Parabolic Differential Equations
and Lyapunov Like Functions*

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1. One of the most important techniques in the theory of nonlinear
differential equations is the direct method of Lyapunov and its extensions. It
depends basically on the fact that a function satisfying the inequality

\[ m'(t) \leq \omega(t, m(t)), \quad m(t_0) = r_0, \]

is majorized by the maximal solution of the equation

\[ r' = \omega(t, r), \quad r(t_0) = r_0. \]

This comparison principle enables one to study various problems of differential
equations [1-3].

The problem of stability of solutions of parabolic equations has been
investigated by Bellman [4], Prodi [5], Narasimhan [6], Mlak [7, 8] and others.
We obtain a number of results in a unified way by the above mentioned
approach. For instance, our results include, bounds on the solutions, unique-
ness, stability and boundedness of solutions. We also indicate that using
Lyapunov like vector functions is useful in some cases. Examples are given
to illustrate some of the results.

2. Let \( I \) denote the interval \( t_0 \leq t < \infty, t_0 \geq 0 \) and \( \mathbb{R}^n \) denote \( n \)-di-
mensional Euclidean space. Let \( D \subset \mathbb{R}^n \) be an open and bounded set. Denote by
\( H \) the cartesian product \( I \times D \). Let \( \overline{H} \) denote the closure of \( H \). The boundary
of \( D \) is denoted by \( \Gamma \). Let \( u = (u_1, \ldots, u_m), \quad p = (p_1, \ldots, p_n) \) and
\( q = (q_1, \ldots, q_n) \). Now let \( F(t, x, u, p, q) \) be a function defined and con-

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tinuous on $H \times R^m \times R^n \times R^{n^2}$. We consider the partial differential system of the type

$$\frac{\partial u^i}{\partial t} = F(t, x, u, u^i_x, u^i_{xx}) \quad (i = 1, 2, \ldots, m)$$

(2.1)

where

$$u^i_x = \left( \frac{\partial u^i}{\partial x_1}, \ldots, \frac{\partial u^i}{\partial x_m} \right)$$

and

$$u^i_{xx} = \left( \frac{\partial^2 u^i}{\partial x_1^2}, \frac{\partial^2 u^i}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u^i}{\partial x_n^2} \right).$$

We shall write, for convenience, the system (2.1) in the following form

$$\frac{\partial u}{\partial t} = F(t, x, u, u^i_x, u^i_{xx})$$

and use similar notation below.

**Definition.** Given an initial function $\phi(t, x)$ which is defined and continuous on $D \cup I \times \Gamma$, a solution of (2.1) is any function $u(t, x)$ satisfying the following properties:

(i) $u(t, x)$ is defined and continuous for $(t, x) \in \bar{H}$;

(ii) $u(t, x) = \phi(t, x)$ for $(t, x) \in D \cup I \times \Gamma$;

(iii) $u(t, x)$ possesses continuous partial derivatives $\partial u/\partial t, u_x, u_{xx}$ in the int $\bar{H}$ and satisfies (2.1) for $(t, x) \in \text{int} \bar{H}$.

We shall consider the following two partial differential systems

$$\frac{\partial u}{\partial t} = f(t, x, u, u^i_x, u^i_{xx}).$$

(2.2)

$$\frac{\partial v}{\partial t} = g(t, x, v, v^i_x, v^i_{xx}),$$

(2.3)

where $f$ and $g$ are vector functions defined and continuous on $\bar{H} \times R^m \times R^n \times R^{n^2}$.

Let us assume, hereafter, that solutions of (2.2) and (2.3) exist as defined above. We establish a number of results on stability and boundedness of solutions of (2.2) and (2.3). Our work constitutes an extension to partial differential systems of our results [1-3] in ordinary and functional differential equations.
Let $G(t, x, m, P, Q)$ be a scalar function defined and continuous on 
$$
\mathcal{H} \times R^+ \times \mathbb{R}^n \times \mathbb{R}^{n^2},
$$
where $R^+$ denotes $[0, \infty)$. Consider the partial differential inequality

$$
\frac{\partial m}{\partial t} \leq G(t, x, m, m_x, m_{xx}), \quad (2.4)
$$

where

$$
m_x = \left( \frac{\partial m}{\partial x_1}, \cdots, \frac{\partial m}{\partial x_m} \right)
$$

and

$$
m_{xx} = \left( \frac{\partial^2 m}{\partial x_1^2}, \frac{\partial^2 m}{\partial x_1 \partial x_2}, \cdots, \frac{\partial^2 m}{\partial x_m^2} \right).
$$

The inequality (2.4) is said to be parabolic, if the following condition holds:

For any system of numbers $Q_{ik}, R_{ik}$ $(i, k = 1, \cdots, n)$; if the quadratic form

$$
\sum_{i,k=1}^n (Q_{ik} - R_{ik}) \lambda_i \lambda_k \geq 0. \quad (2.5)
$$

for arbitrary $\lambda_1, \cdots, \lambda_n$, then

$$
G(t, x, m, P, Q) \geq G(t, x, m, P, R). \quad (2.6)
$$

**Hypotethis 2.7.** Let the inequality (2.4) be parabolic. Suppose further that

$$
G(t, x, m, 0, 0) \leq W(t, m), \quad (2.8)
$$

where $W(t, r)$ is a function defined and continuous on $I \times R^+$. Let $r(t)$ be the maximal solution of the differential equation

$$
r' = W(t, r); \quad r(t_0) = r \geq 0. \quad (2.9)
$$

existing to the right of $t_0$.

The following result plays an important role in our work.

