-1)^p\, x^{p(p-1)/2 + p(j+1) + j^2} t^{j+p}.$$

Letting $p \to p-j$, switching the order of summation, and using property (ii), we rewrite the identity as

$$1 = \sum_{p=0}^{k} F_{k-p,\,p}(x)\, x^{p(p+1)/2} (-1)^p\, t^p \cdot \left[\sum_{j=0}^{p} (-1)^j\, x^{j(j-1)/2} F_{p-j,\,j}(x) \right],$$

where the bracketed expression is precisely $G_p(x, -1)$ which is 0 for $p > 0$ and 1 for $p = 0$. The terms in the outer sum above, therefore, reduce to the $p = 0$ term which is 1 and the identity holds. ∎

LEMMA 4. *For any non-negative integer k,*

$$q^{-k(k+1)/2} t^{[(k+1)/2]} = \sum_{r=0}^{[k/2]} q^{r(2k-2r-1)} \frac{\prod_{i=1}^{2r} (k-2r+i)}{\prod_{i=1}^{r} (2i)} \times q^{-k^2} t^{k-r} \prod_{i=[(k+1)/2]-r+1}^{[(k+1)/2]} \frac{(1-q^{-(2i-1)}t)}{(1-q^{-(2i-1)})}.$$

Proof. The identity to be verified is

$$1 = \sum_{r=0}^{[k/2]} q^{r(2k-2r-1)} \frac{\prod_{i=1}^{2r} (k-2r+i)}{\prod_{i=1}^{r} (2i)} q^{-k(k-1)/2} t^{[k/2]-r} \times \prod_{i=[(k+1)/2]-r+1}^{[(k+1)/2]} \frac{(1-q^{-(2i-1)}t)}{(1-q^{-(2i-1)})}. \tag{3.8}$$

Case 1. When $k = 2c$, equation (3.8) becomes

$$1 = \sum_{r=0}^{c} q^{r(4c-2r-1)} \frac{\prod_{i=1}^{2r} (2c-2r+i)}{\prod_{i=1}^{r} (2i)} q^{-c(2c-1)} t^{c-r} \prod_{i=c-r+1}^{c} \frac{(1-q^{-(2i-1)}t)}{(1-q^{-(2i-1)})}.$$

If we reverse the order of summation by letting $r = c - r = \ell$, simplify, and use the notation from the Gauss identity with $x = q^{-2}$, this identity becomes

$$1 = \sum_{\ell=0}^{c} F_{\ell,\,c-\ell}(x)\, G_{c-\ell}(x, -x^{\ell+1/2} t)\, x^{\ell(\ell-1/2)} t^{\ell}.$$

By Lemma 3 with k equal to c and t equal to $x^{-1/2}t$, this lemma holds when k is even.

Case 2. When $k = 2c-1$, the identity to be proved (3.8) becomes

$$1 = \sum_{r=0}^{c-1} q^{r(4c-2r-3)} \frac{\prod_{i=1}^{2r} (2c-2r-1+i)}{\prod_{i=1}^{r} (2i)} q^{-(c-1)(2c-1)} t^{c-r-1} \times \prod_{i=c-r+1}^{c} \frac{(1-q^{-(2i-1)}t)}{(1-q^{-(2i-1)})}.$$

Let $x=q^{-2}$, change the order of summation by letting $r\to c-r-1=\ell$, and simplify, then the above becomes

$$1=\sum_{\ell=0}^{c-1} F_{\ell,\,c-\ell-1}(x)\,G_{c-\ell-1}(x,\ -x^{\ell+3/2}t)\,x^{\ell(\ell+1/2)}\,t^{\ell}.$$

This identity also follows from Lemma 3 if we set k equal to $c-1$ and t equal to $x^{1/2}t$. Thus, Lemma 4 is true when k is odd and hence for all non-negative integers k. ▮

THEOREM 2. *If m is any non-negative integer,*

$$Z_m(t)=\prod_{i=1}^{[(m+1)/2]}\frac{1-q^{-(2i-1)}}{1-q^{-(2i-1)}t},$$

where $Z_0(t)=1$.

Proof. Start with the recursion formula in section 3.3 and let $k\to m-k$. Then dividing both sides by $Z_m(t)$ and rewriting, leaves the following identity to be proved:

$$\begin{aligned}1=\sum_{k=0}^{m}&\frac{\prod_{i=1}^{m-k}(k+i)}{\prod_{j=1}^{[(m-k)/2]}(2j)}\prod_{i=[(k+1)/2]+1}^{[(m+1)/2]}\frac{(1-q^{-(2i-1)}t)}{(1-q^{-(2i-1)})}\\&\times\left[\sum_{r=0}^{[k/2]}q^{r(2k-2r-1)}\frac{\prod_{p=1}^{2r}(k-2r+p)}{\prod_{l=1}^{r}(2l)}q^{-k^2}t^{k-r}\right.\\&\times\left.\prod_{i=[(k+1)/2]-r+1}^{[(k+1)/2]}\frac{(1-q^{-(2i-1)}t)}{(1-q^{-(2i-1)})}\right].\end{aligned}\tag{3.9}$$

