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Some fundamental algebraic tools for the semantics of computation: Part 3. Indexed categories*

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Abstract

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This paper presents *indexed categories* which model uniformly defined families of categories, and suggests that they are a useful tool for the working computer scientist. An indexed category gives rise to a single *flattened* category as a disjoint union of its component categories plus some additional morphisms. Similarly, an indexed functor (which is a uniform family of functors between the components categories) induces a flattened functor between the corresponding flattened categories. Under certain assumptions, flattened categories are (co)complete if all their components are, and flattened functors have left adjoints if all their components do. Several examples are given. Although this paper is Part 3 of the series "Some fundamental algebraic tools for the semantics of computation", it is entirely independent of Parts 1 and 2.

1. Introduction

Category theory has played an important role in clarifying, generalising, and developing results in both the theory and practice of computing. Many examples

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occur in algebraic specification, which used initiality in the very beginning to explicate the concept of abstract data type [20], and later used final objects [41], left adjoints [40, 8], colimits [5], comma categories [15], 2-categories [14, 16], and sketches [22, 2]. Some early applications of category theory to various topics may be found in the collection [28], and some recent applications to programming language semantics of 2-categories, Kleisli categories, and indexed categories may be found in [30, 31]. Taylor [39] applies indexed category theory to recursive domains and polymorphism.

Institutions [17, 18] use category theory to formalise the concept of logical system. Topics studied here include specification languages (Clear [5, 6], ASL [32], Extended ML [34]), implementations [3, 35], observational equivalence [33], free constructions [36, 38], and model theory [37]. It is hard to see how this work could be done adequately without categorical tools.

This paper is the third in a series [15, 16] intended to introduce fundamental concepts and techniques from category theory to the working computer scientist, but it is entirely independent of the previous parts. Its goal is to present indexed categories. Many-sorted algebras are a prime example with which the reader may already be familiar: for each many-sorted algebraic signature Σ, there is a category $\mathbf{Alg}(\Sigma)$ of Σ-algebras, and a signature morphism $\sigma:\Sigma\to\Sigma'$ induces a functor $\mathbf{Alg}(\sigma):\mathbf{Alg}(\Sigma')\to\mathbf{Alg}(\Sigma)$, which we call a σ-reduct. Thus, there is a functor $\mathbf{Alg}:\mathbf{AlgSig}^{op}\to\mathbf{Cat}$ from the (index) category of signatures to the category of categories. The mathematics literature [25] develops indexed categories "up to coherent isomorphism" and is not very accessible to the average computer scientist. In contrast, this paper develops "strict" indexed categories, which are defined "up to equality", a special case that often arises in theoretical computer science.

Any indexed category gives rise to a "flattened" category by taking the disjoint union of the component categories and adding reduct morphisms. A flattened indexed category has a projection functor, which maps each object to the index of the component category from which it came. This is the "fibred category" [23] presented by the indexed category. Benabou [4] argues that fibred categories formalise the same intuition as indexed categories, but are easier to work with and conceptually simpler. However, his argument does not apply to our strict indexed categories, which are simpler still, and are not proposed for use in foundations, but only as a tool for doing theoretical computer science.

Colimits have been used to "put together" many different kinds of structure, including general systems [11, 19], theories [6, 15, 16], and labelled graphs [9]. The dual concept of limit, particularly the special case of equaliser, has also been applied, for example to study unification in computing and in linguistics [13]. It is especially convenient to use these constructions when every diagram has a (co)limit, i.e. when the category is (co)complete. Section 3 shows that under certain conditions, if all component categories are (co)complete, then so is the flattened category. This simplifies (co)completeness proofs for some categories.

Given two categories indexed over the same category, an indexed functor between them is a family of functors between their component categories that is consistent with the functors induced by the index morphisms. An indexed functor induces a flattened functor between its flattened source and target categories. If all the components of an indexed functor have left adjoints, then so does the flattened functor. This can simplify proofs that some functors have left adjoints. See Section 4.

Although these results may be in the folklore, they seem not to have been previously published[1]. We believe they deserve an exposition for the working computer scientist. We assume familiarity only with basic category theory and universal algebra; such material may be found in [7, 27, 24, 1] and other places; see also [12] for some guidelines for applying category theory. Composition is denoted by "; " (semicolon) in any category, and written in the diagrammatic order; identities are denoted by *id*, possibly with subscripts. Our exposition proceeds in what Benabou [4] calls "naive category theory," without commitment to any particular foundation; indeed, nearly any foundation that has been proposed for category theory is adequate for this paper[2].

2. Indexed categories

It may be surprising to realise that categories over a collection of indices are quite common. In many natural examples, the categories in a family are uniformly defined, in the sense that any index morphism induces a translation functor between the corresponding component categories; moreover, the translation goes in the opposite direction from the index morphism in these examples. Here is a simple example that is still quite typical.

Example 1 (*Many-sorted sets*). Given a set S, there is a category $\mathbf{SSET}(S)$ of S-sorted (or S-indexed) sets, with S-sorted functions as morphisms,

$$\mathbf{SSET}(S) = [S \to \mathbf{Set}],$$

where **Set** is the category of sets, $[S \to \mathbf{Set}]$ is the category of functors from S to **Set** with S viewed as a discrete category and with natural transformations as morphisms under vertical composition (cf. [27, II.4, p. 40]). We may write $X : S \to \mathbf{Set}$ as $\langle X_s \rangle_{s \in S}$, where $X_s = X(s)$ for $s \in S$, and write $g : X \to Y$ in $\mathbf{SSET}(S)$ as $\langle g_s : X_s \to Y_s \rangle_{s \in S}$.

Since indices are sets, index morphisms are functions, and $f : S1 \to S2$ induces a functor $\mathbf{SSET}(f) : \mathbf{SSET}(S2) \to \mathbf{SSET}(S1)$ defined as follows:

- *on objects*: Given $X \in |\mathbf{SSET}(S2)|$, let $\mathbf{SSET}(f)(X) = f; X : S1 \to \mathbf{Set}$ (noting that $X : S2 \to \mathbf{Set}$), i.e. for $s1 \in S1$, let $(\mathbf{SSET}(f)(X))_{s1} = X_{f(s1)}$.
- *on morphisms*: Given $g = \langle g_{s2} : X_{s2} \to Y_{s2} \rangle_{s2 \in S2} : X \to Y$ in $\mathbf{SSET}(S2)$, let $\mathbf{SSET}(f)(g) = \langle g_{f(s1)} : X_{f(s1)} \to Y_{f(s1)} \rangle_{s1 \in S1} : f; X \to f; Y$.

[1] After reading a draft of this paper, John Gray pointed out that Gray [21] develops similar ideas for fibred categories. In particular, his Theorem 4.2 and Proposition 4.1 yield our Theorem 1.

[2] A reader who is nervous about foundations may, for example, check that each of our constructions can be placed at an appropriate level in a hierarchy of universes such as that described in [27].

These induced functors are independent of how index morphisms are decomposed, in the sense that $\mathbf{SSET}(f;f')=\mathbf{SSET}(f');\mathbf{SSET}(f)$; i.e. **SSET** is a (contravariant) functor,

$$\mathbf{SSET}:\mathbf{Set}^{op}\to\mathbf{Cat}.$$

This motivates the following definition.

Definition 1. An *indexed category* $\mathbf{C}$ over an *index category* **Ind** is a functor $\mathbf{Ind}^{op}\to\mathbf{Cat}$. Given an index $i\in|\mathbf{Ind}|$, we may write $\mathbf{C}_i$ for the category $\mathbf{C}(i)$, and given an index morphism $\sigma:i\to j$, we may write $\mathbf{C}_\sigma$ for the functor $\mathbf{C}(\sigma):\mathbf{C}(j)\to\mathbf{C}(i)$. Also, we may call $\mathbf{C}_i$ the ith *component category* of $\mathbf{C}$, and $\mathbf{C}_\sigma$ the *translation functor* induced by σ.

This presents a contravariant functor as a (covariant) functor from the opposite of its source category. While it might seem equally reasonable to present it as a functor from its source category to the opposite of its target category, this would give an unnatural direction to the component morphisms of natural transformations between such functors.

Often, we want to consider the components of an indexed category together in a single "flattened" category obtained by forming a disjoint union of the components and adding some new morphisms based on the index morphisms; this is the so-called "Grothendieck construction" [23].

Example 1 (*continued*). Flattening the indexed category $\mathbf{SSET}:\mathbf{Set}^{op}\to\mathbf{Cat}$ yields the category $\mathbf{SSet}=\mathbf{Flat}(\mathbf{SSET})$ of many-sorted sets, defined as follows:

- *Objects* are many-sorted sets with an explicitly given sort set, i.e. they are pairs $\langle S, X\rangle$, where S is a set (of sorts) and $X:S\to\mathbf{Set}$.
- *Morphisms*: A morphism from $\langle S, X\rangle$ to $\langle S', X'\rangle$ is a pair $\langle f, g\rangle$, where $f:S\to S'$ is a function and $g:X\to f;X'$ is an S-sorted function $\langle g_s:X_s\to X'_{f(s)}\rangle_{s\in S}$.
- *Composition* is defined componentwise, re-indexing the second component. Given $\langle f,g\rangle:\langle S,X\rangle\to\langle S',X'\rangle$ and $\langle f',g'\rangle:\langle S',X'\rangle\to\langle S'',X''\rangle$, let

$$\langle f,g\rangle;\langle f',g'\rangle=\langle f;f',\bar{g}\rangle:\langle S,X\rangle\to\langle S'',X''\rangle,$$

where $\bar{g}=g;\mathbf{SSET}(f)(g')=\langle g_s;g'_{f(s)}:X_s\to X''_{f'(f(s))}\rangle_{s\in S}$.

Definition 2. Given an indexed category $\mathbf{C}:\mathbf{Ind}^{op}\to\mathbf{Cat}$, define the category $\mathbf{Flat}(\mathbf{C})$ as follows:

- *Objects* are pairs $\langle i, a\rangle$, where $i\in|\mathbf{Ind}|$ and $a\in|\mathbf{C}_i|$.
- *Morphisms* from $\langle i,a\rangle$ to $\langle j,b\rangle$ are pairs $\langle\sigma,f\rangle$, where $\sigma:i\to j$ is a morphism in **Ind** and $f:a\to\mathbf{C}_\sigma(b)$ is a morphisms in $\mathbf{C}_i$.

- *Composition*: Given morphisms $\langle\sigma, f\rangle : \langle i, a\rangle \to \langle j, b\rangle$ and $\langle\rho, g\rangle : \langle j, b\rangle \to \langle k, c\rangle$ in **Flat(C)**, let

$$\langle\sigma, f\rangle ; \langle\rho, g\rangle = \langle\sigma ; \rho, f ; C_\sigma(g)\rangle : \langle i, a\rangle \to \langle k, c\rangle.$$

Such a flattened category has a functor extracting the first component of its pairs, which is another important feature of the Grothendieck fibration.

Definition 3. Given an indexed category $\mathbf{C} : \mathbf{Ind}^{op} \to \mathbf{Cat}$, define its *projection functor*

$$\mathbf{Proj}_{\mathbf{C}} : \mathbf{Flat(C)} \to \mathbf{Ind}$$

as follows:
- *on objects*: Given an object $\langle i, a\rangle$ in **Flat(C)**, let $\mathbf{Proj}_{\mathbf{C}}(\langle i, a\rangle) = i$.
- *on morphisms*: Given a morphisms $\langle\sigma, f\rangle$ in **Flat(C)**, let $\mathbf{Proj}_{\mathbf{C}}(\langle\sigma, f\rangle) = \sigma$.

We conclude this section with some further examples.

Example 2 (*Many-sorted algebraic signatures*). Given a set S, the category of S-sorted algebraic signatures is the functor category

$$\mathbf{ALGSIG}(S) = [S^+ \to \mathbf{Set}],$$

where S^+ is the set of all finite nonempty sequences of elements of S, regarded as a discrete category; equivalently, $\mathbf{ALGSIG}(S) = \mathbf{SSET}(S^+)$. Thus, an S-sorted algebraic signature is a family of sets (of operation symbols), one for each finite nonempty sequence of elements of S; such a sequence represents the *rank*, i.e. the arity and result sorts, of the operation symbols in the set that it indexes. An S-sorted algebraic signature morphism is a renaming of operation symbols that preserves their rank.