**Lemma 1.** Let the hypothesis (2.7) hold. Let the function $m(t, x)$ be non-negative, defined and continuous for $(t, x) \in \mathcal{H}$. Assume that $m(t, x)$ has partial derivatives $\partial m/\partial t, m_x, m_{xx}$ for $(t, x) \in \text{int} \mathcal{H}$ and it satisfies the inequality (2.4) for $(t, x) \in \text{int} \mathcal{H}$. If $m(t, x) \leq r(t)$ for $(t, x) \in D \cup I \times \Gamma$, then

$$
m(t, x) \leq r(t) \quad \text{for} \quad (t, x) \in \mathcal{H}. \quad (2.10)
$$
PROOF. Suppose that \( m(t, x) \) has all the properties assumed in the lemma. To prove (2.10), we consider the ordinary differential equation

\[
r' = W(t, r) + \epsilon
\]

(2.11)

which has solutions \( r(t, \epsilon) \), for all sufficiently small \( \epsilon > 0 \), existing as far as \( r(t) \) exists, such that \( r(t_0, \epsilon) = r_0 + \epsilon \). Since \( \lim_{\epsilon \to 0} r(t, \epsilon) = r(t) \) [9], it is enough to prove

\[
m(t, x) < r(t, \epsilon) \quad \text{for} \quad (t, x) \in \mathcal{H}
\]

whenever

\[
m(t, x) < r(t, \epsilon) \quad \text{for} \quad (t, x) \in D \cup I \times \Gamma.
\]

(2.12)

For this purpose, suppose that the set

\[
S = \{(t, x) \in \mathcal{H} : r(t, \epsilon) \leq m(t, x)\}
\]

is nonempty. Let \( S_t \) be the projection of \( S \) on the \( t \)-axis and \( t_1 = \inf S_t \).

We then have \( m(t, x) \leq r(t, \epsilon) \) for \( t_0 \leq t \leq t_1, \ x \in D \). Write

\[
z(x) = r(t_1, \epsilon) - m(t_1, x).
\]

We assert that \( z(x) \) has a minimum equal to zero for some \( x \in D \). If this were not true, since (2.12) holds, one must have \( z(x) > 0 \) for all \( x \in D \). This contradicts the definition of \( t_1 \). Hence there is a point \( x_0 \in D \) such that

\[
r(t_1, \epsilon) = m(t_1, x_0).
\]

(2.13)

It therefore follows that

\[
\frac{\partial m(t, x)}{\partial t} \geq r'(t, \epsilon) \quad \text{at} \quad (t_1, x_0).
\]

(2.14)

Since \( z(x) \) attains an interior minimum at \( (t_1, x_0) \), we obtain

\[
\frac{\partial z(x)}{\partial x} = 0 \quad \text{at} \quad (t_1, x_0)
\]

(2.15)

and the quadratic form

\[
\sum_{i,k=1}^{n} \frac{\partial^2 z(x)}{\partial x_i \partial x_k} \lambda_i \lambda_k \geq 0 \quad \text{at} \quad (t_1, x_0).
\]

(2.16)

From (2.4), (2.11), and (2.14), we get the inequality

\[
G(t_1, x_0, m(t_1, x_0), m_x, m_{xx}) \geq W(t_1, r(t_1, \epsilon)) + \epsilon,
\]

(2.17)
where \( m_x \) and \( m_{xx} \) are to be evaluated at \((t_1, x_0)\). Because (2.14) is parabolic, the relations (2.15) and (2.16) imply, in view of the definition of \( \pi(x) \), the inequality
\[
G(t_1, x_0, m(t_1, x_0), m_x, m_{xx}) \leq G(t_1, x_0, m(t_1, x_0), 0, 0).
\] (2.18)
Further, since the hypothesis (2.7) holds, one obtains from (2.8)
\[
G(t_1, x_0, m(t_1, x_0), 0, 0) \leq W(t_1, m(t_1, x_0)).
\] (2.19)
The inequalities (2.17), (2.18), (2.19) lead to a contradiction because of (2.13). Hence \( S \) is empty and this proves (2.10). The proof is complete.

Let a scalar function \( V(t, x, u, v) \) be defined and continuous on \( \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \). Suppose that it has partial derivatives with respect to \( t \) and the components of \( x, u, \) and \( v \). For convenience, we shall write \( V \) for \( V(t, x, u, v) \) below. We define the function
\[
V^*(t, x, u, v) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial u} \cdot f(t, x, u, u_x^i, u_{xx}^i) + \frac{\partial V}{\partial v} \cdot g(t, x, v, v_x^i, v_{xx}^i),
\] (2.20)
where \( \cdot \) denotes the usual scalar product of vectors. In the following, it is convenient to use the vectors \( V_x, V_{xx} \) of dimensions \( n, n^2 \), respectively defined by
\[
V_x = \left[ \frac{\partial^2 V}{\partial x^\mu} \frac{\partial V}{\partial u_{\mu}} + \frac{\partial^2 V}{\partial x^\mu} \frac{\partial V}{\partial u_{\mu}} + \frac{\partial^2 V}{\partial x^\mu} \frac{\partial V}{\partial u_{\mu}}, \mu = 1, 2, \ldots, m \right]
\]
\[
V_{xx} = \left[ \frac{\partial^2 V}{\partial x^\mu} \frac{\partial V}{\partial u_{\mu}} + \frac{\partial^2 V}{\partial x^\mu} \frac{\partial V}{\partial u_{\mu}} + \frac{\partial^2 V}{\partial x^\mu} \frac{\partial V}{\partial u_{\mu}}, \nu = 1, 2, \ldots, n \right].
\]
With respect to these functions, we state the following theorems.

