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Archimedean Superrigidity of Solvable S-Arithmetic Groups

Dave Witte*

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078

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Let $\mathbb{G}$ be a connected, solvable linear algebraic group over a number field K, let S be a finite set of places of K that contains all the infinite places, and let $\mathscr{O}(S)$ be the ring of S-integers of K. We define a certain closed subgroup $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ of $\mathbb{G}_S = \prod_{v \in S} \mathbb{G}_{K_v}$ that contains $\mathbb{G}_{\mathscr{O}(S)}$, and prove that $\mathbb{G}_{\mathscr{O}(S)}$ is a superrigid lattice in $\overline{\mathbb{G}_{\mathscr{O}(S)}}$, by which we mean that finite-dimensional representations $\alpha : \mathbb{G}_{\mathscr{O}(S)} \to \mathrm{GL}_n(\mathbb{R})$ more or less extend to representations of $\overline{\mathbb{G}_{\mathscr{O}(S)}}$. The subgroup $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ may be a proper subgroup of $\mathbb{G}_S$ for only two reasons. First, it is well known that $\mathbb{G}_{\mathscr{O}(S)}$ is not a lattice in $\mathbb{G}_S$ if $\mathbb{G}$ has nontrivial K-characters, so one passes to a certain subgroup $\mathbb{G}_S^{(1)}$. Second, $\mathbb{G}_{\mathscr{O}(S)}$ may fail to be Zariski dense in $\mathbb{G}_S^{(1)}$ in an appropriate sense; in this sense, the subgroup $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ is the Zariski closure of $\mathbb{G}_{\mathscr{O}(S)}$ in $\mathbb{G}_S^{(1)}$. Furthermore, we note that a superrigidity theorem for many nonsolvable S-arithmetic groups can be proved by combining our main theorem with the Margulis Superrigidity Theorem.

1. INTRODUCTION

Let $\mathbb{G}$ be a solvable linear algebraic group defined over $\mathbb{Q}$. The author recently proved that if the arithmetic subgroup $\mathbb{G}_{\mathbb{Z}}$ is Zariski dense, then it is a superrigid lattice in $\mathbb{G}_{\mathbb{R}}$ (see 1.4), in the sense that any finite-dimensional representation $\alpha : \mathbb{G}_{\mathbb{Z}} \to \mathrm{GL}_n(\mathbb{R})$ more or less extends to a representation of $\mathbb{G}_{\mathbb{R}}$. (A precise definition of superrigidity appears in Definition 1.1 below.) We now prove an appropriate generalization of this result for S-arithmetic subgroups, in place of arithmetic subgroups (see 1.6 and 1.10).

* E-mail address: dwitte@math.okstate.edu.

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(1.1) Definition (cf. [9, Thm. 2, p. 2]). Let Λ be a subgroup of a topological group G. We say that Λ is *superrigid* in G if, for every continuous homomorphism $\alpha : \Lambda \to \mathrm{GL}_n(\mathbb{R})$, there are

- a finite-index open subgroup Λ_1 of Λ,
- a finite-index open subgroup G_1 of G, containing Λ_1, and
- a finite normal subgroup F of $\overline{\Lambda_1^\alpha}$, where $\overline{\Lambda_1^\alpha}$ is the (almost-) Zariski closure of Λ_1^α in $\mathrm{GL}_n(\mathbb{R})$,

such that the induced homomorphism $\alpha_F : \Lambda_1 \to \overline{\Lambda_1^\alpha}/F$ extends to a continuous homomorphism $\sigma : G_1 \to \overline{\Lambda_1^\alpha}/F$.

Note that the only superrigidity under consideration is *archimedean* superrigidity. That is, all representations are over $\mathbb{R}$ (or $\mathbb{C}$). Our results say nothing about representations over other local fields.

(1.2) *Remark.* By considering restrictions and induced representations, one may show, for any finite-index subgroup Λ' of Λ, that Λ is superrigid in G if and only if Λ' is superrigid in G.

(1.3) *Assumption.* For simplicity, we assume that all algebraic groups in this paper are linear. (More precisely, they are subgroups of some special linear group.) Thus, if $\mathbb{G}$ is an algebraic group over $\mathbb{Q}$, then $\mathbb{G}_{\mathbb{Z}}$ is well defined, and is a subgroup of $\mathbb{G}_{\mathbb{Q}}$. Without such an assumption, the definition of arithmetic subgroups of $\mathbb{G}$ is slightly more involved [1, Sect. 7.11, p. 49].

(1.4) Theorem ([13, Thm. 6.8]; see also Sect. 2). *Let $\mathbb{G}$ be a solvable algebraic group over $\mathbb{Q}$. If $\mathbb{G}_{\mathbb{Z}}$ is Zariski dense in $\mathbb{G}$, then $\mathbb{G}_{\mathbb{Z}}$ is a superrigid lattice in $\mathbb{G}_{\mathbb{R}}$.*

If $\mathbb{G}$ has no nontrivial characters defined over $\mathbb{Q}$ (and is connected), then this theorem has a natural generalization to S-arithmetic subgroups.

(1.5) Definition. Let $\mathbb{G}$ be a connected algebraic group over $\mathbb{Q}$, and let S be a finite set of places of $\mathbb{Q}$, containing the infinite place. Define

- $\mathbb{Z}(S) = \{x \in \mathbb{Q} \mid \|x\|_v \leq 1 \text{ for all places } v \notin S\}$;
- $\mathbb{Q}_v$ = the completion of $\mathbb{Q}$ at the place v; and
- $\mathbb{G}_S = \times_{v \in S} \mathbb{G}_{\mathbb{Q}}$.

Via the natural diagonal embedding of $\mathbb{G}_{\mathbb{Q}}$ in $\mathbb{G}_S$, we may think of $\mathbb{G}_{\mathbb{Z}(S)}$ as a subgroup of $\mathbb{G}_S$.

(1.6) Theorem. *Let $\mathbb{G}$ be a connected, solvable algebraic group over $\mathbb{Q}$, and let S be a finite set of places of $\mathbb{Q}$, containing the infinite place. If $\mathbb{G}$ has no nontrivial characters defined over $\mathbb{Q}$, and $\mathbb{G}_{\mathbb{Z}(S)}$ is Zariski dense in $\mathbb{G}$, then $\mathbb{G}_{\mathbb{Z}(S)}$ is a superrigid lattice in $\mathbb{G}_S$.*

If $\mathbb{G}$ does have nontrivial $\mathbb{Q}$-characters, then $\mathbb{G}_{\mathbb{Z}(S)}$ is not a lattice in $\mathbb{G}_S$. However, it is well known that $\mathbb{G}_{\mathbb{Z}(S)}$ is a lattice in a certain subgroup $\mathbb{G}_S^{(1)}$ (see 1.9 and 5.15), and our main theorem implies that $\mathbb{G}_{\mathbb{Z}(S)}$ is superrigid in $\mathbb{G}_S^{(1)}$. More generally, our main theorem shows that an analogous result holds for solvable algebraic groups over any number field, but, in this generality, it may be necessary to replace $\mathbb{G}_S^{(1)}$ with a smaller group $\overline{\mathbb{G}_{\mathscr{O}(S)}}$, which we now define (see 1.9). The idea is that $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ should be the almost-Zariski closure (see 1.8) of $\mathbb{G}_{\mathscr{O}(S)}$ in $\mathbb{G}_S^{(1)}$, but the situation is complicated by the fact that $\mathbb{G}_S^{(1)}$ is not an algebraic group.

(1.7) DEFINITION [13, Defn. 3.2]. A subgroup A of $\mathrm{GL}_n(\mathbb{R})$ is said to be *almost-Zariski closed* if there is a Zariski closed subgroup B of $\mathrm{GL}_n(\mathbb{R})$, such that $B^\circ \subset A \subset B$, where B° is the identity component of B in the topology of $\mathrm{GL}_n(\mathbb{R})$ as a C^∞ manifold (not the Zariski topology). There is little difference between being Zariski closed and almost-Zariski closed, because B° always has finite index in B.

(1.8) DEFINITION [13, Defn. 3.6]. Let A be a subgroup of $\mathrm{GL}_n(\mathbb{R})$. The *almost-Zariski closure* $\overline{A}$ of A is the unique smallest almost-Zariski closed subgroup that contains A. In particular, if A is a subgroup of a Lie group G, we use $\overline{\mathrm{Ad}_G A}$ to denote the almost-Zariski closure of $\mathrm{Ad}_G A$ in the real algebraic group $\mathrm{Aut}(\mathscr{G})$, where $\mathscr{G}$ is the Lie algebra of G.

