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A Sufficient Condition for a Graph to Be Weakly k-Linked

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For a pair (s, t) of vertices of a graph G, let $\lambda_G(s, t)$ denote the maximal number of edge-disjoint paths between s and t. Let (s_1, t_1), (s_2, t_2), (s_3, t_3) be pairs of vertices of G and $k > 2$. It is shown that if $\lambda_G(s_i, t_i) \geqslant 2k + 1$ for each $i = 1, 2, 3$, then there exist $2k + 1$ edge-disjoint paths such that one joins s_1 and t_1, another joins s_2 and t_2 and the others join s_3 and t_3. As a corollary, every $(2k + 1)$-edge-connected graph is weakly $(k + 2)$-linked for $k \geqslant 2$, where a graph is weakly k-linked if for any k vertex pairs (s_i, t_i), $1 \leqslant i \leqslant k$, there exist k edge-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i joins s_i and t_i for $i = 1, 2, \ldots, k$.

1. INTRODUCTION

A graph is weakly k-linked if for any k vertex pairs (s_1, t_1), $(s_2, t_2), \ldots, (s_k, t_k)$ there exist k pairwise edge-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i joins s_i and t_i for $i = 1, 2, \ldots, k$.

Several works have been devoted to characterize a weakly k-linked graph. An obvious necessary condition for a graph G to be weakly k-linked is that G is k-edge-connected. This condition, however, cannot be a sufficient condition if k is even [5].

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Cypher showed that $(k+2)$-edge-connected graphs are weakly k-linked for $k=3, 4, 5$ [2]. Recently it was proved by Okamura that 3-edge-connectedness is sufficient for a graph to be weakly 3-linked [4]. It follows immediately from Menger's theorem that $(2k+1)$-edge-connected graphs are weakly $(k+1)$-linked for $k \geqslant 1$. Thomassen conjectured that if k is odd, k-edge-connected graphs are weakly k-linked and if k is even, $(k+1)$-edge-connected graphs are weakly k-linked [5].

In this paper we show the following: For a pair (s, t) of vertices of a graph G, let $\lambda_G(s, t)$ denote the maximal number of edge-disjoint paths between s and t. Let (s_1, t_1), (s_2, t_2), (s_3, t_3) be pairs of vertices of G and $k \geqslant 2$. Then we show that if $\lambda_G(s_i, t_i) \geqslant 2k+1$ for each $i = 1, 2, 3$, then there exist $2k+1$ edge-disjoint paths such that one joins s_1 and t_1, another joins s_2 and t_2 and the others join s_3 and t_3. This extends Okamura's result and proves a special case of a conjecture of Thomassen.

As a corollary we have that every $(2k+1)$-edge-connected graph is weakly $(k+2)$-linked for $k \geqslant 2$.

Our corollary for $k=2$ says that every 5-edge-connected graph is weakly 4-linked. This result is best possible in the sense that there exists a 4-edge-connected graph which is not weakly 4-linked. Recently the same result for weakly 4-linkedness was obtained, independently, by Enomoto and Saito [3].

2. Preliminaries

Let $G = (V, E)$ be an undirected finite graph, where V is the vertex set and E is the edge set. Multiple edges may exist. Let P be a path between $u \in V$ and $v \in V$. We sometimes say that P runs *from u to v*, though there is no notion of direction for a path. We do not distinguish between P and $E(P)$, the edge set of P, when no confusion arises.

Two paths P_1 and P_2 of G are *edge-disjoint* if they have no common edge. G is *k-edge-connected* if at least k edges must be removed to disconnect G. The following fact is known as Menger's theorem: A graph is k-edge-connected if and only if there are k pairwise edge-disjoint paths between any two vertices of G. A graph is *weakly k-linked* if for any k vertex pairs (s_1, t_1), $(s_2, t_2), \ldots, (s_k, t_k)$ there exist k pairwise edge-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i joins s_i and t_i for $i = 1, 2, \ldots, k$. Throughout this paper "disjoint" means edge-disjoint.