**Theorem 1.** Let the hypothesis (2.7) hold. Suppose that the function \( V^*(t, x, u, v) \) of (2.20) satisfies the condition
\[
V^*(t, x, u, v) \leq G(t, x, V, V_x, V_{xx}).
\] (2.21)
Let \( u(t, x) \) and \( v(t, x) \) be any two solutions of (2.2) and (2.3) such that \( u(t, x) = \phi(t, x), v(t, x) = \psi(t, x) \) for \((t, x) \in D \cup I \times I\). If
\[
V(t, x, \phi(t, x), \psi(t, x)) \leq r(t) \quad \text{for} \quad (t, x) \in D \cup I \times I ;
\] (2.22)
then
\[
V(t, x, u(t, x), v(t, x)) \leq r(t) \quad \text{for} \quad (t, x) \in \mathbb{R}.
\] (2.23)
PROOF. Let \( u(t, x) \) and \( v(t, x) \) be any two solutions of (2.2) and (2.3) satisfying (2.22). Define

\[
m(t, x) = V(t, x, u(t, x), v(t, x)).
\]

Then

\[
m(t, x) = V(t, x, \phi(t, x), \psi(t, x)) \leq r(t) \quad \text{for} \quad (t, x) \in D \cup I \times \Gamma.
\]

Further, because of (2.20), (2.21) and the definition of \( m(t, x) \), one obtains

\[
\frac{\partial m(t, x)}{\partial t} \leq G(t, x, m(t, x), m_x(t, x), m_{xx}(t, x)) \quad \text{for} \quad (t, x) \in \text{int} \bar{\Omega}.
\]

Now a straightforward application of Lemma 1, yields the stated result.

THEOREM 2. Let the assumptions of Theorem 1 hold except that the conditions (2.21) and (2.22) are replaced by

\[
A(t) V^*(t, x, u, v) + V(t, x, u, v) A'(t) \leq G(t, x, A(t) V, A(t) V_x, A(t) V_{xx}).
\]

where \( A(t) > 0 \) is a continuous function on \( I \) and differentiable for each \( t \in I \); and

\[
A(t) V(t, x, \phi(t, x), \psi(t, x)) \leq r(t) \quad \text{for} \quad (t, x) \in D \cup I \times \Gamma.
\]

Then, (2.23) takes the form

\[
A(t) V(t, x, u(t, x), v(t, x)) \leq r(t) \quad \text{for} \quad (t, x) \in \bar{\Omega}.
\]

PROOF. This theorem can be reduced to Theorem 1 by defining

\[
V_1(t, x, u, v) = A(t) V(t, x, u, v)
\]

and verifying that \( V_1(t, x, u, v) \) preserves the properties of \( V(t, x, u, v) \). We leave the details to the reader.

REMARK. Taking \( A(t) \equiv 1 \), we see that Theorem 2 reduces to Theorem 1. Since Theorem 1 is an important tool by itself in the study of various problems of partial differential equations, we have listed it separately. We note that \( W(t, r) \) of (2.8) need not be nonnegative. This has an advantage in obtaining sharper bounds and in considering stability and boundedness results later. For example, taking \( V = |u - v| \) and \( W(t, r) = k(t) r \), where \( k(t) \) is continuous on \( I \), one can get an upper bound from Theorem 1 as follows:

\[
|u(t, x) - v(t, x)| \leq r_0 \exp \left[ \int_{t_0}^t k(s) \, ds \right] \quad \text{for} \quad (t, x) \in \bar{\Omega}.
\]
whenever

$$|\phi(t, x) - \psi(t, x)| \leq r_0 \exp \left[ \int_{t_0}^{t} k(s) \, ds \right] \quad \text{for} \quad (t, x) \in D \cup I \times \Gamma;$$

where

$$\max_{x \in \Delta} |\phi(t_0, x) - \psi(t_0, x)| \leq r_0.$$

If we assume that $V(t, x, u, v) \equiv 0$ if and only if $u = v$, Theorem 1 can be used to get a uniqueness result as follows. We merely state

**UNIQUENESS THEOREM.** Let the hypothesis (2.7) hold with $r_0 = 0$. Suppose further that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial u} \cdot f(t, x, u, u^i_x, u^i_{xx})$$

$$+ \frac{\partial V}{\partial v} \cdot f(t, x, v, v^i_x, v^i_{xx}) \leq G(t, x, V, V_x, V_{xx}).$$

Let the maximal solution $r(t)$ of (2.9) with $r(t_0) = 0$ be identically zero. Then there is at most one solution of (2.2).

3. Suppose that $u(t, x)$ and $v(t, x)$ are any two solutions of (2.2) and (2.3) with the initial functions $\phi(t, x)$ and $\psi(t, x)$ on the boundary $D \cup I \times \Gamma$. Let $|x|$ denote any convenient norm of vector $x$. Define

$$d[\phi(t, \cdot), \psi(t, \cdot)]_\Gamma = \max_{x \in \Gamma} |\phi(t, x) - \psi(t, x)|$$

and

$$d[u(t, \cdot), v(t, \cdot)]_D = \max_{x \in \Delta} |u(t, x) - v(t, x)|.$$

In order to unify our results on stability and boundedness, we list below the following conditions which are natural extensions of the conditions in [1].

(3.1) For each $\epsilon > 0$ and $t_0 \geq 0$, there exists a positive function $\eta(t_0, \epsilon)$ that is continuous in $t_0$ for each $\epsilon$ and such that if

(i) $d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \eta(t_0, \epsilon)$;

(ii) $d[\phi(t, \cdot), \psi(t, \cdot)]_\Gamma < \epsilon, \quad t \geq t_0$;

then

$$d[u(t, \cdot), v(t, \cdot)]_D < \epsilon, \quad t \geq t_0.$$

(3.2) The $\eta$ in (3.1) is independent of $t_0$. 
(3.3) For each $\epsilon > 0$, $\alpha > 0$ and $t_0 \geq 0$, there exists a positive number $T = T(t_0, \epsilon, \alpha)$ such that if

(i) $d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \alpha$;
(ii) $d[\phi(t, \cdot), \psi(t, \cdot)]_r < \epsilon$, $t \geq t_0 + T$,

then

$d[u(t, \cdot), v(t, \cdot)]_D < \epsilon$, $t \geq t_0 + T$.