(1.9) DEFINITION. Let $\mathbb{G}$ be a connected algebraic group over a number field K, and let S be a finite set of places of K, containing the set S_∞ of infinite places. Define

- $\mathscr{O}(S) = \{x \in K \mid \|x\|_v \leq 1 \text{ for all places } v \notin S\}$;
- K_v = the completion of K at the place v;
- $\mathbb{G}_\infty = \times_{v \in S_\infty} \mathbb{G}_{K_v}$;
- $\mathbb{G}_f = \times_{v \in S - S_\infty} \mathbb{G}_{K_v}$;
- $\mathbb{G}_S = \mathbb{G}_\infty \times \mathbb{G}_f$;
- $\mathbb{G}_S^{(1)} = \{x \in \mathbb{G}_S \mid \prod_{v \in S} \|\chi(x_v)\|_v = 1, \text{ for all } K\text{-characters } \chi \text{ of } \mathbb{G}\}$ [10, pp. 263–264]; and
- $\overline{\mathbb{G}_{\mathscr{O}(S)}} = \{x \in \mathbb{G}_S^{(1)} \mid \mathrm{Ad}_{\mathbb{G}_\infty} x \in \overline{\mathrm{Ad}_{\mathbb{G}_\infty} \mathbb{G}_{\mathscr{O}(S)}}\}$.

See Example 5.16 for an example where $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ is a proper subgroup of $\mathbb{G}_S^{(1)}$.

(1.10) MAIN THEOREM (see Sect. 3). *Let $\mathbb{G}$ be a connected, solvable algebraic group over a number field K, and let S be a finite set of places of K, containing all the infinite places. If $\mathbb{G}_{\mathscr{O}(S)}$ is Zariski dense in $\mathbb{G}$, then $\mathbb{G}_{\mathscr{O}(S)}$ is a superrigid lattice in $\overline{\mathbb{G}_{\mathscr{O}(S)}}$.*

The following corollary states that the question of whether S-arithmetic subgroups of a given algebraic group $\mathbb{G}$ are superrigid can essentially be reduced to the same question about the S-arithmetic subgroups of the maximal semisimple quotient of $\mathbb{G}$. The point is that the main theorem implies that solvable S-arithmetic groups are superrigid, which means the radical of $\mathbb{G}$ is under control—all that remains is the semisimple part of $\mathbb{G}$.

(1.11) COROLLARY (see Sect. 4). *Let $\mathbb{G}$ be a connected algebraic group over a number field K, and let S be a finite set of places of K, containing the infinite places. If $\mathbb{G}_{\mathscr{O}(S)}$ is Zariski dense in $\mathbb{G}$, and the image of $\mathbb{G}_{\mathscr{O}(S)}$ in $(\mathbb{G}/\operatorname{Rad}\mathbb{G})_{\mathscr{O}(S)}$ is superrigid, then $\mathbb{G}_{\mathscr{O}(S)}$ is a superrigid lattice in $\overline{\mathbb{G}_{\mathscr{O}(S)}}$.*

G. A. Margulis has shown that S-arithmetic subgroups of semisimple groups of higher S-rank are superrigid.

(1.12) THEOREM (Margulis [9, B(iii), p. 259]). *Let $\mathbb{G}$ be a connected, semisimple algebraic group over a number field K, and let S be a finite set of places of K, containing the infinite places. Assume the S-rank of every simple factor of $\mathbb{G}$ is at least two. Then $\mathbb{G}_{\mathscr{O}(S)}$ is a superrigid lattice in $\mathbb{G}_S$.*

(1.13) COROLLARY. *Let $\mathbb{G}$ be a connected algebraic group over a number field K, and let S be a finite set of places of K, containing the infinite places. If $\mathbb{G}_{\mathscr{O}(S)}$ is Zariski dense in $\mathbb{G}$, and the S-rank of every K-simple factor of $\mathbb{G}/\operatorname{Rad}\mathbb{G}$ is at least two, then $\mathbb{G}_{\mathscr{O}(S)}$ is a superrigid lattice in $\overline{\mathbb{G}_{\mathscr{O}(S)}}$.*

(1.14) *Remark.* Although any connected, noncompact simple Lie group G has no finite-dimensional unitary representations, some of the lattices in G may have finite-dimensional unitary representations. Any such lattice Λ is not a superrigid subgroup of G, in the sense of Definition 1.1. However, the Margulis Superrigidity Theorem [9, Thm. IX.6.16(b), p. 332] asserts that Λ satisfies a weaker definition of superrigidity that considers only representations $\alpha\colon \Lambda \to \mathrm{GL}_n(\mathbb{R})$, such that $\overline{\Lambda^\alpha}$ has no nontrivial connected, compact, semisimple, normal subgroups. If Λ is solvable, then $\overline{\Lambda^\alpha}$ is also solvable, so the assumption on $\overline{\Lambda^\alpha}$ is automatically satisfied and therefore is irrelevant. However, the difference between the two definitions of superrigidity *is* relevant in the context of Corollary 1.11. Fortunately, the corollary is valid for either of the two definitions of superrigidity.

2. AN INSTRUCTIVE PROOF OF A SPECIAL CASE

In this section, we present a simple proof of Theorem 1.4. This special case illustrates many of the ideas involved in the proof of our main theorem. In particular, this case illustrates the importance of the existence of a syndetic hull.

(2.1) Definition ([13, Sect. 5]; cf. [3, p. 6]). Let Γ be a closed subgroup of a Lie group G. A *syndetic hull* of Γ is a connected subgroup B that contains Γ, such that B/Γ is compact.

Although our goal is a proof of Theorem 1.4, which deals only with algebraic groups, it is convenient to prove a more general result that applies to more general Lie groups (see 2.2), because it is easier to work with simply connected groups, but the universal cover of $\mathbb{G}_{\mathbb{R}}$ is often *not* an algebraic group.

The proof we give here is based on the same ideas as [13]. However, the present proof is *much* less complicated because we do not bother to keep track of exactly when it is necessary to pass to a finite-index subgroup or mod out a finite group. In particular, we thereby avoid the need to discuss nilshadows, which play an important role in [13].

(2.2) Theorem. *Let Γ_1 be a lattice in a solvable Lie group G_1, and assume G_1 has only finitely many connected components. If $\overline{\operatorname{Ad}_{G_1}\Gamma_1} = \overline{\operatorname{Ad} G_1}$, then Γ_1 is superrigid in G_1.*

Proof. Replacing G_1 and Γ_1 by finite-index subgroups, we may assume G_1 is connected. Then, by passing to the universal cover, we may assume G_1 is simply connected. Given a homomorphism $\alpha\colon \Gamma_1 \to \mathrm{GL}_n(\mathbb{R})$, let $G_2 = \overline{\Gamma_1^{\alpha}}$. Replacing Γ_1 by a finite-index subgroup, we may assume G_2 is connected, so $[G_1, G_1]$ and $[G_2, G_2]$ are simply connected, nilpotent Lie groups.

Let $G = G_1 \times G_2$ and $\Gamma = \operatorname{graph}(\alpha)$, so Γ is a discrete subgroup of G. Any maximal compact torus of $\overline{\operatorname{Ad} G}$ is of the form $T_1 \times S_1$, where T_1 and S_1 are maximal compact tori of $\overline{\operatorname{Ad} G_1}$ and $\operatorname{Ad} G_2$, respectively. We may assume $T_1 \times S_1$ contains a maximal compact torus T of $\overline{\operatorname{Ad}_G \Gamma}$. Because $\overline{\operatorname{Ad}_{G_1}\Gamma_1} = \overline{\operatorname{Ad} G_1}$, we know that the projection of T into $\overline{\operatorname{Ad} G_1}$ is all of T_1. Therefore, $TS_1 = T_1 \times S_1$ is a maximal compact torus of $\overline{\operatorname{Ad} G}$. Because G_2 is a real algebraic group, there is a compact torus S of G_2 with $\operatorname{Ad}_{G_2} S = S_1$. Hence Lemma 2.8 implies that some finite-index subgroup of Γ has a simply connected syndetic hull X.

It is not difficult (cf. [13, Step 5 of proof of Thm. 6.4, p. 174]) to see that $XG_2 = G_1G_2$ and $X \cap G_2 = e$, from which it follows that X is the graph of a homomorphism $\sigma\colon G_1 \to G_2$. Because X contains a finite-index subgroup of Γ, we know that σ agrees with α on a finite-index subgroup of Γ_1. ∎

By definition, syndetic hulls are connected, so the proof of their existence requires some way to prove that a subgroup is connected. The following proposition plays this role in the proof of Lemma 2.8 below. The proposition is proved in Section 5, but the statement is copied here for ease of reference.