The map $S \mapsto S^+$ extends to a functor $(\_)^+ : \mathbf{Set} \to \mathbf{Set}$, and the indexed category of algebraic signatures is[3]

$$\mathbf{ALGSIG} = (\_)^+ ; \mathbf{SSET} : \mathbf{Set}^{op} \to \mathbf{Cat}.$$

The translation functor $\mathbf{ALGSIG}(f) : \mathbf{ALGSIG}(S') \to \mathbf{ALGSIG}(S)$ induced by a function $f : S \to S'$ extracts an S-sorted algebraic signature from an S'-sorted algebraic signature using f to rename sorts: Given an S'-sorted algebraic signature Σ' and a sequence $s_1 \ldots s_n \in S^+$, the operation symbols of rank $s_1 \ldots s_n$ in the S-sorted algebraic signature $\mathbf{ALGSIG}(f)(\Sigma')$ are exactly the operation symbols of rank $f(s_1) \ldots f(s_n) \in (S')^+$ from Σ'.

Flattening **ALGSIG** gives the usual category of algebraic signatures (e.g. [7]),

$$\mathbf{AlgSig} = \mathbf{Flat(ALGSIG)},$$

[3] This is slightly inaccurate, since it identifies the functor $(\_)^+ : \mathbf{Set} \to \mathbf{Set}$ with its opposite, $((\_)^+)^{op} : \mathbf{Set}^{op} \to \mathbf{Set}^{op}$; although equal as functions, they are different as functors, i.e. as morphisms in **Cat**.

whose objects are pairs $\langle S, \langle \Sigma_r \rangle_{r\in S^+} \rangle$, where S is a set (of sorts) and each Σ_r is a set (of operation symbols of rank r). A morphism from $\langle S, \langle \Sigma_r \rangle_{r\in S^+} \rangle$ to $\langle S', \langle \Sigma'_r \rangle_{r\in (S')^+} \rangle$ is a pair $\langle f, g \rangle$, where $f: S \to S'$ is a sort renaming and $g = \langle g_r : \Sigma_r \to \Sigma'_{f^+(r)} \rangle_{r\in S^+}$ is an operation symbol renaming that preserves rank (as modified by f).

Example 3 (*Many-sorted algebras*). For our purposes, this is perhaps the prototypical indexed category. Given an algebraic signature Σ, then $\mathbf{ALG}(\Sigma)$ has Σ-algebras as its objects and Σ-homomorphisms as its morphisms. Given an algebraic signature morphism $\sigma: \Sigma \to \Sigma'$, then $\mathbf{ALG}(\sigma)$ is the usual *σ-reduct* (or generalised forgetful) functor

$$\_|_\sigma : \mathbf{ALG}(\Sigma') \to \mathbf{ALG}(\Sigma),$$

as defined, for example, in [7]. Thus, the category **AlgSig** of algebraic signature provides indices for the indexed category of many-sorted algebras,

$$\mathbf{ALG} : \mathbf{AlgSig}^{op} \to \mathbf{Cat}.$$

An object in the flattened category $\mathbf{Flat}(\mathbf{ALG})$ of many-sorted algebras is a many-sorted algebra with an explicitly given signature; and a morphism from $\langle \Sigma, A \rangle$ to $\langle \Sigma', A' \rangle$ is a signature morphism $\sigma: \Sigma \to \Sigma'$ and a Σ-homomorphism $h: A \to A'|_\sigma$. Similar "cryptomorphisms" occur in the specification literature, e.g. [26].

Example 4 (*Diagrams*). A *diagram* in a category **T** is a functor to **T** from a small source category, say **G**, which is its *shape*. This is essentially equivalent to the more elementary definition of a diagram as a graph with nodes labelled by objects of **T** and edges labelled by morphisms of **T** having appropriate source and target (e.g. see [15]). Thus, the category $\mathbf{FUNC(T)(G)} = [\mathbf{G} \to \mathbf{T}]$ of functors from **G** to **T** is the category of diagrams with shape **G** in **T**. Then

$$\mathbf{FUNC(T)} : \mathbf{Cat}^{op} \to \mathbf{Cat}$$

is an indexed category with

- *component categories*: $\mathbf{FUNC(T)(G)} = [\mathbf{G} \to \mathbf{T}]$;
- *translation functors*: $\boldsymbol{\Phi}: \mathbf{G} \to \mathbf{G}'$ induces $\mathbf{FUNC(T)}(\boldsymbol{\Phi}) : [\mathbf{G}' \to \mathbf{T}] \to [\mathbf{G} \to \mathbf{T}]$, a functor defined on objects by $\mathbf{FUNC(T)}(\boldsymbol{\Phi})(\mathbf{D}') = \boldsymbol{\Phi} ; \mathbf{D}'$ for $\mathbf{D}' : \mathbf{G}' \to \mathbf{T}$.

Flattening **FUNC(T)** gives the category $\mathbf{Func(T)} = \mathbf{Flat(FUNC(T))}$ of functors into **T**, or diagrams in **T**. A morphism from $\mathbf{D} : \mathbf{G} \to \mathbf{T}$ to $\mathbf{D}' : \mathbf{G}' \to \mathbf{T}$ in **Func(T)** is a functor $\boldsymbol{\Phi} : \mathbf{G} \to \mathbf{G}'$ plus a natural transformation $\alpha : \mathbf{D} \to \boldsymbol{\Phi} ; \mathbf{D}'$ (between functors in $[\mathbf{G} \to \mathbf{T}]$). Goguen [11] applies a similar category in General Systems Theory.

Example 5 (*Theories*). The notion of institution in [17] provides an appropriate framework for considering theories in arbitrary logical systems. An *institution* **I** consists of

(1) a category **Sign** (of *signatures*);

(2) functor $\mathbf{Mod}:\mathbf{Sign}^{op}\to\mathbf{Cat}$ (giving for each $\Sigma\in|\mathbf{Sign}|$ a category $\mathbf{Mod}(\Sigma)$ of *Σ-models*);
(3) a functor $\mathbf{Sen}:\mathbf{Sign}\to\mathbf{Cat}$ (giving for each $\Sigma\in|\mathbf{Sign}|$ a discrete category $\mathbf{Sen}(\Sigma)$ of *Σ-sentences*); and
(4) for each $\Sigma\in|\mathbf{Sign}|$, a (*satisfaction*) relation $\models_\Sigma\subseteq|\mathbf{Mod}(\Sigma)|\times\mathbf{Sen}(\Sigma)$,

such that the following *satisfaction condition* holds for each $\sigma:\Sigma\to\Sigma'$ in **Sign**, each $m'\in|\mathbf{Mod}(\Sigma')|$ and $\varphi\in\mathbf{Sen}(\Sigma)$,

$$m'\models_{\Sigma'}\mathbf{Sen}(\sigma)(\varphi)\;\Leftrightarrow\;\mathbf{Mod}(\sigma)(m')\models_\Sigma\varphi.$$

Given $\sigma:\Sigma\to\Sigma'$, we may write $\mathbf{Sen}(\sigma)$ as just σ and $\mathbf{Mod}(\sigma)$ as $\_|_\sigma$.

This definition involves two indexed categories: **Mod**, indexed by **Sign**, and **Sen**, indexed by $\mathbf{Sign}^{op}$. However, we want to focus here on the indexed category **TH** of theories in **I**, which arises naturally in the study of specifications over **I**. Given $\Sigma\in|\mathbf{Sign}|$, a *Σ-presentation* is a set of Σ-sentences, $\Psi\subseteq\mathbf{Sen}(\Sigma)$. Any such Ψ generates the set of its *logical consequences*,

$$Cl_\Sigma(\Psi)=\{\varphi\in\mathbf{Sen}(\Sigma)\,|\,\text{for all } m\in|\mathbf{Mod}(\Sigma)|,\ m\models\varphi \text{ whenever } m\models\Psi\}.$$

A *Σ-theory* is a Σ-presentation T that is closed under semantic consequence, i.e. such that $T=Cl_\Sigma(T)$. Let $\mathbf{TH}(\Sigma)$ denote the poset category of Σ-theories ordered by inclusion. This extends to an indexed category

$$\mathbf{TH}:\mathbf{Sign}^{op}\to\mathbf{Cat}$$

in which given $\sigma:\Sigma\to\Sigma'$ and a Σ'-theory T',

$$\mathbf{TH}(\sigma)(T')=\{\varphi\in\mathbf{Sen}(\Sigma)\,|\,\sigma(\varphi)\in T'\}.$$

The satisfaction condition implies that this is a Σ-theory, and it is straightforward to check that $\mathbf{TH}(\sigma)$ is a functor, i.e. a monotone map.

Flattening this yields $\mathbf{Th}=\mathbf{Flat}(\mathbf{TH})$, the usual category of theories in an institution **I** [17]: its objects are pairs $\langle\Sigma, T\rangle$, where Σ is a signature and T is a Σ-theory; and its morphisms from $\langle\Sigma, T\rangle$ to $\langle\Sigma', T'\rangle$ are signature morphisms $\sigma:\Sigma\to\Sigma'$ such that $\sigma(\varphi)\in T'$ for all $\varphi\in T$.

We can define a somewhat larger indexed category of presentations. Given Σ, let $\mathbf{PRES}(\Sigma)$ be the poset category of Σ-presentations in **I**. This yields an indexed category

$$\mathbf{PRES}:\mathbf{Sign}^{op}\to\mathbf{Cat},$$

where given $\sigma:\Sigma\to\Sigma'$ in **Sign** and $\Psi'\subseteq\mathbf{Sen}(\Sigma')$,

$$\mathbf{PRES}(\sigma)(\Psi')=\{\varphi\in\mathbf{Sen}(\Sigma)\,|\,\sigma(\varphi)\in\Psi'\}.$$

We can add some further morphisms to the component categories: given Σ, let $\mathbf{PRES}_\models(\Sigma)$ be the category of Σ-presentations preordered by the semantic consequence relation, $\Psi'\models_\Sigma\Psi$ iff $\Psi\subseteq Cl_\Sigma(\Psi')$. This gives an indexed category

$$\mathbf{PRES}_\models:\mathbf{Sign}^{op}\to\mathbf{Cat}.$$

The satisfaction condition implies that $\mathbf{PRES}_{\models}(\sigma):\mathbf{PRES}_{\models}(\Sigma')\to\mathbf{PRES}_{\models}(\Sigma)$, defined just as $\mathbf{PRES}(\sigma)$ above, preserves semantic consequence.

TH is an *indexed subcategory* of **PRES** in a sense that will be made precise in Example 8 of Section 4 below; similarly, **PRES** is an indexed subcategory of $\mathbf{PRES}_{\models}$.

Example 6 (*Institutions*). We first recall the definition of institution morphism from [17]. Given two institutions $\mathbf{I}=\langle\mathbf{Sign},\mathbf{Mod},\mathbf{Sen},\langle\models_{\Sigma}\rangle_{\Sigma\in|\mathbf{Sign}|}\rangle$ and $\mathbf{I}'=\langle\mathbf{Sign}',\mathbf{Mod}',\mathbf{Sen}',\langle\models'_{\Sigma'}\rangle_{\Sigma'\in|\mathbf{Sign}'|}\rangle$, an *institution morphism* from $\mathbf{I}$ to $\mathbf{I}'$ consists of

(1) a functor $\Phi:\mathbf{Sign}\to\mathbf{Sign}'$;
(2) a natural transformation $\beta:\mathbf{Mod}\to\Phi;\mathbf{Mod}'$; and
(3) a natural transformation $\alpha:\Phi;\mathbf{Sen}'\to\mathbf{Sen}$

such that the following *satisfaction condition* holds for each $\Sigma\in|\mathbf{Sign}|$, $m\in|\mathbf{Mod}(\Sigma)|$ and $\varphi'\in\mathbf{Sen}'(\Phi(\Sigma))$,

$$m\models_{\Sigma}\alpha_{\Sigma}(\varphi') \Leftrightarrow \beta_{\Sigma}(m)\models'_{\Phi(\Sigma)}\varphi'.$$

Intuitively, $\mathbf{I}$ is "richer" than $\mathbf{I}'$: Φ extracts simpler $\mathbf{I}'$-signatures from more complex $\mathbf{I}$-signatures; β extracts simpler $\mathbf{I}'$-models from more complex $\mathbf{I}$-models; and α translates $\mathbf{I}'$-sentences to $\mathbf{I}$-sentences, which is possible since $\mathbf{I}$ is more expressive.