(3.4) The $T$ in (3.3) is independent of $t_0$.
(3.5) The conditions (3.1) and (3.3) hold simultaneously.
(3.6) The conditions (3.2) and (3.4) hold simultaneously.
(3.7) For each $\alpha > 0$ and $t_0 \geq 0$, there exists a positive function $\beta(t_0, \alpha)$ that is continuous in $t_0$ for each $\alpha$ and such that if

(i) $d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \alpha$;
(ii) $d[\phi(t, \cdot), \psi(t, \cdot)]_r < \beta(t_0, \alpha)$, $t \geq t_0$,

then

$d[u(t, \cdot), v(t, \cdot)]_D < \beta(t_0, \alpha)$, $t \geq t_0$.

(3.8) The $\beta$ in (3.7) is independent of $t_0$.
(3.9) For each $\alpha > 0$ and $t_0 \geq 0$, there exist positive numbers $N$ and $T = T(t_0, \alpha)$ such that if

(i) $d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \alpha$;
(ii) $d[\phi(t, \cdot), \psi(t, \cdot)]_r < N$, $t \geq t_0 + T$,

then

$d[u(t, \cdot), v(t, \cdot)]_D < N$, $t > t_0 + T$.

(3.10) The $T$ in (3.9) is independent of $t_0$.
(3.11) The conditions (3.7) and (3.9) hold simultaneously.
(3.12) The conditions (3.8) and (3.10) hold simultaneously.

Remark. Corresponding to the conditions above, if we say that the ordinary differential equation (2.9) has the property (3.1a), we mean the following condition is satisfied.

(3.1a) Given $\epsilon > 0$ and $t_0 \geq 0$, there exists a positive function $\eta(t_0, \epsilon)$ that is continuous in $t_0$ for each $\epsilon$ and satisfies the inequality $r(t) < \epsilon$, $t \geq t_0$, provided $r(t_0) \leq \eta(t_0, \epsilon)$.

Conditions (3.2a) to (3.12a) may be formulated similarly.

The following theorems on stability and boundedness are extensions of analogous results in ordinary and functional differential equations [1-3]. We assume that
(3.13) The function $b(r)$ is continuous and non-decreasing in $r$, $b(r) > 0$ for $r > 0$ and $b(|u - v|) \leq V(t, x, u, v)$. On occasion, we may also assume, below, that

$$b(r) \to \infty \quad \text{as} \quad r \to \infty.$$  

(3.14)

**Theorem 3.** Let the assumptions of Theorem 1 hold, together with (3.13). Suppose further that the differential equation (2.9) satisfies one of the conditions (3.1a), (3.2a), (3.3a), (3.4a), (3.5a) and (3.6a); then the systems (2.2) and (2.3) satisfy the corresponding one of the conditions (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6).

**Proof.** For each $\epsilon > 0$, if $|u - v| = \epsilon$, we deduce from (3.13) that

$$b(\epsilon) \leq V(t, x, u, v).$$

Suppose that the differential equation (2.9) has the property (3.1a). Then, given $b(\epsilon) > 0$ and $t_0 > 0$, there exists a positive function $\eta(t_0, \epsilon)$ such that

$$\eta(t_0, \epsilon) = \eta(t_0, \epsilon).$$

(3.15)

if $r(t_0) = r_0 \leq \eta(t_0, \epsilon)$. Suppose that $u(t, x)$ and $v(t, x)$ are any two solutions of (2.2) and (2.3) with the initial functions $\phi(t, x)$ and $\psi(t, x)$ on the boundary $D \cup I \times \Gamma$. Let $r_0 \leq \eta(t_0, \epsilon)$. Then, one obtains from (2.22), the relation

$$V(t_0, x, \phi(t_0, x), \psi(t_0, x)) \leq r_0 \leq \eta(t_0, \epsilon).$$

By (3.13) and the monotonicity of $b(r)$, this implies that

$$|\phi(t_0, x) - \psi(t_0, x)| \leq b^{-1}(\eta(t_0, \epsilon)) \equiv \delta(t_0, \epsilon)$$

for all $x \in D$, which in turn yields

$$d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \delta(t_0, \epsilon).$$

Assume now that there exist solutions $u(t, x)$ and $v(t, x)$ of (2.2) and (2.3) for which

(i) $d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \delta(t_0, \epsilon)$;

(ii) $d[\phi(t, \cdot), \psi(t, \cdot)]_D < \epsilon$, \quad $t \geq t_0$,

have the property that $d[u(t_1, \cdot), v(t_1, \cdot)]_D \geq \epsilon$ for some $t = t_1 > t_0$. Then there exists an $x_0 \in D$, such that $|u(t_1, x_0) - v(t_1, x_0)| = \epsilon$, because of (ii) above. From the relations (2.23), (3.13) and (3.15), we obtain the inequality

$$b(\epsilon) \leq V(t_1, x_0, u(t_1, x_0)) \leq r(t_1) < b(\epsilon),$$

which is a contradiction. This proves (3.1).

The proof of (3.2) is essentially the same, since $\eta(t_0, \epsilon)$ is independent of $t_0$, in this case.
The proofs of other statements are also similar. We shall only indicate the proof of the conclusion (3.3). Since the differential equation (2.9) satisfies (3.3a), given \( b(e) > 0, \alpha > 0 \) and \( t_0 \geq 0 \), there exists a positive number \( T = T(t_0, \alpha, \epsilon) \) such that

\[
r(t) < b(e), \quad t \geq t_0 + T,
\]

if \( r(t_0) = r_0 \leq \alpha \). Suppose that \( u(t, x) \) and \( v(t, x) \) are any two solutions of (2.2) and (2.3) with the initial functions \( \varphi(t, x) \) and \( \psi(t, x) \) on the boundary \( D \cup I \times \Gamma \). Choosing \( r_0 \leq \alpha \), one obtains, as before,

\[
d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq b^{-1}(\alpha) \equiv \gamma.
\]

