Sign-changing and constant-sign solutions for $p$-Laplacian problems with jumping nonlinearities

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By variational methods, we prove the existence of a sign-changing solution for the $p$-Laplacian equation under Dirichlet boundary condition with jumping nonlinearity having relation to the Fučík spectrum of $p$-Laplacian. We also provide the multiple existence results for the $p$-Laplacian problems.

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1. Introduction and statements of main results

1.1. Introduction

In this paper, we consider the existence of a sign-changing solution for the following $p$-Laplacian equation

$$
\begin{cases}
-\Delta_p u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
(P)
$$

where $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary $\partial \Omega$. Here we say that $u \in W^{1,p}_0(\Omega)$ is a (weak) solution of (P) if

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doi:10.1016/j.jde.2010.08.017
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\Omega} f(x, u) \varphi \, dx
\]

holds for any \( \varphi \in W^{1,p}_0(\Omega) \).

Throughout this paper, we assume that the nonlinear term \( f \) satisfies the following assumption \( (F) \):\( (F) \) \( f \) is a Carathéodory function on \( \Omega \times \mathbb{R} \) and satisfies the following conditions for some constants \( a_0, b_0, a \) and \( b \):

\[
f(x, u) = \begin{cases} 
a_0 u_+^{p-1} - b_0 u_-^{p-1} + g_0(x, u), \\
a u_+^{p-1} - b u_-^{p-1} + g(x, u),
\end{cases}
\]

\[
g_0(x, u) = o(|u|^{p-1}) \quad \text{as } |u| \to 0, \quad \text{uniformly in a.e. } x \in \Omega,
\]

\[
g(x, u) = o(|u|^{p-1}) \quad \text{as } |u| \to \infty, \quad \text{uniformly in a.e. } x \in \Omega
\]

where \( u_\pm = \max\{\pm u, 0\} \) and \( g \) is bounded on any bounded set. Because \( f \) satisfies \( f(x, 0) = 0 \) for a.e. \( x \in \Omega \) under \( (F) \), we consider the case where \( (P) \) has a trivial solution. Note that the nonlinear term \( f \) has the growth condition \( |f(x, t)| \leq C_0 |t|^{p-1} \) for every \( t \in \mathbb{R} \), a.e. \( x \in \Omega \), where \( C_0 \) is a positive constant under the assumption \( (F) \) above.

We easily see that the nonlinear term \( f \) as in \( (F) \) has a relation to the Fučík spectrum of the \( \nabla \)Laplacian which has been considered by Fučík [11] \( (p = 2) \) and by many authors (cf. [8,6,9]). We say that \( (a, b) \in \mathbb{R}^2 \) is in the Fučík spectrum of the \( \nabla \)Laplacian on \( W^{1,p}_0(\Omega) \) \( (1 < p < \infty) \) if the equation

\[
-\Delta_p u = au_+^{p-1} - bu_-^{p-1}, \quad u \in W^{1,p}_0(\Omega),
\]

has a non-trivial weak solution and we denote the Fučík spectrum of the \( \nabla \)Laplacian by \( \Sigma_p \). In the case of \( a = b = \lambda \in \mathbb{R} \), Eq. (1) reads

\[
-\Delta_p u = \lambda |u|^{p-2} u.
\]

Hence \( (\lambda, \lambda) \) belongs to \( \Sigma_p \) if and only if \( \lambda \) is an eigenvalue of \( -\Delta_p \), i.e., there exists a non-zero weak solution \( u \in W^{1,p}_0(\Omega) \) to \( -\Delta_p u = \lambda |u|^{p-2} u \). The set of all eigenvalues of \( -\Delta_p \) is, as usual, denoted by \( \sigma(-\Delta_p) \).

It is well known that the first eigenvalue \( \lambda_1 \) of \( -\Delta_p \) is positive, simple, and has a positive eigenfunction \( \varphi_1 \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \cap C^1(\Omega) \) with \( \int_{\Omega} \varphi_1^p \, dx = 1 \) (see [12, Proposition 1.5.19]). Therefore, \( \Sigma_p \) contains the lines \( \{\lambda_1\} \times \mathbb{R} \) and \( \mathbb{R} \times \{\lambda_1\} \) since \( \varphi_1 \) or \( -\varphi_1 \) becomes a solution to (1) with \( (a, b) = (\lambda_1, 0) \) or \( (a, \lambda_1) \), respectively. Furthermore, [6] showed that there exists a Lipschitz continuous curve contained in \( \Sigma_p \) which is called the first non-trivial curve \( \mathcal{C} \). The construction of the curve \( \mathcal{C} \) is carried out as follows in [6]:

For \( s \geq 0 \), we define

\[
J_s(u) := \int_{\Omega} |\nabla u|^p \, dx - s \int_{\Omega} u_+^p \, dx \quad \text{for } u \in S,
\]

\[
S := \left\{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} |u|^p \, dx = 1 \right\},
\]

\[
\Sigma := \{ \gamma \in C([0, 1], S) : \gamma(0) = \varphi_1, \ \gamma(1) = -\varphi_1 \}.
\]

Here, we consider \( J_s \) as a functional on the manifold \( S \) and the set \( C([0, 1], S) \) denotes the continuous functions from \([0, 1]\) to \( S \) with the topology induced by the \( W^{1,p}_0(\Omega) \) norm. Finally, we set

\[
c(s) := \inf_{\gamma \in \Sigma} \max_{t \in [0, 1]} J_s(\gamma(t)).
\]
Then, it was proved in [6] that \( c(s) \) is a critical value of \( J_s \) with \( c(0) = \lambda_2 \) (where \( \lambda_2 \) is the second eigenvalue of \( p \)-Laplacian), \( c(s) \) is strictly decreasing and \( c(s) > \lambda_1 \) for every \( s \geq 0 \). Similarly, we define for \( s \geq 0 \),

\[
\tilde{c}(s) := \inf_{\gamma \in \Sigma} \max_{t \in [0,1]} \tilde{J}_s(\gamma(t)).
\]  

(3)

Because \(-\varphi_1\) is a global minimum point of \( \tilde{J}_s \) and \( \varphi_1 \) is a strict local minimum point of \( \tilde{J}_s \), \( \tilde{c}(s) \) is also proved to be a critical value of \( \tilde{J}_s \) by a similar argument as in [6]. Moreover, we note that \( \tilde{c}(s) = c(s) \) for every \( s \geq 0 \), and hence \( \tilde{c}(s) > \lambda_1 \) for every \( s \geq 0 \) (see Remark 3 in [16]).

Then, \( \mathcal{C} \) is defined as follows:

\[
\mathcal{C} := \{(c(s) + s, c(s)) : s \geq 0\} \cup \{(\tilde{c}(s), \tilde{c}(s) + s) : s \geq 0\}.
\]

In addition, in [9] and [6], it is shown that

\[
D_i \cap \Sigma_p = \emptyset \quad (i = 1, 2^\pm, 3),
\]  

(4)

where \( D_i \) are subsets defined as follows (see Fig. 1)

\[
D_1 = \{(a, b) \mid a, b < \lambda_1\},
\]

\[
D_2^+ = \{(a, b) \mid b < \lambda_1 < a\},
\]

\[
D_2^- = \{(a, b) \mid a < \lambda_1 < b\},
\]

\[
D_3 = \{(a, b) \mid (a, b) \notin D_1 \cup D_2^+ \cup D_2^- \text{ and } (a, b) \text{ lies below the curve } \mathcal{C}\},
\]

\[
D_4 = \{(a, b) \mid (a, b) \text{ lies above the curve } \mathcal{C}\}.
\]

For the nonlinearity \( f \) as in \((F)\), it is known that we can obtain a non-trivial solution of \((P)\) from the location of \((a_0, b_0)\) and \((a, b)\) in \( \mathbb{R}^2 \) (cf. [9,13,17,21]). Roughly speaking, Dancer and Perera (see [9]) and Jiang (see [13]) proved that if \( (\lambda_1) \times \mathbb{R} \) (resp. \( \mathbb{R} \times (\lambda_1) \)) lies between \((a_0, b_0)\) and \((a, b)\), then \((P)\) has a positive (resp. negative) solution, and if the first non-trivial curve \( \mathcal{C} \) lies between \((a_0, b_0)\)
and $(a, b)$, then (P) has a non-trivial solution. However, they did not clarify whether the non-trivial solution is sign-changing or not. Thus, the main purpose of this paper is to show that (P) has at least one sign-changing solution in the case of $(a_0, b_0) \in D_4$ and $(a, b) \notin D_4$. Note that the results of [9] and [13] cannot treat the resonant case $(a_0, b_0) \in \Sigma_p$ or $(a, b) \in \Sigma_p$. In this paper, we also give existence results generalizing a part of [17] in the resonant case. In addition, we pay attention to the result of Zhang et al. [21]. They considered the case where $f(x, u) = f(u)$ and $f$ has the sign condition $f(u)u \geq 0$, and then gave the multiple existence results in the condition $(a_0, b_0) \in D_4 \cup \mathscr{C}$ and $(a, b) \in D_1$ or the condition $(a_0, b_0) \in D_1$ and $(a, b) \in D_4 \setminus \Sigma_p$. To obtain solutions, we do not need the sign-condition for $f$, but we have to assume the local sign condition in the only case of $(a_0, b_0) \in \mathscr{C}$ (i.e. resonant case).

Finally, we would like to remark that the present paper is a development of [4] and [17] by the present authors.

1.2. Statement of results

Setting $F(x, u) := \int_0^u f(x, s) \, ds$ and $G_0(x, u) := \int_0^u g_0(x, s) \, ds$, we can now state relevant conditions on $f(x, u)$ and $g_0(x, u)$, which are not necessarily simultaneously assumed in our results.

\[(F++) \ pF(x, t) - f(x, t)t \to +\infty \text{ as } t \to +\infty, \text{ uniformly in a.e. } x \in \Omega;\]
\[(F--) \ pF(x, t) - f(x, t)t \to -\infty \text{ as } t \to -\infty, \text{ uniformly in a.e. } x \in \Omega;\]
\[(F++) \ pF(x, t) - f(x, t)t \to -\infty \text{ as } t \to -\infty, \text{ uniformly in a.e. } x \in \Omega;\]
\[(F--) \ pF(x, t) - f(x, t)t \to +\infty \text{ as } t \to +\infty, \text{ uniformly in a.e. } x \in \Omega.\]

\[(G_0) \text{ there exists a } \delta_0 > 0 \text{ such that } G_0(x, u) > 0 \text{ for } 0 < |u| \leq \delta_0, \text{ a.e. } x \in \Omega.\]

Now, we state our existence results.

**Theorem 1.** Assume either $(a_0, b_0) \in D_4$ or $(a_0, b_0) \in \mathscr{C}$ and $(G_0)$. If one of the following conditions holds, then (P) has a positive solution, a negative solution and a sign-changing solution.