By Lemma 4, the expression in brackets becomes $q^{-k(k+1)/2}t^{[(k+1)/2]}$. Split the above sum into sums over even and odd k:

$$\begin{aligned}1=&\sum_{j=0}^{[m/2]}\prod_{i=1}^{[m/2]-j}\frac{(2(i+j))}{(2i)}\prod_{i=j+1}^{[(m+1)/2]}(1-q^{-(2i-1)}t)\,q^{-j(2j+1)}t^{j}\\&+\sum_{j=1}^{[(m+1)/2]}\frac{\prod_{i=1}^{[m/2]-j+1}(2(i+j-1))}{\prod_{i=1}^{[(m+1)/2]-j}(2i)}\prod_{i=j+1}^{r}(1-q^{-(2i-1)}t)\,q^{-j(2j-1)}t^{j}.\end{aligned}\tag{3.10}$$

If we let $r=[(m+1)/2]$, equation (3.10) becomes

$$\begin{aligned}1=&\prod_{i=1}^{r}(1-q^{-(2i-1)}t)+\sum_{j=1}^{r}q^{-j(2j-1)}t^{j}\frac{\prod_{i=j+1}^{r}(1-q^{-(2i-1)}t)}{\prod_{i=1}^{r-j}(1-q^{-2i})}\\&\times\left[\prod_{i=1}^{r-j}(2(i+j))\,q^{-2j}+\prod_{i=1}^{r-j+1}(2(i+j-1))\right].\end{aligned}$$

The expression in brackets above reduces to $\prod_{i=1}^{r-j}(2(i+j))$, and if $x=q^{-2}$ the identity above simplifies to

$$1=\sum_{j=0}^{r}\prod_{i=1}^{r-j}\frac{1-x^{i+j}}{1-x^{i}}\prod_{p=j+1}^{r}(1-x^{p-1/2}t)\,x^{j(j-1/2)}t^{j},$$

where $r=[(m+1)/2]$. This identity follows from Lemma 3 with $k=r$ and t, in the Lemma, equal to $x^{-1/2}t$ and Theorem 2 holds. ∎

4. Quaternion Case

Let $X=H_m(D)$ be the $(2m^2-m)$-dimensional vector space over K of Hermitian matrices of degree m with entries in D where D is the unique quaternion division algebra over a p-adic local field K. The reduced norm of D, $n(\xi)=\xi\xi'$, maps $D^\times=D-\{0\}$ surjectively to $K^\times=K-\{0\}$ where $'$ is the involution on the quaternions. The inverse image of O_K under the norm map is O_D, the maximal compact subring of D. If π_D is picked so that $n(\pi_D)=\pi_K$ then $\pi_D O_D$ is the ideal of non-units in O_D. The reduced trace, $\mathrm{tr}(\xi)=\xi+\xi'$, maps ξO_D to O_K if and only if ξ is in $\pi_D^{-1}O_D$ [11]. If $f(x)$ is the generic norm of $H_m(O_D)$, as a Jordan algebra, (i.e. the determinant), we will calculate

$$Z(t)=\int_{X^0}|f(x)|_K^s\,dx,$$

where $C^0=O_D$, $X^0=H_m(C^0)$, dx is the Haar measure on O_D normalized so that $\mu(O_D)=\mu(O_K)=1$, $d(\pi_D x)=q^{-2}\,dx$, and G^0 is the image of $GL_m(C^o)$ in $GL(X^0)$ under the map $g\to$ "$x\to g\cdot x=gx{}^tg'$", in which ${}^tg'$ is the Hermitian adjoint of g. We remark that there is a K-linear isomorphism, θ, from $H_m(D)$ to $\mathrm{Alt}_{2m}(K)$ such that $\det(x)=Pf(\theta x)$ for $x\in H_m(D)$ and where $Pf(\theta x)$ is the Pfaffian of θx [8]. Hence, we are computing the local zeta function of the Pfaffian of a subalgebra of Mat_{2m}.

4.1. *Orbital Decomposition*

Since $O_K/\pi_K O_K$ is isomorphic to the finite field with q elements, $\mathbf{F}_q$, and $O_D/\pi_D O_D$ is isomorphic to the finite field with q^2 elements, $\mathbf{F}_{q^2}$, if we let $H_m(\pi_D^{-1}O_D)$ denote the set of Hermitian matrices of X^0 with diagonal entries in O_K and off-diagonal entries in $\pi_D^{-1}O_D$, we have the isomorphism

$$H_m(O_D)/\pi_K H_m(\pi_D^{-1}O_D)\cong H_m(\mathbf{F}_{q^2}).$$

Splitting $\pi_K H_m(\pi_D^{-1} O_D)$ into its alternating and Hermitian parts, we have that

$$\pi_K H_m(\pi_D^{-1} O_D)/\pi_K H_m(O_D) \cong \pi_D \operatorname{Alt}_m(\mathbf{F}_{q^2}).$$

As in section 3.1, we determine the orbital structure of $H_m(\mathbf{F}_{q^2})$ and $\operatorname{Alt}_m(\mathbf{F}_{q^2})$ under the action of $\bar{G} = GL_m(\mathbf{F}_{q^2})$ where $\bar{G} = G^0 \bmod \pi_D$. By the diagonalization of quadratic forms over $\mathbf{F}_q^2$ (where the norm map from $\mathbf{F}_{q^2}$ to $\mathbf{F}_q$ is surjective) and the orbital structure of skew-symmetric or alternating matrices, we have the following decompositions into disjoint orbits:

$$H_m(\mathbf{F}_{q^2}) = \bigcup_{k=0}^{m} \bar{G} \cdot \begin{pmatrix} 1_k & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.1}$$

and

$$\operatorname{Alt}_m(\mathbf{F}_q) = \{0\} \cup \left\{ \bigcup_{k=1}^{[m/2]} \bar{G} \cdot \begin{pmatrix} E_k & 0 \\ 0 & 0 \end{pmatrix} \right\}, \tag{4.2}$$

where $[m/2]$ and E_k are as in section 3.1.