Institutions and institution morphisms, with composition defined componentwise in a rather straightforward manner, form a category [17]. We wish to describe it using indexed categories. It costs no more to generalise from logical systems in which the meanings of sentences in models are **true** or **false**, to semantic systems in which the meanings of sentences in models lie in an arbitrary category **V**. Following [18][4] after [29], the category **Room**(**V**) of **V**-*rooms* is the comma category

$$\mathbf{Room}(\mathbf{V})=(|\_\_|\downarrow\mathbf{FUNC}_{Disc}(\mathbf{V})),$$

where $|\_\_|:\mathbf{Cat}\to\mathbf{Cat}$ is the discretisation functor and $\mathbf{FUNC}_{Disc}(\mathbf{V}):\mathbf{DCat}^{op}\to\mathbf{Cat}$ is the indexed category of functors into **V** restricted to discrete categories in **DCat** as source (see Example 4). Thus, a **V**-room is a triple $\langle\mathbf{M},\mathbf{R},S\rangle$, where $\mathbf{M}$ is a category, S is a discrete category, and $\mathbf{R}:|\mathbf{M}|\to[S\to\mathbf{V}]$. A **V**-*room morphism* $\langle\mathbf{f},g\rangle:\langle\mathbf{M},\mathbf{R},S\rangle\to\langle\mathbf{M}',\mathbf{R}',S'\rangle$ consists of a functor $\mathbf{f}:\mathbf{M}\to\mathbf{M}'$ and a function $g:S'\to S$ such that the following diagram commutes in **Cat**.

$$\begin{array}{ccc}
|\mathbf{M}| & \xrightarrow{\mathbf{R}} & [S\to\mathbf{V}] \\
{\scriptstyle |\mathbf{f}|}\downarrow & & \downarrow{\scriptstyle g;(__)} \\
|\mathbf{M}'| & \xrightarrow[\mathbf{R}']{} & [S'\to\mathbf{V}]
\end{array}$$

[4] Goguen and Burstall [18, Proposition 16] define the category of **V**-rooms to be the comma category $(|\_\_|^{op}\downarrow\mathbf{V}^{-})$, where $|\_\_|^{op}:\mathbf{Cat}^{op}\to\mathbf{Cat}^{op}$ is the opposite of the discretisation functor and $\mathbf{V}^{-}:\mathbf{DCat}\to\mathbf{Cat}^{op}$ is the opposite of our $\mathbf{FUNC}_{Disc}(\mathbf{V}):\mathbf{DCat}^{op}\to\mathbf{Cat}$. Consequently, a **V**-room is a triple $\langle\mathbf{M},\mathbf{R},S\rangle$, where $\mathbf{M}$ is a category, S is a descrete category, and $\mathbf{R}:|\mathbf{M}|\to[S\to\mathbf{V}]$ is a morphism in $\mathbf{Cat}^{op}$, i.e. $\mathbf{R}$ is a functor from $[S\to\mathbf{V}]$ to $|\mathbf{M}|$. This is a bug since $\mathbf{R}$ should go the opposite way.

That is, $\mathbf{R}'(\mathbf{f}(m)) = g;\mathbf{R}(m)$ for all $m \in |\mathbf{M}|$, i.e.

$$\mathbf{R}'(\mathbf{f}(m))(s') = \mathbf{R}(m)(g(s'))$$

for all $m \in |\mathbf{M}|$ and $s' \in S'$ (a ghost of the satisfaction condition).

The category of *generalised institutions* [18] with signature category **Sign** is the functor category

$$\mathbf{INS}(\mathbf{Sign}) = [\mathbf{Sign}^{op} \to \mathbf{Room}(\mathbf{V})].$$

This extends to an indexed category

$$\mathbf{INS} : \mathbf{Cat}^{op} \to \mathbf{Cat},$$

where the translation functor $\mathbf{INS}(\boldsymbol{\Phi}) : \mathbf{INS}(\mathbf{Sign}') \to \mathbf{INS}(\mathbf{Sign})$ is defined on objects by $\mathbf{INS}(\boldsymbol{\Phi})(\mathbf{I}') = \boldsymbol{\Phi}^{op};\mathbf{I}'$ for $\boldsymbol{\Phi} : \mathbf{Sign} \to \mathbf{Sign}'$ a functor and $\mathbf{I}' : \mathbf{Sign}'^{op} \to \mathbf{Room}(\mathbf{V})$. This naturally extends to morphisms in $\mathbf{INS}(\mathbf{Sign}')$. Finally, the flattened category of generalised institutions is $\mathbf{Ins} = \mathbf{Flat}(\mathbf{INS})$. The reader may check that if **V** is **Bool**, the category with exactly two morphisms, both identities, then this definition coincides with the explicit definitions of institution and institution morphism given above.

3. Completeness of flattened categories

This section studies how limits and colimits in a flattened category relate to the corresponding constructions in its index and component categories. Given a shape category **G**, a category **T** is **G**-*(co)complete* if any diagram of shape **G** has a (co)limit in **T**, and a functor is **G**-*(co)continuous* if it preserves the (co)limits of all diagrams of shape **G**. Then **T** is *(co)complete* if it is **G**-(co)complete for all small **G**. Similarly, a functor is *(co)continuous* if it preserves all small (co)limits.

3.1. Limits

There is no hope for constructing limits in a flattened category unless its index and component categories have limits. The only additional assumption needed is continuity of the translation functors.

Theorem 1. *If* $\mathbf{C} : \mathbf{Ind}^{op} \to \mathbf{Cat}$ *is an indexed category such that*
(1) **Ind** *is complete,*
(2) $\mathbf{C}_i$ *is complete for all indices* $i \in |\mathbf{Ind}|$, *and*
(3) $\mathbf{C}_\sigma : \mathbf{C}_j \to \mathbf{C}_i$ *is continuous for all index morphisms* $\sigma : i \to j$,
then $\mathbf{Flat}(\mathbf{C})$ *is complete.*

Proof. It suffices to prove that $\mathbf{Flat}(\mathbf{C})$ has all products and equalisers (cf. [27, Th.V.2.1, p. 109]).

Products: Given a family $\langle i_n, a_n \rangle$ for $n \in N$ of objects in $\mathbf{Flat}(\mathbf{C})$, let i be a product in **Ind** of the i_n with projections $\pi_n : i \to i_n$ for $n \in N$, and let a be a product in $\mathbf{C}_i$ of $\mathbf{C}_{\pi_n}(a_n)$

for $n \in N$ with projections $f_n : a \to \mathbf{C}_{\pi_n}(a_n)$ for $n \in N$. Then we claim that $\langle i, a \rangle$ with projections $\langle \pi_n, f_n \rangle : \langle i, a \rangle \to \langle i_n, a_n \rangle$ is a product in $\mathbf{Flat(C)}$ of the $\langle i_n, a_n \rangle$ for $n \in N$.

Given an object $\langle j, b \rangle$ in $\mathbf{Flat(C)}$ with morphisms $\langle \sigma_n, g_n \rangle : \langle j, b \rangle \to \langle i_n, a_n \rangle$ in $\mathbf{Flat(C)}$ for $n \in N$, there exists a unique index morphism $\sigma : j \to i$ such that $\sigma ; \pi_n = \sigma_n$ in **Ind** for all $n \in N$. Moreover, continuity of $\mathbf{C}_\sigma$ guarantees that $\mathbf{C}_\sigma(a)$ with projections $\mathbf{C}_\sigma(f_n) : \mathbf{C}_\sigma(a) \to \mathbf{C}_\sigma(\mathbf{C}_{\pi_n}(a_n))$ for $n \in N$ is a product in $\mathbf{C}_j$ of $\mathbf{C}_\sigma(\mathbf{C}_{\pi_n}(a_n)) = \mathbf{C}_{\sigma_n}(a_n)$ for $n \in N$. Hence, there exists a unique morphism $g : b \to \mathbf{C}_\sigma(a)$ such that $g ; \mathbf{C}_\sigma(f_n) = g_n$ in $\mathbf{C}_j$ for each $n \in N$. Then $\langle \sigma, g \rangle : \langle j, b \rangle \to \langle i, a \rangle$ is a unique morphism in $\mathbf{Flat(C)}$ such that $\langle \sigma, g \rangle ; \langle \pi_n, f_n \rangle = \langle \sigma_n, g_n \rangle$ for each $n \in N$.

Equalisers: Given morphisms $\langle \sigma 1, f 1 \rangle$, $\langle \sigma 2, f 2 \rangle : \langle i, a \rangle \to \langle j, b \rangle$ in $\mathbf{Flat(C)}$, let $\sigma : k \to i$ be an equaliser of $\sigma 1$, $\sigma 2 : i \to j$ in **Ind**. Notice that $\mathbf{C}_\sigma(\mathbf{C}_{\sigma_1}(b)) = \mathbf{C}_{\sigma ; \sigma 1}(b) = \mathbf{C}_{\sigma ; \sigma 2}(b) = \mathbf{C}_\sigma(\mathbf{C}_{\sigma 2}(b))$. Let $f : c \to \mathbf{C}_\sigma(a)$ be an equaliser of $\mathbf{C}_\sigma(f1)$, $\mathbf{C}_\sigma(f2) : \mathbf{C}_\sigma(a) \to \mathbf{C}_\sigma(\mathbf{C}_{\sigma 1}(b))$ in $\mathbf{C}_k$. We claim that $\langle \sigma, f \rangle : \langle k, c \rangle \to \langle i, a \rangle$ is an equaliser of $\langle \sigma 1, f 1 \rangle$, $\langle \sigma 2, f 2 \rangle$ in $\mathbf{Flat(C)}$. First observe that by construction we have

$$\begin{aligned} \langle \sigma, f \rangle ; \langle \sigma 1, f 1 \rangle &= \langle \sigma ; \sigma 1, f ; \mathbf{C}_\sigma(f 1) \rangle \\ &= \langle \sigma ; \sigma 2, f ; \mathbf{C}_\sigma(f 2) \rangle \\ &= \langle \sigma, f \rangle ; \langle \sigma 2, f 2 \rangle. \end{aligned}$$

Next consider $\langle \rho, g \rangle : \langle m, d \rangle \to \langle i, a \rangle$ such that

$$\langle \rho, g \rangle ; \langle \sigma 1, f 1 \rangle = \langle \rho, g \rangle ; \langle \sigma 2, f 2 \rangle$$

in $\mathbf{Flat(C)}$, i.e. $\rho ; \sigma 1 = \rho ; \sigma 2$ in **Ind** and $g ; \mathbf{C}_\rho(f 1) = g ; \mathbf{C}_\rho(f 2)$ in $\mathbf{C}_m$. By construction, there exists a unique index morphism $\theta : m \to k$ such that $\theta ; \sigma = \rho$ in **Ind**. Moreover, since $\mathbf{C}_\theta$ is continuous, $\mathbf{C}_\theta(f) : \mathbf{C}_\theta(c) \to \mathbf{C}_\theta(\mathbf{C}_\sigma(a)) = \mathbf{C}_\rho(a)$ is an equaliser of $\mathbf{C}_\theta(\mathbf{C}_\sigma(f 1)) = \mathbf{C}_\rho(f 1)$ and $\mathbf{C}_\theta(\mathbf{C}_\sigma(f 2)) = \mathbf{C}_\rho(f 2) : \mathbf{C}_\rho(a) \to \mathbf{C}_{\theta ; \sigma ; \sigma 1}(b)$ in $\mathbf{C}_m$. Hence, there is a unique morphism $h : d \to \mathbf{C}_\theta(c)$ such that $h ; \mathbf{C}_\theta(f) = g$ in $\mathbf{C}_m$. Therefore, $\langle \theta, h \rangle : \langle m, d \rangle \to \langle k, c \rangle$ is a unique morphism in $\mathbf{Flat(C)}$ such that $\langle \theta, h \rangle ; \langle \sigma, f \rangle = \langle \rho, g \rangle$. □

A sharper result can be proved in much the same way: a diagram $\mathbf{D} : \mathbf{G} \to \mathbf{Flat(C)}$ has a limit in $\mathbf{Flat(C)}$ whenever $\mathbf{D} ; \mathbf{Proj_C} : \mathbf{G} \to \mathbf{Ind}$ has a limit in **Ind** such that the component category corresponding to the limit index is **G**-complete and the translation functors induced by index morphisms into the limit index are **G**-continuous.

3.2. *Colimits*

The construction of colimits in a flattened category is not quite so simple since the proof of Theorem 1 does not directly dualise. This is because in constructing limits, it was easy to translate the objects (and morphisms) of component categories *against* index morphisms using translation functors, whereas the analogous construction for colimits requires translation *along* index morphisms. The following property provides this capability.

Definition 4. An indexed category $\mathbf{C}:\mathbf{Ind}^{op}\to\mathbf{Cat}$ is *locally reversible* if for each index morphism $\sigma:i\to j$ in **Ind**, the translation functor $\mathbf{C}_\sigma:\mathbf{C}_j\to\mathbf{C}_i$ has a left adjoint. Given $\sigma:i\to j$ in **Ind**, let us denote an arbitrary but fixed left adjoint to $\mathbf{C}_\sigma:\mathbf{C}_j\to\mathbf{C}_i$ by $\mathbf{F}_\sigma:\mathbf{C}_i\to\mathbf{C}_j$ and denote the unit of this adjunction by $\eta^\sigma:\mathbf{id}_{\mathbf{C}_i}\to\mathbf{F}_\sigma;\mathbf{C}_\sigma$.