In order to state Cypher's lemma, we introduce his notation. Let (s_i, t_i) be vertex pairs of G for $i = 1, 2, \ldots, k$. We use a sequence of integers $n_1, n_2, \ldots, n_k$ to indicate that there are n_i disjoint paths from s_i to t_i for $i = 1, 2, \ldots, k$. If these paths for several pairs are all disjoint, we indicate this by using parentheses to group the associated integers. If an integer n appears i times successively in the sequence, the n's can be abbreviated to $n^{(i)}$. It is easy to

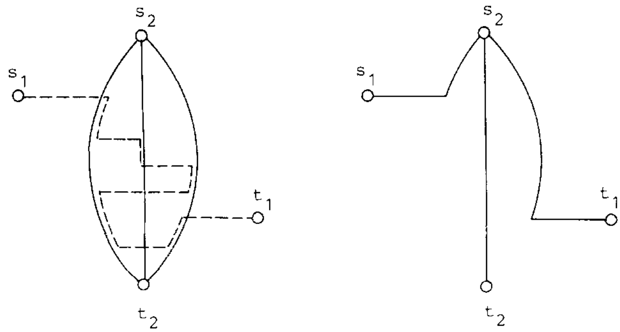

FIG. 1. $1, 3 \to (1, 1)$.

see that if there is a path from s_1 to t_1 and three disjoint paths from s_2 to t_2, then we can always find two disjoint paths, one from s_1 to t_1 and the other from s_2 to t_2. We write this as "$1, 3 \to (1, 1)$" or "$1, 3 \to (1^{(2)})$", where "$\to$" means "imply". See Fig. 1. This simple example shows an essential method of our discussion of the next section, where we construct desired paths from path segments which have already been guaranteed to exist.

The following lemma was given by Cypher.

LEMMA 1. [2]. $2p+1$, $(1^{(p)}, q) \to (1^{(p+1)}, q-2)$, *where* $p \geqslant 1$ *and* $q \geqslant 3$.

In the next section we shall prove

THEOREM. $2k + 1^{(3)} \to (1, 1, 2k-1)$, *where* $k \geqslant 2$.

Using Cypher's lemma, we have Corollaries 1 and 2.

COROLLARY 1. $2k + 1^{(k+2)} \to (1^{(k+2)})$, *where* $k \geqslant 1$.

Proof.

$$\begin{aligned} 2k + 1^{(k+2)} &\to 2k + 1^{(k-1)}, (1, 1, 2k-1) \\ &\to 2k + 1^{(k-2)}, (1, 1, 1, 2k-3) \\ &\vdots \\ &\to (1^{(k+2)}). \end{aligned}$$

COROLLARY 2. *If a graph* G *is* $(2k+1)$*-edge-connected,* G *is weakly* $(k+2)$*-linked, where* $k \geqslant 2$.

3. Proof of Theorem

In this section we shall prove our Theorem, i.e., $2k+1^{(3)} \to (1, 1, 2k-1)$. Let (s_1, t_1), (s_2, t_2), and (s_3, t_3) be three vertex pairs of G, and let $P^j_1, P^j_2, \ldots, P^j_{2k+1}$ be pairwise disjoint paths from s_j to t_j for $j = 1, 2, 3$. In particular we call each P^3_i a *rib*. We begin with the $2k+1$ ribs and examine how the paths from s_1 to t_1 intersect the ribs using a marking procedure as follows. The procedure consists of three steps. We denote by $\bar{P}$ the set $\{P^1_i \mid 1 \leqslant i \leqslant 2k+1\}$.

Step 1. Let G' be the subgraph of G that consists of the $2k+1$ ribs. For each P^1_i, $1 \leqslant i \leqslant 2k+1$, we do the following:

Proceed along P^1_i from s_1. If we reach t_1 without encountering an edge of G', i.e., P^1_i contains no edge of G', then output "yes" and stop. Otherwise mark with "$\bar{s}_1$" the first edge of G' encountered. Starting from t_1, proceed along P^1_i until encountering an edge of G' and mark the edge "$\bar{t}_1$." Note that an edge may have both "$\bar{s}_1$" and "$\bar{t}_1$."