4.2. *Cardinality of the Orbits*, $|\bar{G} \cdot \overline{x_0}|$.

From Dickson [5, pages 78, 134, 94], we have the formulae

$$|GL_m(\mathbf{F}_{q^2})| = q^{2m^2} \prod_{i=1}^{m} (1 - q^{-2i}), \qquad |U_m(\mathbf{F}_{q^2})| = q^{m^2} \prod_{i=1}^{m} (1 - (-q)^{-i})$$

$$|Sp_{2r}(\mathbf{F}_{q^2})| = q^{2r(2r+1)} \prod_{i=1}^{r} (1 - q^{-4i}).$$

To simplify the formulae in this section, we will again use the notation $(a)_\pm = (1 \pm q^{-a})$. In addition we make the convention that if there is no sign in the subscript a minus sign will be assumed (i.e. $(a) = (a)_-$). To compute the cardinality of the orbits in (4.1), let $\bar{x}_0 = \left(\begin{smallmatrix} 1_k & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Computing the fixer of $\bar{x}_0$, we find $|\bar{H}| = |GL_{m-k}(\mathbf{F}_{q^2})|\, q^{2k(m-k)}\, |U_k(\mathbf{F}_{q^2})|$. By Dickson's formulae,

$$|\bar{G} \cdot \overline{x_0}| = q^{k(2m-k)} \prod_{i=1}^{k} \frac{(1 - q^{-2(m-k+i)})}{(1 - (-q)^{-i})}. \tag{4.3}$$

To compute the cardinality of the orbits of (4.2), let $\overline{x_0} = \pi_D \left(\begin{smallmatrix} E_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Computing the fixer of $\overline{x_0}$, we find $|\bar{H}| = |GL_{m-2r}(\mathbf{F}_{q^2})|\, q^{4r(m-2r)}\, |Sp_{2r}(\mathbf{F}_{q^2})|$. By Dickson's formulae,

$$|\bar{G} \cdot \overline{x_0}| = q^{2r(2m-2r-1)} \frac{\prod_{i=1}^{2r} (2m - 4r + 2i)}{\prod_{l=1}^{r} (4l)}. \tag{4.4}$$

4.3. *Two Partial Integrals and a Recursion Formula for* $Z_m(t)$.

LEMMA 5 (First partial Integral). *If m is a positive integer*, $0 \leqslant k \leqslant m$, *and*

$$L_{m,k} = \int_{\left(\begin{smallmatrix} E_r & 0 \\ 0 & 0 \end{smallmatrix}\right) + \pi_K H_m(\pi_D^{-1} O_D)} |\det(x)|_K^s \, dx,$$

then $L_{m,k} = q^{-(2m-1)} L_{m-1,k-1}$.

Remark. Notice that by repeated application of Lemma 5,

$$L_{m,k} = q^{-k(2m-k)} \int_{\pi_K H_{m-k}(\pi_D^{-1} O_D)} |\det(x)|_K^s \, dx. \tag{4.5}$$

Proof. If the α_i are all 1 and D is substituted for K', the proof is identical to the proof of Lemma 1 up until the last two lines. The difference in the results comes from the change in measure due to the quaternion algebra. Following the proof of Lemma 1, we see that

$$L_{m,k} = L_{m-1,k-1} \operatorname{vol}(1 + \pi_K O_K) \operatorname{vol}(\pi_D O_D^{m-1}) = q^{-(2m-1)} L_{m-1,k-1}. \quad \blacksquare$$

LEMMA 6 (Second Partial Integral). *If m is a positive integer*, $0 \leqslant r \leqslant [m/2]$, *and*

$$M_{m,2r} = \int_{\pi_D\left(\begin{smallmatrix} E_r & 0 \\ 0 & 0 \end{smallmatrix}\right) + \pi_K H_m(O_D)} |\det(x)|_K^s \, dx,$$

then $M_{m,2r} = q^{-m(2m-1)} t^{m-r} Z_{m-2r}(t)$.

Proof. The proof follows that of Lemma 2. The differences are again due to the change in measure in the quaternion case. For example, for $X = H_m(O_D)$ we have that $d(\pi_K x) = q^{-m(2m-1)} \, dx$. In the quaternion situation equation (3.7) in Lemma 2 becomes

$$\begin{aligned} M_{m,2r} &= q^{-(2m-1)} t \operatorname{vol}(-e_1 + \pi_D O_D^{m-1}) \operatorname{vol}(\pi_K O_K) \operatorname{vol}(\pi_K O_D^{m-2}) \, M_{m-2,2r-2} \\ &= q^{-(8m-10)} t M_{m-2,2r-2} \end{aligned}$$

and by the inductive hypothesis $M_{m,2r} = q^{-m(2m-1)} t^{m-r} Z_{m-2r}(t)$. ∎

THEOREM 3 (Recursion Formula for $Z_m(t)$). *If m is a positive integer then*

$$Z_m(t) = \sum_{k=0}^{m} \left[\prod_{i=1}^{k} \frac{(1-q^{-2(m-k+i)})}{(1-(-q)^{-i})} \right] \sum_{r=0}^{[(m-k)/2]} q^{2r(2(m-k)-2r-1)}$$
$$\times \left[\frac{\prod_{i=1}^{2r} (2(m-k-2r+i))}{\prod_{i=1}^{r} (4i)} \right] q^{-(m-k)(2(m-k)-1)} t^{m-k-r} Z_{m-k-2r}(t).$$

Proof. The proof follows that of Theorem 1, using formulae (4.3) and (4.4) for the orbits and Lemmas 5 and 6 for the partial integrals. ∎

4.4. *Closed Form Expression for $Z_m(t)$ in the Quaternion Case*

We will use Gauss' identity and the notation developed in section 3.4 to prove Lemmas 7, 6, and 9 which we will need in the proof of Theorem 4.