This does not require $\mathbf{C}$ to be "globally reversible" in the sense that the family of left adjoints forms an indexed (by $\mathbf{Ind}^{op}$) category. In general, $\mathbf{F}_{\sigma;\rho}\neq\mathbf{F}_\sigma;\mathbf{F}_\rho$. However, the following fact holds.

Fact 1. *Given a locally reversible indexed category* $\mathbf{C}:\mathbf{Ind}^{op}\to\mathbf{Cat}$ *and index morphisms* $\sigma:i\to j$ *and* $\rho:j\to k$, *there is a natural isomorphism*

$$\iota_{\sigma,\rho}:\mathbf{F}_{\sigma;\rho}\to\mathbf{F}_\sigma;\mathbf{F}_\rho.$$

Proof. $\mathbf{F}_\sigma;\mathbf{F}_\rho$ is left adjoint to $\mathbf{C}_{\sigma;\rho}=\mathbf{C}_\rho;\mathbf{C}_\sigma$ (cf. [27, Th. IV.8.1, p. 101]) and any two left adjoints to the same functor are naturally isomorphic (cf. [27, Cor. IV.1.1, p. 83]). In fact, given $a\in|\mathbf{C}_i|$, then $\iota_{\sigma,\rho}(a):\mathbf{F}_{\sigma;\rho}(a)\to\mathbf{F}_\rho(\mathbf{F}_\sigma(a))$ is given by

$$\iota_{\sigma,\rho}(a)=(\eta^\sigma(a);\mathbf{C}_\sigma(\eta^\rho(\mathbf{F}_\sigma(a))))^{\#}$$

and its inverse by

$$\iota_{\sigma,\rho}^{-1}(a)=((\eta^{\sigma;\rho}(a))^{\#})^{\#}:\mathbf{F}_\rho(\mathbf{F}_\sigma(a))\to\mathbf{F}_{\sigma;\rho}(a),$$

where $f^{\#}$ denotes the morphism "adjoint" to f (the reader may determine the adjunctions to which the sharps in this formula refer). □

Definition 5. Given a locally reversible indexed category $\mathbf{C}:\mathbf{Ind}^{op}\to\mathbf{Cat}$ and an index morphism $\rho:i\to j$, any morphism $\langle\sigma,g\rangle:\langle k,a\rangle\to\langle i,b\rangle$ (with the same i) in $\mathbf{Flat(C)}$ "lifts along ρ" to a morphism in $\mathbf{C}_j$ given by

$$L_\rho(\langle\sigma,g\rangle)=\iota_{\sigma,\rho}(a);\mathbf{F}_\rho(g^{\#}):\mathbf{F}_{\sigma;\rho}(a)\to\mathbf{F}_\rho(b).$$

Lemma 1. *Under the notation and assumptions of Definition* 5, *given an index morphism* $\theta:j\to m$ *in* **Ind** *and given a morphism* $\langle\rho;\theta,f\rangle:\langle i,b\rangle\to\langle m,c\rangle$ *in* $\mathbf{Flat(C)}$, *then* $f^{\#}:\mathbf{F}_\sigma(b)\to\mathbf{C}_\theta(c)$ *is a morphism in* $\mathbf{C}_j$ *such that in* $\mathbf{Flat(C)}$,

$$\langle\sigma;\rho,\eta^{\sigma;\rho}(a)\rangle;\langle\theta,L_\rho(\langle\sigma,g\rangle);f^{\#}\rangle=\langle\sigma,g\rangle;\langle\rho;\theta,f\rangle:\langle k,a\rangle\to\langle m,c\rangle.$$

Proof. We check that in $\mathbf{C}_k$

$$\eta^{\sigma;\rho}(a);\mathbf{C}_{\sigma;\rho}(L_\rho(\langle\sigma,g\rangle);f^{\#})=g;\mathbf{C}_\sigma(f):a\to\mathbf{C}_{\sigma;\rho;\theta}(c)$$

as follows.

$$
\begin{aligned}
&\eta^{\sigma;\rho}(a);\mathbf{C}_{\sigma;\rho}(L_\rho(\langle\sigma,g\rangle);f^{\#}) && \text{(Definition 5)}\\
&=\eta^{\sigma;\rho}(a);\mathbf{C}_{\sigma;\rho}(\iota_{\sigma,\rho}(a));\mathbf{C}_{\sigma;\rho}(\mathbf{F}_\rho(g^{\#});f^{\#}) && \text{(proof of Fact 1)}\\
&=\eta^{\sigma}(a);\mathbf{C}_{\sigma}(\eta^{\rho}(\mathbf{F}_\sigma(a));\mathbf{C}_{\sigma;\rho}(\mathbf{F}_\rho(g^{\#});f^{\#}) && (\mathbf{C}_{\sigma;\rho}=\mathbf{C}_\rho;\mathbf{C}_\sigma)\\
&=\eta^{\sigma}(a);\mathbf{C}_{\sigma}(\eta^{\rho}(\mathbf{F}_\sigma(a));\mathbf{C}_{\rho}(\mathbf{F}_\rho(g^{\#}));\mathbf{C}_\rho(f^{\#})) && \text{(naturality of } \eta^\rho)\\
&=\eta^{\sigma}(a);\mathbf{C}_{\sigma}(g^{\#};\eta^{\rho}(b);\mathbf{C}_\rho(f^{\#})) && (f=\eta^\rho(b);\mathbf{C}_\rho(f^{\#}))\\
&=\eta^{\sigma}(a);\mathbf{C}_{\sigma}(g^{\#});\mathbf{C}_\sigma(f) && (g=\eta^\sigma(a);\mathbf{C}_\sigma(g^{\#}))\\
&=g;\mathbf{C}_\sigma(f). \qquad \square
\end{aligned}
$$

Corollary 1. *Under the notation and assumptions of Definition* 5

$$\eta^{\sigma;\rho}(a);\mathbf{C}_{\sigma;\rho}(L_\rho(\langle\sigma,g\rangle))=g;\mathbf{C}_\sigma(\eta^\rho(b))$$

Proof. By Lemma 1, since $\eta^\rho(b)^{\#}=\mathrm{id}_{\mathbf{F}_\rho(b)}$. $\square$

We are now ready for the main result.

Theorem 2. *If* $\mathbf{C}:\mathbf{Ind}^{op}\to\mathbf{Cat}$ *is an indexed category such that*

(1) **Ind** *is cocomplete*;

(2) $\mathbf{C}_i$ *is cocomplete for all* $i\in|\mathbf{Ind}|$; *and*

(3) **C** *is locally reversible*,

then **Flat(C)** *is cocomplete.*

Proof. Dually to the proof of Theorem 1, it suffices to prove that **Flat(C)** has all coproducts and coequalisers.

Coproducts: Given a family $\langle i_n, a_n\rangle$ for $n\in N$ of objects in **Flat(C)**, let i with injections $\rho_n: i_n\to i$ be a coproduct in **Ind** of the i_n for $n\in N$, and let a be a coproduct in $\mathbf{C}_i$ of the $\mathbf{F}_{\rho_n}(a_n)$ for $n\in N$ with injections $f_n^{\#}:\mathbf{F}_{\rho_n}(a_n)\to a$ for $n\in N$. Now define $f_n=\eta^{\rho_n}(a_n);\mathbf{C}_{\rho_n}(f_n^{\#}):a_n\to\mathbf{C}_{\rho_n}(a)$ for $n\in N$. Then we claim that $\langle i, a\rangle$ with injections $\langle\rho_n, f_n\rangle:\langle i_n, a_n\rangle\to\langle i, a\rangle$ for $n\in N$, is a coproduct in **Flat(C)** of the $\langle i_n, a_n\rangle$ for $n\in N$.

Given an object $\langle j, b\rangle$ and morphisms $\langle\sigma_n, g_n\rangle:\langle i_n, a_n\rangle\to\langle j, b\rangle$ in **Flat(C)** for $n\in N$, there exists a unique index morphism $\sigma: i\to j$ such that $\rho_n;\sigma=\sigma_n$ in **Ind** for all $n\in N$. Moreover, there is a unique $g:a\to\mathbf{C}_\sigma(b)$ such that $f_n^{\#};g=g_n^{\#}:\mathbf{F}_{\rho_n}(a_n)\to\mathbf{C}_\sigma(b)$ for all $n\in N$ ($g_n^{\#}$ is well defined since $g_n:a_n\to\mathbf{C}_{\rho_n}(\mathbf{C}_\sigma(b))$). Now because

$$
\begin{aligned}
f_n;\mathbf{C}_{\rho_n}(g)&=\eta^{\rho_n}(a_n);\mathbf{C}_{\rho_n}(f_n^{\#});\mathbf{C}_{\rho_n}(g)\\
&=\eta^{\rho_n}(a_n);\mathbf{C}_{\rho_n}(f_n^{\#};g)\\
&=\eta^{\rho_n}(a_n);\mathbf{C}_{\rho_n}(g_n^{\#})\\
&=g_n
\end{aligned}
$$

in $\mathbf{C}_{i_n}$, it follows that $\langle\sigma, g\rangle:\langle i, a\rangle\to\langle j, b\rangle$ satisfies $\langle\rho_n, f_n\rangle;\langle\sigma, g\rangle=\langle\sigma_n, g_n\rangle$ in **Flat(C)** for all $n\in N$. Moreover, $\langle\sigma, g\rangle$ is the only morphism in **Flat(C)** with this property: The uniqueness of σ is obvious, and the uniqueness of g follows by its construction from the fact that given $g':a\to\mathbf{C}_\sigma(b)$ with $f_n;\mathbf{C}_{\rho_n}(g')=g_n$ for all $n\in N$, then $f_n^\#;g'=g_n^\#$ for all $n\in N$ and, thus, $g=g'$.

Coequalisers: Given morphisms $\langle\sigma 1, f1\rangle$, $\langle\sigma 2, f2\rangle:\langle i, a\rangle\to\langle j, b\rangle$ in **Flat(C)**, let $\sigma:j\to k$ be a coequaliser of $\sigma 1$, $\sigma 2:i\to j$ in **Ind**. Then in $\mathbf{C}_k$ there are morphisms (cf. Definition 5)

$$L_\sigma(\langle\sigma 1, f1\rangle),\ L_\sigma(\langle\sigma 2, f2\rangle):\mathbf{F}_{\sigma 1;\sigma}(a)\to\mathbf{F}_\sigma(b).$$

Let $f^\#:\mathbf{F}_\sigma(b)\to c$ be their coequaliser in $\mathbf{C}_k$ and let $f=\eta^\sigma(b);\mathbf{C}_\sigma(f^\#):b\to\mathbf{C}_\sigma(c)$ in $\mathbf{C}_j$. We now claim that $\langle\sigma, f\rangle:\langle j, b\rangle\to\langle k, c\rangle$ is a coequaliser in **Flat(C)** of the morphisms $\langle\sigma 1, f1\rangle$, $\langle\sigma 2, f2\rangle:\langle i, a\rangle\to\langle j, b\rangle$. First notice that by Lemma 1, in **Flat(C)** we have

$$\begin{aligned}\langle\sigma 1, f1\rangle;\langle\sigma, f\rangle&=\langle\sigma 1;\sigma, \eta^{\sigma 1;\sigma}(a)\rangle;\langle id_k, L_\sigma(\langle\sigma 1, f1\rangle);f^\#\rangle\\&=\langle\sigma 2;\sigma, \eta^{\sigma 2;\sigma}(a)\rangle;\langle id_k, L_\sigma(\langle\sigma 2, f2\rangle);f^\#\rangle\\&=\langle\sigma 2, f2\rangle;\langle\sigma, f\rangle.\end{aligned}$$

Now consider a morphism $\langle\rho, g\rangle:\langle j, b\rangle\to\langle m, d\rangle$ such that in **Flat(C)**

$$\langle\sigma 1, f1\rangle;\langle\rho, g\rangle=\langle\sigma 2, f2\rangle;\langle\rho, g\rangle,$$

i.e. such that $\sigma 1;\rho=\sigma 2;\rho$ in **Ind** and $f1;\mathbf{C}_{\sigma 1}(g)=f2;\mathbf{C}_{\sigma 2}(g)$ in $\mathbf{C}_i$. Then by construction, there exists a unique index morphism θ: $k\to m$ such that $\sigma;\theta=\rho$ in **Ind**. Moreover, by Lemma 1

$$\begin{aligned}\eta^{\sigma 1;\sigma}(a);\mathbf{C}_{\sigma 1;\sigma}(L_\sigma(\langle\sigma 1, f1\rangle);g^\#)&=f1;\mathbf{C}_{\sigma 1}(g)\\&=f2;\mathbf{C}_{\sigma 2}(g)\\&=\eta^{\sigma 2;\sigma}(a);\mathbf{C}_{\sigma 2;\sigma}(L_\sigma(\langle\sigma 2, f2\rangle;g^\#)\end{aligned}$$

in $\mathbf{C}_i$ (recall that $\sigma 1;\sigma=\sigma 2;\sigma$ and that $g^\#:\mathbf{F}_\sigma(\sigma)\to\mathbf{C}_\theta(d)$). Hence, the properties of adjunction imply $L_\sigma(\langle\sigma 2, f2\rangle);g^\#=L_\sigma(\langle\sigma 1, f1\rangle);g^\#$. Thus, there exists a unique morphism $h:c\to\mathbf{C}_\theta(d)$ such that $f^\#;h=g^\#$ in $\mathbf{C}_k$.