Let \( \{t_k\} \) be a divergent sequence. Suppose that (i) \( d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \gamma \) and (ii) \( d[\phi(t, \cdot), \psi(t, \cdot)]_D \leq \epsilon \) for \( t \geq t_0 + T \). Assume now that there exist solutions \( u(t, x) \) and \( v(t, x) \), whose initial functions \( \varphi(t, x) \) satisfy (i) and (ii), having the property \( d[u(t_k, \cdot), v(t_k, \cdot)]_D \geq \epsilon \). Then, there exist \( x_k \in D \) such that \( |u(t_k, x_k) - v(t_k, x_k)| = \epsilon \). This, together with the relations (2.23), (3.13) and (3.16), implies the following contradiction

\[
b(\epsilon) \leq V(t_k, x_k, u(t_k, x_k), v(t_k, x_k)) \leq r(t_k) < b(\epsilon).
\]

Hence the conclusion (3.3) holds and this completes the proof.

**Theorem 4.** Let the assumptions of Theorem 1 hold, together with (3.13) and (3.14). Suppose further that the differential equation (2.9) satisfies one of the conditions (3.7a), (3.8a), (3.9a), (3.10a), (3.11a) and (3.12a); then the systems (2.2) and (2.3) satisfy the corresponding one of the conditions (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12).

**Proof.** Suppose that the differential equation (2.9) has the property (3.7a). Then, corresponding to \( \alpha > 0 \) and \( t_0 \geq 0 \), there exists a positive function \( \beta(t_0, \alpha) \), that is continuous in \( t_0 \) for each \( \alpha \) and satisfies

\[
r(t) < \beta(t_0, \alpha)
\]

if \( r_0 \leq \alpha \) and \( t \geq t_0 \). Since \( b(r) \rightarrow \infty \) as \( r \rightarrow \infty \), there exists an \( L = L(t_0, \alpha) \) such that

\[
b(L) > \beta(t_0, \alpha).
\]

Assume now that \( u(t, x) \) and \( v(t, x) \) are any solutions of (2.2) and (2.3) with the initial functions \( \phi(t, x) \) and \( \psi(t, x) \) on the boundary \( D \cup I \times \Gamma \). Let \( r_0 \leq \alpha \). Then it follows from (3.13) that

\[
b[\phi(t_0, x) - \psi(t_0, x)] \leq V(t_0, x, \phi(r, x), \psi(t_0, x)) \leq r_0 \leq \alpha.
\]
Since \( b(r) \) is nonnegative and increasing,
\[
| \phi(t_0, x) - \psi(t_0, x) | \leq b^{-1}(\alpha) = \gamma,
\]
x \in D, which implies that
\[
dx[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \gamma.
\]
If there exist solutions \( u(t, x) \) and \( v(t, x) \) for which
\[
(i) \quad d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \gamma.
\]
\[
(ii) \quad d[\phi(t, \cdot), \psi(t, \cdot)]_R \leq L \quad \text{for} \quad t \geq t_0,
\]
which have the property that \( d[u(t_1, \cdot), v(t_1, \cdot)]_D \geq L \) for some \( t = t_1 \geq t_0 \),
then there is an \( x_0 \in D \) such that
\[
| u(t_1, x_0) - v(t_1, x_0) | = L
\]
because of (ii) above. In view of the relations (2.23), (3.17), and (3.18), we are
led to the contradiction
\[
b(L) \leq V(t_1, x_0, u(t_1, x_0), v(t_1, x_0)) \leq r(t_1) \leq \beta(t_0, \alpha) < b(L)
\]
as before and this proves the conclusion (3.7).

By following the proof of Theorem 3 and that given above, we can easily
construct proofs of the remaining statements. We omit the details.

**Theorem 5.** Let the assumptions of Theorem 2 hold, together with (3.13).
Suppose that the differential equation (2.9) satisfies one of the conditions (3.1a)
and (3.2a). Let \( A(t) \to \infty \) as \( t \to \infty \). Then, the systems (2.2) and (2.3) satisfy
the corresponding one of the condition (3.3) and (3.4). If, in addition, \( A(t) \geq 1 \),
then, the systems (2.2) and (2.3) have the properties (3.5) and (3.6) respec-
tively.

**Proof.** For any \( \epsilon > 0 \), if \( | u - v | = \epsilon \), it follows from (3.13) that
\( b(\epsilon) \leq V(t, x, u, v) \). If the equation (2.9) satisfies (3.1a), then given \( b(\epsilon) > 0 \)
and \( t_0 \geq 0 \), there exists a positive number \( \eta(t_0, \epsilon) \) such that
\[
r(t) < b(\epsilon) \tag{3.19}
\]
provided \( r_0 \leq \eta(t_0, x) \) and \( t \geq t_0 \). Suppose that \( u(t, x) \) and \( v(t, x) \) are any
solutions of (2.2) and (2.3) with the initial functions \( \phi(t, x) \) and \( \psi(t, x) \) on
the boundary \( D \cup I \times \Gamma \). Choosing \( r_0 \leq \eta(t_0, \epsilon) \), we have, from (2.22a)
\[
A(t_0) V(t_0, x, \phi(t_0, x), \psi(t_0, x)) \leq r_0 \leq \eta(t_0, \epsilon).
\]
Because of (3.13), this means that
\[
d[\phi(t_0, x), \psi(t_0, x)]_0 \leq b^{-1}(\eta(t_0, \epsilon)/A(t_0)) = \alpha.
\]
Let \( \{t_k\} \) be a divergent sequence. Suppose that

(i) \[
d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_0 \leq \alpha;
\]
and that there exists a \( T = T(t_0, \alpha, \epsilon) \) such that

(ii) \[
d[\phi(t, \cdot), \psi(t, \cdot)]_r < \epsilon \quad \text{for} \quad t \geq t_0 + T.
\]