(2.3) PROPOSITION (see 5.20). *Let G be a connected, solvable Lie group, let $\rho: G \to \mathrm{GL}_n(\mathbb{R})$ be a continuous homomorphism, and let A be an almost-Zariski closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. If there is a compact, abelian subgroup S of G and a compact torus T of A such that $S^\rho T$ is abelian and contains a maximal compact torus of $\overline{G^\rho}$, then the inverse is image $\rho^{-1}(A)$ has only finitely many connected components.*

(2.4) PROPOSITION [11, Prop. 2.5, p. 31]. *Let Γ be a closed subgroup of a simply connected, nilpotent Lie group G. Then Γ has a unique syndetic hull in G.*

(2.5) LEMMA [13, Cor. 5.18]. *If Γ is a discrete subgroup of a connected, abelian Lie group G, then some finite-index subgroup of Γ has a simply connected syndetic hull in G.*

Proof. Let $\tilde{G}$ be the universal cover of G. Replacing Γ by a finite-index subgroup, we may assume Γ is torsion-free (hence free abelian), so there is a subgroup $\tilde{\Gamma}$ of $\tilde{G}$ that maps isomorphically onto Γ under the covering map $\tilde{G} \to G$. Let $\tilde{X}$ be the unique syndetic hull of $\tilde{\Gamma}$ in $\tilde{G}$ (see 2.4), and let X be the image of $\tilde{X}$ in G. Then X is a simply connected syndetic hull of Γ, as desired. ∎

(2.6) PROPOSITION [13, Cor. 3.14]. *Let A be a connected Lie subgroup of a connected Lie group G. Then the normalizer of A in $\overline{\mathrm{Ad}\, G}$ is almost-Zariski closed.*

(2.7) PROPOSITION [13, Cor. 3.10]. *If A is a subgroup of a connected, solvable Lie group G such that $[G, G]$ is simply connected, then the centralizer of A in $\overline{\mathrm{Ad}\, G}$ is almost-Zariski closed.*

The following lemma, which establishes the existence of an appropriate syndetic hull, is a crucial ingredient in the proof of Theorem 2.2. A generalization of this lemma is used in the proof of our main theorem (see 3.5).

(2.8) LEMMA. *Let Γ be a discrete subgroup of a connected, solvable Lie group G, such that $[G, G]$ is simply connected. If there is a compact, abelian subgroup S of G and a compact torus T of $\overline{\mathrm{Ad}_G \Gamma}$, such that $(\mathrm{Ad}_G S)T$ is a maximal compact torus of $\overline{\mathrm{Ad}\, G}$, then some finite-index subgroup of Γ has a simply connected syndetic hull in G.*

Proof. Let $H = \mathrm{Ad}_G^{-1}(\overline{\mathrm{Ad}_G \Gamma})$. Proposition 2.3 implies that H has only finitely many connected components, so H° contains a finite-index subgroup of Γ. Thus, there is no harm in replacing G with H°, which means we may assume $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}\, G}$.

From Proposition 2.4, we know that $[\Gamma, \Gamma]$ has a unique syndetic hull U in $[G, G]$. The uniqueness implies that Γ normalizes U. Then, because U is connected and $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}\, G}$, we conclude that all of G normalizes U (see 2.6). Thus, there is no harm in modding out U, so we may assume $[\Gamma, \Gamma] = e$, that is, Γ is abelian. So Γ centralizes Γ. Because $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}\, G}$, this implies all of G centralizes Γ (see 2.7), so $\mathrm{Ad}_G \Gamma$ is trivial. Because $\overline{\mathrm{Ad}_G \Gamma} = \overline{\mathrm{Ad}\, G}$, this means $\mathrm{Ad}\, G$ is trivial, so G is abelian. The desired conclusion now follows from Lemma 2.5. ∎

3. PROOF OF THE MAIN THEOREM

This section presents a proof of the main theorem. However, instead of the theorem as stated in Section 1, we prove a more general version that applies to groups whose algebraic structure is similar to that of $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ (see 3.6). We are not particularly interested in this generalization for its own sake; rather, the intention is to clarify the main ideas of the proof by separating out the crucial hypotheses. The following proposition shows that Main Theorem 1.10 is indeed a special case of Theorem 3.6.

(3.1) PROPOSITION. *Let $\mathbb{G}$ be a connected, solvable algebraic group over a number field K, and let S be a finite set of places of K, containing all the infinite places, such that $\mathbb{G}_{\mathscr{O}(S)}$ is Zariski dense in $\mathbb{G}$. Let*

- $\phi: \mathbb{G}_S \to \mathbb{G}_\infty$ *be the projection with kernel* $\mathbb{G}_f$;
- $G = \overline{\mathbb{G}_{\mathscr{O}(S)}}$ (*see* 1.9);
- $G_\infty = G^\phi$;
- $G_f = \mathbb{G}_f$; *and*
- $\Lambda =$ *any finite-index subgroup of* $\mathbb{G}_{\mathscr{O}(S)}$.

Then

1. G_∞ *is a solvable Lie group such that* $[G'_\infty, G'_\infty] = [G'_\infty, (G'_\infty)^\circ]$ *is simply connected and nilpotent, for every finite-index subgroup* G'_∞ *of* G_∞;
2. G_f *is a locally compact, totally disconnected, solvable group, such that* $[G_f, G_f]$ *has no infinite discrete subgroups*;
3. G *is an open subgroup of the direct product* $G_\infty \times G_f$;
4. Λ *is a lattice in* G;
5. $\overline{\mathrm{Ad}_{G_\infty} \Lambda^\phi}$ *is a finite-index subgroup of* $\overline{\mathrm{Ad}\, G_\infty}$; *and*
6. $[\Lambda, \Lambda]$ *is cocompact in* $[G, G]$.

Proof. (1) The definition of G_∞ implies that it is a (not necessarily closed) Lie subgroup of $\mathbb{G}_\infty$; so G_∞ is a solvable Lie group. We may write $\mathbb{G}$ as a semidirect product $\mathbb{G} = \mathbb{T} \ltimes \mathbb{U}$, where $\mathbb{T}$ is a torus and $\mathbb{U}$ is the unipotent radical. The unipotent group $\mathbb{U}$ has no nontrivial characters, so we have $\mathbb{U}_S \subset \mathbb{G}_S^{(1)}$; in particular, $\mathbb{U}_\infty \subset \mathbb{G}_S^{(1)}$. It is well known that $\mathbb{U}_{\mathscr{O}}$ is a lattice in $\mathbb{U}_\infty$ (see 5.15), so the Borel Density Theorem (see 5.14) implies that

$$\mathrm{Ad}_{G_\infty} \mathbb{U}_\infty \subset \overline{\mathrm{Ad}_{\mathbb{G}_\infty} \mathbb{U}_{\mathscr{O}}} \subset \overline{\mathrm{Ad}_{G_\infty} \mathbb{G}_{\mathscr{O}(S)}}.$$

Therefore, $\mathbb{U}_\infty \subset \overline{\mathbb{G}_{\mathscr{O}(S)}}$, so $\mathbb{U}_\infty \subset G_\infty$. Because $\mathbb{U}_\infty$ is connected (see 5.1), this implies $\mathbb{U}_\infty \subset G'_\infty$, so $G'_\infty = T_\infty \ltimes \mathbb{U}_\infty$, where $T_\infty = \mathbb{T}_\infty \cap G'_\infty$. Then, because $\mathbb{T}$ is abelian, this implies that $[G'_\infty, G'_\infty] = [T_\infty, \mathbb{U}_\infty][\mathbb{U}_\infty, \mathbb{U}_\infty]$. Because $\mathbb{U}_\infty$ is connected, this implies that $[G'_\infty, G'_\infty]$ is connected. Therefore, being a connected subgroup of $\mathbb{U}_\infty$, the commutator subgroup $[G'_\infty, G'_\infty]$ is simply connected (see 5.2).

(2) The fact that $[G_f, G_f]$ has no infinite discrete subgroups follows from the observation that $[\mathbb{G}, \mathbb{G}] \subset \mathbb{U}$ is unipotent.

(3) Because the range of each nonarchimedean valuation is a discrete set (and the group of K-characters is finitely generated), there is an open set $F \subset \mathbb{G}_f$ such that $\|\chi(x_v)\|_v = 1$ for all $x \in F$, all K-characters χ, and all $v \in S - S_\infty$; hence $F \subset \mathbb{G}_S^{(1)}$. Since $\mathrm{Ad}_{G_\infty} F \subset \mathrm{Ad}_{G_\infty} \mathbb{G}_f = e$, then the definition of $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ implies that $F \subset \mathbb{G}_{\mathscr{O}(S)}$.

Let H be the identity component of G_∞. Because the range of each nonarchimedean valuation is a discrete set, and H is contained in the identity component of $(\mathbb{G}_S^{(1)})^\phi$, we must have $\prod_{v \in S_\infty} \|\chi(x_v)\|_v = 1$, for every $x \in H$, so $H \subset \mathbb{G}_S^{(1)}$. Furthermore, because $H \subset G_\infty = (\overline{\mathbb{G}_{\mathscr{O}(S)}})^\phi$, we must have

$$\mathrm{Ad}_{\mathbb{G}_\infty} H \subset \mathrm{Ad}_{\mathbb{G}_\infty} G_\infty \subset \overline{\mathrm{Ad}_{\mathbb{G}_\infty} \mathbb{G}_{\mathscr{O}(S)}}.$$

Therefore, $H \subset \overline{\mathbb{G}_{\mathscr{O}(S)}}$.