(i) $(a, b) \in D_1$;
(ii) $a = \lambda_1$, $b < \lambda_1$ and $(F++)$;
(iii) $b = \lambda_1$, $a < \lambda_1$ and $(F--)$;
(iv) $a = b = \lambda_1$, $(F++)$ and $(F--)$.

**Theorem 2.** Assume either $(a_0, b_0) \in D_4$ or $(a_0, b_0) \in \mathscr{C}$ and $(G_0)$. If one of the following conditions holds, then (P) has a positive solution and at least one sign-changing solution.

(i) $(a, b) \in D_2^-$;
(ii) $a = \lambda_1$, $b > \lambda_1$ and $(F++)$;
(iii) $b = \lambda_1$, $a < \lambda_1$ and $(F++)$;
(iv) $a = b = \lambda_1$, $(F++)$ and $(F--)$.

**Theorem 3.** Assume either $(a_0, b_0) \in D_4$ or $(a_0, b_0) \in \mathscr{C}$ and $(G_0)$. If one of the following conditions holds, then (P) has a negative solution and at least one sign-changing solution.

(i) $(a, b) \in D_2^+$;
(ii) $a = \lambda_1$, $b < \lambda_1$ and $(F++)$;
(iii) $b = \lambda_1$, $a > \lambda_1$ and $(F--)$;
(iv) $a = b = \lambda_1$, $(F++)$ and $(F--)$.

**Theorem 4.** Assume either $(a_0, b_0) \in D_4$ or $(a_0, b_0) \in \mathscr{C}$ and $(G_0)$. If one of the following conditions holds, then (P) has at least one sign-changing solution.

(i) $(a, b) \in D_3$;
belongs to int
Remark 6. It is well known that if
Proposition 9. there exists a greatest negative solution w
Proposition 8. in the special case of
2.1. Extremal constant-sign solutions
2. Preliminaries

In this section, we use the super-subsolution method. So, we recall the definition of super- and sub-solutions.

**Definition 5.** A function \( u \in W_0^{1, p}(\Omega) \) is called a sub-solution (resp. super-solution) of \( (P) \) if
\[
\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx - \int_\Omega f(x, u) \varphi \, dx \leq 0 \quad \text{(resp. } \geq 0)\]
for any \( \varphi \in W_0^{1, p}(\Omega) \) satisfying \( \varphi(x) \geq 0 \) (a.e. \( x \in \Omega \)).

**Remark 6.** It is well known that if \( u \in W_0^{1, p}(\Omega) \) is a positive (resp. negative) solution of \( (P) \), then \( u \) belongs to \( \text{int}(C_0^1(\overline{\Omega})) \) (resp. \( -\text{int}(C_0^1(\overline{\Omega})) \)) from the nonlinear regularity theory (see [1,10]) and the nonlinear strong maximum principle due to [20], where \( \text{int}(C_0^1(\overline{\Omega})) \) denotes the interior of the positive cone
\[
C_0^1(\overline{\Omega}) = \{ u \in C_0^1(\overline{\Omega}) ; \, u(x) \geq 0 \text{ for every } x \in \Omega \}
\]
of the Banach space \( C_0^1(\overline{\Omega}) \), that is,
\[
\text{int}(C_0^1(\overline{\Omega})) = \left\{ u \in C_0^1(\overline{\Omega}) ; \, u(x) > 0 \text{ for } \forall x \in \Omega, \, \frac{\partial u}{\partial \nu}(x) < 0 \text{ for } \forall x \in \partial \Omega \right\}
\]
(where \( \nu \) is an outer normal).

Since the following two results are essentially proved in [4], we omit the proof.

**Proposition 7.** (See [4, Theorem 2.1.]) Let \( a_0 > \lambda_1 \) and \( \overline{v} \in \text{int}(C_0^1(\overline{\Omega})) \) be a super-solution of \( (P) \). Then, there exists a smallest positive solution \( v_0 \in \text{int}(C_0^1(\overline{\Omega})) \) of \( (P) \) within the order interval \([0, \overline{v}]\).

**Proposition 8.** (See [4, Theorem 2.1.]) Let \( b_0 > \lambda_1 \) and \( \overline{w} \in \text{int}(C_0^1(\overline{\Omega})) \) be a sub-solution of \( (P) \). Then, there exists a greatest negative solution \( w_0 \in \text{int}(C_0^1(\overline{\Omega})) \) of \( (P) \) within the order interval \([\overline{w}, 0]\).

Under the local sign condition for \( g_0 \), we can show the existence of an extremal positive solution in the special case of \( a_0 = \lambda_1 \).

**Proposition 9.** Let \( a_0 = \lambda_1 \) and \( \overline{v} \in \text{int}(C_0^1(\overline{\Omega})) \) be a super-solution of \( (P) \). In addition, we assume that \( g_0 \) satisfies the following local sign condition \( (g_0^+) \):
\[
(g_0^+) \text{ there exist a } \delta_0 > 0 \text{ and a measurable subset } \Omega' \text{ of } \Omega \text{ with positive measure } |\Omega'| > 0 \text{ such that}
\]
\[
g_0(x, u) \geq 0 \text{ for every } u \in [0, \delta_0], \, \text{a.e. } x \in \Omega,
\]
\[
g_0(x, u) > 0 \text{ for every } u \in (0, \delta_0], \, \text{a.e. } x \in \Omega'.
\]
Then, every positive solution $v$ of (P) satisfies $\|v\|_{\infty} > \delta_0$. Moreover, there exists a smallest positive solution $v_0 \in \text{int}(C^1_0(\Omega)_+) \text{ of (P) within the order interval } [0, \tilde{v}].$

**Proof.** First, we prove, by contradiction, that if $u$ is a positive solution (note $u \in \text{int}(C^1_0(\Omega)_+)$ by Remark 6), then $\|u\|_{\infty} > \delta_0$ holds. So, we assume that there exists a positive solution $u$ satisfying $\|u\|_{\infty} \leq \delta_0$. Setting $\gamma(x) := g_0(x, u(x))/u(x)^{p-1}$, we obtain $\gamma \in L^\infty(\Omega)$ and $\gamma(x) > 0$ for a.e. $x \in \Omega$ from (F) and $(g_0+)$.

It is known that the first eigenvalue $\lambda_1(\alpha)$ of the following nonlinear eigenvalue problem with weight $\alpha \in L^\infty(\Omega)$ satisfying $\alpha \not\equiv 0$,

$$
\begin{align*}
-\Delta_p u &= \lambda \alpha(x)|u|^{p-2}u \quad \text{in } \Omega, \\
\ u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

admits the variational characterization

$$
\lambda_1(\alpha) = \inf_{\Omega} \left\{ \int |\nabla u|^p \, dx; \; u \in W^{1,p}_0(\Omega) \text{ and } \int_{\Omega} \alpha(x)|u|^p \, dx = 1 \right\}
$$

and also that any eigenfunction associated to a positive eigenvalue different from $\lambda_1(\alpha)$ changes sign (see [5]).

Using this fact, because $u$ is a positive solution for $-\Delta_p u = a_0u^{p-1} + \gamma(x)u^{b-1}$, the first eigenvalue $\lambda_1(a_0 + \gamma(\cdot))$ with weight $a_0 + \gamma(\cdot) \in L^\infty(\Omega)$ is equal to 1.

On the other hand, the following inequality

$$
1 = \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} (a_0 + \gamma(x))u^p \, dx} \leq \frac{\int_{\Omega} |\nabla \varphi_1|^p \, dx}{\int_{\Omega} (a_0 + \gamma(x))\varphi_1^p \, dx} = \frac{\lambda_1}{a_0 + \int_{\Omega} \gamma(x)\varphi_1^p \, dx} < \frac{\lambda_1}{a_0} = 1
$$

implies a contradiction.

Next, we prove the existence of a smallest positive solution in $[0, \tilde{v}]$. For $\varepsilon > 0$ with $\|\varepsilon \varphi_1\|_{\infty} \leq \delta_0$, $\varepsilon \varphi_1$ is a sub-solution for (P) because $-\Delta_p (\varepsilon \varphi_1) = a_0(\varepsilon \varphi_1)^{p-1} < a_0(\varepsilon \varphi_1)^{p-1} + g_0(x, \varepsilon \varphi_1)$. Since for sufficiently large $n$, we have $\varphi_1/n(x) \leq \varphi(x)$ for every $x \in \Omega$, by the standard method of supersolution (cf. [3]), there exist a smallest solution and a greatest solution of (P) within the order interval $[\frac{1}{\varepsilon} \varphi_1, \tilde{v}]$. Let $u_n$ denote the smallest positive solution in $[\frac{1}{\varepsilon} \varphi_1, \tilde{v}]$ for sufficiently large $n$. Then, since $u_n \in \text{int}(C^1(\overline{\Omega}))$ is the smallest solution in $[\frac{1}{\varepsilon} \varphi_1, \tilde{v}]$, we may assume that $u_n(x) \downarrow u(x)$ at each point $x \in \Omega$ (monotone decreasing) for some function $u$ with $0 \leq u \leq \tilde{v}$. To prove $u \not\equiv 0$ by contradiction, we assume $u = 0$. Choose $\varepsilon > 0$ such that $2\varepsilon < \delta_0$. Then, for every $x \in \overline{\Omega}$, there exists $n_x \in \mathbb{N}$ such that $u_n(x) \leq \varepsilon$ if $n \geq n_x$ because $u_n(x) \downarrow 0$. And there exists a neighborhood $U_x \subset \mathbb{R}^N$ of $x \in \overline{\Omega}$ such that $u_{n_x}(y) \leq 2\varepsilon$ for every $y \in U_x$ (where we may consider $u_{n_x}(y) = 0$ if $y \not\in \Omega$). We may assume $\overline{\Omega} \subset \bigcup U_{x_1} \cup \cdots \cup U_{x_l}$ from the compactness of $\overline{\Omega}$ by choosing finitely many points $x_j$ ($1 \leq j \leq l$) and we set $N := \max_{1 \leq j \leq l} n_{x_j}$. For every $x \in \overline{\Omega}$, we have $x \in U_{x_j}$ for some $j$, and hence $u_N(x) \leq u_{n_{x_j}}(x) \leq 2\varepsilon$. Hence, $\max_{x \in \overline{\Omega}} u_N(x) \leq 2\varepsilon < \delta_0$. This contradicts to the fact $\|u_N\|_{\infty} > \delta_0$ because $u_N$ is a positive solution of (P) in $[\varphi_1/N, \tilde{v}]$. \qed

Because the following result can be shown by a similar argument as above, we omit the proof here.