Step 2. As soon as there exists a rib with more than two edges marked "$\bar{s}_1$", we do the following:

Call this rib R and assume that R contains in order the $\bar{s}_1$-marked edges $e_1, e_2, \ldots, e_n$. Let $P[s_1; e_i] \in \bar{P}$ be the path along which we had proceeded when e_i was marked "$\bar{s}_1$." Define $e_i[s_1]$ to be the end-vertex of e_i that is closer to s_1 on $P[s_1; e_i]$. Analogously $P[t_1; e]$ and $e[t_1]$ are defined for a $\bar{t}_1$-marked edge e. If the subpath of R between $e_1[s_1]$ and $e_n[s_1]$ contains an edge marked "$\bar{t}_1$," output "yes" and stop. Otherwise for each e_i, $2 \leqslant i \leqslant n-1$, alter the label of e_i to "$\bar{\bar{s}}_1$" and proceed further along $P[s_1; e_i]$ from e_i toward t_1 until encountering an edge e of G'. Mark e "$\bar{s}_1$." See Fig. 2.

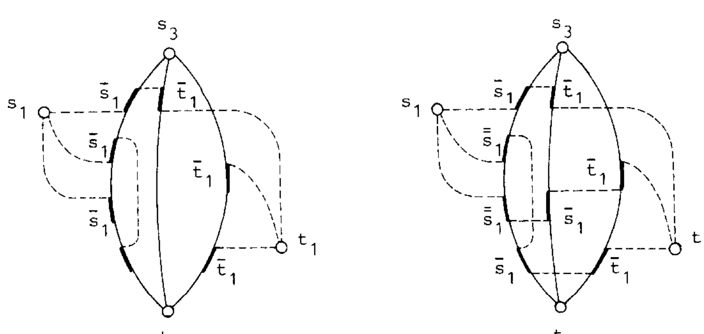

FIG. 2. Marking configurations.

Step 3. As soon as there exists a rib with more than two edges marked "$\bar{t}_1$," we do the following:

Call this rib R and assume that R contains in order the $\bar{t}_1$-marked edges $e_1, e_2, \ldots, e_n$. If the subgraph of R between $e_1[t_1]$ and $e_n[t_1]$ contains an edge marked "$\bar{s}_1$," output "yes" and stop. Otherwise for each e_i, $2 \leqslant i \leqslant n-1$, alter the label of e_i to "$\bar{\bar{t}}_1$" and proceed further along $P[t_1; e_i]$ from e_i toward s_1 until encountering an edge e of G'. Mark e "$\bar{t}_1$." Let R' be the rib containing the edge e. If R' contains an edge marked "$\bar{\bar{s}}_1$" and the subpath of R' between $e_x[s_1]$ and $e_y[s_1]$ contains e, then output "yes" and stop, where e_x and e_y are the $\bar{s}_1$-marked edges of R'. Note that if a rib has a $\bar{\bar{s}}_1$-marked edge, there are exactly two $\bar{s}_1$-marked edges in the rib and all $\bar{\bar{s}}_1$-marked edges of the rib lie between these two $\bar{s}_1$-marked edges.

LEMMA 2. *If the marking procedure produces an output of "yes," then there are $2k+2$ pairwise disjoint paths, one from s_1 to t_1 and the others from s_3 to t_3.*

Proof. Suppose the procedure outputs "yes" in Step 2. When this happens, there is a rib with more than two $\bar{s}_1$-marked edges $e_1, e_2, \ldots, e_n$ and a $\bar{t}_1$-marked edge e lies between $e_1[s_1]$ and $e_n[s_1]$. Therefore we have a structure shown in Fig. 3. p_a, P_b, and P_c are portions of $P[s_1; e_1]$, $P[s_1; e_n]$, and $P[s_1; e_i]$, respectively, where e_i $(2 \leqslant i \leqslant n-1)$ is a $\bar{s}_1$-marked edge closest to e on R. P_d is a portion of $P[t_1; e]$. Now we reroute R using P_a and P_b so that the new route of R runs through s_1. P_c, P_d, and possibly, a portion of R together give a path from s_1 to t_1. P_a, P_b, and P_c might have, on their half ways, $\bar{\bar{s}}_1$-marked edges, i.e., edges of other ribs.