Assume now that there exist solutions \( u(t, x) \) and \( v(t, x) \) of (2.2) and (2.3), whose initial functions \( \phi(t, x) \) and \( \psi(t, x) \) satisfy (i) and (ii) above, have the property that \( d[u(t_k, \cdot), v(t_k, \cdot)]_D \geq \epsilon \). Then, there exist \( x_k \in D \), such that

\[
|u(t_k, x_k) - v(t_k, x_k)| = \epsilon.
\]

This, together with (2.23a), (3.13), and (3.19), yields the inequality

\[
A(t_k) b(\epsilon) \leq A(t_k) V(t_k, x_k, u(t_k, x_k), v(t_k, x_k)) \leq r(t_k) < b(\epsilon).
\]

Since \( A(t_k) \to \infty \) as \( t_k \to \infty \), \( A(t_k) > 1 \) for large \( k \). As \( b(\epsilon) > 0 \), this is a contradiction. Hence the conclusion (3.3) follows. If \( A(t) \geq 1 \), then, in analogy to the proof of Theorem 3, we find that the systems (2.2) and (2.3) satisfy (3.1). This implies that they have the property (3.5). The proof of the other cases is similar. We leave the details.

**Theorem 6.** Let the assumptions of Theorem 2 hold, together with (3.13) and (3.14). Let \( A(t) \to \infty \) as \( t \to \infty \). Suppose that the differential equation (2.9) satisfies one of the conditions (3.7a) and (3.8a). Then the systems (2.2) and (2.3) satisfy the corresponding one of the conditions (3.9) and (3.10). If, in addition, \( A(t) \geq 1 \), the systems (2.2) and (2.3) satisfy (3.11) and (3.12) respectively.

**Proof.** We first show that (3.9) is implied by (3.7a). Let \( u(t, x) \) and \( v(t, x) \) be any solutions of (2.2) and (2.3) such that

\[
A(t) V(t, x, \phi(t, x), \psi(t, x)) \leq r(t) \quad \text{for} \quad (t, x) \in D \cup I \times \Gamma
\]

(3.20)

where \( \phi(t, x) \) and \( \psi(t, x) \) are initial functions on the boundary \( D \cup I \times \Gamma \).

Then, we have from Theorem 2 that

\[
A(t) V(t, x, u(t, x), v(t, x)) \leq r(t) \quad \text{for} \quad (t, x) \in \bar{H}.
\]

(3.21)

Since the equation (2.9) satisfies (3.7a), given \( \alpha > 0 \) and \( t_0 \geq 0 \), there exists a positive number \( \beta(t_0, \alpha) \) such that \( r(t) < \beta(t_0, \alpha) \) if \( r_0 \leq \alpha \). Since \( b(r) \to \infty \) as \( r \to \infty \), there exists an \( L \) such that

\[
b(L) > \beta(t_0, \alpha).
\]

(3.22)

Now choosing \( r_0 \leq \alpha \), we obtain from (3.13) and (3.20) that

\[
d[\phi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq b^{-1}(\alpha) \equiv \nu.
\]
Suppose that the initial functions satisfy (i) \( d[\psi(t_0, \cdot), \psi(t_0, \cdot)]_D \leq \nu; \)
(ii) there exists a \( T = T(t_0, \nu) \) such that \( d[\psi(t, \cdot), \psi(t, \cdot)]_D < L \) for \( t > t_0 + T. \)
Let \( \{t_k\} \) be a divergent sequence. If possible, let \( d[u(t_k, \cdot), v(t_k, \cdot)]_D > L. \)
Then, as before, there exist \( x_k \in D \) such that \( |u(t_k, x_k) - v(t_k, x_k)| = L. \)
From the relations (3.13), (3.21), and (3.22), it follows that
\[
A(t_k) b(L) \leq A(t_k) V(t_k, x_k, u(t_k, x_k), v(t_k, x_k)) \leq r(t_k) \leq \beta(t_0, \alpha) < b(L).
\]
This is a contradiction, since \( A(t_k) \to \infty \) as \( t_k \to \infty \) and \( b(L) > 0. \) This proves (3.9). If \( A(t) \geq 1, \) then in analogy to the proof of Theorem 4, we find that the systems (2.2) and (2.3) satisfy (3.7). This implies that they have the property (3.11). Similar conclusions hold for the other case—and the proof is complete.

4. We now extend the preceding results to perturbed systems. Corresponding to (2.2) and (2.3), let us consider the systems
\[
\frac{\partial u}{\partial t} = f(t, x, u, u_x, u_{xx}) + F_1(t, x, u, v) \tag{4.1}
\]
\[
\frac{\partial v}{\partial t} = g(t, x, v, v_x, v_{xx}) + G_1(t, x, u, v), \tag{4.2}
\]
where \( F_1 \) and \( G_1 \) are perturbations. If the solutions of (4.1) and (4.2) satisfy the conditions (3.1) to (3.12) for all the perturbations \( F \) and \( G \) for which
\[
|F_1(t, x, u, v)| + |G_1(t, x, u, v)| \leq \eta V(t, x, u, v) \quad (\eta > 0), \tag{4.3}
\]
we say that the systems (2.2) and (2.3) satisfy the conditions (3.1) to (3.12) weakly.