**Proposition 10.** Let $b_0 = \lambda_1$ and $\tilde{v} \in -\text{int}(C^1(\overline{\Omega}))$ be a sub-solution of (P). In addition, we assume that $g_0$ satisfies the following local sign condition $(g_0-)$.\hspace{1cm}
(g₀−) there exist a δ₀ > 0 and a measurable subset Ω’ of Ω with |Ω’| > 0 such that

\[ g₀(x, u) \leq 0 \quad \text{for every } u \in [−δ₀, 0], \text{ a.e. } x \in Ω, \]
\[ g₀(x, u) < 0 \quad \text{for every } u \in [−δ₀, 0), \text{ a.e. } x \in Ω’. \]

Then, every negative solution w of (P) satisfies \( \|w\|_∞ > δ₀ \). Moreover, there exists a greatest negative solution \( w₀ \in − \text{int}(C₀(Ω)_+) \) of (P) within the order interval \([\hat{w} , 0] \).

Because we can obtain a constant-sign solution of (P) under suitable conditions on \( a, b \) and \( f \) from [9] and [18], we can easily show the following result by using the propositions above.

**Corollary 11.** If one of the following cases holds, then (P) has an extremal positive (resp. negative) solution.

(i) \( a < λ₁ < a₀ \) (resp. \( b < λ₁ < b₀ \));
(ii) \( a = λ₁ < a₀ \) and \( (F+−) \) (resp. \( b = λ₁ < b₀ \) and \( (F−−) \));
(iii) \( a < λ₁ = a₀ \) and \( (g₀+) \) (resp. \( b < λ₁ = b₀ \) and \( (g₀−) \));
(iv) \( a = a₀ = λ₁, \) \( (F+−) \) and \( (g₀+) \) (resp. \( b = b₀ = λ₁, \) \( (F−−) \) and \( (g₀−) \)).

**Proof.** We only show the existence of an extremal positive solution since an extremal negative solution can be considered by the same argument.

In the case (i) and other cases, we can get at least one positive solution \( \hat{v} \) of (P) from [9] or [18], respectively. Noting \( \hat{v} \in \text{int}(C₀(Ω)_+) \) by Remark 6, we can apply Proposition 7 or Proposition 9 as \( \hat{v} \) is a super-solution in each case. Thus, we obtain a smallest positive solution \( v₀ \in \text{int}(C₀(Ω)_+) \) within the order interval \([0, \hat{v}] \). □

**2.2. Variational characterization of extremal solutions**

By using \( C₀(Ω) \) functions \( v \) and \( w \) with \( w \leq v \), we introduce several functions instead of \( f \) as follows:

\[
\begin{align*}
f_{[w,v]}(x, t) := & \begin{cases} f(x, v(x)) & \text{if } t \geq v(x), \\ f(x, t) & \text{if } w(x) < t < v(x), \\ f(x, w(x)) & \text{if } t \leq w(x), \end{cases} \\
F_{[w,v]}(x, u) := & \int_0^u f_{[w,v]}(x, t) \, dt,
\end{align*}
\]

\[
\begin{align*}
f_{[−∞,v]}(x, t) := & \begin{cases} f(x, v(x)) & \text{if } t \geq v(x), \\ f(x, t) & \text{if } t < v(x), \end{cases} \\
F_{[−∞,v]}(x, u) := & \int_0^u f_{[−∞,v]}(x, t) \, dt,
\end{align*}
\]

\[
\begin{align*}
f_{[w,∞]}(x, t) := & \begin{cases} f(x, t) & \text{if } w(x) < t, \\ f(x, w(x)) & \text{if } t \leq w(x), \end{cases} \\
F_{[w,∞]}(x, u) := & \int_0^u f_{[w,∞]}(x, t) \, dt.
\end{align*}
\]

For the above nonlinearity \( f_{[y,z]} \) (where \( [y, z] = [w, v] \) or \( [−∞, v] \) or \( [w, ∞] \)), we define the \( C^1 \) functional on \( W₀^{1,p}(Ω) \) by

\[
E_{[y,z]}(u) := \int_Ω |\nabla u|^p \, dx − p \int_Ω f_{[y,z]}(x, u) \, dx. \tag{5}
\]

It is easily seen that \( f_{[y,z]}(x, t) = f(x, t) \) for every \( y(x) \leq t \leq z(x) \).
Remark 12. Let \( v \) and \( w \) be \( C^1_0(\overline{\Omega}) \) functions satisfying \( w \leq v \). It follows from \( f_{[w,v]} \in L^\infty(\Omega) \) that \( E_{[w,v]} \) is coercive and bounded from below on \( W^{1,p}_0(\Omega) \). Moreover, it is easy to see that \( E_{[w,v]} \) is weakly lower semi-continuous. Hence, by the standard argument (cf. [14, Theorem 1.1]), \( E_{[w,v]} \) has a global minimum point.

In the sequel, \( K(E) \) denotes the set of all critical points of a \( C^1 \) functional \( E \) on \( W^{1,p}_0(\Omega) \). Now, we shall characterize extremal solutions by using the above \( C^1 \) functionals on \( W^{1,p}_0(\Omega) \).

Proposition 13. Let \( v_0 \in \text{int}(C^1_0(\overline{\Omega})_+) \) and \( w_0 \in -\text{int}(C^1_0(\overline{\Omega})_+) \) be extremal constant-sign solutions of (P). Then the following assertions hold:

- (i) \( K(E_{[0,v_0]}) = \{0, v_0\} \).
- (ii) \( K(E_{[w_0,0]}) = \{0, w_0\} \).
- (iii) If \( u \in K(E_{[w_0,v_0]}) - \{0, v_0, w_0\} \), then \( u \) is a sign-changing solution of (P).
- (iv) If \( u \in E_{[0,\infty]} \) \( \setminus \{0\} \), then \( u \) is a positive solution of (P).
- (v) If \( u \in E_{[-\infty,0]} \) \( \setminus \{0\} \), then \( u \) is a negative solution of (P).
- (vi) If \( u \in K(E_{[v_0,\infty]}) \setminus \{0, v_0\} \), then \( u \) is a negative solution or a sign-changing solution of (P).
- (vii) If \( u \in K(E_{[w_0,\infty]}) \setminus \{0, w_0\} \), then \( u \) is a positive solution or a sign-changing solution of (P).

Proof. (i) It is easy to see that \( \{0, v_0\} \subset K(E_{[0,v_0]}) \) because \( v_0 \) is a positive solution of (P).

Let \( v \) be a non-trivial critical point of \( E_{[0,v_0]} \). At first, we shall prove \( 0 \leq v \leq v_0 \). The equation

\[
0 = [E'_{[0,v_0]}(v), v] = -p \int_\Omega |\nabla v_+|^p \, dx
\]

shows \( v_+ = 0 \), which yields \( v \geq 0 \). On the other hand, noting the following equation

\[
0 = \frac{1}{p} [E'_{[v_0,0]}(v) - E'_{[0,v_0]}(v_0), (v - v_0)_+]
\]

\[
= \int_\Omega (|\nabla v|^p - |\nabla v_0|^p - 2|\nabla v_0|^p) \nabla (v - v_0)_+ \, dx
\]

\[
- \int_\Omega (f_{[0,v_0]}(x,v_0) - f_{[0,v_0]}(x,v))(v - v_0)_+ \, dx
\]

\[
= \int_\Omega (|\nabla v|^p - |\nabla v_0|^p - 2|\nabla v_0|^p) \nabla (v - v_0)_+ \, dx,
\]

we have \( (v - v_0)_+ = 0 \), and hence \( v \leq v_0 \) holds. Consequently, these imply that \( v \) is a non-negative solution of (P). Then, because of \( v \neq 0 \), we have \( v > 0 \) in \( \Omega \) by Harnack inequality (cf. [19]), whence \( v \in \text{int}(C^1_0(\overline{\Omega})_+) \) holds (see Remark 6). Since \( v_0 \) is a smallest positive solution within some order interval \( \{0, v_0\} \subset \{0, \tilde{v}\} \) (where \( \tilde{v} \in \text{int}(C^1_0(\overline{\Omega})_+) \) is a super-solution), \( v = v_0 \) holds.

(ii) It is easily seen that \( \{0, w_0\} \subset K(E_{[w_0,0]}) \). By a similar argument as in (i), if \( 0 \neq w \in K(E_{[w_0,0]}) \) holds, then \( w \in -\text{int}(C^1_0(\overline{\Omega})_+) \) is a negative solution of (P) within the order interval \([w_0,0] \). Thus, we have \( w = w_0 \) provided \( 0 \neq w \in K(E_{[w_0,0]}) \) because \( w_0 \) is a largest negative solution of (P) within the order interval \([w_0,0] \).

(iii) Let \( u \) be a non-trivial critical point of \( E_{[w_0,v_0]} \) different from \( v_0 \) and \( w_0 \). By taking \( \psi = (u - v_0)_+ \) as a test function in the equation \( 0 = [E_{[w_0,v_0]}(u) - E'_{[w_0,v_0]}(v_0), \varphi] \), we have \( u \leq v_0 \). Similarly, we obtain \( w_0 \leq u \) by considering \( 0 = [E_{[w_0,v_0]}(u) - E'_{[w_0,v_0]}(w_0), -(u - w_0)_-] \). Thus, \( u \) is a solution of (P) satisfying \( w_0 \leq u \leq v_0 \).
If \( u \) is a non-negative solution, then \( u > 0 \) in \( \Omega \) by Harnack inequality (note \( u \neq 0 \)), and hence \( u \in \text{int}(C^1_0(\overline{\Omega}^+)) \). This contradicts to the minimality of \( v_0 \) in the order interval \([0, v_0]\) (note \( u \neq v_0 \)). Similarly, if we assume that \( u \) is a non-positive solution, then we can get a contradiction. Therefore, if \( u \in K(E_{[w_0, v_0]} \setminus [0, v_0, w_0]) \), then \( u \) is a sign-changing solution.

(iv) Let \( u \in K(E_{[0, \infty]} \setminus [0]) \). Then, we have \( 0 = \langle E'_{[0, \infty]}(u), u \rangle = -p \int_\Omega |\nabla u|_p^p \, dx \), and hence \( u \geq 0 \). It follows from Harnack inequality that \( u > 0 \) in \( \Omega \) by \( u \neq 0 \). So, \( u \) is a positive solution of (P).

(v) By a similar argument as in (iv), we obtain \( u < 0 \) in \( \Omega \) provided \( u \in K(E_{[-\infty, 0]} \setminus [0]) \). Therefore, if \( u \in K(E_{[-\infty, 0]} \setminus [0]) \), then \( u \) is a negative solution of (P).

(vi) Let \( u \in K(E_{[-\infty, v_0]} \setminus [0, v_0]) \). By considering \( 0 = \langle E'_{[-\infty, v_0]}(u) - E'_{[-\infty, v_0]}(v_0), (u - v_0)_+ \rangle \), we have \( u \leq v_0 \). Thus, \( u \) is a non-trivial solution of (P). If we assume that \( u \geq 0 \), then \( u \) is a positive solution of (P) within the order interval \([0, v_0]\) by the same argument as in (iii). This yields a contradiction because \( u \neq v_0 \) and \( v_0 \) is the smallest positive solution in \([0, v_0]\) also.