LEMMA 7. *For s, ℓ non-negative integers such that $0 < \ell < s$*

$$0 = \sum_{j=0}^{s} (-1)^j x^{j(j-1)/2} \frac{G_{s-\ell}(x, x^{s-j+1}t)}{\prod_{k=1}^{s-j} (1-x^k) \prod_{k=1}^{j} (1-x^k)}. \tag{4.6}$$

Proof. The right-hand side of the identity to be proved becomes

$$\frac{1}{\prod_{k=1}^{s} (1-x^k)} \left[\sum_{j=0}^{s} (-1)^j x^{j(j-1)/2} F_{j,s-j}(x)\, G_{s-\ell}(x, x^{s-j+1}t) \right].$$

Applying Gauss' identity and reversing the order of summation, we have left to show that the expression in brackets above is 0:

$$[\cdot] = \sum_{i=1}^{s-\ell} x^{i(i-1)/2 + i(s+1)} F_{i,s-\ell-i}(x)\, t^i \cdot \left[\sum_{j=0}^{s} (-1)^j x^{j(j-1)/2} F_{j,s-j}(x)\, x^{-ij} \right]. \tag{4.7}$$

The bracketed expression above can be rewritten as $G_s(x, -x^{-i}) = \prod_{k=1}^{s} (1-x^{k-i-1})$. The terms in the outer sum are all 0 when $s > i > 0$ which is always the case by the hypothesis. ∎

LEMMA 8. *For an integer $k > 0$, the following identity holds*:

$$x^{k^2} \ell^k G_k(x, t) = \sum_{j=0}^{k} (-1)^j x^{j(j-1)/2} F_{k-j,j}(x)$$
$$\times G_j(x, -x^{k-j+1}\ell)\, G_{k-j}(x, x^k \ell t). \tag{4.8}$$

Proof. Compare coefficients of t^i. Using Gauss' identity, the left-hand side of (4.8) is $x^{k^2} \ell^k \sum_{i=0}^{k} F_{i,k-i}(x)\, x^{i(i-1)/2} t^i$ and we see that the coefficient of t^i is $x^{k^2} \ell^k F_{i,k-i}(x)\, x^{i(i-1)/2}$. The Gauss identity allows the right-hand side of (4.8) to be rewritten as $\sum_{j=0}^{k} B_j \sum_{i=0}^{k-j} A_{i,j} \cdot t^i$ where

$B_j = (-1)^j x^{j(j-1)/2} F_{k-j,j}(x)\, G_j(x, -x^{k-j+1}\ell)$ and $A_{i,j} = F_{i,k-j-i}(x) \times x^{i(i-1)/2+ki}\ell^i$. Changing the order of summation, the right-hand side becomes $\sum_{i=0}^{k} (\sum_{j=0}^{k-i} B_j A_{i,j})\, t^i$ in which the coefficient of t^i is $\sum_{j=0}^{k-i} B_j A_{i,j}$. Using the second property of the Gauss identity which says that

$$F_{j,k-j}(x) \cdot F_{k-j-i,i}(x) = F_{j,k-j-i}(x) \cdot F_{k-i,i}(x),$$

the coefficient of t^i on the right-hand side of (4.8) becomes precisely

$$F_{i,k-i}(x)\, x^{i(i-1)/2+ki}\ell^i \sum_{j=0}^{k-i} (-1)^j x^{j(j-1)/2} F_{k-j-i,j}(x) \cdot G_j(x, -x^{k-j+1}\ell).$$

Equating coefficients of t^i from both sides of (4.8), we have left to show that

$$x^{k^2-ki}\ell^{k-i} = \sum_{j=0}^{k-i} (-1)^j x^{j(j-1)/2} F_{k-j-i,j}(x) \cdot G_j(x, -x^{k-j+1}\ell).$$

Using Gauss' identity and property (ii) once more,

$$\begin{aligned} x^{k^2-ki}\ell^{k-i} &= \sum_{j=0}^{k-i} \sum_{p+q=j} F_{k-p-q-i,p+q}(x)\, F_{p,q}(x)\, x^{kp+q(q-1)/2}\ell^p(-1)^q \\ &= \sum_{p=0}^{k-i} F_{p,k-i-p}(x)\, x^{kp}\ell^p \\ &\quad \cdot \left[\sum_{q=0}^{k-i-p} F_{q,k-i-p-q}(x)\, x^{q(q-1)/2}(-1)^q \right], \end{aligned} \tag{4.9}$$

where the bracketed expression is precisely

$$G_{k-i-p}(x, -1) = \begin{cases} 1 & \text{if } k-i-p=0 \\ 0 & \text{otherwise.} \end{cases}$$

The terms in the outer sum of (4.9) are, therefore, all 0 except when $p = k-i$ and the sum above reduces to $x^{k(k-i)}\ell^{k-i}$ which is exactly the left-hand side. ∎