So $H \times F$ is an open subset of $G_\infty \times G_f$ contained in $\overline{\mathbb{G}_{\mathscr{O}(S)}} = G$. This establishes (3).

(4) Because $\mathbb{G}_{\mathscr{O}(S)}$ is a lattice in $\mathbb{G}_S^{(1)}$ (see 5.15) and G is a closed subgroup of $\mathbb{G}_S^{(1)}$ that contains $\mathbb{G}_{\mathscr{O}(S)}$, we see that $\mathbb{G}_{\mathscr{O}(S)}$ is a lattice in G, as desired.

(5) The desired conclusion is immediate from the fact that $\overline{\mathbb{G}_{\mathscr{O}(S)}} = G$.

(6) Let $\mathbb{U} = \mathbb{U}^1 \supset \mathbb{U}^2 \supset \cdots \supset \mathbb{U}^n = e$ be the descending central series of $\mathbb{U}$, and let $\mathbb{U}^i_\Lambda = \Lambda \cap \mathbb{U}^i_S$, a finite-index subgroup of $\mathbb{U}^i_{\mathscr{O}(S)}$. From Lemma 3.3 below, we see that $[\mathbb{U}_\Lambda, \mathbb{U}^{n-2}_\Lambda]$ is cocompact in $\mathbb{U}^{n-1}_S$. Then, by modding out $\mathbb{U}^{n-1}$ and applying the lemma again, we see that $[\mathbb{U}_\Lambda, \mathbb{U}^{n-3}_\Lambda]$ is cocompact in $\mathbb{U}^{n-2}_S$. Continuing in this manner, we see that $[\mathbb{U}_\Lambda, \mathbb{U}_\Lambda]$ is cocompact in $[\mathbb{U}, \mathbb{U}]_S$. Hence, there is no harm in modding out $[\mathbb{U}, \mathbb{U}]$, so we may assume $\mathbb{U}$ is abelian. Then $[\mathbb{G}, \mathbb{U}, \mathbb{U}] = e$, so we conclude from Lemma

3.3 below that $[\Lambda, \mathbb{U}_\Lambda]$ is cocompact in $[\Lambda, \mathbb{U}]_S$. Hence, there is no harm in modding out $[\Lambda, \mathbb{U}]$, which means that we may assume Λ centralizes $\mathbb{U}$. Because Λ is Zariski dense in $\mathbb{G}$, this implies that $\mathbb{G}$ centralizes $\mathbb{U}$, so $\mathbb{G} = \mathbb{T} \times \mathbb{U}$ is abelian, so (6) is trivially true. ∎

(3.2) *Remark.* From the proof, we see that it would suffice to make the weaker assumption that $\mathrm{Ad}_{\mathbb{G}} \mathbb{G}_{\mathscr{O}(S)}$ is Zariski dense in $\mathrm{Ad}\, \mathbb{G}$ in place of the assumption that $\mathbb{G}_{\mathscr{O}(S)}$ is Zariski dense in $\mathbb{G}$.

(3.3) LEMMA. *Let $\mathbb{G}$ be an algebraic group over a number field K, let S be a finite set of places of K that contains all the infinite places, and let Λ be a finite-index subgroup of $\mathbb{G}_{\mathscr{O}(S)}$. If $\mathbb{X}$ and $\mathbb{V}$ are connected K-subgroups of $\mathbb{G}$, such that $\mathbb{V}$ is unipotent, $\mathbb{X}$ normalizes $\mathbb{V}$, and $[\mathbb{X}, \mathbb{V}, \mathbb{V}] = e$, then $[\Lambda \cap \mathbb{X}_S, \Lambda \cap \mathbb{V}_S]$ is cocompact in $[\Lambda \cap \mathbb{X}_S, \mathbb{V}]_S$.*

Proof. From the ascending chain condition on Zariski closed, connected subgroups, we know there is a finite subset $\{x_1, \ldots, x_m\}$ of $\Lambda \cap \mathbb{X}_S$ such that $[\Lambda \cap \mathbb{X}_S, \mathbb{V}] = [x_1, \mathbb{V}][x_2, \mathbb{V}] \cdots [x_m, \mathbb{V}]$. The image of an S-arithmetic subgroup under a K-epimorphism is an S-arithmetic subgroup [10, Thm. 5.9, p. 269], so we see that $[x_i, \Lambda \cap \mathbb{V}_S]$ is a finite-index subgroup of $[x_i, \mathbb{V}]_{\mathscr{O}(S)}$, which is cocompact in $[x_i, \mathbb{V}]_S$ (see 5.15). ∎

(3.4) DEFINITION [9, Sect. I, p. 8]. Let Γ and Λ be subgroups of a group G. We say that Λ *commensurabilizes* Γ if $\Gamma \cap (\lambda^{-1}\Gamma\lambda)$ is a finite-index subgroup of both Γ and $\lambda^{-1}\Gamma\lambda$, for every $\lambda \in \Lambda$.

(3.5) LEMMA. *Suppose G is a solvable Lie group, and $[G^\circ, G]$ is simply connected. Let Γ be a discrete subgroup of G° and let Λ be a subgroup of G that contains Γ and commensurabilizes it. Suppose some subgroup of Γ has a syndetic hull U, such that*

- *U is simply connected;*
- *$U \subset \mathrm{nil}\, G$;*
- *U contains $[\Lambda, \Lambda]$; and*
- *U is normalized by Λ.*

If there is a compact, abelian subgroup S of G° and a compact torus T of $\overline{\mathrm{Ad}_G \Lambda}$, such that $(\mathrm{Ad}_G S)T$ is a maximal compact torus of $\overline{\mathrm{Ad}\, G}$, then some finite-index subgroup of Γ has a simply connected syndetic hull in G that contains U and is normalized by Λ.

Proof. Because $\overline{\mathrm{Ad}\, G^\circ}$ is a normal subgroup of $\overline{\mathrm{Ad}\, G}$, we know that $(\mathrm{Ad}_G S)T$ contains a maximal compact torus of $\overline{\mathrm{Ad}\, G^\circ}$ (see 5.9), so $\mathrm{Ad}_G^{-1}(\overline{\mathrm{Ad}_G \Lambda})$ has only finitely many connected components (see 5.20). In other words, letting $H = \mathrm{Ad}_G^{-1}(\overline{\mathrm{Ad}_G \Lambda})$, we know that H° is a finite-index subgroup of $H \cap G^\circ$, so H° contains a finite-index subgroup of Γ. Also, because $\operatorname{Ad}_G U$ is unipotent (see 5.3) and Γ contains a cocompact subgroup of U, we conclude from the Borel Density Theorem (5.14) that $\operatorname{Ad}_G U \subset \overline{\operatorname{Ad}_G \Gamma} \subset \overline{\operatorname{Ad}_G \Lambda}$, so $U \subset H$. Hence, there is no harm in replacing G with H, so we may assume $\overline{\operatorname{Ad}_G \Lambda} = \overline{\operatorname{Ad} G}$.

Because Λ normalizes U, and $\overline{\operatorname{Ad}_G \Lambda} = \overline{\operatorname{Ad} G}$, we conclude that all of G normalizes U (see 5.18). Then there is no harm in modding out U, so $[\Lambda, \Lambda] = e$; that is, Λ is abelian.

Because $\Gamma \subset \Lambda$, this implies that Λ centralizes Γ, so all of G centralizes Γ (see 5.17). In particular, G° centralizes Γ, which means $\Gamma \subset Z(G^\circ)$. Furthermore, $\operatorname{Ad} G^\circ \subset \overline{\operatorname{Ad}_G \Lambda}$ is abelian, so G° is nilpotent. This implies that $Z(G^\circ)$ is connected (see 5.4). Hence, some finite-index subgroup of Γ has a simply connected syndetic hull X in $Z(G^\circ)$ (see 2.5). All that remains is to show that X is normalized by Λ.

Let

- $\tilde{G}^\circ$ be the universal cover of G°,
- $\tilde{X}$ be the connected subgroup of $\tilde{G}^\circ$ that covers X,
- $\tilde{\Gamma}$ be the inverse image of Γ in $\tilde{X}$, and
- Z be the kernel of the covering map $\tilde{G}^\circ \to G$.