(vii) Let \( u \in K(E_{[w_0, \infty]} \setminus [0, w_0]) \). By the same argument as in (iii), it can be seen that \( u \geq w_0 \) and \( u \) is a non-trivial solution of (P). If we assume \( u \leq 0 \), then \( u \) is a negative solution of (P) within the order interval \([w_0, 0]\). Thus, we can get a contradiction, similarly. \( \Box \)

To close this subsection, we state the second variational characterization of extremal solutions, which plays an important role in our proofs.

**Lemma 14.** Let \( v_0 \in \text{int}(C^1_0(\overline{\Omega}^+)) \) and \( w_0 \in -\text{int}(C^1_0(\overline{\Omega}^+)) \) be extremal constant-sign solutions for (P). If the infimum of \( E_{[0, v_0]} \) (resp. \( E_{[w_0, 0]} \)) is negative, then \( v_0 \) (resp. \( w_0 \)) is a local minimum point of both \( E_{[w_0, v_0]} \) and \( E_{[-\infty, v_0]} \) (resp. \( E_{[w_0, v_0]} \) and \( E_{[w_0, \infty]} \)).

**Proof.** Let \( \inf_{W^1_p(\Omega)} E_{[0, v_0]} < 0 \). Then, \( v_0 \) is a global minimum point of \( E_{[0, v_0]} \) because of \( K(E_{[0, v_0]}) = [0, v_0] \) from Proposition 13 and the existence of a global minimizer (see Remark 12). By the definitions of \( E_{[0, v_0]} \) and \( E_{[w_0, v_0]} \), we have \( E_{[0, v_0]}(u) = E_{[w_0, v_0]}(u) = E_{[-\infty, v_0]}(u) \) for every \( u \in W^1_{1,p}(\Omega) \) with \( u \geq 0 \). Since \( v_0 \) is a global minimizer of \( E_{[0, v_0]} \) and \( v_0 \in \text{int}(C^1_0(\overline{\Omega}^+)) \), \( v_0 \) is a local minimum point of \( E_{[w_0, v_0]} \) and \( E_{[-\infty, v_0]} \) in \( C^1_0(\overline{\Omega}) \). It follows from [2, Theorem 1.1] that \( v_0 \) is a local minimum point of \( E_{[w_0, v_0]} \) and \( E_{[-\infty, v_0]} \) in \( W^1_{1,p}(\Omega) \). We omit the proof concerning \( w_0 \) since it can be done similarly. \( \Box \)

### 2.3. The Cerami condition

It is well known that the Palais–Smale condition and the Cerami condition imply the compactness of the critical set at any level \( c \in \mathbb{R} \), and they play an important role in minimax argument. Here, we recall the definition of the Cerami condition.

**Definition 15.** A \( C^1 \) functional \( I \) on a Banach space \( X \) is said to satisfy the Cerami condition at \( c \in \mathbb{R} \) if any sequence \( \{u_n\} \subset X \) satisfying

\[
I(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{X^*} \to 0 \quad \text{as} \ n \to \infty
\]

has a convergent subsequence. We say that \( I \) satisfies the Cerami condition if \( I \) satisfies the Cerami condition at any \( c \in \mathbb{R} \).

**Remark 16.** We note that the Cerami condition is weaker than the usual Palais–Smale condition. Moreover, we also note that if \( I \) satisfies the Palais–Smale or the Cerami condition, then \( I \) has the deformation property (see the following lemma and [15]).

The result of the following Lemma 17 is contained in [15, Lemma 3.4] if \( X \) is a Hilbert space. We omit the proof here because we can prove it by the same argument as in [15, Lemma 3.4] using the
fact that there exists a locally Lipschitz continuous pseudo-gradient vector field for a $C^1$ functional on a Banach space (see [14, Lemma 6.1]).

**Lemma 17.** Let $I$ be a $C^1$ functional on a Banach space $X$ and suppose that $I$ satisfies the Cerami condition at any level $c \in [a, b]$ and $I$ has no critical value in $(a, b)$. Assume that $K(I) \cap I^{-1}(c)$ consists only of isolated points (or $K(I) \cap I^{-1}(c) = \emptyset$). Denote the set $\{u \in X; I(u) \leq c\}$ by $I^c$ for every $c \in \mathbb{R}$. Then, there exists an $\eta \in C([0, 1] \times X, X)$ satisfying the following:

(i) $\eta(t, u)$ is nonincreasing in $t$ for every $u \in X$;
(ii) $\eta(t, u) = u$ for any $u \in I^a$, $t \in [0, 1]$;
(iii) $\eta(0, u) = u$ and $\eta(1, u) \in I^a$ for any $I^b \setminus (K(I) \cap I^{-1}(b))$.

that is, $I^a$ is a strong deformation retract of $I^b \setminus (K(I) \cap I^{-1}(b))$.

Now we introduce the assumption $(g0)$ for the nonlinear term $g$ to prepare the results concerning the Cerami condition.

$(g0)$ $g$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying $g(x, t) = o(|t|^{p-1})$ as $|t| \to \infty$, uniformly in a.e. $x \in \Omega$ and $g$ is bounded on any bounded set.

For $a, b \in \mathbb{R}$ and the nonlinear term $g$ satisfying $(g0)$, we introduce a $C^1$ functional on $W_0^{1, p}(\Omega)$ as follows:

$$I_{(a, b)}(u) = \int_\Omega |\nabla u|^p \, dx - a \int_\Omega u_+^p \, dx - b \int_\Omega u_-^p \, dx - p \int_\Omega G(x, u) \, dx,$$

where $G(x, u) = \int_0^u g(x, t) \, dt$. Then, we have the following result concerning the Cerami condition or the Palais–Smale condition for the functional above.

**Proposition 18.** (See [17, Proposition 14.]) Let $g$ satisfy $(g0)$. Then the following assertions hold:

(i) If $(a, b) \notin \Sigma_p$, then $I_{(a, b)}$ satisfies the Palais–Smale condition;
(ii) if either $(F++)$ and $(F--)$ or $(F+-)$ and $(F-)$ hold, then $I_{(a, b)}$ satisfies the Cerami condition.

Because the Cerami condition is important to the mountain pass argument, we shall give some results concerning it for functionals defined in the previous subsection.

**Lemma 19.** Let $v_0$ and $w_0$ be $C^1_0(\Omega)$ functions with $w_0 \leq 0 \leq v_0$. Then the following assertions hold:

(i) $E_{[w_0, v_0]}$ satisfies the Palais–Smale condition.
(ii) If $a \neq \lambda_1$, then $E_{[-\infty, v_0]}$ satisfies the Palais–Smale condition.
(iii) If $a \neq \lambda_1$, then $E_{[w_0, \infty]}$ satisfies the Palais–Smale condition.
(iv) If $(F-+)$ or $(F-)$ holds, then $E_{[-\infty, v_0]}$ satisfies the Cerami condition.
(v) If $(F++)$ or $(F-)$ holds, then $E_{[w_0, \infty]}$ satisfies the Cerami condition.

where $E_{[w_0, v_0]}, E_{[-\infty, v_0]}$ and $E_{[w_0, \infty]}$ are the functionals defined in Section 2.2.

**Proof.** Case (i): We note $I_{[w_0, v_0]} \in L^\infty(\Omega)$ and so $I_{[w_0, v_0]}(x, u) = o(|u|^{p-1})$ as $|u| \to \infty$. Therefore, it follows from Proposition 18 that $E_{[w_0, v_0]}$ satisfies the Palais–Smale condition since $(0, 0) \notin \Sigma_p$ (see (4)).

Case (ii): $I_{[-\infty, v_0]}(x, u) = 0 \cdot u_+^{p-1} - bu_-^{p-1} + o(|u|^{p-1})$ as $|u| \to \infty$ and hence $E_{[-\infty, v_0]}$ satisfies the Palais–Smale condition from Proposition 18 since $(0, b) \notin \Sigma_p$ (see (4)) provided $b \neq \lambda_1$. 

Case (iii): $f_{[w_0,\infty)}(x, u) = au^{p - 1} - 0 \cdot u^{p - 1} + o(|u|^{p - 1})$ as $|u| \to \infty$ and hence $E_{[w_0,\infty)}$ satisfies the Palais–Smale condition from Proposition 18 since $(a, 0) \notin \Sigma_p$ (see (4)) provided $a \neq \lambda_1$.

Case (iv): Noting that $f_{[-\infty, v_0]}(x, u) = 0 \cdot u^{p - 1} - bu^{p - 1} + o(|u|^{p - 1})$ as $|u| \to \infty$, we may assume $b = \lambda_1$ because if $b \neq \lambda_1$, then $E_{[-\infty, v_0]}$ satisfies the Palais–Smale condition by case (ii).

Let $\{u_n\}$ be a Cerami sequence for $E_{[-\infty, v_0]}$ at level $c$, namely

$$E_{[-\infty, v_0]}(u_n) \to c$$ and $$\left(1 + \|u_n\|\right)\left\|E'_{[-\infty, v_0]}(u_n)\right\|_{W^{-1, p'}} \to 0 \quad \text{as} \quad n \to \infty,$$

where $\| \cdot \|$ denotes the norm defined by $\|u\| := (\int_\Omega |\nabla u|^p \, dx)^{1/p}$ for $u \in W^{1,p}_0(\Omega)$. It is sufficient to show the boundedness of $\{u_n\}$ since in that case the convergence follows from a standard argument. We prove the boundedness of $\{u_n\}$ by contradiction. Thus, we may assume $\|u_n\| \to \infty (n \to \infty)$ by choosing a subsequence. Set $w_n := u_n/\|u_n\|$. Then, taking a subsequence, we may suppose that there exists a $w_0 \in W^{1,p}_0(\Omega)$ such that

$$w_n \rightharpoonup w_0 \quad \text{in} \quad W^{1,p}_0(\Omega), \quad w_n \to w_0 \quad \text{in} \quad L^p$$

and $w_n(x) \to w_0(x)$ for a.e. $x \in \Omega$ as $n \to \infty$. Then, because we have the following condition

$$\left\|E'_{[-\infty, v_0]}(u_n)\right\|_{W^{-1, p'}}/\|u_n\|^{p - 1} \to 0,$$

it follows from [17, Lemma 13] that $w_n$ strongly converges to $w_0$ being a non-trivial solution to $-\Delta_p u = -\lambda_1 u^{p - 1}$. Hence $w_0 = -\varphi_1/|\varphi_1|$ since $\lambda_1$ is a simple eigenvalue and hence $u_n(x) \to -\infty$ for a.e. $x \in \Omega$ (recall $\varphi_1 > 0$ in $\Omega$). Furthermore, the following inequality as $n \to \infty$ implies the boundedness of $\|u_n^+\|:

$$o(1) = \frac{1}{p} \left\{E'_{[-\infty, v_0]}(u_n), u_n^+\right\} = \|u_n^+\|^p - \int_\Omega f_{[-\infty, v_0]}(u_n)u_n^+ \, dx$$

$$\geqslant \|u_n^+\|^p - \|f_{[0,v_0]}\|_{L^\infty(\Omega)} \|u_n^+\|_{L^1(\Omega)}.$$