The following analogous theorems for weak stability and boundedness may then be stated. Assume that
\[
\left| \frac{\partial V}{\partial u} \right|, \quad \left| \frac{\partial V}{\partial v} \right| \leq k, \quad (k = k(t, x, u, v)). \tag{4.4}
\]

**Theorem 7.** Let the assumptions of Theorem 1 hold, except that the condition (2.21) is replaced by
\[
V^*(t, x, u, v) + \alpha V(t, x, u, v) \leq G(t, x, V, V_x, V_{xx}), \tag{4.5}
\]
where \( \alpha = K \eta \) \((K \text{ is the constant defined in (4.4))}. \) Suppose that (3.13) holds. Then if the differential equation (2.9) satisfies one of the conditions (3.1a), (3.2a), (3.3a), (3.4a), (3.5a) and (3.6a), the systems (2.2) and (2.3) satisfy weakly the corresponding one of the conditions (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6).
Theorem 8. Let the assumptions in the first sentence of Theorem 7 hold, together with (3.13) and (3.14). Suppose that the differential equation (2.9) satisfies one of the conditions (3.7a), (3.8a), (3.9a), (3.10a), (3.11a), and (3.12a). Then the systems (2.2) and (2.3) satisfy weakly the corresponding one of the conditions (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12).

Proof of Theorems 7 and 8. Define
\[ V^*(t, x, u, v) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial u} \cdot (f(t, x, u, u_x, u_{xx}) + F_1(t, x, u, v)) \]
\[ + \frac{\partial V}{\partial v} \cdot (g(t, x, v, v_x, v_{xx}) + G_1(t, x, u, v)). \]

Using (4.3), (4.4), and (4.5) and noting that \( \alpha = K \eta \), we obtain the inequality
\[ V^*(t, x, u, v) \leq G(t, x, V, V_x, V_{xx}). \]

If \( u(t, x) \) and \( v(t, x) \) are any two solutions of (4.1) and (4.2) with the initial functions \( \phi(t, x) \) and \( \psi(t, x) \) on the boundary \( D \cup I \times I' \), we can obtain the desired results by applying directly the proofs of Theorems 1, 3, and 4. We omit the details.

Theorem 9. Suppose that the assumption of Theorem 1 hold, except that the condition (2.21) is replaced by
\[ V^*(t, x, u, v) + \alpha V(t, x, u, v) \leq G(t, x, V e^{\beta t}, V_x e^{\beta t}, V_{xx} e^{\beta t}) e^{-\beta t} \]
(4.6)
where \( \beta \) is positive and satisfies the inequality \( \alpha \geq K \eta + \beta \). Let the assumption (3.13) hold. Then, if the differential equation (2.9) satisfies one of the conditions (3.1a) and (3.2a), the systems (2.2) and (2.3) satisfy weakly the corresponding one of the conditions (3.3) and (3.4). If \( e^{\beta t} \) in (4.6) is replaced by \( e^{\beta(t-t_0)} \), the systems satisfy (3.5) and (3.6) respectively.

Theorem 10. Let the assumptions in the first sentence of Theorem 9 hold, together with (3.14). Let the differential equation (2.2) satisfy the condition (3.7a) or (3.8a). Then, the systems (2.2) and (2.3) satisfy weakly the condition (3.9) or (3.10). If \( e^{\beta t} \) in (4.6) is replaced by \( e^{\beta(t-t_0)} \), the systems have the property (3.11) or (3.12) respectively.

Proof of Theorem 9 and 10. Proceeding as in the proof of Theorems 7 and 8 we obtain the inequality
\[ V^*(t, x, u, v) + \beta V(t, x, u, v) \leq G(t, x, V e^{\beta t}, V_x e^{\beta t}, V_{xx} e^{\beta t}) e^{-\beta t}. \]
This is similar to condition (2.21a) of Theorem 2 with $A(t) = e^{at}$. Hence one obtains from Theorem 2,

$$V(t, x, u(t, x), v(t, x)) e^{at} \leq r(t) \quad \text{for} \quad (t, x) \in \mathcal{R},$$

where $u(t, x)$ and $v(t, x)$ are any two solutions of (4.1) and (4.2) with the initial functions $\phi(t, x)$ and $\psi(t, x)$ satisfying

$$V(t, x, \phi(t, x) \cdot \psi(t, x)) e^{at} \leq r(t) \quad \text{for} \quad (t, x) \in D \cup I \times \Gamma.$$

Now, following the proofs of Theorems 5 and 6, we can establish the results. We leave the details to the reader.

5. Let us replace (2.4) by the parabolic inequalities

$$\frac{\partial z_i}{\partial t} \leq G_i(t, x, z_1, \ldots, z_m, z_i', z_{i,x}),$$

(5.1)

where each $G_i$ is defined and continuous on $H \times R^m_+ \times R^n \times R^{n^2}$. Let the functions $W_i(t, z_1, \ldots, z_m)$ be defined and continuous on $I \times R^m_+$ and for each $i$, let $W_i(t, z_1, \ldots, z_m)$ be nondecreasing in $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$. Then, it is known [10] that the ordinary differential system

$$r_i' = W_i(t, r_1, \ldots, r_m); \quad r_i(t_0) \leq r_i^0 \geq 0, \quad (5.2)$$

has the maximal solutions $r_i(t)$ existing to the right of $t_0$. Hence, replacing (2.8) by

$$G_i(t, x, z_1, \ldots, z_m, 0, 0) \leq W_i(t, z_1, \ldots, z_m), \quad (5.3)$$

and considering $V_i(t, x, u_i, v_i)$ instead of $V(t, x, u_i, v_i)$ such that

$$V(t, x, u, v) = \sum_{i=1}^m V_i(t, x, u_i, v_i), \quad (5.4)$$

one can prove the following theorem analogous to Theorem 1, using an argument similar to that of Theorem 1 and the notion of maximal solution of the system (5.2).