To ensure that P_a, P_b, and P_c are disjoint from other ribs, we do "s_1-shuntings" as follows. Let R be a rib with a $\bar{\bar{s}}_1$-marked edge, and let e_x and e_y be the outermost $\bar{s}_1$-marked edges on R. Note that at any one point in the

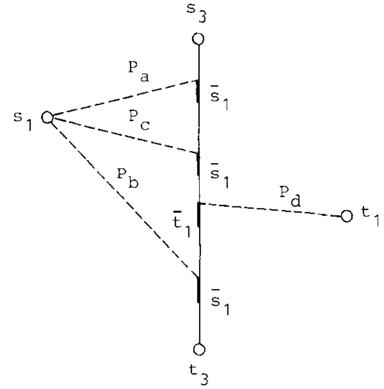

FIG. 3. Structure for Step 2.

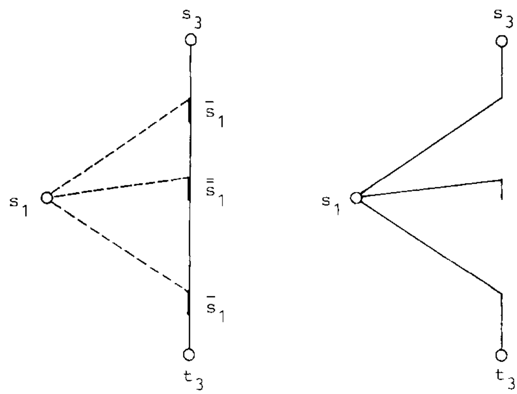

FIG. 4. s_1-shunting.

execution of the marking procedure, a $\bar{\bar{s}}_1$-marked edge cannot be an outermost marked edge on a rib. We say that we do a *s_1-shunting* for R if we reroute R as follows. The new route starts at s_3, proceeds along R to $e_x[s_1]$, follows $P[s_1; e_x]$ to s_1, and then follows $P[s_1; e_y]$ to $e_y[s_1]$ and proceeds along R to t_3, where we assume e_x is closer to s_3 than e_y on R. See Fig. 4. A t_1-shunting is analogously defined.

Returning to our proof, we repeat s_1-shuntings as often as we can. Now we have new $2k + 1$ ribs from s_3 to t_3; some ribs have new routes through s_1 and the others remain untouched. It is easy to see that these new ribs are pairwise disjoint and P_c and P_d are disjoint from these new ribs. Thus we have the $2k + 2$ desired paths.

The similar argument deals with the case when "yes" is produced in Step 3. The only exception is that not only s_1-shuntings but t_1-shuntings will be done. There are two places in Step 3 where "yes" is produced. Suppose that "yes" is produced when a $\bar{s}_1$-marked edge is ascertained to be in a rib R with more than two $\bar{t}_1$-marked edges. We do s_1-shuntings as often as we can, and then do t_1-shuntings as well. Suppose that "yes" is produced when an edge e newly marked "$\bar{t}_1$" is ascertained to be in a rib R' with a $\bar{\bar{s}}_1$-marked edge. In this case, before doing shuntings we must put the marking configuration of "$\bar{s}_1$" and "$\bar{\bar{s}}_1$" on the ribs back to that of the first moment when more than two $\bar{s}_1$-marks were attached to R' so that e lay among them. Note that the marking configuration of "$\bar{t}_1$" and "$\bar{\bar{t}}_1$" on the ribs is not changed. It follows easily that in either case there are $2k + 2$ desired paths.

Q.E.D.

Hence if the marking procedure produces an output of "yes," Theorem is immediately verified by observing

$$2k + 1^{(3)} \to 2k + 1(1, 2k + 1) \to (1, 1, 2k - 1).$$

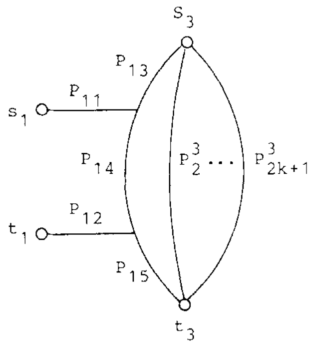

FIG. 5. The $2k+5$ disjoint paths.