Thus, because $u_n^+(x) \to 0$ for a.e. $x \in \Omega$ and $|pF_{[0,v_0]}(x, t) - f_{[0,v_0]}(x,t)t| \leqslant Ct_+$ for a.e. $x \in \Omega$, $t \in \mathbb{R}$, we obtain

$$\lim_{n \to \infty} \int_\Omega pF_{[0,v_0]}(x, u_n) - f_{[0,v_0]}(x, u_n)u_n \, dx$$

$$= \lim_{n \to \infty} \int_\Omega pF_{[0,v_0]}(x, u_n^+) - f_{[0,v_0]}(x, u_n^+)u_n^+ \, dx = 0$$

by Lebesgue’s theorem. Now, we note

$$pF_{[-\infty, v_0]}(x, t) - f_{[-\infty, v_0]}(x, t)t$$

$$= \left(pF_{[-\infty, 0]}(x, t) - f_{[-\infty, 0]}(x, t)t\right) + \left(pF_{[0,v_0]}(x, t) - f_{[0,v_0]}(x, t)t\right)$$

holds for every $t \in \mathbb{R}$, a.e. $x \in \Omega$. Moreover, we have

$$\text{ess inf}\left\{ pF_{[-\infty, 0]}(x, t) - f_{[-\infty, 0]}(x, t)t; \, x \in \Omega, \, t \in \mathbb{R} \right\} > -\infty$$

(resp. $\text{ess inf}\left\{ f_{[-\infty, 0]}(x, t)t - pF_{[-\infty, 0]}(x, t); \, x \in \Omega, \, t \in \mathbb{R} \right\} > -\infty$),
from \((F++)(\text{resp.} F--)\), and hence

\[
\liminf_{n \to \infty} \int_{\Omega} p F_{[-\infty,0]}(x, u_n) - f_{[-\infty,0]}(x, u_n) u_n \, dx = +\infty
\]

\[
\text{(resp. } \liminf_{n \to \infty} \int_{\Omega} f_{[-\infty,0]}(x, u_n) u_n - p F_{[-\infty,0]}(x, u_n) \, dx = +\infty \).
\]

by Fatou’s lemma. Therefore, we obtain a contradiction by taking the limit superior (resp. inferior) with respect to \(n\) in the following relation

\[
c + o(1) = E_{[-\infty,v_0]}(u_n) - \frac{1}{p} (E'_{[-\infty,v_0]}(u_n), u_n)
\]

\[
= - \int_{\Omega} p F_{[-\infty,v_0]}(x, u_n) - f_{[-\infty,v_0]}(x, u_n) u_n \, dx
\]

\[
\text{(resp. } \int_{\Omega} f_{[-\infty,v_0]}(x, u_n) u_n - p F_{[-\infty,v_0]}(x, u_n) \, dx \).
\]

Case (v): Noting that \(f_{[w_0,\infty]}(x, u) = a u^{p-1}_+ - 0 \cdot u^{p-1}_- + o(|u|^{p-1})\) as \(|u| \to \infty\), we may assume \(a = \lambda_1\). In fact, if \(a \neq \lambda_1\), then \(E_{[w_0,\infty]}\) satisfies the Palais–Smale condition by case (iii). Let \((u_n)\) be a Cerami sequence for \(E_{[w_0,\infty]}\) at level \(c\). Then, by a similar argument to case (iv), we can prove the boundedness of \([u_n]\). Hence, the convergence of a Cerami subsequence follows from a standard argument. \(\square\)

3. Proofs

Throughout this section, we make use of \(W_0^{1,p}(\Omega)\) endowed with norm \(\|u\| := (\int_\Omega |\nabla u|^p \, dx)^{1/p}\) for \(u \in W_0^{1,p}(\Omega)\) and let \(\|\cdot\|_q\) denote the usual \(L^q(\Omega)\) norm for \(1 \leq q \leq \infty\).

We state the following result concerning the construction of a suitable path related to the application of the mountain pass theorem.

**Lemma 20.** Assume that either \((a_0, b_0) \in D_4\) or \((a_0, b_0) \in \mathcal{C}\) and \((G_0)\) holds. Then, for every \(v_0 \in \text{int}(C_1^{1}(\Omega)_+)\) and \(w_0 \in - \text{int}(C_1^{1}(\Omega)_+)\), there exist \(u_0 = u_0(a_0, b_0) \in C_0^{1}(\Omega)\) and \(\tau_0 = \tau_0(u_0, v_0, w_0) > 0\) such that

\[
w_0(x) \leq \tau_0 tu_0+ - \tau_0 (1-t) u_0-(x) \leq v_0(x) \text{ for every } t \in [0, 1], x \in \Omega
\]

and

\[
\max_{t \in [0,1]} E_{[w_0,v_0]}(\tau_0 tu_0+ - \tau_0 (1-t) u_0-) < 0.
\]

**Proof.** Firstly, we deal with the case of \((a_0, b_0) \in D_4\). From the characterization of the first non-trivial curve \(\mathcal{C}\) (i.e. \(\mathcal{C}\) is a strictly decreasing Lipschitz continuous curve due to [6]), there exists an \((a_0, \beta_0) \in \mathcal{C}\) such that \(a_0 < a_0\) and \(\beta_0 > b_0\). Because of \((a_0, \beta_0) \in \mathcal{C} \subset \Sigma_p\), we obtain some \(0 \neq u_0 \in W_0^{1,p}(\Omega)\) that is a solution for \(-\Delta_p u = a_0 u^{p-1}_+ - \beta_0 u^{p-1}_-\) in \(\Omega\) (note that \(u_0\) has just two nodal domains (cf. [7]) and so \(u_0\) changes sign). Then \(u_0 \in C_0^{1}(\Omega)\) holds by the nonlinear regularity theory.
(see \cite{1,10}). Therefore, it follows from \( v_0 \in \text{int}(C^1_0(\Omega)_+) \) and \( w_0 \in -\text{int}(C^1_0(\Omega)_+) \) that there exists an \( r_0 > 0 \) satisfying

\[
v_0(x) - ru_0(x) > 0 \quad \text{for every } |r| \leq r_0, \ x \in \Omega,
\]

\[
w_0(x) - ru_0(x) < 0 \quad \text{for every } |r| \leq r_0, \ x \in \Omega.
\]

Thus, for every \( r \in (0, r_0] \), we have

\[
w_0(x) \leq rtu_{0+}(x) - r(1-t)u_{0-}(x) \leq v_0(x) \quad \text{for every } t \in [0, 1), \ x \in \Omega,
\]

and hence

\[
E_{[w_0, v_0]}(rtu_{0+} - r(1-t)u_{0-}) = (rt)^p \int_{\Omega} |\nabla u_{0+}|^p \, dx + r^p (1-t)^p \int_{\Omega} |\nabla u_{0-}|^p \, dx \nonumber
\]

\[ - p \int_{\Omega} F(x, rtu_{0+} - r(1-t)u_{0-}) \, dx.
\]

We note that

\[
\int_{\Omega} F(x, rtu_{0+} - r(1-t)u_{0-}) \, dx = (rt)^p a_0 \|u_{0+}\|^p_p + r^p (1-t)^p b_0 \|u_{0-}\|^p_p
\]

\[
+ o(r^p) t^p \|u_{0+}\|^p_p + o(r^p) (1-t)^p \|u_{0-}\|^p_p
\]

as \( r \to +0 \) holds from (F). Therefore, we obtain for every \( t \in [0, 1) \),

\[
E_{[w_0, v_0]}(rtu_{0+} - r(1-t)u_{0-}) \nonumber
\]

\[
= (rt)^p (a_0 - a_0) \|u_{0+}\|^p_p + r^p (1-t)^p (\beta_0 - b_0) \|u_{0-}\|^p_p
\]

\[
+ o(r^p) t^p \|u_{0+}\|^p_p + o(r^p) (1-t)^p \|u_{0-}\|^p_p
\]

\[
\leq (rt)^p (a_0 - a_0 + o(1)) \|u_{0+}\|^p_p + r^p (1-t)^p (\beta_0 - b_0 + o(1)) \|u_{0-}\|^p_p
\]

as \( r \to +0 \) by using the fact that \( \|u_{0+}\|^p = a_0 \|u_{0+}\|^p_p \) and \( \|u_{0-}\|^p = \beta_0 \|u_{0-}\|^p_p \), which shows that

\[
\max_{t \in [0, 1]} E_{[w_0, v_0]}(rtu_{0+} - r(1-t)u_{0-}) < 0
\]

for sufficiently small \( r > 0 \) because of \( a_0 - a_0 < 0 \) and \( \beta_0 - b_0 < 0 \).

Next, we treat the case of \( (a_0, b_0) \in \mathcal{C} \) and \( (G_0) \). From \( (a_0, b_0) \in \mathcal{C} \subset \Sigma_p \) and the nonlinear regularity theorem (cf. \cite{1,10}), we obtain a solution \( 0 \neq u_0 \in C^1_0(\Omega) \) for \( -\Delta_p u = a_0 u^{p-1} - b_0 u^{p-1} \) in \( \Omega \).

By the same argument as above, there exists an \( r_0 > 0 \) satisfying

\[
w_0(x) \leq rtu_{0+}(x) - r(1-t)u_{0-}(x) \leq v_0(x) \quad \text{for every } r \in (0, r_0), \ t \in [0, 1], \ x \in \Omega
\]

and
for every \( r \in (0, r_0] \) and \( t \in [0, 1] \). Put \( \tau_0 := \min[\delta_0/\|u_0\|_{\infty}, r_0] > 0 \) (where \( \delta_0 \) is a positive constant described in (G0)), then

\[
E_{[w_0, v_0]}(\tau_0 t u_0 - \tau_0 (1 - t) u_0) = -p \int_{\Omega} G_0(x, \tau_0 t u_0) \, dx - p \int_{\Omega} G_0(x, -\tau_0 (1 - t) u_0) \, dx < 0
\]

holds for every \( t \in [0, 1] \) by (G0). \( \square \)

3.1. Proof of Theorem 1

Now, we start to prove Theorem 1.

Proof of Theorem 1. At first, we note that there exist extremal solutions \( v_0 \in \text{int}(C^1_0(\overline{\Omega})) \) and \( w_0 \in -\text{int}(C^1_0(\overline{\Omega})) \) of (P) from Corollary 11.