LEMMA 9. *For any non-negative integer n,*

$$\begin{aligned} q^{-n^2} t^{[(n+1)/2]} \frac{A_n^*(t)}{\prod_{i=1}^{n} (1-q^{-(2i-1)}t)} &= \sum_{r=0}^{[n/2]} q^{2r(2n-2r-1)} \frac{\prod_{i=1}^{2r} (1-q^{-2(n-2r+i)})}{\prod_{i=1}^{r} (1-q^{-4i})} \\ &\quad \times q^{-n(2n-1)} t^{n-r} \frac{A_{n-2r}(t)}{\prod_{i=1}^{n-2r} (1-q^{-(2i-1)}t)}, \end{aligned}$$

where

$$A_n^*(t)=\sum_{i=0}^{[n/2]}(-1)^i q^{-i(2(i+\gamma_n)-1)}t^i \frac{\prod_{j=i+1}^{n}(1-q^{-2j})}{\prod_{j=1}^{[n/2]-i}(1-q^{-4j})\prod_{j=1}^{i+\gamma_n}(1+q^{-(2i-1)})}$$

with $\gamma_n=n-2[n/2]$ (*i.e.* $\gamma_n=0$ *if* n *is even and* 1 *if* n *is odd*) *and where*

$$A_n(t)=\sum_{i=0}^{[n/2]}(-1)^i q^{-2i^2}t^i \frac{\prod_{j=[n/2]-i+1}^{n}(1-q^{-2j})}{\prod_{j=1}^{[(n+1)/2]-i}(1+q^{-(2j-1)})\prod_{j=1}^{i}(1-q^{-4j})}.$$

Remark. Notice that if we let $u=q^{-1}$ and $v=t$ and think of $A_n(t)=A_n(u,v)$ and $A_n^*(t)=A_n^*(u,v)$ as functions of both u and v then we have the following relation between the two polynomials:

$$A_n(u^{-1},v^{-1})=(-1)^n u^{-n^2}v^{[n/2]}A_n^*(u,v).$$

Proof. We must verify the identity

$$A_n^*(t)=\sum_{r=0}^{[n/2]} q^{-n(n-1)+2r(2n-2r-1)}t^{[n/2]}\frac{\prod_{i=1}^{2r}(1-q^{-2(n-2r+i)})}{\prod_{i=1}^{r}(1-q^{-4i})}$$
$$\times\prod_{i=n-2r+1}^{n}(1-q^{-(2i-1)}t)\,A_{n-2r}(t).$$

If we reverse the order of summation by letting r go to $[n/2]-r$ and expand the right-hand side of the identity above in powers of t, we find that

$$A_n^*(t)=\sum_{r=0}^{[n/2]}C_r\sum_{i=0}^{2[n/2]-r}(-1)^i t^{r+i}\sum_{k=0}^{i}B_{i-k,r}D_{k,r},$$

where

$$C_r=q^{-2r(2r+2\gamma_n-1)}\frac{\prod_{s=1}^{2[n/2]-2r}(4r+2\gamma_n+2s)}{\prod_{s=1}^{[n/2]-r}(4s)},$$

$$B_{i-k,r}=q^{-(i-k)(i-k+4r+2\gamma_n)}\prod_{s=1}^{2[n/2]-2r-i+k}\frac{(2i-2k+2s)}{(2s)},$$

and

$$D_{k,r}=q^{-2k^2}\frac{\prod_{s=r-k+1}^{2r+\gamma_n}(2s)}{\prod_{s=1}^{r-k+\gamma_n}(2s-1)+\prod_{s=1}^{k}(4s)}.$$

Rewriting in powers of t, the identity to be proven becomes

$$A_n^*(t) = \sum_{p=0}^{[n/2]} \Xi_1(p)(-1)^p t^p + \sum_{p=[n/2]+1}^{2[n/2]} \Xi_2(p)(-1)^p t^p,$$

where

$$\Xi_1(p) = \sum_{i=0}^{p} (-1)^i \sum_{j=0}^{[(p-i)/2]} (-1)^j C_{i+j} B_{p-i-2j,\, i+j} D_{j,\, i+j}$$

and

$$\Xi_2(p) = \sum_{i=0}^{2[n/2]-p} (-1)^i \sum_{j=0}^{[(p-i)/2]} (-1)^j C_{i+j} B_{p-i-2j,\, i+j} D_{j,\, i+j}.$$

To prove this identity we have just to show that $\Xi_1(p)$ equals the coefficient of $(-1)^p t^p$ in $A_n^*(t)$ for $0 \leqslant p \leqslant [n/2]$ and that $\Xi_2(p)$ is 0 for $[n/2]+1 \leqslant p \leqslant 2[n/2]$. To obtain the latter result, we show that the inner sum in the expression for $\Xi_2(p)$ is 0 for all i and for $[n/2]+1 \leqslant p \leqslant 2[n/2]$. Hence, we need to show that $0 = \sum_{j=0}^{[(p-i)/2]} (-1)^j C_{i+j} B_{p-i-2j,\, i+j} D_{j,\, i+j}$ or that

$$0 = W \sum_{j=0}^{[(p-i)/2]} (-1)^j q^{-2j(j-1)} \frac{\prod_{k=p-i-2j+1}^{2[n/2]-2i-2j} (2k)}{\prod_{k=1}^{[n/2]-i-j} (4k) \prod_{k=1}^{j} (4k)} \tag{4.10}$$