Because Λ centralizes Γ, we know $[\tilde{\Gamma}, \Lambda] \subset Z$. On the other hand, because $[G^\circ, G]$ is simply connected, we know that $Z \cap [\tilde{G}^\circ, G] = e$. Therefore, $[\tilde{\Gamma}, \Lambda] = e$, so Λ normalizes $\tilde{\Gamma}$. Now $\tilde{G}^\circ$ is a simply connected, nilpotent Lie group, so $\tilde{X}$ is the unique syndetic hull of $\tilde{\Gamma}$ in $\tilde{G}^\circ$ (see 2.4), so we conclude that Λ normalizes $\tilde{X}$. Hence, Λ normalizes X, as desired. ∎

(3.6) THEOREM. *Suppose*

- G_∞ *is a solvable Lie group such that* $[G'_\infty, G'_\infty] = [G'_\infty, (G'_\infty)^\circ]$ *is simply connected and nilpotent, for every finite-index subgroup* G'_∞ *of* G_∞; *and*
- G_f *is a locally compact, totally disconnected, solvable group, such that* $[G_f, G_f]$ *has no infinite discrete subgroups.*

Let

- G *be an open subgroup of the direct product* $G_\infty \times G_f$, *and*
- $\phi\colon G_\infty \times G_f \to G_\infty$ *be the projection with kernel* G_f,

and let Λ *be a lattice in* G *such that*

- $\overline{\operatorname{Ad}_{G_\infty} \Lambda^\phi}$ *is a finite-index subgroup of* $\overline{\operatorname{Ad} G_\infty}$, *and*
- $[\Lambda_1, \Lambda_1]$ *is cocompact in* $[G, G]$, *for every finite-index subgroup* Λ_1 *of* Λ

Then Λ *is superrigid in* G.

More precisely, if $\alpha\colon \Lambda \to \mathrm{GL}_n(\mathbb{R})$ *is a homomorphism such that*

$$\overline{\left\{\left(\mathrm{Ad}_{G_\infty}\lambda, \lambda^\alpha\right) \mid \lambda \in \Lambda\right\}}$$

is connected, then there is a finite subgroup F *of* $Z(\overline{\Lambda^\alpha})$ *such that the induced homomorphism* $\alpha_F\colon \Lambda \to \overline{\Lambda^\alpha}/F$ *extends to a continuous homomorphism* $\sigma\colon G_1 \to \overline{\Lambda^\alpha}/F$, *for some finite-index subgroup* G_1 *of* G.

Proof. Replacing G and G_∞ by finite-index subgroups, we may assume $\overline{\mathrm{Ad}_{G_\infty}\Lambda^\phi} = \overline{\mathrm{Ad}\,G_\infty}$. Assume for simplicity that G_∞° is simply connected. (In the situation of Proposition 3.1, this may be achieved by passing to a universal cover.) Let K be a compact open subgroup of G_f contained in G. Let $\Gamma = \Lambda \cap (G_\infty^\circ K)$; note that Λ commensurabilizes Γ. We may assume, by replacing K with a finite-index subgroup, that $\Gamma \cap K = e$ (see 5.11), so ϕ is faithful on Γ.

Step 1. *There is a unique homomorphism* $\beta\colon [G_\infty, G_\infty] \to [\overline{\Lambda^\alpha}, \overline{\Lambda^\alpha}]$ *such that* $\phi\beta$ *agrees with* α *on* $\Gamma \cap [\Lambda, \Lambda]$. Let $\Gamma_1 = \Gamma \cap [\Lambda, \Lambda]$. Because ϕ is faithful on Γ, the homomorphism $\alpha|_{\Gamma_1}\colon\Gamma_1 \to [\overline{\Lambda^\alpha}, \overline{\Lambda^\alpha}]$ induces a homomorphism $\overline{\alpha}\colon \Gamma_1^\phi \to [\overline{\Lambda^\alpha}, \overline{\Lambda^\alpha}]$. Because Γ_1^ϕ is a lattice in $[G_\infty, G_\infty]$ (see 5.22), we know that $\overline{\alpha}$ extends to a unique homomorphism $\beta\colon [G_\infty, G_\infty] \to [\overline{\Lambda^\alpha}, \overline{\Lambda^\alpha}]$ (see 5.12).

Step 2. $\phi\beta$ *also agrees with* α *on* $[\Lambda, \Lambda]$. Because $[G_f, G_f]$ has no infinite discrete subgroups, we see that for every $\lambda \in [\Lambda, \Lambda]$, there is some $n \in \mathbb{Z}^+$ with $\lambda^n \in G_\infty \times K$. Because G_∞/G_∞° is abelian, we also know $\lambda \in G_\infty^\circ \times G_f$, so we conclude that $\lambda^n \in G_\infty^\circ K$; therefore, $\lambda^n \in \Gamma$. Therefore, $\phi\beta$ agrees with α on Λ^n. Because nth roots are unique in a unipotent Lie group such as $[\overline{\Lambda^\alpha}, \overline{\Lambda^\alpha}]$, we conclude that $\phi\beta$ agrees with α on λ, as desired.

Step 3. β *extends to a homomorphism* $\rho\colon G_\infty^\circ \to \overline{\Lambda^\alpha}$ *such that* $\phi\rho$ *agrees with* α *on a finite-index subgroup of* Γ, *and we have* $g^{\lambda^\phi \rho} = g^{\rho\lambda^\alpha}$ *for all* $g \in G_\infty^\circ$ *and* $\lambda \in \Lambda$. Let

- $\hat{G} = G_\infty \times \overline{\Lambda^\alpha}$;
- $\hat{\Gamma} = \{(\gamma^\phi, \gamma^\alpha) \mid \gamma \in \Gamma\}$;
- $\hat{\Lambda} = \{(\lambda^\phi, \lambda^\alpha) \mid \lambda \in \Lambda\}$; and
- $U = \{(u, u^\beta) \mid u \in [G_\infty, G_\infty]\}$.

From Lemma 3.5, we see that some finite-index subgroup of $\hat{\Gamma}$ has a simply connected syndetic hull $\hat{X}$ in $\hat{G}$, such that $\hat{X}$ contains U and is normalized by $\hat{\Lambda}$. It is not difficult to see that $\hat{X}$ is the graph of a homomorphism $\rho\colon G_\infty^\circ \to \overline{\Lambda^\alpha}$ (see Step 5 of the proof of [13, Thm. 6.4]). Because $\hat{\Lambda}$ normalizes $\hat{X}$, we have $g^{\lambda^\phi \rho} = g^{\rho\lambda^\alpha}$.

Step 4. *Completion of the proof*. Let $L = \operatorname{graph}(\alpha)$, $\hat{X} = \operatorname{graph}(\rho)$, and $X = \hat{X}K[G_f, G_f]$. (So L, $\hat{X}$, and X are subgroups of $G \times \overline{\Lambda^\alpha}$.) From Step 3, we see that L normalizes X; hence XL is a subgroup of $G \times \overline{\Lambda^\alpha}$. Let $F = XL \cap (e \times \overline{\Lambda^\alpha})$ and $H = G_\infty^\circ K[G_f, G_f]$. It suffices to show that F is a finite, normal subgroup of $\overline{\Lambda^\alpha}$, for then XL/F is the graph of a well-defined homomorphism

$$\sigma : H\Lambda \to \overline{\Lambda^\alpha}/F,$$

and $H\Lambda$ is a finite-index subgroup of G, because it is an open subgroup that contains the lattice Λ.

Because L normalizes X, L, and $(e \times \overline{\Lambda^\alpha})$, it is obvious that Λ^α normalizes F, so we only need to show that F is finite. Because $[\Lambda, \Lambda]$ is cocompact in $[G, G]$ and Γ is a lattice in $G_\infty^\circ K$ we know that $\Gamma[\Lambda, \Lambda]$ is a finite-index subgroup of $\Lambda \cap H$. Then, from Steps 2 and 3, we conclude that $\phi\rho$ agrees with α on a finite-index subgroup of $\Lambda \cap H$, so X contains a finite-index subgroup of $L \cap (H \times \overline{\Lambda^\alpha})$. On the other hand, because $X \subset H \times \overline{\Lambda^\alpha}$, we have

$$F = XL \cap \left(e \times \overline{\Lambda^\alpha}\right) = \left(X\left(L \cap \left(H \times \overline{\Lambda^\alpha}\right)\right)\right) \cap \left(e \times \overline{\Lambda^\alpha}\right).$$

Therefore, $X \cap (e \times \overline{\Lambda^\alpha})$ contains a finite-index subgroup of F. Because $X \cap (e \times \overline{\Lambda^\alpha}) = e$, this implies that F is finite, as desired. ∎

4. APPLICATION TO NONSOLVABLE GROUPS

In this section, we prove Corollary 1.11. As described in Remark 1.14, we prove two versions of this corollary (see 4.2).

(4.1) Definition (cf. 1.1). Let Λ be a subgroup of a topological group G. Let us say that Λ is *weakly superrigid* in G if, for every continuous homomorphism $\alpha : \Lambda \to \mathrm{GL}_n(\mathbb{R})$, *such that* $\overline{\Lambda^\alpha}$ *has no nontrivial connected, compact, semisimple, normal subgroups*, there are

- a finite-index open subgroup Λ_1 of Λ.
- a finite-index open subgroup G_1 of G, containing Λ_1, and
- a finite, normal subgroup F of $\overline{\Lambda_1^\alpha}$ where $\overline{\Lambda_1^\alpha}$ is the (almost-) Zariski closure of Λ_1^α in $\mathrm{GL}_n(\mathbb{R})$,

such that the induced homomorphism $\alpha_F : \Lambda_1 \to \overline{\Lambda_1^\alpha}/F$ extends to a continuous homomorphism $\sigma : G_1 \to \overline{\Lambda_1^\alpha}/F$.