Let us start to show the existence of a sign-changing solution of (P). It is sufficient to prove the existence of a non-trivial critical point of \( E_{[w_0, v_0]} \) different from \( v_0 \) and \( w_0 \) by Proposition 13, where \( E_{[w_0, v_0]} \) is the functional defined by (5) for \( v_0 \) and \( w_0 \). Moreover, we may assume that a global minimum point \( u \) of \( E_{[w_0, v_0]} \) is equal to either \( v_0 \) or \( w_0 \) (see Remark 12 for the existence of a global minimizer). Indeed, from Lemma 20, the infimum of \( E_{[w_0, v_0]} \) is negative, whence \( u \) is a non-trivial critical point of \( E_{[w_0, v_0]} \). In addition, if \( u \) is different from both \( v_0 \) and \( w_0 \), then \( u \) is a sign-changing solution of (P) by Proposition 13. Here, we treat only the case where \( v_0 \) is the global minimum point of \( E_{[w_0, v_0]} \) since the case \( u = w_0 \) can be treated by a similar argument. Then, we obtain the following inequality:

\[
E_{[w_0, v_0]}(v_0) = \min_{W^{1,p}_{0}(\Omega)} E_{[w_0, v_0]}(w_0) = E_{[w_0, 0]}(w_0) = \min_{W^{1,p}_{0}(\Omega)} E_{[w_0, 0]} < 0.
\]

Indeed, Lemma 20 shows \( \min_{W^{1,p}_{0}(\Omega)} E_{[w_0, 0]} < 0 \) (see Remark 12 for the existence of a global minimum point) since \( E_{[w_0, 0]}(\tau_0 u_0 - \tau_0 (1 - t) u_0) < 0 \), for some \( u_0 = u_0(a_0, b_0) \in C^1_0(\overline{\Omega}) \) and \( \tau_0 = \tau_0(u_0, v_0, w_0) > 0 \). Furthermore, by using the fact that \( K(E_{[w_0, 0]}(0, w_0)) \) and \( w_0 \) is the global minimum point of \( E_{[w_0, v_0]} \).

Now, recall that \( E_{[w_0, v_0]} \) satisfies the Palais–Smale condition by Lemma 19. Because \( w_0 \) is a local minimum point of \( E_{[w_0, v_0]} \) from Lemma 14, there exists an \( r > 0 \) such that \( 2r < \min(\|w_0 - v_0\|, \|w_0\|) \) and

\[
E_{[w_0, v_0]}(w_0) = \inf_{B_{2r}(w_0)} E_{[w_0, v_0]} < 0.
\]
where $B_r(w_0) = \{ z \in W^{1,p}_0(\Omega) : \| z - w_0 \| < r \}$. Moreover, we may suppose that
\[
\inf_{\partial B_r(w_0)} E_{[w_0,v_0]} > E_{[w_0,v_0]}(w_0) \tag{6}
\]
holds. Indeed, if $\inf_{\partial B_r(w_0)} E_{[w_0,v_0]} = E_{[w_0,v_0]}(w_0)$ holds, by using the quantitative deformation theorem, we can show that $E_{[w_0,v_0]}$ has another non-trivial critical point $z_0 \in \partial B_r(w_0)$ (see Appendix in [16] for details). So, assuming (6), we define
\[
\Gamma := \{ \gamma \in C([0,1], W^{1,p}_0(\Omega)) : \gamma(0) = w_0, \gamma(1) = v_0 \}
\]
and
\[
c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_{[w_0,v_0]}(\gamma(t)).
\]
Then
\[
c \geq \inf_{\partial B_r(w_0)} E_{[w_0,v_0]} > E_{[w_0,v_0]}(w_0) \geq E_{[w_0,v_0]}(v_0) \tag{7}
\]
holds. Thus, the mountain pass theorem guarantees that $c$ is a critical value of $E_{[w_0,v_0]}$.

Let us show that $c < 0$ holds to prove the existence of a non-trivial critical point of $E_{[w_0,v_0]}$ other than $w_0$ and $v_0$ (note (7)), which implies the existence of a sign-changing solution for (P) by Proposition 13. For our purpose, it suffices to produce a $\gamma_0 \in \Gamma$ such that $\max_{t \in [0,1]} E_{[w_0,v_0]}(\gamma_0(t)) < 0$. Note that $E_{[0,v_0]}$ and $E_{[w_0,0]}$ satisfy the Palais–Smale condition from Lemma 19.

Recall that from Lemma 20, there exist $0 \neq u_0 = u_0(0, b_0) \in C^1_0(\Omega)$ and $\tau_0 = \tau_0(u_0, v_0, w_0) > 0$ such that
\[
\max_{t \in [0,1]} E_{[w_0,v_0]}(\tau_0 t u_0 + (-\tau_0 (1 - t)) u_0) < 0. \tag{8}
\]
Since $E_{[0,v_0]}$ has no critical value in $(E_{[0,v_0]}(v_0), 0)$ and $v_0$ is the unique global minimum point of $E_{[0,v_0]}$ by Proposition 13, Lemma 17 yields the existence of a $\xi \in C([0,1], W^{1,p}_0(\Omega))$ satisfying the following:
\[
\begin{cases}
\xi(0) = \tau_0 u_{0+} \text{ and } \xi(1) = v_0, \\
E_{[0,v_0]}(\xi(t)) \leq E_{[0,v_0]}(\xi(0)) = E_{[w_0,v_0]}(\tau_0 u_{0+}) < 0 \quad \text{for every } t \in [0,1].
\end{cases} \tag{9}
\]
Similarly, applying Lemma 17 to $E_{[w_0,0]}$, we obtain an $\eta \in C([0,1], W^{1,p}_0(\Omega))$ satisfying
\[
\begin{cases}
\eta(0) = -\tau_0 u_{0-} \text{ and } \eta(1) = w_0, \\
E_{[w_0,0]}(\eta(t)) \leq E_{[w_0,0]}(\eta(0)) = E_{[w_0,v_0]}(-\tau_0 u_{0-}) < 0 \quad \text{for every } t \in [0,1].
\end{cases} \tag{10}
\]
Then we note that
\[
\begin{cases}
\xi(0)_+ = \tau_0 u_{0+}, \quad \xi(1)_+ = v_0, \quad -\eta(0)_- = -\tau_0 u_{0-}, \quad -\eta(1)_- = w_0, \\
E_{[w_0,v_0]}(\xi(t)_+) \leq E_{[0,v_0]}(\xi(t)_+) \leq E_{[0,v_0]}(\xi(t)) < 0, \\
E_{[w_0,v_0]}(-\eta(t)_-) \leq E_{[w_0,v_0]}(-\eta(t)_-) \leq E_{[w_0,v_0]}(\eta(t)) < 0
\end{cases} \tag{11}
\]
hold for every $t \in [0,1]$. Therefore, by setting
solution. By the same argument as in Theorem 1, we can obtain a sign-changing solution of
we have an extremal negative solution
To get a sign-changing solution of
We note that
Therefore, assuming (12), we define
(2-ii)
Case (2-ii): In this case, we note that
We note that
v
other than
Case (2-i): Take a negative solution
As we will see later,
In all cases of this theorem, there exists an extremal positive solution
3.2. Proof of Theorem 2
In this subsection, we provide the proof of Theorem 2.
Proof of Theorem 2. In all cases of this theorem, there exists an extremal positive solution \( v_0 \in \text{int}(C_0^1(\Omega)_+) \) by Corollary 11. Here, we prove the existence of a sign-changing solution by dividing into two cases as follows:

(2-i) \( (P) \) has at least one negative solution;
(2-ii) \( (P) \) has no negative solutions.

Case (2-i): Take a negative solution \( \tilde{w} \) of \( (P) \), and then \( \tilde{w} \in \text{int}(C_0^1(\Omega)_+) \) (see Remark 6). Thus, we have an extremal negative solution \( w_0 \in \text{int}(C_0^1(\Omega)_+) \) by applying Proposition 8 to \( \tilde{w} \) as a sub-solution. By the same argument as in Theorem 1, we can obtain a sign-changing solution of \( (P) \).

Case (2-ii): In this case, we note that \( E_{[-\infty,0]} \) has no non-trivial critical points by Proposition 13. To get a sign-changing solution of \( (P) \), it suffices to show the existence of a non-trivial critical point of \( E_{[-\infty,0]} \) other than \( v_0 \).

We note that \( v_0 \) is the unique global minimum point of \( E_{[0,v_0]} \) (see Remark 12 for the existence of a global minimum point of \( E_{[0,v_0]} \) because of \( \text{K}(E_{[0,v_0]}) = \{0,v_0\} \) and \( \min_{W_0^{1,p}(\Omega)} E_{[0,v_0]} \leq E_{[0,v_0]}(\tau_0 u_{0+}) = E_{[-\varphi_1,v_0]}(\tau_0 u_{0+}) < 0 \), where \( u_0 = u_0(a_0,b_0) \) is a \( C_0^1(\Omega) \) function and \( \tau_0 = \tau_0(u_0,v_0,-\varphi_1) \) is a positive constant obtained by Lemma 19. Then, \( v_0 \) is a local minimum point of \( E_{[-\infty,0]} \) by Lemma 14, whence there exists an \( r > 0 \) such that \( 2r < \|v_0\| \) and

\[
E_{[0,v_0]}(v_0) = E_{[-\infty,v_0]}(v_0) = \inf_{B_{2r}(v_0)} E_{[-\infty,v_0]} < 0.
\]

Moreover, we may assume that

\[
\inf_{\partial B_{r}(v_0)} E_{[-\infty,v_0]} > E_{[-\infty,v_0]}(v_0)
\] (12)

because if \( \inf_{\partial B_{r}(v_0)} E_{[-\infty,v_0]} = E_{[-\infty,v_0]}(v_0) \), then we can show that \( E_{[-\infty,v_0]} \) has another critical point \( z_0 \in \partial B_{r}(v_0) \) by using the quantitative deformation theorem (see Appendix in [16] for details). Therefore, assuming (12), we define

\[
\Gamma := \{ \gamma \in C([0,1], W_0^{1,p}(\Omega))\mid \gamma(0) = v_0, E_{[-\infty,v_0]}(\gamma(1)) < E_{[-\infty,v_0]}(v_0) \}.
\]

As we will see later, \( \Gamma \neq \emptyset \) and hence the following value \( c \) is well defined:

\[
c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_{[-\infty,v_0]}(\gamma(t)).
\]