Now $\langle\theta, h\rangle:\langle k, c\rangle\to\langle m, d\rangle$ satisfies $\langle\sigma, f\rangle;\langle\theta, h\rangle=\langle\rho, g\rangle$ in **Flat(C)**, since in $\mathbf{C}_j$ we have $f;\mathbf{C}_\sigma(h)=\eta^\sigma(b);C_\sigma(f^\#;h)=\eta^\sigma(b);C_\sigma(g^\#)=g$. Moreover, $\langle\theta, h\rangle$ is the only morphism in **Flat(C)** with this property: the uniqueness of θ is obvious; and the uniqueness of h follows from its construction (if $f;\mathbf{C}_\sigma(h')=g$ for some $h':c\to\mathbf{C}_\theta(d)$, then $f^\#;h'=g^\#$, and thus $h=h'$). □

A sharper result can be proved in much the same way: a diagram $\mathbf{D}:\mathbf{G}\to\mathbf{Flat(C)}$ has a colimit in **Flat(C)** whenever $\mathbf{D};\mathbf{Proj_C}:\mathbf{G}\to\mathbf{Ind}$ has a colimit in **Ind** such that the component category corresponding to the colimit index is **G**-cocomplete and all the translation functors induced by the index morphisms in the colimiting cocone have left adjoints.

3.3. Applications

We can use these theorems to check the completeness and/or cocompleteness for some interesting categories. The results are already known, but our proofs are more direct.

Example 1 (*continued*). Consider again the indexed category $\mathbf{SSET}:\mathbf{Set}^{op}\to\mathbf{Cat}$ of many-sorted sets. It is well known that for any set S, the category $\mathbf{SSET}(S)$ of S-sorted sets is both complete and cocomplete, and of course the index category **Set** is also both complete and cocomplete. Moreover, it is not hard to see that the functor $\mathbf{SSET}(f):\mathbf{SSET}(S')\to\mathbf{SSET}(S)$ is continuous for any index morphism (i.e. function) $f:S\to S'$, and that it has a left adjoint (sending an S-sorted set $\langle X_s\rangle_{s\in S}$ to the S'-sorted set $\langle \biguplus\{X_s \mid f(s)=s'\}\rangle_{s'\in S'}$ where $\biguplus$ denotes disjoint union). Thus, Theorems 1 and 2 imply that the (flattened) category of many-sorted sets $\mathbf{SSet}=\mathbf{Flat}(\mathbf{SSET})$ is both complete and cocomplete.

Example 2 (*continued*). Consider the indexed category $\mathbf{ALGSIG}:\mathbf{Set}^{op}\to\mathbf{Cat}$ of many-sorted algebraic signatures. Again, the index category and all component categories are both complete and cocomplete, and the translation functors are continuous and have left adjoints (this follows from the definition $\mathbf{ALGSIG}=(\_\_)^+;\mathbf{SSET}$ since **SSET** has all these properties). Thus, the category of algebraic signatures $\mathbf{AlgSig}=\mathbf{Flat}(\mathbf{ALGSIG})$ is both complete and cocomplete.

Example 3 (*continued*). Consider the indexed category $\mathbf{ALG}:\mathbf{AlgSig}^{op}\to\mathbf{Cat}$ of many-sorted algebras. Again, the index category is complete and cocomplete (by Example 2 above), as are all component categories, and the translation (forgetful) functors are continuous and have left adjoints (the existence of left adjoints to these forgetful functors is a nontrivial, but familiar, property; see [7] for an expository presentation). Also, cocompleteness of the category of Σ-algebras is not quite obvious: to form a coproduct of Σ-algebras, form their disjoint union and then freely complete it to a Σ-algebra; coequalisers are not very hard. Theorems 1 and 2 now imply that the category $\mathbf{Flat}(\mathbf{ALG})$ of many-sorted algebras is both complete and cocomplete. This provides an appropriate framework for operations like the amalgamated union of algebras over different signatures, as used for example in [10].

Example 4 (*continued*). Let **T** be any category and consider again the indexed category $\mathbf{FUNC}(\mathbf{T}):\mathbf{Cat}^{op}\to\mathbf{Cat}$ of functors into (or diagrams in) **T**. The index category **Cat** is both complete and cocomplete. If **T** is complete, then so are all the component categories. For, given $\mathbf{G}\in|\mathbf{Cat}|$, limits in $\mathbf{FUNC}(\mathbf{T})(\mathbf{G})=[\mathbf{G}\to\mathbf{T}]$ are constructed "pointwise" as limits in **T** "parameterised" by (objects of) **G** (cf. [27, V.3, p. 112]). Moreover, the translation functors in **FUNC(T)** preserve limits constructed in this way. Thus, **Func(T)** = **Flat(FUNC(T))** is complete whenever **T** is.

Dually, if **T** is cocomplete, then the component categories are also cocomplete and the translation functors are cocontinuous. But to apply Theorem 2, we need the translation functors to have left adjoints; unfortunately, in general they do not.

It is interesting to compare this with Kan extensions (cf. [27, X]). Given a functor $\boldsymbol{\Phi}:\mathbf{G}\to\mathbf{G}'$ and a diagram $\mathbf{F}:\mathbf{G}\to\mathbf{T}$, then a *left Kan extension* of **F** along $\boldsymbol{\Phi}$ is an object $\mathbf{F}'\in|\mathbf{FUNC(T)(G')}|$ free over $\mathbf{F}\in|\mathbf{FUNC(T)(G)}|$ with respect to the functor $\mathbf{FUNC(T)}(\boldsymbol{\Phi}):\mathbf{FUNC(T)(G')}\to\mathbf{FUNC(T)(G)}$, with unit morphism $\eta_{\mathbf{F}}:\mathbf{F}\to\boldsymbol{\Phi};\mathbf{F}'$, a natural transformation between functors in $[\mathbf{G}\to\mathbf{T}]$. If every diagram $\mathbf{F}:\mathbf{G}\to\mathbf{T}$ has a left Kan extension along $\boldsymbol{\Phi}$, then the translation functor $\mathbf{FUNC(T)}(\boldsymbol{\Phi}):\mathbf{FUNC(T)(G')}\to\mathbf{FUNC(T)(G)}$ has a left adjoint. Dualising the construction of a right Kan extension [27, Th.X.1, pp. 233–4], we obtain the following proposition.

Proposition 1. *Given* $\boldsymbol{\Phi}:\mathbf{G}\to\mathbf{G}'$, *and* $\mathbf{F}:\mathbf{G}\to\mathbf{T}$, *and* $n'\in|\mathbf{G}'|$, *let* $(\boldsymbol{\Phi}\downarrow n')$ *be the comma category of objects* $\boldsymbol{\Phi}$*-over* n' (*cf.* [27, pp. 46–7]), *and let* $\mathbf{P}_{n'}:(\boldsymbol{\Phi}\downarrow n')\to\mathbf{G}$ *be the obvious projection functor, and let* $\mathbf{D}_{n'}=\mathbf{P}_{n'};\mathbf{F}:(\boldsymbol{\Phi}\downarrow n')\to\mathbf{T}$. *Now suppose that for each* $n'\in|\mathbf{G}'|$, *the diagram* $\mathbf{D}_{n'}:(\boldsymbol{\Phi}\downarrow n')\to\mathbf{T}$ *has a colimit* $\mathbf{F}'(n')\in|\mathbf{T}|$. *Then the assignment* $n'\mapsto\mathbf{F}'(n')$ *extends to a functor* $\mathbf{F}':\mathbf{G}'\to\mathbf{T}$, *using the colimit property of* $\mathbf{F}'(n')$ *for* $n'\in|\mathbf{G}'|$ *in the usual way. Moreover, there is a natural transformation* $\eta_{\mathbf{F}}:\mathbf{F}\to\boldsymbol{\Phi};\mathbf{F}'$ *such that* $\eta_{\mathbf{F},n}:\mathbf{F}\to\mathbf{F}'(\boldsymbol{\Phi}(n))$ *is the morphism in the colimiting cocone for* $\mathbf{F}'(\boldsymbol{\Phi}(n))$ *corresponding to the object* $\langle n, id_{\boldsymbol{\Phi}(n)}\rangle\in|(\boldsymbol{\Phi}\downarrow\boldsymbol{\Phi}(n))|$ *for each* $n\in|\mathbf{G}|$. *Finally,* $\mathbf{F}'$ *with the unit* $\eta_{\mathbf{F}}$ *is a left Kan extension of* **F** *along* $\boldsymbol{\Phi}$.

Proposition 2. *Given a functor* $\boldsymbol{\Phi}:\mathbf{G}\to\mathbf{G}'$ *with* **G** *small and a cocomplete category* **T**, *any functor* $\mathbf{F}:\mathbf{G}\to\mathbf{T}$ *has a left Kan extension along* $\boldsymbol{\Phi}$.

Even though the category of all diagrams in **T** need not be cocomplete when **T** is, the category of small diagrams has this property.

Proposition 3. *Let* **SCat** *be the category of all small categories, let* **T** *be a category, and let*

$$\mathbf{SFUNC(T)}:\mathbf{SCat}^{op}\to\mathbf{Cat}$$

be the indexed category of small diagrams in **T**, *defined as the restriction of* **FUNC(T)** *to* $\mathbf{SCat}^{op}$. *Then the category* **SFunc(T)** = **Flat(SFUNC(T))** *of small diagrams in* **T** *is cocomplete whenever* **T** *is.*

Example 5 (*continued*). Given an institution **I**, consider the indexed category of theories in **I**, $\mathbf{TH}:\mathbf{Sign}^{op}\to\mathbf{Cat}$. Given $\Sigma\in|\mathbf{Sign}|$, clearly $\mathbf{TH}_\Sigma$ is a complete lattice, i.e. is complete and cocomplete as a category. Moreover, it is not hard to see that given a signature morphism $\sigma:\Sigma\to\Sigma'$, then $\mathbf{TH}_\sigma:\mathbf{TH}_{\Sigma'}\to\mathbf{TH}_\Sigma$ has a left adjoint which maps a Σ-theory T to the Σ'-theory generated by the set $\{\sigma(\varphi) \mid \varphi \in T\}$ of Σ'-sentences. Thus, Theorem 2 implies that the flattened category **Th** = **Flat(TH)** of theories in **I** is cocomplete whenever the category **Sign** of signatures is cocomplete. It is even easier to see that the categories **Pres** = **Flat(PRES)** and $\mathbf{Pres}_{\models} = \mathbf{Flat}(\mathbf{PRES}_{\models})$ are cocomplete whenever **Sign** is. A similar result holds for completeness, but is less interesting.

Example 6 (*continued*). Given an arbitrary category **V**, consider the indexed category $\mathbf{INS}: \mathbf{Cat}^{op} \to \mathbf{Cat}$ of institutions. Recall that $\mathbf{INS}(\mathbf{Sign}) = [\mathbf{Sign}^{op} \to \mathbf{Room}(\mathbf{V})]$ for $\mathbf{Sign} \in |\mathbf{Cat}|$. Arguments as in Example 4 above show that **Ins** = **Flat(INS)** is complete provided that the category **Room(V)** is complete. For this we can use the following general result on comma categories (its dual is stated in [3], and proved in detail in [37]; a slightly weaker result is given in [27, Lemma in V.6] and [15, Prop. 2]).