The first implication is from Lemma 2 and the second one is from Lemma 1. Therefore we deal with, henceforth, the case when the procedure terminates without producing "yes."

Since the marking procedure terminates without "yes," there are exactly $2k+1$ $\bar{s}_1$-marks and exactly $2k+1$ $\bar{t}_1$-marks attached on the ribs and each rib has at most two $\bar{s}_1$-marks and at most two $\bar{t}_1$-marks. Therefore there is at least one rib that has both a $\bar{s}_1$-mark and a $\bar{t}_1$-mark. Let P^3_1 be such a rib.

We now execute s_1-shuntings and t_1-shuntings as often as we can. Then we have new $2k+1$ pairwise disjoint ribs. We call these new ribs $P^3_1,\ldots,P^3_{2k+1}$ again, since no confusion arises. We have also two disjoint paths, P_{11} from s_1 to a vertex of P^3_1 and P_{12} from t_1 to a vertex of P^3_1, such that these paths are disjoint from the new ribs. P_{11} and P_{12} may be of length 0. Though there may be other paths emanating from s_1 or t_1, which were not used by shuntings, we can ignore them in our proof.

Let v_1 and v_2 be the vertices of P^3_1 shared also by P_{11} and P_{12}, respectively. We assume, by symmetry, that v_1 is closer to s_3 than v_2 on P^3_1. We decompose P^3_1 into three pieces, P_{13} from s_3 to v_1, P_{14} from v_1 to v_2 and P_{15} from v_2 to t_3. See Fig. 5. Now we have $2k+5$ pairwise disjoint paths $P_{11},\ldots,P_{15}$, $P^3_2,\ldots,P^3_{2k+1}$. In the following we examine, using the marking procedure, how the paths from s_2 to t_2 intersect these $2k+5$ paths.

Execute the marking procedure with the $2k+5$ paths instead of the previous ribs. This time we proceed along $P^2_1,\ldots,P^2_{2k+1}$ instead of $P^1_1,\ldots,P^1_{2k+1}$. $\bar{s}_2(\bar{\bar{s}}_2)$-marks and $\bar{t}_2(\bar{\bar{t}}_2)$-marks are attached to edges of the $2k+5$ paths. Note that we treat each P_{1j}, $1 \leqslant j \leqslant 5$, as if it were a rib, i.e., we possibly attach marks on P_{1j} as well as on P^3_i, $2 \leqslant i \leqslant 2k+1$. The similar argument for $\bar{s}_1(\bar{t}_1)$-marking shows that if the procedure produces an output of "yes," then we immediately have a $(1,1,2k-1)$-solution. Thus we have only to consider the case when the procedure terminates without producing "yes." In this case each P_{1j}, and also each P^3_i, has at most two $\bar{s}_2$-marks and at most two $\bar{t}_2$-marks.

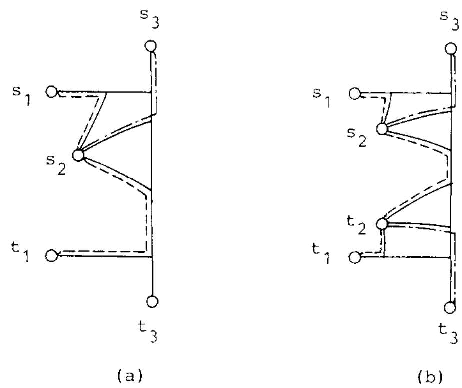

FIG. 6. Illustrations for Case 1.

Let C_1 be the graph composed of $P_{11},..., P_{15}$, and let $C_i = P_i^3$ for $i = 2, 3,..., 2k+1$. We call each C_i a *complex*. First we consider the case when C_1 has either more than two $\bar{s}_2$-marks or more than two $\bar{t}_2$-marks. In the following discussion we assume that $s_2(t_2)$-shuntings are done whenever they are necessary.

Case 1. C_1 has either more than two $\bar{s}_2$-marks or more than two $\bar{t}_2$-marks.