where the W above is made up of all the factors independent of j. Then

$$W = (-1)^i q^{-\alpha_n} \frac{\prod_{k=i+1}^{n} (2k)}{\prod_{k=1}^{i+\gamma_n} (2k-1)_+ \prod_{k=1}^{2[n/2]-p-i} (2k)},$$

where $\alpha_n = (p+i)^2 - 2i$ if n is even and $(p+i)^2 + 2p$ if n is odd. We have left to show that the sum on the right-hand side of equation (4.10) is 0. If we write the product in the numerator of each term as a product over even k times a product over odd k and let $x = q^{-4}$, the sum on the right-hand side of (4.10) can be rewritten in the form

$$\sum_{j=0}^{[(p-i)/2]} (-1)^j x^{j(j-1)/2} \frac{\prod_{k=1}^{[(p-i)/2]-(p-[n/2])} (1 - x^{[(p-i)/2]-j+k\pm 1/2})}{\prod_{k=1}^{[(p-i)/2]-j} (1-x^k) \prod_{k=1}^{j} (1-x^k)}, \tag{4.11}$$

where $\pm\frac{1}{2} = -\frac{1}{2}$ if $p-i$ is even and $+\frac{1}{2}$ if $p-i$ is odd. By applying Lemma 7 with $s = [(p-i)/2]$, $\ell = p-[n/2]$, and $t = -x^{\pm 1/2}$, the sum above is identically 0 and we have shown that $\Xi_2(p) = 0$.

To complete the proof of Lemma 9, it remains to be shown that $\Xi_1(p)$ equals the coefficient of $(-1)^p t^p$ in the definition of $A_n^*(t)$ when $0 \leqslant p \leqslant [n/2]$. Simplifying the expression for $\Xi_1(p)$ much as we did that for $\Xi_2(p)$, we see that

$$\Xi_1(p) = \sum_{i=0}^{p} (-1)^i q^{-\alpha_n} \frac{\prod_{k=i+1}^{n} (2k)}{\prod_{k=1}^{i+\gamma_n} (2k-1)_+ \prod_{k=1}^{2[n/2]-p-i} (2k)}$$
$$\times \left[\sum_{j=0}^{[(p-i)/2]} (-1)^j q^{-2j(j-1)} \frac{\prod_{k=p-i-2j+1}^{2[n/2]-2i-2j} (2k)}{\prod_{k=1}^{[n/2]-i-j} (4k) \prod_{k=1}^{j} (4k)} \right].$$

Using the notation from the Gauss identity and by taking out a factor of $1/\prod_{k=1}^{[(p-i)/2]} (4k)$,

$$\Xi_1(p) = \sum_{i=0}^{p} (-1)^i q^{-\alpha_n} \frac{\prod_{k=i+1}^{n} (2k)}{\prod_{k=1}^{i+\gamma_n} (2k-1)_+ \prod_{k=1}^{2[n/2]-p-i} (2k) \prod_{k=1}^{[(p-i)/2]} (4k)}$$
$$\times \left[\sum_{j=0}^{[(p-i)/2]} (-1)^j q^{-2j(j-1)} F_{[(p-i)/2]-j,j}(q^{-4}) \right.$$
$$\left. \times \prod_{k=[(p-i+1)/2]-j+1}^{[n/2]-i-j} (4k-2) \right].$$

Now the expression in brackets above has the product representation

$$[\cdot] = q^{(p-i)(p-i-1)} \prod_{k=[(p-i+1)/2]+i}^{p-1} \left(4\left[\frac{n}{2}\right] - 4k\right) \prod_{k=[(p-i+1)/2]+1}^{[n/2]-[(p-i)/2]-i} (4k-2).$$

To show that this relation holds, we use the notation from the Gauss identity, let $x = q^{-4}$, and show that

$$x^{(p-i)(p-i-1)/4} \prod_{k=[(p-i+1)/2]+i}^{p-1} (1 - x^{[n/2]-k}) \prod_{k=[(p-i+1)/2]+1}^{[n/2]-[(p-i)/2]-i} (1 - x^{k-1/2})$$
$$= \sum_{j=0}^{[(p-i)/2]} (-1)^j x^{j(j-1)/2} F_{[(p-i)/2]-j,j}(x)$$
$$\times \prod_{k=[(p-i+1)/2]-j+1}^{[n/2]-i-j} (1 - x^{k-1/2}).$$

Rewriting the first product on the left-hand side and dividing both sides by the second product, we rewrite the identity for the inner sum as

$$x^{(p-i)(p-i-1)/4} \prod_{k=1}^{[(p-i)/2]} (1-x^{k-1+[n/2]-2[(p-i)/2]\pm 1})$$
$$= \sum_{j=0}^{[(p-i)/2]} (-1)^j x^{j(j-1)/2} F_{[(p-i)/2]-j,j}(x)$$
$$\times \prod_{k=1}^{j} (1-x^{k+[(p-i)/2]-j+1/2})$$
$$\times \prod_{k=1}^{[(p-i)/2]-j} (1-x^{k-[(p-i)/2]+[n/2]-i+1/2}),$$

where ± 1 is $+1$ when $p-i$ is odd and -1 when $p-i$ is even. By applying Lemma 8 with $k=[(p-i)/2]$, $\ell = x^{\pm 1/2}$, and $t=-x^{[n/2]-2[(p-i)/2]\pm 1}$, the identity for the inner sum holds and our expression for $\Xi_1(p)$ becomes