(4.2) COROLLARY (CF. 1.11). *Let $\mathbb{G}$ be a connected algebraic group over a number field K, and let S be a finite set of places of K, containing the infinite places. If $\mathbb{G}_{\mathscr{O}(S)}$ is Zariski dense in $\mathbb{G}$, and the image of $\mathbb{G}_{\mathscr{O}(S)}$ in $\overline{(\mathbb{G}/\operatorname{Rad}\mathbb{G})_{\mathscr{O}(S)}}$ is superrigid (or weakly superrigid), then $\mathbb{G}_{\mathscr{O}(S)}$ is a superrigid lattice in $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ (or a weakly superrigid lattice in $\overline{\mathbb{G}_{\mathscr{O}(S)}}$, respectively).*

Proof (cf. proof of [13, Thm. 9.9]). Suppose $\alpha: \mathbb{G}_{\mathscr{O}(S)} \to \mathrm{GL}_n(\mathbb{R})$ is a homomorphism. (If the goal is to prove that $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ is *weakly* superrigid, assume $\overline{\mathbb{G}^{\alpha}_{\mathscr{O}(S)}}$ has no nontrivial connected, compact, semisimple, normal subgroups.) Let Γ be a finite-index subgroup of $\mathbb{G}_{\mathscr{O}(S)}$, and let $H = \overline{\Gamma^{\alpha}}$. Replacing Γ by a finite-index subgroup, we may assume H is connected and that there is a Levi subgroup $\mathbb{L}$ of $\mathbb{G}$ such that $\Gamma = (\mathbb{L}_S \cap \Gamma)((\operatorname{Rad}\mathbb{G})_S \cap \Gamma)$. Let L and R be finite-index subgroups of $\overline{\mathbb{L}_{\mathscr{O}(S)}}$ and $\overline{(\operatorname{Rad}\mathbb{G})_{\mathscr{O}(S)}}$, respectively. Let $\alpha_L = \alpha|_{L\cap\Gamma}$ and $\alpha_R = \alpha|_{R\cap\Gamma}$, and let $L_H = \overline{(L\cap\Gamma)^{\alpha}}$. Because $L \cap \Gamma$ is a superrigid lattice in L (or weakly superrigid lattice, respectively), it must be the case that L_H is semisimple, so L_H is a Levi subgroup of H, and (after passing to a finite-index subgroup) there are a finite (or compact, respectively), normal subgroup F of $C_{L_H}(L^{\beta_L})$, and a continuous homomorphism $\beta_L: L \to L_H/F$, such that

$$\gamma^{\beta_L} = \gamma^{\alpha}F, \qquad \forall \gamma \in L \cap \Gamma.$$

From Theorem 1.10, we also know that α_R extends to a homomorphism

$$\beta_R: R \to \overline{(R\cap\Gamma)^{\alpha}} = \operatorname{Rad} H.$$

The semisimple group $\mathbb{L}$ has no nontrivial K-characters, so $\mathbb{L}_S^{(1)} = \mathbb{L}_S$. Therefore, $\mathbb{L}_f \subset \overline{\mathbb{L}_{\mathscr{O}(S)}}$, so L contains a finite-index subgroup of $\mathbb{L}_f$. Thus, we may assume $L = L_\infty \times L_f$, where $L_\infty = \mathbb{L}_\infty \cap L$ and $L_f = \mathbb{L}_f \cap L$. Because L_f is totally disconnected and has no open normal subgroups of infinite index, we know that $L_f^{\beta_L}$ is finite; therefore, replacing L by a finite-index subgroup, we may assume β_L is trivial on L_f.

Case 1. *$C_H(\operatorname{Rad} H)$ has no nontrivial, compact, solvable, normal subgroups.* (Note that if $\mathbb{G}_{\mathscr{O}(S)}$ has no nontrivial connected, compact, semisimple, normal subgroups, then this implies that $C_H(\operatorname{Rad} H)$ has no compact normal subgroups at all, solvable or not.) In this case, the extension β_R is unique (cf. [13, Cor. 6.11]), so graph(α) normalizes graph(β_R).

Write $\operatorname{Rad}\mathbb{G} = \mathbb{T} \ltimes \mathbb{U}$, where $\mathbb{T}$ is a torus and $\mathbb{U}$ is the unipotent radical. We may assume $R = T \ltimes U$, where T and U are finite-index subgroups of $\overline{\mathbb{T}_{\mathscr{O}(S)}}$ and $\mathbb{U}_S$, respectively. Because graph(α) normalizes graph(β_R), and $\mathbb{L}$ centralizes $\mathbb{T}$, we see that graph(α_L) centralizes graph($\beta_R|_T$), so L_H centralizes T^{β_R}.

Let $\phi\colon L \to L_\infty$ be the projection with kernel L_f, let $A = \{(\gamma^\phi, \gamma^\alpha) \mid \gamma \in L \cap \Gamma\}$, and let $\overline{A}$ be the almost-Zariski closure of A in $L_\infty \times L_H$ (see 4.3).

Let $U_\infty = \mathbb{U}_\infty \cap U$ and $U_f = \mathbb{U}_f \cap U$, and assume $U = U_\infty \times U_f$. Because $C_H(\operatorname{Rad} H)$ has no nontrivial, compact, solvable, normal subgroups, it is not difficult to show that $U_f^{\beta_R}$ must be trivial. Then, because graph(α) normalizes graph(β_R), we see that A must normalize graph$(\beta_R|_{U_\infty})$. Thus, $\overline{A}$ normalizes graph$(\beta_R|_{U_\infty})$, (see 5.18). Therefore, $\overline{A} \cap (e \times L_H)$ centralizes $U_\infty^{\beta_R} = U^{\beta_R}$. Because $\overline{A} \cap (e \times L_H) \subset L_H$ also centralizes T^{β_R}, we conclude that $\overline{A} \cap (e \times L_H)$ centralizes $\overline{(TU)^{\beta_R}} = \overline{R^{\beta_R}} = \operatorname{Rad} H$ (see 5.17). Then, since

$$\overline{A} \cap (e \times L_H) \subset \text{graph}(\beta_L) \cap (e \times L_H) = e \times F,$$

and $C_H(\operatorname{Rad} H)$ has no nontrivial compact, (solvable) normal subgroups, we conclude that $\overline{A} \cap (e \times L_H)$ is trivial. This means that $\overline{A}$ is the graph of a well-defined homomorphism $\beta_L'\colon L_\infty \to L_H$ and, from the definition of A, we see that $\phi\beta_L'$ extends α_L. Hence, we may assume $\beta_L = \beta_L'$ (and F is trivial). Then graph(β_L) normalizes both graph$(\beta_R|_{U_\infty})$ and graph$(\beta_R|_{U_f})$ (the latter because $\beta_R|_{U_f}$ is trivial), and, being contained in $L \times L_H$, centralizes graph$(\beta_R|_T)$. Hence, graph(β_L) normalizes graph(β_R), so the function

$$\beta\colon L \ltimes R \to H\colon (l, r) \mapsto l^{\beta_L} r^{\beta_R}$$

is a homomorphism. The completes the proof in this case.

Case 2. *The general case*. Let C be the (unique) maximal compact, solvable, normal subgroup of $C_H(\operatorname{Rad} H)$ (which we no longer assume to be trivial). From Case 1, we know there is a finite-index subgroup G of $\overline{\mathbb{G}_{\mathscr{O}(S)}}$ and a homomorphism $\overline{\beta}\colon G \to H/C$ such that $\overline{\beta}$ extends to the homomorphism induced by α. Now C° is a compact torus (see 5.6), so the Levi decomposition implies that there is a normal subgroup J of H such that $JC = H$ and $J \cap C$ is finite. There is no harm in modding out this finite intersection, so we may assume $J \cap C$ is trivial. Then H/C is naturally isomorphic to J, so we can thin of $\overline{\beta}$ as a homomorphism from G to J. From the definition of $\overline{\beta}$, we have $\gamma^\alpha \in \gamma^\beta C$, for all $\gamma \in \Gamma$. Since C is central in H (see 5.7), this implies that there is a homomorphism $\sigma\colon \Gamma \to C$ such that $\gamma^\alpha = \gamma^\beta \gamma^\sigma$, for all $\gamma \in \Gamma$. Because C is abelian, we know that σ is trivial on $[\Gamma, \Gamma]$; in particular, σ is trivial on a finite-index subgroup of $L \cap \Gamma$ and a finite-index subgroup of $[\Gamma, R \cap \Gamma]$. From the proof of Proposition 3.1(6), we see that $[\Gamma, R \cap \Gamma]$ is a cocompact subgroup of $[G, R]$, so, replacing Γ by a subgroup of finite index, we may assume that σ is trivial on $(L \cap \Gamma)([G, R] \cap \Gamma) = [G, G] \cap \Gamma$. Thus, σ extends to a homomorphism $\tau\colon G \to C$ (for example, this follows by applying the main theorem (1.10) to the abelian group $\mathbb{G}/[\mathbb{G},\mathbb{G}]$). Then the homomorphism $g \mapsto g^{\beta} g^{\tau}$ extends α, as desired. ∎

(4.3) *Remark.* $\mathbb{G}_\infty$ is of the form $\mathbb{A}_{\mathbb{C}} \times \mathbb{B}_{\mathbb{R}}$, where $\mathbb{A}$ and $\mathbb{B}$ are algebraic groups defined over $\mathbb{C}$ and $\mathbb{R}$, respectively. By restriction of scalars, the $\mathbb{C}$-points of an n-dimensional algebraic group defined over $\mathbb{C}$ can be viewed as the $\mathbb{R}$-points of a $2n$-dimensional algebraic group defined over $\mathbb{R}$. Thus, we see that $\mathbb{G}_\infty$ can be viewed as the $\mathbb{R}$-points of an algebraic group defined over $\mathbb{R}$. Therefore, in a natural way, $\mathbb{G}_\infty$ has a Zariski topology.