**Theorem 1*. Let the hypothesis (2.7) hold corresponding to the relations (5.2) and (5.3). Suppose further that

$$V_i^+(t, x, u_i, v_i) = \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial u_i} f_i(t, x, u, u_i', u_{ix}) + \frac{\partial V_i}{\partial v_i} g_i(t, x, v, v_i', v_{ix})$$

$$\leq G_i(t, x, V_1, \ldots, V_m, V_i, V_{ix}), \quad (5.5)$$

where...
Then
\[ V_i(t, x, \phi_i(t, x), \psi_i(t, x)) \leq r_i(t) \quad \text{for} \quad (t, x) \in D \cup I \times \Gamma, \]
implies
\[ V_i(t, x, u_i(t, x), v_i(t, x)) \leq r_i(t) \quad \text{for} \quad (t, x) \in \mathcal{H}. \]

Corresponding to this change, since the conditions (3.1a) to (3.12a) are to be satisfied for \( \sum_{i=1}^{m} r_i(t) \), the proofs of Theorems 3 and 4 are much the same. One can formulate theorems analogous to Theorem 2 and its applications—Theorems 5 and 6. We do not attempt to go into details.

6. We shall give some examples in this section. Let \( L(u) \) denote the following differential form

\[ L(u) = \sum_{\nu, k} a_{\nu, k}(x) u_{\nu x_k} + \sum_{\nu} b_{\nu}(x) u_{x\nu} \]  \hspace{1cm} (6.1)

where the coefficients \( a_{\nu, k}(x) \) and \( b_{\nu}(x) \) are continuous in \( D + \Gamma \) and the quadratic form \( \sum_{\nu, k} a_{\nu, k} \xi_{\nu} \xi_k \geq 0 \) for \( x \in D + \Gamma \) and \( \xi_{\nu}, \xi_k \) real. Let \( F_i(t, x, u_1, \cdots u_m) \) be continuous on \( \mathcal{H} \times \mathbb{R}^m \). Consider the system

\[ \frac{\partial u_i}{\partial t} = L(u_i) + F_i(t, x, u_1, \cdots, u_m). \]  \hspace{1cm} (6.2)

Assume that \( F_i(t, x, 0, \cdots, 0) \equiv 0 \) and

\[ \sum_i u_i F_i(t, x, u_1, \cdots, u_m) \leq \lambda(t) \sum_i u_i^2, \]

where \( \lambda(t) \) is continuous on \( I \). Taking \( V(t, x, u) = \sum_i u_i^2 \) and making use of the fact that \( \sum_{\nu, k} a_{\nu, k} \xi_{\nu} \xi_k \geq 0 \), we obtain the inequality

\[ \frac{\partial V}{\partial t} \leq L(V) + 2\lambda(t) V = G(t, x, V, V_x, V_{xx}). \]

Since

\[ G(t, x, V, 0, 0) \leq 2\lambda(t) V = W(t, V), \]

it follows from Theorem 1 that

\[ \sum_i u_i^2(t, x) \leq r_0 \exp \left[ 2 \int_{t_0}^{t} \lambda(s) \, ds \right] \quad \text{for} \quad (t, x) \in \mathcal{H} \]

whenever

\[ \sum \phi_i^2(t, x) \leq r_0 \exp \left[ 2 \int_{t_0}^{t} \lambda(s) \, ds \right] \quad \text{for} \quad (t, x) \in D \cup I \times \Gamma. \]
If, in addition, \( \int_{t_0}^{\infty} \lambda(s) \, ds < \infty \), the application of Theorem 3 yields the stability of identically zero solution of (6.2). On the other hand, taking

\[ V_i(t, x, u) = \sum_i u_i^2 A(t) \]

where \( A(t) = \exp[-2 \int_{t_0}^{t} \lambda(s) \, ds] \), one can apply Theorem 2 with \( w = 0 \). The assumption \( \int_{t_0}^{\infty} \lambda(s) \, ds = -\infty \), implies the asymptotic stability of identically zero solution of (6.2) because of Theorem 5.

The approach indicated in Section 5 makes it possible to consider more general systems of the type

\[ \frac{\partial u_i}{\partial t} = L_i(u_i) + F_i(t, x, u_1, \ldots, u_m), \quad (i = 1, 2, \ldots, m), \quad \text{(6.3)} \]

where \( L_i(x) = \sum_{r,k} a_{r,k}^i(x) \phi_{\sigma_r, \sigma_k} + \sum_r b^i_r(x) \phi_{\sigma_r} \)

with \( \sum a_{r,k}^i \phi_{\sigma_r, \sigma_k} \geq 0 \) as above. Suppose that \( u_i F_i \leq \sum_{i=1}^{m} c_{i\mu} u_i^2 \) where \( c_{i\mu} \geq 0 \) for \( i \neq \mu \). Taking \( V_i(t, x, u_i) = u_i^2 \) such that \( V = \sum_i u_i^2 \), we get

\[ \frac{\partial V_i}{\partial t} \leq L_i(V_i) + 2 \sum_{\mu} c_{i\mu} V_\mu = G_i(t, x, V_1, \ldots, V_m, V_i^i, V_\mu^i). \]

Since

\[ G_i(t, x, V_1, \ldots, V_m, 0, 0) \leq 2 \sum_{\mu} c_{i\mu} V_\mu = W_i(t, V_1, \ldots, V_m), \]

we obtain from Theorem 1* that

\[ V_i(t, x, u_i(t, x)) \leq r_i(t) \quad \text{for} \quad (t, x) \in \bar{R}, \]

whenever

\[ V_i(t, x, \phi_i(t, x)) \leq r_i(t) \quad \text{for} \quad (t, x) \in D \cup I \times \Gamma, \]

where \( r_i(t) \) is the solution of \( r'_i = \sum_{\mu} c_{i\mu} r_\mu \), \( r_i(t_0) = r_0^i \). Observe that the monotonic assumptions on \( \psi_i \) are satisfied since \( c_{i\mu} \geq 0 \) for \( i \neq \mu \). This is also a necessary and sufficient condition for the nonnegativity of all the elements of \( e^{c(t-t_0)} \) for \( t \geq t_0 \), where \( e^{c(t-t_0)} \) is the solution of the matrix equation

\[ X' = CX, \quad X(t_0) = I. \]

Now it is easy to see that the stability properties of the system (6.3) depend on the stability properties of the linear differential equation.
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