$$\Xi_1(p) = \sum_{i=0}^{p} (-1)^i q^{-p(2p\pm 1)-i(2i\pm 1)}$$
$$\times \frac{\prod_{k=i+1}^{n}(2k)\prod_{k=[(p-i+1)/2]+i}^{p-1}(4[n/2]-4k)\prod_{k=[(p-i+1)/2]+1}^{[n/2]-[(p-i)/2]-i}(4k-2)}{\prod_{k=1}^{2[n/2]-p-i}(2k)\prod_{k=1}^{i+\gamma_n}(2k-1)_+\prod_{k=1}^{[(p-i)/2]}(4k)},$$

where $\pm 1 = -1$ if n is even and $+1$ if n is odd. Simplifying we see that

$$\Xi_1(p) = q^{-p(2p\pm 1)} \frac{\prod_{k=0}^{p-1}(4[n/2]-4k)\prod_{k=1}^{[n/2]-\gamma_n}(4k-2)}{\prod_{k=1}^{p+\gamma_n}(2k-1)_+}$$
$$\times \left[\sum_{i=0}^{p} (-1)^i q^{-i(2i\pm 1)} \frac{\prod_{k=i+1+\gamma_n}^{p+\gamma_n}(2k-1)_+}{\prod_{k=1}^{p-i}(2k)\prod_{k=1}^{i}(2k)}\right]. \tag{4.12}$$

Setting $\Xi_1(p)$ equal to the coefficient of $(-1)^p t^p$ in $A_n^*(t)$ and simplifying, we will have proven Lemma 9 when we have shown that

$$\sum_{i=0}^{p} (-1)^i q^{-i(2i\pm 1)} \frac{\prod_{k=i+1}^{p}(2k\pm 1)_+\prod_{k=1}^{p}(2k)}{\prod_{k=1}^{p-i}(2k)\prod_{k=1}^{i}(2k)} = 1.$$

Letting $x=q^{-2}$ and using the notation from the Gauss identity, the expression above becomes:

$$\sum_{i=0}^{p} (-1)^i x^{i(i\pm 1/2)} \prod_{k=i+1}^{p} (1+x^{k\pm 1/2}) F_{i,p-i}(x) = 1$$

This identity follows from Lemma 3 with $k=p$ and $t=-x^{\pm 1/2}$ and Lemma 9 is proved. ∎

THEOREM 4. *If m is any non-negative integer then*

$$Z_m(t) = \frac{A_m(t)}{\prod_{i=1}^{m} (1-q^{-(2i-1)}t)},$$

where $A_m(t)$ is defined as in Lemma 9 and $Z_0(t)=1$.

Proof. Start with the recursion formula in section 4.3. Substituting in for $Z_{m-k-2r}(t)$, the identity to be proved becomes

$$\frac{A_m(t)}{\prod_{i=1}^{m} (1-q^{-(2i-1)})}$$

$$= \sum_{k=0}^{m} \prod_{i=1}^{k} \frac{(2(m-k+i))}{(1-(-q)^{-i})} \left[\sum_{r=0}^{[(m-k)/2]} q^{2r(2(m-k)-2r-1)} \right.$$

$$\times \frac{\prod_{i=1}^{2r} (2(m-k-2r+i))}{\prod_{i=1}^{r} (4i)} q^{-(m-k)(2(m-k)-1)} t^{m-k-1}$$

$$\left. \times \frac{A_{m-k-2r}(t)}{\prod_{i=1}^{m-k-2r} (1-q^{-(2i-1)})} \right].$$

By Lemma 9, the expression in brackets above is exactly

$$[\cdot] = q^{-(m-k)^2} t^{[(m-k+1)/2]} \frac{A^*_{m-k}(t)}{\prod_{i=1}^{m-k} (1-q^{-(2i-1)}t)}$$

and the identity to be proved becomes

$$A_m(t) = \sum_{k=0}^{m} q^{-(m-k)^2} t^{[(m-k+1)/2]} \prod_{i=1}^{k} \frac{(2(m-k+i))}{1-(-q)^{-i}}$$

$$\times \prod_{i=m-k+1}^{m} (1-q^{-(2i-1)}t)\, A^*_{m-k}(t)$$

If we reverse the order of summation by letting $k \to m-k$ and start to expand the right side of the identity above in powers of t, we get that

$$A_m(t) = \sum_{k=0}^{m} C_k \sum_{i=0}^{m-[(k+1)/2]} (-1)^i\, t^{i+[(k+1)/2]} \sum_{\ell=0}^{i} B_{i-\ell,k} D_{\ell,k},$$

where

$$C_k = \prod_{i=1}^{m-k} \frac{(2k-2i)}{(1-(-q)^{-i})} q^{-k^2},$$

$$B_{i-\ell,k} = q^{-(i-\ell)(i-\ell+2k)} \prod_{j=1}^{m-k-(i-\ell)} \frac{(2i-\ell+j)}{(2j)},$$

$$D_{\ell,k} = q^{-\ell(2\ell \pm 1)} \frac{\prod_{j=\ell+1}^{k}(2j)}{\prod_{j=1}^{[k/2]-\ell}(4j)\prod_{j=1}^{\ell+\gamma_k}(2j-1)_+},$$

where ± 1 is -1 when k is even and $+1$ when k is odd and $\gamma_k = 0$ when k is even and 1 when k is odd. With these definitions in mind, we can now fully expand our identity in powers of t.

$$A_m(t) = \sum_{p=0}^{m} (-1)^p t^p \left[\sum_{j=0}^{p} (-1)^j \sum_{i=0}^{p-j} (-1)^{[(i+1)/2]} \times C_{2j+i} B_{p-i-j,\,2j+i} D_{[i/2],\,2j+i} \right].$$

We can prove this identity by showing that the coefficients of t on both sides of the equal sign agree. The manipulations are analogous to those in the proof of Lemma 9. ▌

4.5. *A Functional Equation for* $Z_m(t)$.