5. MISCELLANEOUS FACTS FROM LIE THEORY

In this section, we collect for convenient reference a number of facts that are used in Section 3.

All locally compact groups (including all Lie groups) in this paper are assumed to be second countable.

5A. Connected Subgroups

(5.1) LEMMA (cf. [6, Thm. VIII.1.1, p. 107]). *Every almost-Zariski closed, unipotent subgroup of* $\mathrm{GL}_n(\mathbb{R})$ *is connected and simply connected.*

(5.2) LEMMA [5, Thm. XII.2.2, p. 137]. *Every connected subgroup of any simply connected, solvable Lie group G is closed and simply connected.*

(5.3) LEMMA (cf. [2, Cor. I.5.3.7, p. 47]). *If A is a connected subgroup of a Lie group G, then $A \subset \operatorname{nil} G$ if and only if $\overline{\mathrm{Ad}_G A}$ is unipotent. In particular, $\overline{\mathrm{Ad}_G \operatorname{nil} G}$ is unipotent.*

(5.4) LEMMA [5, Thm. XVI.1.1, p. 188]. *If G is a connected, nilpotent Lie group, then $Z(G)$ is connected.*

(5.5) LEMMA [13, Lem. 3.20]. *Every connected, unipotent Lie subgroup of* $\mathrm{GL}_n(\mathbb{R})$ *is Zariski closed.*

5B. Compact Subgroups

(5.6) PROPOSITION [8, Satz 4; 5, Thm. XIII.1.3, p. 144]. *If a Lie group G is compact, connected, and solvable, then G is abelian. Hence $G \cong \mathbb{T}^n$, for some n.*

(5.7) LEMMA (cf. [8, Satz 5]). *If G is a connected Lie group, then every compact subgroup of $\operatorname{nil} G$ is central in G.*

(5.8) PROPOSITION [5, Thm. XV.3.1, pp. 180–181, and see p. 186]. *If G is a Lie group that has only finitely many connected components, then G has a maximal compact subgroup K, and every compact subgroup of G is contained in a conjugate of K.*

(5.9) COROLLARY. *Let G be a Lie group that has only finitely many connected components. If H is a closed, normal subgroup of G, and K is a maximal compact subgroup of G, then $H \cap K$ is a maximal compact subgroup of H.*

(5.10) COROLLARY. *Let G be a connected, solvable Lie group. Then all the maximal compact tori of $\overline{\operatorname{Ad} G}$ are conjugate under $\overline{\operatorname{Ad} G}$.*

Proof. Write $\overline{\operatorname{Ad} G} = T \ltimes U$, where T is a maximal torus of $\overline{\operatorname{Ad} G}$, and U is the unipotent radical. Let $M = [T(\operatorname{Ad} G)] \cap U$. Then $\operatorname{Ad} G \subset TM$ and TM is almost-Zariski closed (see 5.5), so we must have $TM = \overline{\operatorname{Ad} G}$. Thus, $M = U$, so $T(\operatorname{Ad} G) = \overline{\operatorname{Ad} G}$. All the maximal compact tori of $\overline{\operatorname{Ad} G}$ are conjugate under $\overline{\operatorname{Ad} G}$ (see 5.8), so, because T normalizes (indeed, centralizes) the maximal compact torus that it contains, this implies that the maximal compact tori are conjugate under $\operatorname{Ad} G$. ∎

(5.11) LEMMA [4, Thm. 7.6, p. 61]. *If K is a totally disconnected, compact group, then K is residually finite.*

5C. Representations of Nilpotent Groups

(5.12) THEOREM [11, Thm. 2.11, p. 33]. *Let G and H be simply connected, nilpotent Lie groups, and let Γ be a lattice in G. Then every homomorphism from Γ to H extends to a unique continuous homomorphism from G to H.*

(5.13) COROLLARY. *Let G be a simply connected, nilpotent Lie group. Then the trivial automorphism is the only automorphism of G that centralizes a cocompact subgroup of G.*

The following is a useful special case of the Borel Density Theorem.

(5.14) PROPOSITION [11, Thm. 3.2, p. 45]. *If Γ is a lattice in a Lie group G, and $\phi\colon G \to \mathrm{GL}_n(\mathbb{R})$ is a representation such that G^ϕ is unipotent, then $\Gamma^\phi = G^\phi$.*

5D. S-Arithmetic Groups

(5.15) THEOREM [10, Thm. 5.6, p. 264]. *Let $\mathbb{G}$ be a connected, solvable algebraic group over a number field K, and let S be a finite set of places of K, containing all the infinite places. Then $\mathbb{G}_{\mathscr{O}(S)}$ is a lattice in $\mathbb{G}_S^{(1)}$ (see Defn.* 1.9).

(5.16) EXAMPLE. $\overline{\mathbb{G}_{\mathcal{O}(S)}}$ may be a proper subgroup of $\mathbb{G}_S^{(1)}$, even if $\mathbb{G}_{\mathcal{O}(S)}$ is Zariski dense in $\mathbb{G}$. To see this, it suffices to construct an anisotropic torus $\mathbb{T}$ over a number field K, such that $\mathbb{T}_{\mathcal{O}(S)}$ is not Zariski dense in $\mathbb{T}_\infty$ (see 4.3). For then the desired example is obtained by forming a semidirect product $\mathbb{G} = \mathbb{T} \ltimes \mathbb{U}$, such that $C_{\mathbb{T}}(\mathbb{U})$ is trivial, and letting $S = S_\infty$.

Let p and q be two distinct primes, with $q \equiv 3 \pmod 4$, let $K = \mathbb{Q}(i, \sqrt{p})$, and let $\mathbb{T} = SO(x^2 + qy^2)$. Then $\mathbb{T}$ is defined over K (in fact, it is defined over $\mathbb{Q}$) and is K-anisotropic (because q is not a square in K). Now K is Galois over $\mathbb{Q}$ and has two places, both complex, so

$$\mathbb{T}_\infty = SO\big(x^2 + qy^2\big)_{\mathbb{C}} \times SO\big(x^2 + qy^2\big)_{\mathbb{C}}.$$

Let σ be the nontrivial Galois automorphism of K that fixes i, and let τ denote complex conjugation. Define $\phi\colon \mathbb{T}_\infty \to \mathbb{T}_{\mathbb{C}}$ by $(u, v)^\phi = uu^\tau vv^\tau$. Then we have

$$(\mathbb{T}_{\mathcal{O}})^\phi = \left\{(u, u^\sigma)^\phi \mid u \in \mathbb{T}_{\mathcal{O}}\right\} = \left\{uu^\tau u^\sigma u^{\sigma\tau} \mid u \in \mathbb{T}_{\mathcal{O}}\right\}.$$

Clearly, then, each element of $(\mathbb{T}_{\mathcal{O}})^\phi$ belongs to $\mathbb{T}_{\mathcal{O}}$ and is fixed by the Galois group of K. Therefore, $(\mathbb{T}_{\mathcal{O}})^\phi \subset \mathbb{T}_{\mathbb{Z}}$. But $\mathbb{T}_{\mathbb{Z}}$ is finite, because it is a discrete subset of the compact group $\mathbb{T}_{\mathbb{R}}$, so we conclude that $(\mathbb{T}_{\mathcal{O}})^\phi$ is not Zariski dense in $\mathbb{T}$. Because ϕ is a surjective morphism of algebraic groups (where $\mathbb{T}_\infty$ is endowed with the Zariski topology described in Remark 4.3), this implies that $\mathbb{T}_{\mathcal{O}}$ is not Zariski dense in $\mathbb{T}_\infty$.

5E. Centralizers and Normalizers are Almost-Zariski Closed

The following two propositions are proved in [13] only in the case where G is connected. The general case follows by applying essentially the same proofs to the connected group $G^\circ \ltimes \overline{\operatorname{Ad} G^\circ}$.