Lemma 2. *Given categories* **A**, **B**, **K** *and functors* $\mathbf{F}: \mathbf{A} \to \mathbf{K}$ *and* $\mathbf{G}: \mathbf{B} \to \mathbf{K}$, *if* **A** *and* **B** *are complete and if* $\mathbf{G}: \mathbf{B} \to \mathbf{K}$ *is continuous, then* $(\mathbf{F} \downarrow \mathbf{G})$ *is complete.*

Recall that we defined $\mathbf{Room}(\mathbf{V}) = (|\_\_| \downarrow \mathbf{FUNC}_{Disc}(\mathbf{V}))$, where $|\_\_|: \mathbf{Cat} \to \mathbf{Cat}$ and $\mathbf{FUNC}_{Disc}(\mathbf{V}): \mathbf{DCat}^{op} \to \mathbf{Cat}$. Since **Cat** is complete and **DCat**, the category of discrete categories, is cocomplete (hence, $\mathbf{DCat}^{op}$ is complete), the only thing to check is the continuity of $\mathbf{FUNC}_{Disc}(\mathbf{V})$. This follows from the construction of colimits in **DCat** and limits in **Cat**: The coproduct in **DCat** of any family of discrete categories $\mathbf{S}_n$ for $n \in N$ is just their disjoint union $\mathbf{S} = \biguplus_{n \in N} \mathbf{S}_n$. It is not hard to see that the functor category $[\mathbf{S} \to \mathbf{V}]$ is (isomorphic to) the product of the categories $[S_n \to \mathbf{V}]$, for $n \in N$. Then, the coequaliser in **DCat** of any two functors $\mathbf{F}, \mathbf{G}: \mathbf{S1} \to \mathbf{S2}$ is given as the natural quotient functor $\mathbf{H}: \mathbf{S2} \to \mathbf{S2}/\equiv$, where $\equiv$ is the least equivalence on (objects of) **S2** such that $\mathbf{F}(s) \equiv \mathbf{G}(s)$ for all $s \in \mathbf{S1}$; and $\mathbf{S2}/\equiv$ is the quotient (discrete) category. Again, it is not hard to see that the functor category $[(\mathbf{S2}/\equiv) \to \mathbf{V}]$ is isomorphic to the subcategory of $[\mathbf{S2} \to \mathbf{V}]$ that contains as objects all functors $\mathbf{D}: \mathbf{S2} \to \mathbf{V}$ such that $\mathbf{F}; \mathbf{D} = \mathbf{G}; \mathbf{D}$, and similarly for morphisms. The isomorphism is given by the functor

$$\mathbf{FUNC}_{Disc}(\mathbf{V})(\mathbf{H}): [(\mathbf{S2}/\equiv) \to \mathbf{V}] \to [\mathbf{S2} \to \mathbf{V}].$$

Thus, $\mathbf{FUNC}_{Disc}(\mathbf{V})(\mathbf{H})$ is an equaliser in **Cat** of the functors $\mathbf{FUNC}_{Disc}(\mathbf{V})(\mathbf{F})$ and $\mathbf{FUNC}_{Disc}(\mathbf{V})(\mathbf{G})$.

Summing up, $\mathbf{FUNC}_{Disc}(\mathbf{V})$ maps coproducts in **DCat** to products in **Cat** and coequalisers in **DCat** to equalisers in **Cat**. Hence, $\mathbf{FUNC}_{Disc}(\mathbf{V})$ is continuous as a functor from $\mathbf{DCat}^{op}$ to **Cat**. Thus, by Lemma 2, **Room(V)** is complete and, thus, the category **Ins** of institutions is complete.

Since morphisms in **Ins** have richer institutions as their source, limits, not colimits, are appropriate for "putting institutions together," and, hence, the completeness of **Ins** is relevant.

4. Indexed functors

Definition 6. An *indexed functor* **F** from one **Ind**-indexed category $\mathbf{C}:\mathbf{Ind}^{op}\to\mathbf{Cat}$ to another $\mathbf{D}:\mathbf{Ind}^{op}\to\mathbf{Cat}$ is a natural transformation $\mathbf{F}:\mathbf{C}\to\mathbf{D}$, i.e. for each $i\in|\mathbf{Ind}|$, a functor $\mathbf{F}_i:\mathbf{C}_i\to\mathbf{D}_i$ such that $\mathbf{F}_j;\mathbf{D}_\sigma=\mathbf{C}_\sigma;\mathbf{F}_i$ for each $\sigma:i\to j$ in **Ind**.

Ind: **Cat:**

$$
\begin{array}{ccc}
i & & \\
\Big\downarrow \sigma & & \\
j & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{C}_i & \xrightarrow{\mathbf{F}_i} & \mathbf{D}_i \\
\mathbf{C}_\sigma \Big\uparrow & & \Big\uparrow \mathbf{D}_\sigma \\
\mathbf{C}_j & \xrightarrow[\mathbf{F}_j]{} & \mathbf{D}_j
\end{array}
$$

This gives a category **INDEXEDCAT(Ind)** of **Ind**-indexed categories, with the obvious vertical composition of morphisms.

Example 7 (*Powerset functor*). Given a set S, let us define the S-sorted powerset functor $\mathbf{P}_S:\mathbf{SSET}(S)\to\mathbf{SSET}(S)$ as follows: $\mathbf{P}_S$ maps an S-sorted set $\langle X_s\rangle_{s\in S}$ to the S-sorted set $\langle 2^{X_s}\rangle_{s\in S}$ of the powersets of its components; and $\mathbf{P}_S$ maps an S-sorted function $\langle g_s:X_s\to Y_s\rangle_{s\in S}$ to the S-sorted family $\langle 2^g_s:2^{X_s}\to 2^{Y_s}\rangle_{s\in S}$ of the corresponding image functions, $2^g_s(A)=\{g_s(x)\mid x\in A\}$ for any $A\subseteq X_s$ and $s\in S$. It is not hard to see that $\mathbf{P}=\langle\mathbf{P}_S\rangle_{S\in|\mathbf{Set}|}$ forms an indexed functor $\mathbf{P}:\mathbf{SSET}\to\mathbf{SSET}$.

Example 8. Recall that Example 5 defined three indexed categories

$$
\begin{array}{ll}
\mathbf{TH}: & \mathbf{Sign}^{op}\to\mathbf{Cat}, \\
\mathbf{PRES}: & \mathbf{Sign}^{op}\to\mathbf{Cat}, \\
\mathbf{PRES}_\models: & \mathbf{Sign}^{op}\to\mathbf{Cat},
\end{array}
$$

where $\mathbf{TH}_\Sigma$ is a subcategory of $\mathbf{PRES}_\Sigma$ for each $\Sigma\in|\mathbf{Sign}|$, which in turn is a subcategory of $(\mathbf{PRES}_\models)_\Sigma$. It is not hard to see that the families of inclusion functors, from $\mathbf{TH}_\Sigma$ to $\mathbf{PRES}_\Sigma$ and from $\mathbf{PRES}_\Sigma$ to $(\mathbf{PRES}_\models)_\Sigma$ indexed by signatures $\Sigma\in|\mathbf{Sign}|$ form indexed functors, from **TH** to **PRES** and from **PRES** to $\mathbf{PRES}_\models$.

This motivates the following definition. An indexed category $\mathbf{C}:\mathbf{Ind}^{op}\to\mathbf{Cat}$ is an *indexed subcategory* of $\mathbf{D}:\mathbf{Ind}^{op}\to\mathbf{Cat}$ (they must have the same category of indices) iff $\mathbf{D}_i$ is a subcategory of $\mathbf{C}_i$ for each $i\in|\mathbf{Ind}|$, and the family of inclusion functors forms an indexed functor from **D** to **C**. This can be somewhat generalised by considering indexed subcategories **D** over a subcategory of indices of **C**.

Flattening extends from indexed categories to indexed functors.

Definition 7. Let **Ind** be a category. Then the *flattened functor*,

$$\mathbf{Flat_{Ind}} : \mathbf{INDEXEDCAT(Ind)} \to \mathbf{Cat},$$

is defined as follows:

- *on objects*: Given $\mathbf{C} : \mathbf{Ind}^{op} \to \mathbf{Cat}$, then $\mathbf{Flat_{Ind}(C)}$ is the flattened category of Definition 2.
- *on morphisms*: Given an **Ind**-indexed functor $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ (for $\mathbf{C}, \mathbf{D} : \mathbf{Ind}^{op} \to \mathbf{Cat}$), then the functor $\mathbf{Flat_{Ind}(F)} : \mathbf{Flat_{Ind}(C)} \to \mathbf{Flat_{Ind}(D)}$ is defined as follows:
  - *on objects*: Given $\langle i, a \rangle \in |\mathbf{Flat_{Ind}(C)}|$, let $\mathbf{Flat_{Ind}(F)}(\langle i, a \rangle) = \langle i, \mathbf{F}_i(a) \rangle$.
  - *on morphisms*: Given a morphism $\langle \sigma, f \rangle : \langle i, a \rangle \to \langle j, b \rangle$ in $\mathbf{Flat_{Ind}(C)}$, let $\mathbf{Flat_{Ind}(F)}(\langle \sigma, f \rangle) = \langle \sigma, \mathbf{F}_i(f) \rangle : \langle i, \mathbf{F}_i(a) \rangle \to \langle j, \mathbf{F}_j(b) \rangle$ in $\mathbf{Flat_{Ind}(D)}$, recalling that $\mathbf{D}_\sigma(\mathbf{F}_j(b)) = \mathbf{F}_i(\mathbf{C}_\sigma(b))$.

We may write **Flat** instead of $\mathbf{Flat_{Ind}}$. It is straightforward to show that it is a functor.

Intuitively, flattened indexed functors leave the first element of their arguments unchanged, but use it to select the appropriate component category for the indexed functor to operate upon. In a sense, flattening an indexed functor forms the disjoint union of its components. The similarity of Definition 6 to the definitions of Example 4 (the category of functors into a fixed target category) suggests the following:

Example 9 (*Indexed categories*). The indexed category of indexed categories is defined by

$$\mathbf{INDEXEDCAT} = \mathbf{OP}; \mathbf{FUNC(Cat)} : \mathbf{Cat}^{op} \to \mathbf{Cat},$$

where $\mathbf{OP} : \mathbf{Cat}^{op} \to \mathbf{Cat}^{op}$ maps a category $\mathbf{K}$ to its opposite $\mathbf{K}^{op}$, and maps a functor $\mathbf{F} : \mathbf{K} \to \mathbf{M}$ to its opposite $\mathbf{F}^{op} : \mathbf{K}^{op} \to \mathbf{M}^{op}$. (It makes a nice puzzle to define $\mathbf{OP} = ((\_)^{op})^{op}$.) Thus, given $\mathbf{Ind} \in |\mathbf{Cat}|$, let

$$\mathbf{INDEXEDCAT(Ind)} = [\mathbf{Ind}^{op} \to \mathbf{Cat}]$$

as in Definition 6, and given $\boldsymbol{\Phi} : \mathbf{Ind} \to \mathbf{Ind}'$ and $\mathbf{C}' : (\mathbf{Ind}')^{op} \to \mathbf{Cat}$, let

$$\mathbf{INDEXEDCAT}(\boldsymbol{\Phi})(\mathbf{C}') = \boldsymbol{\Phi}^{op}; \mathbf{C}' : \mathbf{Ind}^{op} \to \mathbf{Cat}.$$

Flattening yields the category $\mathbf{IndexedCat} = \mathbf{Flat(INDEXEDCAT)}$ of indexed categories, with its objects an index category and an indexed category over it, and its morphism from $\langle \mathbf{Ind1}, \mathbf{C1} : \mathbf{Ind1}^{op} \to \mathbf{Cat} \rangle$ to $\langle \mathbf{Ind2}, \mathbf{C2} : \mathbf{Ind2}^{op} \to \mathbf{Cat} \rangle$ pairs $\langle \boldsymbol{\Phi}, \mathbf{F} \rangle$, where $\boldsymbol{\Phi} : \mathbf{Ind1} \to \mathbf{Ind2}$ is a functor and $\mathbf{F} : \mathbf{C1} \to \boldsymbol{\Phi}^{op}; \mathbf{C2}$ is a natural transformation.

For example, let us consider the relationship between the indexed categories of many-sorted algebras (Example 3) and of many-sorted sets (Example 1). First, there is a functor $\mathbf{Sorts} : \mathbf{AlgSig} \to \mathbf{Set}$, which maps a signature to its set of sorts (in fact, this is the projection functor of Definition 3). Then, given an algebraic signature Σ, there is a forgetful functor (e.g. [7])

$$\mathbf{U}_\Sigma : \mathbf{Alg}(\Sigma) \to \mathbf{SSET(Sorts}(\Sigma)),$$

which maps a Σ-algebra to its many-sorted carrier. It is not hard to check that the family $\mathbf{U}=\langle \mathbf{U}_\Sigma\rangle_{\Sigma\in|\mathbf{AlgSig}|}$ forms a natural transformation $\mathbf{U}:\mathbf{ALG}\to\mathbf{Sorts}^{op};\mathbf{SSET}$, so that $\langle\mathbf{Sorts},\mathbf{U}\rangle:\langle\mathbf{AlgSig},\mathbf{ALG}\rangle\to\langle\mathbf{Set},\mathbf{SSET}\rangle$ is a morphism of indexed categories.

Let us note that $\mathbf{Flat}=\langle\mathbf{Flat}_{\mathbf{Ind}}\rangle_{\mathbf{Ind}\in|\mathbf{Cat}|}$ as defined in Definition 7 is also an indexed functor, from the **Cat**-indexed category **INDEXEDCAT** to the constant **Cat**-indexed category that assigns the category **Cat** to each index (and the identity functor on **Cat** to each index morphism).