If $X = H_m(D)$ and $\Phi_{X^0}(x)$ is the characteristic function of $H_m(O_D)$, then $Z_m(t) = \int_X |\det(x)|_K^s \Phi_{X^0}(x)\, dx$ where $\Phi_{X^0}(x)$ is a element in the Schwartz–Bruhat space of locally constant functions on X with compact support. We construct the Fourier transform of $\Phi_{X^0}(x)$. Take the symmetric, non-degenerate, K-bilinear form $\mathrm{tr}(xy)$ on $X \times X$. Choose a non-trivial character of K, ψ_0, such that $\psi_0 = 1$ on O_K and $\psi_0 \neq 1$ on $\pi_K^{-1} O_K$ and let $d'x$ be the unique self-dual Haar measure corresponding to ψ_0. Then $\psi^*_{X^0}(x) = \int_X \psi_0(\mathrm{tr}(xy))\, \Phi_{X^0}(y)\, d'y$ is the Fourier transform of Φ_{X^0}. As $\psi^*_{X^0}(x) = \int_{X^0} \psi_0(\mathrm{tr}(xy))\, d'y$ and $\mathrm{tr}(\xi O_D) \in O_K$ if and only if $\xi \in \pi_D^{-1} O_D$,

$$\psi^*_{X^0}(x) = \begin{cases} \mathrm{vol}(X^0) & x \in H_m(\pi_D^{-1} O_D) \\ 0 & \text{otherwise} \end{cases}$$

Since $H_m(\pi_D^{-1} O_D)$ is the group of annihilators of X^0, denote it by $(X^0)_* = X^0_*$. With this notation, $\psi^*_{X^0}(x) = \mathrm{vol}(X^0)\, \psi_{X^0_*}(x)$, where $\psi_{X^0_*}$ is the characteristic function of X^0_*. Define

$$Z^*_m(t) = \int_X |\det(x)|_K^s\, \psi^*_{X^0}(x)\, dx;$$

then $Z^*_m(t) = \mathrm{Vol}(X^0) \int_{X^0_*} |\det(x)|^s_K \, dx$. By a change of variables $x = \pi_K y$,

$$\int_{\pi_K X^0_*} |\det(x)|^s_K \, dx = q^{-m(2m-1)} \, t^m \int_{X^0_*} |\det(x)|^S_K \, dx$$

and since $\mathrm{vol}(X^0) = \mathrm{Vol}(H_m(O_D)) = q^{-m(m-1)/2}$,

$$Z^*_m(t) = q^{m(3m-1)/2} t^{-m} \int_{\pi_K H_m(\pi_D^{-1} O_D)} |\det(x)|^S_K \, dx.$$

The inner sum in the recursion formula in Theorem 3 gave a recursive formula for the integral above. In Lemma 8, the closed form expression for this recursive formula was found to be

$$\int_{\pi_K H_m(\pi_D^{-1} O_D)} |\det(x)|^s_K \, dx = q^{-m^2} t^{[(m+1)/2]} \frac{A^*_m(t)}{\prod_{i=1}^m (1 - q^{-(2i-1)} t)}.$$

Using our expression for $Z^*_m(t)$ above we see that

$$Z^*_m(t) = q^{m(m-1)/2} t^{-[m/2]} \frac{A^*_m(t)}{\prod_{i=1}^m (1 - q^{-(2i-1)} t)}.$$

The quaternion division algebra is an arithmetic "prehomogeneous vector space" [18, 19, 10]. If, for the moment, the notations D and X refer to their tensor products with the universal field, then $\det(x)$ is an irreducible polynomial on X such that the identity component of its group of similarities is transitive on $Y = X - \det^{-1}(0)$. Then G_K is transitive on Y_K; this transitivity follows from the subjectivity of the reduced norm of D_K. Therefore, returning to the original notation and dropping the subscripts K, a general theorem [11, Theorem 1] states that

$$Z(\omega)^* \, (\Phi) = \gamma(\omega) \, Z(\omega_{2m-1} \omega^{-1}) \, (\Phi)$$

for some $\gamma(\omega)$ where ω is a quasicharacter of $K^* = K - \{0\}$. Taking $\omega = \omega_s$ where $\omega_s = |\cdot|^s_K$ and $\Phi = \phi_{X^o}$ and using the formulae for $Z^*_m(t)$ and $Z_m(t)$, it is easy to verify that:

$$Z^*_m(q^{2m-1} t) = (-1)^{m(m-1)/2} \prod_{i=1}^m \frac{(1 - q^{-(2i-1)} t^{-1})}{(1 - q^{2i-2} t)} \, Z_m(t^{-1}).$$

In other words, the $\gamma(\omega_s)$ above becomes the product of m Tate local gamma factors [20] up to a factor of $(-1)^{m(m-1)/2}$. This value for $\gamma(\omega_s)$ has been obtained using a different method in [11].

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