We note that \( \gamma(1) \notin B_{2r}(v_0) \), and hence \( \gamma([0,1]) \cap \partial B_{r}(v_0) \neq \emptyset \) for every \( \gamma \in \Gamma \). Therefore, \( c \) satisfies

\[
c \geq \inf_{\partial B_{r}(v_0)} E_{[-\infty,v_0]} > E_{[-\infty,v_0]}(v_0).
\] (13)
Recall that by Lemma 19, $E_{[-\infty ,v_0]}$ satisfies the Palais–Smale condition or the Cerami condition in the cases of (i)–(ii) or the cases of (iii)–(iv), respectively. Here, we shall prove that $c$ is a critical value of $E_{[-\infty ,v_0]}$ by contradiction. So, we assume that $c$ is a regular value of $E_{[-\infty ,v_0]}$. Then, there exists $0 < \varepsilon_0 < \inf_{\partial B_r(v_0)} E_{[-\infty ,v_0]} - E_{[-\infty ,v_0]}(v_0)$ such that $[c - \varepsilon_0 , c + \varepsilon_0]$ contains no critical values of $E_{[-\infty ,v_0]}$ since $E_{[-\infty ,v_0]}$ satisfies the Cerami condition. Hence, Lemma 17 yields the existence of a $\theta \in C(W_0^{1,p}(\Omega), W_0^{1,p}(\Omega))$ satisfying

\[
\begin{cases}
E_{[-\infty ,v_0]}(\theta(u)) \leq c - \varepsilon_0 & \text{if } u \in E_{[-\infty ,v_0]}^{-1}((c - \varepsilon_0 , c + \varepsilon_0)), \\
\theta(u) = u & \text{if } u \in E_{[-\infty ,v_0]}^{-1}((\infty , c - \varepsilon_0)).
\end{cases}
\]

Moreover, from the definition of $c$, there exists a $\gamma \in \Gamma$ such that $\max_{t \in [0,1]} E_{[-\infty ,v_0]}(\theta(\gamma(t))) < c - \varepsilon_0$. Then, $\theta \circ \gamma \in \Gamma$ holds (since $\theta(\gamma(0)) = v_0$ and $\theta(\gamma(1)) = \gamma(1)$ from (13)) and

\[
\max_{t \in [0,1]} E_{[-\infty ,v_0]}(\theta(\gamma(t))) \leq c - \varepsilon_0,
\]

which contradicts the definition of $c$.

We claim that $\Gamma \neq \emptyset$ and $(E_{[-\infty ,v_0]}(v_0) < c) < 0$ ensuring the existence of a critical point of $E_{[\infty ,v_0]}$ other than $v_0$ or 0. To prove this claim, it suffices to produce a $v_0 \in \Gamma$ such that $\max_{t \in [0,1]} E_{[-\infty ,v_0]}(\theta(t)) < 0$. Recall that $E_{[-\infty ,v_0]}$ satisfies the Palais–Smale condition or the Cerami condition from Lemma 19 in the cases of (i)–(ii) or cases of (iii)–(iv), respectively. And also note that $E_{[\infty ,v_0]}$ satisfies the Palais–Smale condition by Lemma 19.

Now, applying Lemma 20 to $-\varphi_1 \in \text{int}(C^1_0(\overline{\Omega}))$ and $v_0 \in \text{int}(C^1_0(\overline{\Omega}))$, we obtain $0 \neq u_0 = u_0(a_0,b_0) \in C^1_0(\overline{\Omega})$ and $t_0 = t_0(u_0, v_0, -\varphi_1) > 0$ such that

\[
-\varphi_1(x) \leq t_0 u_{0+}(x) - t_0 (1-t) u_{0-}(x) \leq v_0(x) \quad \text{for every } t \in [0,1], x \in \Omega
\]

and

\[
\max_{t \in [0,1]} E_{[-\varphi_1,v_0]}(t_0 u_{0+} - t_0 (1-t) u_{0-}) < 0.
\]

These imply

\[
\max_{t \in [0,1]} E_{[-\varphi_1,v_0]}(t_0 u_{0+} - t_0 (1-t) u_{0-}) < 0. \tag{14}
\]

Applying Lemma 17 to $E_{[-\infty ,0]}$, we obtain an $\eta \in C([0,1], W_0^{1,p}(\Omega))$ satisfying

\[
\begin{cases}
\eta(0) = -t_0 u_{0-} \quad \text{and} \quad E_{[-\infty ,0]}(\eta(1)) < E_{[-\infty ,v_0]}(v_0), \\
E_{[-\infty ,0]}(\eta(t)) \leq E_{[-\infty ,0]}(\eta(0)) = E_{[-\infty ,v_0]}(-t_0 u_{0-}) < 0 \quad \text{for every } t \in [0,1]
\end{cases}
\]

since $\inf_{W_{0}^{1,p}(\Omega)} E_{[-\infty ,0]} = -\infty$ holds (see Appendix A) and $E_{[-\infty ,0]}$ is assumed to have no critical values in $(-\infty , 0)$. Moreover, Lemma 17 yields a $\xi \in C([0,1], W_0^{1,p}(\Omega))$ satisfying the following:

\[
\begin{cases}
\xi(0) = t_0 u_{0+} \quad \text{and} \quad \xi(1) = v_0, \\
E_{[0,v_0]}(\xi(t)) \leq E_{[0,v_0]}(\xi(0)) = E_{[-\infty ,v_0]}(t_0 u_{0+}) < 0 \quad \text{for every } t \in [0,1]
\end{cases}
\]

because $v_0$ is the unique global minimum point $E_{[0,v_0]}$ and $K(E_{[0,v_0]}) = \{0, v_0\}$. Concerning $\xi$ and $\eta$, we note that
an extremal positive solution $v$

Proof of Theorem 1, we can get a sign-changing solution of same argument as in the proof of Theorem 2, we may assume that there exists an $M$

Moreover, we not et that and

Applying Lemma 20 to $\phi$

Assuming (18), we define the set

Proof of Theorem 3.

3.3. Proof of Theorem 3

we have

Case (3-i): Let $\tilde{v}$ be a positive solution and then $\tilde{v} \in \text{int}(C^1_0(\Omega)_+)$ (see Remark 6). Hence, we have an extremal positive solution $v_0 \in \text{int}(C^1_0(\Omega)_+)$ by Proposition 7. By the same argument as in the proof of Theorem 1, we can get a sign-changing solution of (P).

Case (3-ii): In this case, it follows from Proposition 13 that $E_{[0, \infty]}$ has no non-trivial critical points. Moreover, we note that $w_0$ is the unique global minimum point of $E_{[w_0, 0]}$ because $K(E_{[0, w_0]}) = \{0, w_0\}$ and $\inf_{w_0 \in \text{int}(\Omega)} E_{[w_0, 0]} \leq E_{[0, w_0]}(-\tau_0 u_0) = E_{[w_0, \varphi_1]}(-\tau_0 u_0) < 0$, where $u_0 = u_0(a_0, b_0)$ is a $C^1_0(\Omega)$ function and $\tau_0 = \tau_0(u_0, \varphi_1, w_0)$ is a positive constant obtained by Lemma 20. Then, by the same argument as in the proof of Theorem 2, we may assume that there exists an $r$ satisfying $0 < 2r < \|w_0\|$,.

\[
E_{[0, w_0]}(w_0) = E_{[w_0, \infty]}(w_0) = \inf_{B_{2r}(w_0)} E_{[w_0, \infty]} < 0 \quad \text{and} \quad E_{[w_0, \infty]}(w_0) < \inf_{\partial B_r(w_0)} E_{[w_0, \infty]}.
\]

Assuming (18), we define the set $\Gamma$ as follows:

Applying Lemma 20 to $\varphi_1 \in \text{int}(C^1_0(\Omega)_+)$ and $w_0 \in -\text{int}(C^1_0(\Omega)_+)$, we obtain $0 \neq u_0 = u_0(a_0, b_0) \in C^1_0(\Omega)$ and $\tau_0 = \tau_0(u_0, \varphi_1, w_0) > 0$ such that

$w_0(x) \leq \tau_0 t u_0(x) - \tau_0(1 - t) u_0(x) \leq \varphi_1(x)$ for every $t \in [0, 1]$, $x \in \Omega$

and

$$\max_{t \in [0, 1]} E_{[w_0, \varphi_1]}(\tau_0 t u_0(x) - \tau_0(1 - t) u_0(x)) < 0.$$
Then, we have

$$\max_{t \in [0, 1]} E_{[w_0, \infty]}(\tau_0 t u_{0_+} - \tau_0 (1 - t) u_{0_-}) < 0.$$  

Because $K(E_{[w_0, 0]}) = \{0, w_0\}$ and $E_{[0, \infty]}$ has no critical values in $(0, \infty)$, we can produce a $\gamma_0 \in \Gamma$ such that $\max_{t \in [0, 1]} E_{[w_0, \infty]}(\gamma(t)) < 0$ by replacing $E_{[0, w_0]}$ and $E_{[-\infty, 0]}$ with $E_{[w_0, 0]}$ and $E_{[0, \infty]}$ in the argument of the proof of Theorem 2, respectively. Hence, from the discussion above and the Cerami condition for $E_{[w_0, \infty]}$, the minimax value

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} E_{[w_0, \infty]}(\gamma(t))$$

is a negative critical value of $E_{[w_0, \infty]}$ with $E_{[w_0, \infty]}(w_0) < c$ (note (18)). This shows the existence of a sign-changing solution of (P) from Proposition 13 because (P) was supposed to have no positive solutions. \qed

3.4. Proof of Theorem 4

**Proof of Theorem 4.** We prove the existence of a sign-changing solution of (P) by considering the situation of constant sign solutions, which is divided into the following cases:

- **(4-i)** (P) has at least one positive solution and at least one negative solution;
- **(4-ii)** (P) has at least one positive solution but no negative solutions;
- **(4-iii)** (P) has no positive solutions but at least one negative solution;
- **(4-iv)** (P) has no positive solutions and no negative solutions.

**Case (4-i):** Let $\bar{v}$ and $\bar{w}$ be a positive solution and a negative solution of (P), respectively. Then from $\bar{v} \in \text{int}(C_0^1(\Omega)_+)$ and $\bar{w} \in -\text{int}(C_0^1(\Omega)_+)$ (see Remark 6) we obtain extremal solutions $v_0 \in \text{int}(C_0^1(\Omega)_+)$ and $w_0 \in -\text{int}(C_0^1(\Omega)_+)$ by Propositions 7 and 8, respectively. Therefore, by the same argument as in the proof of Theorem 1, we can get a sign-changing solution of (P).

**Case (4-ii):** Choose a positive solution $\bar{v}$ of (P). Then, it follows from Proposition 7 that there exists an extremal positive solution $v_0 \in \text{int}(C_0^1(\Omega)_+)$ of (P). Because (P) is assumed to have no negative solutions, by the same argument as in case (2-ii) of Theorem 2, we can obtain a sign-changing solution of (P).

**Case (4-iii):** We let $\bar{w}$ be a negative solution of (P). Then we have an extremal negative solution $w_0 \in -\text{int}(C_0^1(\Omega)_+)$ by Proposition 8. Since (P) is assumed to have no positive solutions, by the same argument as in case (3-ii) of Theorem 3, we have a sign-changing solution of (P).