Part of our original motivation for looking more carefully at indexed categories was to reduce a family of adjunctions (between component categories) to a single adjunction (between flattened categories); a somewhat parallel motive appears in "getting a charter from a parchment" [18].

Definition 8. Let $\mathbf{U}:\mathbf{C}\to\mathbf{D}$ be an **Ind**-indexed functor. Then U *has a left adjoint locally* iff $\mathbf{U}_i:\mathbf{C}_i\to\mathbf{D}_i$ has a left adjoint for each index $i\in|\mathbf{Ind}|$.

Theorem 3. *Given an* **Ind**-*indexed functor* $\mathbf{U}:\mathbf{C}\to\mathbf{D}$, *which has a left adjoint locally, then* $\mathbf{Flat(U)}:\mathbf{Flat(C)}\to\mathbf{Flat(D)}$ *has a left adjoint.*

Proof. Given an object $\langle i, a\rangle$ in **Flat(C)**, then $\mathbf{U}_i:\mathbf{C}_i\to\mathbf{D}_i$ has (let us say) left adjoint $\mathbf{F}_i:\mathbf{D}_i\to\mathbf{C}_i$ with unit $\eta_i:\mathbf{id}_{\mathbf{C}_i}\to\mathbf{F}_i;\mathbf{U}_i$. Now we claim that $\langle i,\mathbf{F}_i(a)\rangle$ is a free object in **Flat(D)** over $\langle i, a\rangle$ with respect to the functor **Flat(U)**, having as its unit $\langle id_i, \eta_i(a)\rangle:\langle i,a\rangle\to\langle i,\mathbf{U}_i(\mathbf{F}_i(a))\rangle=\mathbf{Flat(U)}(\langle i,\mathbf{F}_i(a)\rangle)$. For, let $\langle j,b\rangle$ be an object in **Flat(D)**, let $\langle\sigma,f\rangle:\langle i,a\rangle\to\mathbf{Flat(U)}(\langle j,b\rangle)=\langle j,\mathbf{U}_i(b)\rangle$ be a morphism in **Flat(C)**, and let $f^{\#}:\mathbf{F}_i(c)\to b$ be the unique morphism in $\mathbf{D}_i$ such that $\eta_i(a);\mathbf{U}_i(f^{\#})=f$ in $\mathbf{C}_i$. Then $\langle\sigma,f^{\#}\rangle:\langle i,\mathbf{F}_i(a)\rangle\to\langle j,b\rangle$ is the only morphism in **Flat(D)** such that $\langle id_i, \eta_i(a)\rangle;\mathbf{Flat(U)}(\langle\sigma,f^{\#}\rangle)=\langle\sigma,f\rangle$ in **Flat(C)**. □

Example 10. The **AlgSig**-indexed forgetful functor $\mathbf{U}:\mathbf{ALG}\to\mathbf{Sorts}^{op};\mathbf{SSET}$ was defined in Example 9, and it is well known that each $\mathbf{U}_\Sigma:\mathbf{ALG}(\Sigma)\to\mathbf{SSET}(\mathbf{Sorts}(\Sigma))$ has a left adjoint. Theorem 3 implies that the flattening of these forgetful functors,

$$\mathbf{Flat(U)}:\mathbf{Flat(ALG)}\to\mathbf{Flat}(\mathbf{Sorts}^{op};\mathbf{SSET}),$$

has a left adjoint obtained by flattening the local left adjoints.

Example 11. There is a **Sign**-indexed inclusion functor from the indexed category **TH** of theories to the indexed category **PRES** of presentations in an arbitrary institution **I** (cf. Example 8). It is clear from the definitions in Example 5 (where these categories were defined) that for each signature $\Sigma\in|\mathbf{Sign}|$, the inclusion functor from $\mathbf{TH}_\Sigma$ to $\mathbf{PRES}_\Sigma$ has a left adjoint (i.e. $\mathbf{TH}_\Sigma$ is a reflexive subcategory of $\mathbf{PRES}_\Sigma$ in the sense of [27, V.3, pp. 88–9]). In fact, the left adjoint is the closure operator $Cl_\Sigma:\mathbf{PRES}_\Sigma\to\mathbf{TH}_\Sigma$ defined in Example 5. Theorem 3 now implies that the category $\mathbf{Th}=\mathbf{Flat(TH)}$ of theories in **I** is a reflective subcategory of **Pres** = **Flat(PRES)**, the category of presentations in **I**.

Theorem 3 suggests a different way to prove the cocompleteness of flattened categories. Given a shape category **G** and a target category **T**, the *diagonal functor*

$$\Delta^{\mathbf{G}}_{\mathbf{T}}:\mathbf{T}\to[\mathbf{G}\to\mathbf{T}]$$

is defined as follows:

- *on objects*: Given $t\in|\mathbf{T}|$, $\Delta^{\mathbf{G}}_{\mathbf{T}}(t)$ be the "constant" diagram, i.e. the functor that maps each object of **G** to t and each morphism in **G** to the identity on t.
- *on morphisms*: Given $f:t1\to t2$ in **T**, let $\Delta^{\mathbf{G}}_{\mathbf{T}}(f):\Delta^{\mathbf{G}}_{\mathbf{T}}(t1)\to\Delta^{\mathbf{G}}_{\mathbf{T}}(t2)$ be the "constant" natural transformation, $\Delta^{\mathbf{G}}_{\mathbf{T}}(f)_n=f$ for each $n\in|\mathbf{G}|$.

Fact 2. *Given categories* **G** *and* **T**, *then* **T** *is* **G**-*cocomplete iff the diagonal functor* $\Delta^{\mathbf{G}}_{\mathbf{T}}:\mathbf{T}\to[\mathbf{G}\to\mathbf{T}]$ *has a left adjoint.*

Proof. Given a diagram $\mathbf{D}:\mathbf{G}\to\mathbf{T}$, the free object over **D** with respect to $\Delta^{\mathbf{G}}_{\mathbf{T}}$ is a colimit of **D**; the unit is the colimiting cocone on **D**; and vice versa, the colimit of **D** is a free object over **D** with respect to $\Delta^{\mathbf{G}}_{\mathbf{T}}$. □

Now we follow this hint in proving a slightly stronger form of Theorem 2.

Theorem 2′. *Given a category* **G**, *let* $\mathbf{C}:\mathbf{Ind}^{op}\to\mathbf{Cat}$ *be an indexed category such that*

(1) **Ind** *is* **G**-*cocomplete*;

(2) $\mathbf{C}_i$ *is* **G**-*cocomplete for all* $i\in|\mathbf{Ind}|$; *and*

(3) **G** *is locally reversible.*

Then **Flat(C)** *is* **G**-*cocomplete.*

Proof. **C** gives rise to an **Ind**-indexed category $\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}$ of **G**-diagrams in **C** as follows:

- *Component categories*: Given $i\in|\mathbf{Ind}|$, then $\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}(i)=[\mathbf{G}\to\mathbf{C}_i]$.
- *Translation functors*: Given $\sigma:i\to j$ in **Ind**, define the functor $\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}(\sigma):[\mathbf{G}\to\mathbf{C}_j]\to[\mathbf{G}\to\mathbf{C}_i]$ on objects by $\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}(\sigma)(\mathbf{D})=\mathbf{D};\mathbf{C}_\sigma$ for $\mathbf{D}:\mathbf{G}\to\mathbf{C}_j$; it extends to morphisms in $[\mathbf{G}\to\mathbf{C}_j]$ in the obvious way.

Now, we have the diagonal **Ind**-indexed functor

$$\Delta^{\mathbf{G}}_{\mathbf{C}}:\mathbf{C}\to\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}$$

defined by $(\Delta^{\mathbf{G}}_{\mathbf{C}})_i=\Delta^{\mathbf{G}}_{\mathbf{C}_i}:\mathbf{C}_i\to[\mathbf{G}\to\mathbf{C}_i]$ for $i\in|\mathbf{Ind}|$. (It is not hard to check that this is indeed an indexed functor.) Moreover, by assumption 2 and Fact 2, $\Delta^{\mathbf{G}}_{\mathbf{C}_i}$ has a left adjoint for each $i\in|\mathbf{Ind}|$. Hence, by Theorem 3,

$$\mathbf{Flat}(\Delta^{\mathbf{G}}_{\mathbf{C}}):\mathbf{Flat}(\mathbf{C})\to\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$$

has a left adjoint. We can identify $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ with a subcategory of $[\mathbf{G}\to\mathbf{Flat}(\mathbf{C})]$ which, roughly, contains the **G**-diagrams in $\mathbf{Flat}(\mathbf{C})$ that fit entirely into one of the component categories of **C**: a diagram $\mathbf{D}:\mathbf{G}\to\mathbf{Flat}(\mathbf{C})$ is in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ iff $\mathbf{D};\mathbf{Proj}_{\mathbf{C}}:\mathbf{G}\to\mathbf{Ind}$ is a constant functor, and a diagram morphism δ is in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ iff δ horizontally composed with $\mathbf{Proj}_{\mathbf{C}}$ yields a constant natural transformation.

The corresponding faithful functor $\mathbf{J}:\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})\to[\mathbf{G}\to\mathbf{Flat}(\mathbf{C})]$ may be defined as follows:

- *on objects*: Given $\langle i,\mathbf{D}\rangle\in|\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})|$ (i.e. $i\in|\mathbf{Ind}|$ and $\mathbf{D}:\mathbf{G}\to\mathbf{C}_i$), the **G**-diagram $\mathbf{J}(\langle i,\mathbf{D}\rangle):\mathbf{G}\to\mathbf{Flat}(\mathbf{C})$ is defined as follows:
  - *on objects*: $\mathbf{J}(\langle i,\mathbf{D}\rangle)(n)=\langle i,\mathbf{D}(n)\rangle$ for $n\in|\mathbf{G}|$.
  - *on morphisms*: $\mathbf{J}(\langle i,\mathbf{D}\rangle)(e)=\langle id_i,\mathbf{D}(e)\rangle$ for any morphism e in **G**.
- *on morphisms*: Given a morphism $\langle\gamma,\alpha\rangle:\langle i,\mathbf{D}\rangle\to\langle j,\mathbf{E}\rangle$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$, where $\gamma:i\to j$ is an index morphism and $\alpha:\mathbf{D}\to\mathbf{E};\mathbf{C}_\gamma$ is a morphism in $[\mathbf{G}\to\mathbf{C}_i]$, then $\mathbf{J}(\langle\gamma,\alpha\rangle):\mathbf{J}(\langle i,\mathbf{D}\rangle)\to\mathbf{J}(\langle j,\mathbf{E}\rangle)$ is the natural transformation defined by $\mathbf{J}(\langle\gamma,\alpha\rangle)(n)=\langle\gamma,\alpha(n)\rangle:\langle i,\mathbf{D}(n)\rangle\to\langle j,\mathbf{E}(n)\rangle$ for $n\in|\mathbf{G}|$.

It is not hard to see that $\mathbf{J}(\langle\gamma,\alpha\rangle)$ is indeed a natural transformation, and that **J** is a faithful functor.

The following identifies $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ with its image under **J** in $[\mathbf{G}\to\mathbf{Flat}(\mathbf{C})]$ and refers to **J** as an inclusion functor. Unfortunately, $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ is in general a *proper* subcategory of $[\mathbf{G}\to\mathbf{Flat}(\mathbf{C})]$, and so the proof of Theorem 2′ is not yet finished. One can directly check that

$$\Delta^{\mathbf{G}}_{\mathbf{Flat}(\mathbf{C})}=\mathbf{Flat}(\Delta^{\mathbf{G}}_{\mathbf{C}});\mathbf{J}.$$

Since we already know that $\mathbf{Flat}(\Delta^{\mathbf{G}}_{\mathbf{C}})$ has a left adjoint, to show that $\Delta^{\mathbf{G}}_{\mathbf{Flat}(\mathbf{C})}$ has a left adjoint it is enough to prove that **J** has a left adjoint (cf. [27, Th. V.8.1, p. 101]). Thus, the following lemma will complete the proof.

Lemma 2. *The inclusion functor* **J** *has a left adjoint, i.e.* $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ *is a reflexive subcategory of* $[\mathbf{G}\to\mathbf{Flat}(\mathbf{C})]$ (*cf.* [27, V.3, pp. 88–9] *for the definition and basic facts about reflexive subcategories*).

Proof of Lemma 2. Given a **G**-diagram $\mathbf{D}:\mathbf{G}\to\mathbf{Flat}(\mathbf{C})$, we are to find its reflection in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$, i.e. a **G**-diagram $\mathbf{R}(\mathbf{D}):\mathbf{G}\to\mathbf{Flat}(\mathbf{C})$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ together with a diagram morphism $\eta_{\mathbf{D}}:\mathbf{D}\to\mathbf{R}(\mathbf{D})$ such that for any diagram $\mathbf{D}'$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ and morphism $\delta:\mathbf{D}\to\mathbf{D}'$ there exists a unique $\delta^{\#}:\mathbf{R}(\mathbf{D})\to\mathbf{D}'$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ such that $\eta_{\mathbf{D}};\delta^{\#}=\delta$ in $[\mathbf{G}\to\mathbf{Flat}(\mathbf{C})]$.