We assume, by symmetry, that C_1 has more than two $\bar{s}_2$-marks. We claim that there are two disjoint paths, one from s_1 to t_1 and the other from s_2 to either s_3 or t_3, such that they are disjoint from $C_2, C_3,..., C_{2k+1}$. Clearly the claim holds if either P_{13} or P_{15} has a $\bar{s}_2$-mark. Thus we assume all $\bar{s}_2$-marks of C_1 are on P_{11}, P_{12} or P_{14}. Since each of these three paths has at most two $\bar{s}_2$-marks, we always find the two required paths in the manner illustrated in Fig. 6a, where a broken line indicates the path from s_1 to t_1 and a dot and dash line the path from s_2 to s_3.

Let P be the path from s_1 to t_1 found in the above claim. If there is a $\bar{t}_2$-mark not on P, we immediately have a $(1, 1, 2k-1)$-solution. Thus we consider only the case when all $\bar{t}_2$-marks are on P. Then we can reroute P using two paths, each from t_2 to an end-vertex of a $\bar{t}_2$-marked edge, such that there is a path from t_2 to either s_3 or t_3 which is disjoint from the rerouted path P. See Fig. 6b. Hence we have a $(1, 1, 2k-1)$-solution.

Next we consider the remaining case.

Case 2. C_1 has at most two $\bar{s}_2$-marks and at most two $\bar{t}_2$-marks.

In this case every complex has at most two $\bar{s}_2$-marks and at most two $\bar{t}_2$-marks. Thus it follows immediately that at least one complex has both a $\bar{s}_2$-mark and a $\bar{t}_2$-mark. If a complex other than C_1 is such a complex, we have

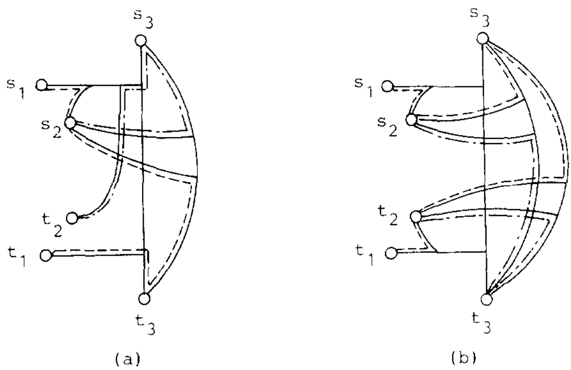

FIG. 7. Constructions of the two paths for Case 2.

a $(1, 1, 2k-1)$-solution. Therefore we assume that only C_1 is such a complex. Observe, under this assumption, that there are two complexes different from C_1 such that one has two $\bar{s}_2$-marks and the other has two $\bar{t}_2$-marks, since $k \geqslant 2$.

If either a $\bar{s}_2$-mark or a $\bar{t}_2$-mark is on P_{13} or P_{15}, then we have a $(1, 1, 2k-1)$-solution. Thus we can assume that all $\bar{s}_2$-marks and all $\bar{t}_2$-marks of C_1 are on P, the path from s_1 to t_1 composed of P_{11}, P_{12} and P_{14}. Let e_x and e_y be a $\bar{s}_2$-marked edge and a $\bar{t}_2$-marked edge of P, respectively. By symmetry, there are four essentially distinct configurations of e_x and e_y on the path; (Case 2a) $e_x, e_y \in P_{11}$, (Case 2b) $e_x \in P_{11}$, $e_y \in P_{14}$, (Case 2c) $e_x, e_y \in P_{14}$ and (Case 2d) $e_x \in P_{11}$, $e_y \in P_{12}$. Case 2c requires no further discussion. For each remaining case we can always constuct two disjoint paths, one from s_1 to t_1 and the other from s_2 to t_2, at the cost of at most two ribs. Figure 7a illustrates the construction of the desired paths for Cases 2a and 2b, and Fig. 7b for Case 2d. In these figures a broken line indicates the path from s_1 to t_1 and a dot and dash line the path from s_2 to t_2. Therefore we have a $(1, 1, 2k-1)$-solutions. This completes the proof of theorem.

Remark. The referee notified us that Corollary 2 also follows from Okamura's result combined with the following theorem of *W*. Mader (which has not yet been published): If x and y are vertices in a $(k+2)$-edge-connected graph G, then G has a path P between x and y such that $G - E(p)$ is k-edge-connected.

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