(5.17) PROPOSITION [13, Col. 3.10]. *Let A be a subgroup of G°, where G is a solvable Lie group such that $[G^\circ, G]$ is simply connected. Then the centralizer of A in $\overline{\operatorname{Ad} G}$ is almost-Zariski closed.*

(5.18) PROPOSITION [13, Cor. 3.14]. *Let A be a connected Lie subgroup of a Lie group G. Then the normalizer of A in $\overline{\operatorname{Ad} G}$ is almost-Zariski closed. In particular, $\overline{\operatorname{Ad}_G A}$ normalizes A.*

5F. Virtual Connectivity of an Inverse Image

(5.19) PROPOSITION [13, Lem. 5.6]. *Suppose G is a connected, solvable Lie group, and A is an almost-Zariski closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. If $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$ is a continuous homomorphism, such that A contains a* *maximal compact torus of* $\overline{G^\rho}$, *the the inverse image* $\rho^{-1}(A)$ *is a connected subgroup of* G.

(5.20) COROLLARY. *Let* G *be a connected, solvable Lie group, let* $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$ *be a continuous homomorphism, and let* A *be an almost-Zariski closed subgroup of* $\mathrm{GL}_n(\mathbb{R})$. *If there is a compact, abelian subgroup* S *of* G *and a compact torus* T *of* A, *such that* $S^\rho T$ *is abelian and contains a maximal compact torus of* $\overline{G^\rho}$, *then the inverse image* $\rho^{-1}(A)$ *has only finitely many connected components.*

Proof. By replacing A with $A \cap \overline{G^\rho}$, and T with $(T \cap \overline{G^\rho})^\circ$, we may assume $A \subset \overline{G^\rho}$. By replacing T with a larger torus (and replacing S with a conjugate that commutes with this larger torus (see 5.10)), we may assume T is a maximal compact torus of A. Let H be the (unique) almost-Zariski closed, connected subgroup of $\overline{G^\rho}$ that has T as a maximal compact torus and satisfies $HS^\rho = \overline{G^\rho}$; let $S_1 = S^\varphi/(H \cap S^\rho)$. Now H contains the commutator subgroup of $\overline{G^\rho}$, so there is a natural homomorphism $\overline{G^\rho} \to \overline{G^\rho}/H \cong S_1$. Let $\sigma\colon G \to S_1$ be the composition of ρ with this homomorphism.

Let K be the kernel of σ; we begin by showing that K has only finitely many connected components. Replacing S by a subgroup, we may assume $K \cap S$ is finite. The fibration $K \cap S \to S \xrightarrow{\sigma} S_1$ yields the following long exact sequence of homotopy groups [12, Cor. IV.8.6, p. 187]:

$$\pi_1(S) \to \pi_1(S_1) \to \pi_0(K \cap S).$$

Because $K \cap S$ is finite, we conclude that the cokernel of the map $\pi_1(S) \to \pi_1(S_1)$ is finite. Because $S \subset G$, this implies that the cokernel of the map $\pi_1(G) \xrightarrow{\sigma_*} \pi_1(S_1)$ is finite. Thus, from the long exact sequence

$$\pi_1(G) \xrightarrow{\sigma_*} \pi_1(S_1) \to \pi_0(K) \to \pi_0(G) = 0,$$

obtained from the fibration $K \to G \to S_1$, we conclude that $\pi_0(K)$ is finite, as desired.

Because $A \subset H$, it is easy to see that $\rho^{-1}(A) \subset K$. Thus, the conclusion of the preceding paragraph implies that K° contains a finite-index subgroup of $\rho^{-1}(A)$, so there is no harm in replacing G with K, so we may assume $G^\sigma = e$, which means $S^\rho \subset H$, so A contains a maximal compact torus of $\overline{G^\rho}$. Hence, the proposition implies that $\rho^{-1}(A)$ is connected. ∎

(5.21) *Remark.* $\rho^{-1}(A)$ need not be connected, even if the restriction of ρ to T is faithful. For example, in $\mathbb{T}^3$, let

- $G = \{(e^{it}, e^{i\lambda t})\} \times \mathbb{T}$ (where λ is irrational),
- $A = \mathbb{T} \times \{(e^{2is}, e^{is})\}$,
- $T = \{(1,1)\} \times \mathbb{T}$, and
- $\rho =$ the inclusion $G \hookrightarrow \mathbb{T}^3$.

Then $\rho^{-1}(A) = G \cap A$ has two components:

$$\left\{\left(e^{it}, e^{i\lambda t}, e^{i\lambda t/2}\right)\right\} \qquad \text{and} \qquad \left\{\left(e^{it}, e^{i\lambda t}, -e^{i\lambda t/2}\right)\right\}.$$

5G. The Commutator Subgroup of a Lattice

(5.22) LEMMA. *Suppose*

- G *is a solvable Lie group such that* G° *is simply connected, and* $[G,G] = [G,G^\circ]$ *is nilpotent*;
- Γ *is a subgroup of an abstract group* Λ, *such that* Λ *commensurabilizes* Γ; *and*
- $\phi\colon \Lambda \to G$ *is a homomorphism such that* $\overline{\mathrm{Ad}_G \Lambda^\phi} = \overline{\mathrm{Ad}\, G}$, *the restriction of* ϕ *to* Γ *is faithful, and* Γ^ϕ *is a lattice in* G°.

Then $(\Gamma \cap [\Lambda,\Lambda])^\phi$ *is a lattice in* $[G,G]$.

Proof. Let U be the unique syndetic hull of $(\Gamma \cap [\Lambda,\Lambda])^\phi$ in $[G,G]$ (see 2.4). Since Λ commensurabilizes $\Gamma \cap [\Lambda,\Lambda]$, the uniqueness of U implies that Λ^ϕ normalizes U. Because $\overline{\mathrm{Ad}_G \Lambda^\phi} = \overline{\mathrm{Ad}\, G}$, this implies that U is a normal subgroup of G (see 5.18). There is no harm in modding out U, so we may assume $\Gamma \cap [\Lambda,\Lambda] = e$.

Let us now show that

(∗) each $\lambda \in \Lambda$ centralizes a finite-index subgroup of Γ.

Because Λ commensurabilizes Γ, there is some finite-index subgroup N of Γ with $N^\lambda \subset \Gamma$. Then $[N,\lambda] \subset \Gamma \cap [\Lambda,\Lambda] = e$, so λ centralizes N.

By replacing Γ with a subgroup of finite index, we may assume that $\overline{\mathrm{Ad}_G \Gamma^\phi}$ is connected, in which case, (∗) implies that $\mathrm{Ad}_G \Lambda^\phi$ centralizes $\overline{\mathrm{Ad}_G \Gamma \phi}$. Because $\overline{\mathrm{Ad}_G \Lambda^\phi} = \overline{\mathrm{Ad}\, G}$, this implies that $\overline{\mathrm{Ad}_G \Gamma^\phi} \subset Z(\overline{\mathrm{Ad}\, G})$. Therefore, because G°/Γ^ϕ is compact, we see that the image of Γ° in $\overline{\mathrm{Ad}\, G}/Z(\overline{\mathrm{Ad}\, G})$ is compact. But compact, connected Lie groups are abelian (see 5.6), so we conclude that G° is nilpotent.

Therefore, (∗) implies that Λ centralizes G° (see 5.17). Hence $[G,G] = [G,G^\circ] = e$, so the desired conclusion is trivially true. ∎

6. ERRATA TO [13]

- In the second sentence of the abstract, Γ should be assumed to be discrete.
- The reference for Lemma 3.21 should be to [6], which was mistakenly omitted from the bibliography.
- The reference for Proposition 5.2 and Corollary 5.3 should be [R, Thm. 2.1, p. 29] and [R, Prop. 2.5, p. 31], respectively.
- In the second paragraph of Step 5 of the proof of Theorem 6.4, the reference should be to Proposition 5.4, not 5.2.
- In the statement of Proposition 6.10, one must assume $G_1^{\beta} \subset \overline{\Gamma_1^{\alpha}}$.
- The proof of Proposition 6.10 should begin by noting that, because $G_1^{\beta_1}$ is connected, there is no harm in assuming G_2 is connected.
- There is an error in the proof of Theorem 9.9. Lines 4–6 of page 191 (immediately following the displayed equation) should be replaced with the following: "and $C_{L_H}(\operatorname{Rad} H)$ has no compact, connected, normal subgroups, we conclude that $F = \overline{\operatorname{graph} \alpha} \cap (e \times L_H)$ is finite. Thus, $(\overline{\operatorname{graph} \alpha} \cap (L \times L_H))/F$ is the graph of a well-defined homomorphism $\overline{\beta}'_L \colon L \to L_H/F$. Because L is algebraically simply connected, we can lift $\overline{\beta}'_L$ to a homomorphism $\beta'_L \colon L \to L_H$. Note that β'_L agrees with α on a finite-index subgroup of $L \cap \Gamma$ (cf. the argument in the final paragraph of this proof). Therefore, by replacing β_L with β'_L, we may assume σ is trivial. In other words, β_L extends α_L." ∎

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