So, given an arbitrary diagram $\mathbf{D}:\mathbf{G}\to\mathbf{Flat}(\mathbf{C})$, where $\mathbf{D}(n)=\langle i_n,a_n\rangle$ for $n\in|\mathbf{G}|$, and $\mathbf{D}(e)=\langle\sigma_e,f_e\rangle:\langle i_n,a_n\rangle\to\langle i_m,a_m\rangle$ for $e:n\to m$ in **G**, let i be a colimit in **Ind** of $\mathbf{D};\mathbf{Proj}_{\mathbf{C}}:\mathbf{G}\to\mathbf{Ind}$, with injections $\rho_n:i_n\to i$ for $n\in|\mathbf{G}|$ (**Ind** is **G**-cocomplete by assumption 1). Now define $\mathbf{R}(\mathbf{D}):\mathbf{G}\to\mathbf{Flat}(\mathbf{C})$ as follows:

- *on objects*: $\mathbf{R}(\mathbf{D})(n)=\langle i,\mathbf{F}_{\rho_n}(a_n)\rangle$ for $n\in|\mathbf{G}|$.
- *on morphisms*: $\mathbf{R}(\mathbf{D})(e)=\langle id_i,L_{\rho_m}(\langle\sigma_e,f_e\rangle)\rangle:\langle i,\mathbf{F}_{\rho_n}(a_n)\rangle\to\langle i,\mathbf{F}_{\rho_m}(a_m)\rangle$ for $e:n\to m$ in **G**.

Recall that indeed $L_{\rho_m}(\langle\sigma_e, f_e\rangle): \mathbf{F}_{\sigma_e;\rho_m}(a_n) = \mathbf{F}_{\rho_n}(a_n) \to \mathbf{F}_{\rho_m}(a_m)$ (see Definition 5).

Let us check that $\mathbf{R}(\mathbf{D})$ is a functor, i.e. it preserves identities and composition. It is obvious that it preserves identities (Definition 5 implies that $L_{\rho_n}(\langle id_n, id_{a_n}\rangle) = \mathbf{F}_{\rho_n}(id_{a_n}) = id_{\mathbf{F}_{\rho_n}(a_n)}$). For composition, given $e: n \to m$ and $d: m \to k$ in $\mathbf{G}$, we have to show that in $\mathbf{C}_i$

$$L_{\rho_m}(\langle\sigma_e, f_e\rangle); L_{\rho_k}(\langle\sigma_d, f_d\rangle) = L_{\rho_k}(\langle\sigma_e, f_e\rangle; \langle\sigma_d, f_d\rangle).$$

This may be checked by going back to $\mathbf{C}_{i_n}$. On the one hand, in $\mathbf{C}_{i_n}$ we have

$$\begin{aligned}
&\eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_k}(\langle\sigma_e, f_e\rangle; \langle\sigma_d, f_d\rangle)) \\
&\quad = \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_k}(\langle\sigma_e;\sigma_d, f_e; \mathbf{C}_{\sigma_e}(f_d)\rangle)) \quad \text{(Corollary 1, } \rho_n = \sigma_e; \sigma_d; \rho_k) \\
&\quad = f_e; \mathbf{C}_{\sigma_e}(f_d); \mathbf{C}_{\sigma_e;\sigma_d}(\eta^{\rho_k}(a_k));
\end{aligned}$$

on the other hand, in $\mathbf{C}_{i_n}$ we have

$$\begin{aligned}
&\eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_m}(\langle\sigma_e, f_e\rangle); L_{\rho_k}(\langle\sigma_d, f_d\rangle)) && \text{(Corollary 1, } \rho_n = \sigma_e; \rho_m) \\
&\quad = f_e; \mathbf{C}_{\sigma_e}(\eta^{\rho_m}(a_m)); \mathbf{C}_{\sigma_e}(\mathbf{C}_{\rho_m}(L_{\rho_k}(\langle\sigma_d, f_d\rangle))) && \text{(Corollary 1, } \rho_m = \sigma_d; \rho_k) \\
&\quad = f_e; \mathbf{C}_{\sigma_e}(f_d); \mathbf{C}_{\sigma_e}(\mathbf{C}_{\sigma_d}(\eta^{\rho_k}(a_k))).
\end{aligned}$$

Hence, in $\mathbf{C}_{i_n}$

$$\eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_m}(\langle\sigma_e, f_e\rangle); L_{\rho_k}(\langle\sigma_d, f_d\rangle)) = \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_k}(\langle\sigma_e, f_e\rangle; \langle\sigma_d, f_d\rangle)),$$

which by properties of adjunctions implies that indeed

$$L_{\rho_m}(\langle\sigma_e, f_e\rangle); L_{\rho_k}(\langle\sigma_d, f_d\rangle) = L_{\rho_k}(\langle\sigma_e, f_e\rangle; \langle\sigma_d, f_d\rangle).$$

Clearly, $\mathbf{R}(\mathbf{D})$ is in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$. Having defined $\mathbf{R}(\mathbf{D})$ as above, there is an obvious way to define $\eta_{\mathbf{D}}: \mathbf{D} \to \mathbf{R}(\mathbf{D})$: for $n \in |\mathbf{G}|$, let $\eta_{\mathbf{D}}(n) = \langle\rho_n, \eta^{\rho_n}(a_n)\rangle : \langle i_n, a_n\rangle \to \langle i, \mathbf{F}_{\rho_n}(a_n)\rangle$. We have to check that $\eta_{\mathbf{D}}$ is a natural transformation. Given $e: n \to m$ in $\mathbf{G}$, we need to show that

$$\mathbf{D}(e); \eta_{\mathbf{D}}(m) = \eta_{\mathbf{D}}(n); \mathbf{R}(\mathbf{D})(e),$$

i.e. that

$$\langle\sigma_e, f_e\rangle; \langle\rho_m, \eta^{\rho_m}(a_m)\rangle = \langle\rho_n, \eta^{\rho_n}(a_n)\rangle; \langle id_i, L_{\rho_m}(\langle\sigma_e, f_e\rangle)\rangle.$$

Since $\sigma_e; \rho_m = \rho_n$ by construction, the only thing to check is that

$$f_e; \mathbf{C}_{\sigma_e}(\eta^{\rho_m}(a_m)) = \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_m}(\langle\sigma_e, f_e\rangle)),$$

which follows directly from Corollary 1. Now we claim that $\mathbf{R}(\mathbf{D})$ is a reflection of $\mathbf{D}$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ with unit $\eta_{\mathbf{D}}: \mathbf{D} \to \mathbf{R}(\mathbf{D})$. Given a diagram $\mathbf{D}'$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ and a diagram morphism $\delta: \mathbf{D} \to \mathbf{D}'$, say that $\mathbf{D}'(n) = \langle j, b_n\rangle$ for $n \in |\mathbf{G}|$, and $\mathbf{D}'(e) = \langle id_j, g_e\rangle$

for $e:n\to m$ in $\mathbf{G}$ with $g_e:b_n\to b_m$ in $\mathbf{C}_j$ (such an index $j\in|\mathbf{Ind}|$ exists since $\mathbf{D}'$ is in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$). Also, say that $\delta(n)=\langle\theta_n, h_n\rangle:\langle i_n, a_n\rangle\to\langle j, b_n\rangle$ for $n\in|\mathbf{G}|$.

By construction, there exists a unique index morphism $\gamma:i\to j$ such that $\rho_n;\gamma=\theta_n$ for each $n\in|\mathbf{G}|$. We now define the diagram morphism $\delta^{\#}:\mathbf{R}(\mathbf{D})\to\mathbf{D}'$ by $\delta^{\#}(n)=\langle\gamma, h_n^{\#}\rangle:\langle i, \mathbf{F}_{\rho_n}(a_n)\rangle\to\langle j, b_n\rangle$ for $n\in|\mathbf{G}|$, where $h_n^{\#}:\mathbf{F}_{\rho_n}(a_n)\to\mathbf{C}_\gamma(b_n)$ is the unique morphism in $\mathbf{C}_i$ that satisfies $\eta^{\rho_n}(a_n);\mathbf{C}_{\rho_n}(h_n^{\#})=h_n:a_n\to\mathbf{C}_{\rho_n}(\mathbf{C}_\gamma(b_n))$. First, let us check that $\delta^{\#}$ is indeed a morphism in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$; the nontrivial part is to verify that $\delta^{\#}$ is a natural transformation, i.e. for any $e:n\to m$ in $\mathbf{G}$ that

$$\delta^{\#}(n);\mathbf{D}'(e)=\mathbf{R}(\mathbf{D})(e);\delta^{\#}(m),$$

or, equivalently, that

$$\langle\gamma, h_n^{\#}\rangle;\langle id_j, g_e\rangle=\langle id_i, L_{\rho_m}(\langle\sigma_e, f_e\rangle)\rangle;\langle\gamma, h_m^{\#}\rangle.$$

We must prove that in $\mathbf{C}_i$

$$h_n^{\#};\mathbf{C}_\gamma(g_e)=L_{\rho_m}(\langle\sigma_e, f_e\rangle);h_m^{\#}.$$

To see this, note that by construction in $\mathbf{C}_{i_n}$

$$\eta^{\rho_n}(a_n);C_{\rho_n}(h_n^{\#};\mathbf{C}_\gamma(g_e))=h_n;C_{\theta_n}(g_e),$$

and by Lemma 1 (since $\rho_n=\sigma_e;\rho_m$)

$$\eta^{\rho_n}(a_n);\mathbf{C}_{\rho_n}(L_{\rho_m}(\langle\sigma_e, f_e\rangle);h_m^{\#})=f_e;\mathbf{C}_{\sigma_e}(h_m).$$

However, since $\delta:\mathbf{D}\to\mathbf{D}'$ is a natural transformation,

$$\mathbf{D}(e);\delta(m)=\delta(n);\mathbf{D}'(e),$$

i.e.

$$\langle\sigma_e, f_e\rangle;\langle\theta_m, h_m\rangle=\langle\theta_n, h_n\rangle;\langle id_j, g_e\rangle,$$

which implies that

$$f_e;\mathbf{C}_{\sigma_e}(h_m)=h_n;\mathbf{C}_{\theta_n}(g_e).$$

Hence, putting these equations together,

$$\eta^{\rho_n}(a_n);\mathbf{C}_{\rho_n}(h_n^{\#};\mathbf{C}_\gamma(g_e))=\eta^{\rho_n}(a_n);\mathbf{C}_{\rho_n}(L_{\rho_m}(\langle\sigma_e, f_e\rangle);h_m^{\#}).$$

Thus indeed,

$$h_n^{\#};\mathbf{C}_\gamma(g_e)=L_{\rho_m}(\langle\sigma_e, f_e\rangle);h_m^{\#}.$$

We now claim that $\delta^{\#}:\mathbf{R}(\mathbf{D})\to\mathbf{D}'$ is a unique morphism in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ such that $\eta_{\mathbf{D}};\delta^{\#}=\delta$. First, we have to verify that $\eta_{\mathbf{D}}(n);\delta^{\#}(n)=\delta(n)$ for $n\in|\mathbf{G}|$, i.e. that

$$\langle\rho_n, \eta^{\rho_n}(a_n)\rangle;\langle\gamma, h_n^{\#}\rangle=\langle\theta_n, h_n\rangle,$$

or equivalently, that

$$\langle \rho_n; \gamma, \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(h_n^{\#})\rangle = \langle \theta_n, h_n \rangle,$$

which is clearly true. Moreover, the construction guarantees that $\delta^{\#}(n)$ is the only morphism in **Flat(C)** such that $\mathbf{Proj}_{\mathbf{C}}(\delta^{\#}(n)) = \gamma$ and $\eta_{\mathbf{D}}(n); \delta^{\#}(n) = \delta(n)$. Since the uniqueness of γ is obvious, this gives the uniqueness of $\delta^{\#}$ and completes the proof of Lemma 2 and, hence, of Theorem 2′. □

We do not apologise for giving a second proof of this theorem; on the contrary, we feel its details are worth examining, especially the "reflection lemma" (Lemma 2).

5. Summary

This paper has presented indexed categories and given examples supporting the view that they are a useful tool for structuring and clarifying certain constructions and proofs in computer science. Given an indexed category **C**, we have constructed a "flattened" category **Flat(C)** containing the components of **C**. We have also introduced indexed functors, and shown how to flatten them. Finally, we have shown that flattening preserves the important properties of completeness, cocompleteness, and existence of left adjoints.

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