**Case (4-iv):** First, we treat the case of $(a_0, b_0) \in D_4$. In this case, we already know that (P) has at least one non-trivial solution in the case (i) and other cases from [9] or [17] of Theorem 4, respectively (see also the next subsection). Therefore, (P) is has at least one sign-changing solution because (P) is assumed to have no positive solutions and no negative solutions.

Next, we consider the case of $(a_0, b_0) \in \mathcal{C}$ and $(G_0)$. For our assertion, it suffices to show the existence of a non-trivial critical point of the $C^1$ functional $E$ defined by

$$E(u) = \int_{\Omega} |\nabla u|^p \, dx - p \int_{\Omega} F(x, u) \, dx \quad \text{for } u \in W^{1,p}_0(\Omega).$$

(19)

Roughly speaking, by using Lemma 20 and the same argument as in [17, Theorem 5] and [17, Theorem 6], we can get at least one non-trivial critical point of $E$, whence (P) has at least one sign-changing solution. The details are presented in the following subsection. \qed
3.5. Existence of the non-trivial critical point of $E$ in (19)

For $(\alpha, \beta) \in \mathbb{R}^2$, we introduce the following two sets:

$$
\Psi(\alpha, \beta) := \{ u \in W_0^{1,p}(\Omega); \|u\|^p \geq \alpha\|u_+\|^p + \beta\|u_-\|^p \} \quad (20)
$$

and

$$
\Gamma(\alpha, \beta) := \left\{ \gamma \in C([0,1], W_0^{1,p}(\Omega)); \gamma(0) = \gamma(0) \notin \Psi(\alpha, \beta) \text{ and } \gamma(1) = -\gamma(1) \notin \Psi(\alpha, \beta) \right\}. \quad (21)
$$

Then, the following linking result is proved in [17].

Lemma 21. (See [17, Lemma 21].) Let $(\alpha, \beta) \in \mathcal{C}$. Then

$$
\gamma([0,1]) \cap \Psi(\alpha, \beta) \neq \emptyset \quad \text{for every } \gamma \in \Gamma(\alpha, \beta).
$$

By using the linking result above, we start to show the existence of a non-trivial critical point of $E$ defined by (19) in the cases (i) and (ii) of Theorem 4 under assumption (4-iv).

Proof of Theorem 4 in the cases of (i) and (ii) under assumption (4-iv). We choose $(\alpha, \beta) \in \mathcal{C}$ as follows:

\[
\begin{align*}
&\begin{cases}
    a < \alpha \text{ and } b < \beta & \text{in the case } (a, b) \in D_3, \\
    \alpha = c(a) + a \text{ and } \beta = c(a) & \text{in the case } a \geq b = \lambda_1,
    \\
    \alpha = \tilde{c}(b) \text{ and } \beta = \tilde{c}(b) + b & \text{in the case } \lambda_1 = a < b,
\end{cases}
\end{align*}
\]

where $c(a)$ and $\tilde{c}(b)$ are the positive constants defined by (2) or (3), respectively. Now, we show that $E$ is bounded from below on $\Psi(\alpha, \beta)$, where $\Psi(\alpha, \beta)$ is the subset of $W_0^{1,p}(\Omega)$ defined by (20). Indeed, for every $u \in \Psi(\alpha, \beta)$,

$$
E(u) \geq (\alpha - a)\|u_+\|^p + (\beta - b)\|u_-\|^p - p \int_\Omega G(x, u) \, dx
$$

holds. Because $\alpha > a$, $\beta > b$ (in all cases) and $\int_\Omega G(x, u) \, dx = o(\|u\|^p_p)$ as $\|u\|_p \to \infty$, we have

$$
M := \inf\{E(u); \ u \in \Psi(\alpha, \beta)\} > -\infty.
$$

Noting $(\alpha, \beta) \in \mathcal{C}$, we set

$$
\Gamma := \left\{ \gamma \in \Gamma(\alpha, \beta); \ E(\gamma(0)), E(\gamma(1)) < M - 1 \right\},
$$

$$
c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)),
$$

where $\Gamma(\alpha, \beta)$ is the set defined by (21). Now we shall show that $\Gamma \neq \emptyset$ and $c$ is negative. For this purpose, it suffices to produce a $\gamma_0 \in \Gamma$ such that $\max_{t \in [0,1]} E(\gamma_0(t)) < 0$.

If there exists such a $\gamma_0 \in \Gamma$, we can show that $c$ is a negative critical value of $E$. Indeed, from Lemma 21, we have
\[
\gamma([0, 1]) \cap \Psi(\alpha, \beta) \neq \emptyset \quad \text{for every } \gamma \in \Gamma
\]

(note \( \Gamma \subset \Gamma(\alpha, \beta) \)), and hence \( c \geq M \) holds. Then, we can prove that \( c \) is a critical value of \( E \) since \( E \) satisfies the Palais–Smale condition or the Cerami condition in the case of \((a, b) \in D_3 \subset \mathbb{R}^2 \setminus \Sigma_p \) and other cases from Proposition 18, respectively.

Now, we start to construct a \( \gamma_0 \in \Gamma \) such that \( \max_{t \in [0, 1]} E(\gamma_0(t)) < 0 \). From Lemma 20, there exist \( u_0 = u_0(a_0, b_0) \in C_p(\Omega) \) and \( \tau_0 = \tau_0(u_0, \varphi_1, -\varphi_1) > 0 \) such that \(-\varphi_1(x) \leq \tau_0 u_0(x) - \tau_0(1-t)u_{0-}(x) \leq \varphi_1(x) \) for every \( t \in [0, 1] \), \( x \in \Omega \) and

\[
\max_{t \in [0, 1]} E([-\varphi_1, \varphi]) (\tau_0 u_0 - \tau_0(1-t)u_{0-}) < 0.
\]

Thus, we have

\[
\max_{t \in [0, 1]} E(\tau_0 u_0 - \tau_0(1-t)u_{0-}) < 0. \tag{22}
\]

Recall that \( E_{[-\infty, 0]} \) and \( E_{[0, \infty]} \) satisfy the Palais–Smale condition or the Cerami condition in each case by Lemma 19. We also recall that \( E_{[-\infty, 0]} \) and \( E_{[0, \infty]} \) have no non-trivial critical points in the case (4-iv) from Proposition 13. Hence, we have

\[
\inf_{W^{1,p}_0(\Omega)} E_{[-\infty, 0]} = \inf_{W^{1,p}_0(\Omega)} E_{[0, \infty]} = -\infty \quad \text{(see Appendix A)}.
\]

Therefore, we can obtain a \( \xi \in C([0, 1], W^{1,p}_0(\Omega)) \) and an \( \eta \in C([0, 1], W^{1,p}_0(\Omega)) \) satisfying

\[
\begin{cases}
\xi(0) = \tau_0 u_{0+}, & \eta(0) = -\tau_0 u_{0-}, \quad E_{[0, \infty]}(\xi(1)) < M - 1, \quad E_{[-\infty, 0]}(\eta(1)) < M - 1, \\
E_{[0, \infty]}(\xi(t)) \leq E_{[0, \infty]}(\xi(0)) = E(\tau_0 u_{0+}) < 0 \quad \text{for every } t \in [0, 1], \\
E_{[-\infty, 0]}(\eta(t)) \leq E_{[-\infty, 0]}(\eta(0)) = E(-\tau_0 u_{0-}) < 0 \quad \text{for every } t \in [0, 1]
\end{cases} \tag{23}
\]

by applying Lemma 17 to \( E_{[0, \infty]} \) and \( E_{[-\infty, 0]} \). Let us note that

\[
\begin{cases}
\xi(0)_+ = \tau_0 u_{0+}, & -(\eta(0))_- = -\tau_0 u_{0-}, \\
E(\xi(t)_+) = E_{[0, \infty]}(\xi(t)) \leq E_{[0, \infty]}(\xi(t)) < 0, \\
E(-(\eta(t))_-) = E_{[-\infty, 0]}(\eta(t)) \leq E_{[-\infty, 0]}(\eta(t)) < 0
\end{cases} \tag{24}
\]

hold for every \( t \in [0, 1] \). This also yields that

\[
\xi(1)_+ \notin \Psi(\alpha, \beta) \quad \text{and} \quad -(\eta(1))_- \notin \Psi(\alpha, \beta) \tag{25}
\]

from the definition of \( M \). Therefore, by setting

\[
\gamma_0(t) = \begin{cases}
-(\eta(1 - 3t))_- & \text{if } 0 \leq t \leq 1/3, \\
\tau_0(3t - 1)u_{0+} - \tau_0(2 - 3t)u_{0-} & \text{if } 1/3 \leq t \leq 2/3, \\
\xi(3t - 2)_+ & \text{if } 2/3 \leq t \leq 1,
\end{cases}
\]

we have \( \gamma_0 \in \Gamma \) such that \( \max_{t \in [0, 1]} E(\gamma_0(t)) < 0 \) by (22)-(25). \( \square \)

Finally, we treat the last case (iii) of Theorem 4.

**Proof of Theorem 4 in the case of (iii) under assumption (4-iv).** We define the following approximate functionals on \( W^{1,p}_0(\Omega) \):

\[
\gamma([0, 1]) \cap \Psi(\alpha, \beta) \neq \emptyset \quad \text{for every } \gamma \in \Gamma
\]
for \(u \in W_{0}^{1,p}(\Omega)\) and \(n \in \mathbb{N}\). Then, for every \(n \in \mathbb{N}\), we have

\[
M_n := \inf_{\varphi(a,b)} E^n > -\infty
\]

because \(\int_{\Omega} G(x,u)\,dx = o(\|u\|^p_p)\) as \(\|u\|^p_p \to \infty\). From Lemma 20, there exist \(u_0 = u_0(a_0, b_0) \in C_{0}^{1}(\overline{\Omega})\) and \(\tau_0 = \tau_0(u_0, \varphi_1, -\varphi_1) > 0\) such that

\[
-\varphi_1(x) \leq \tau_0 tu_{0+}(x) - \tau_0 (1-t) u_{0-}(x) \leq \varphi_1(x) \quad \text{for every} \ t \in [0,1], \ x \in \Omega
\]

and

\[
\max_{t \in [0,1]} E_{[-\varphi_1, \varphi_1]}(\tau_0 tu_{0+} - \tau_0 (1-t) u_{0-}) < 0.
\]

Hence

\[
-d := \max_{t \in [0,1]} E(\tau_0 tu_{0+} - \tau_0 (1-t) u_{0-}) < 0
\]

holds. Choose a natural number \(n_0\) satisfying \(n_0 > \|\tau_0 u_0\|^p_p/d\). Here, we claim that for every \(n \geq \max(n_0, 1/(a - \lambda_1), 1/(b - \lambda_1))\), \(E_n\) has at least one critical value in \((-\infty, -d + \|\tau_0 u_0\|^p_p/n_0]\). In fact, since critical points of \(E_{0,[0,\infty]}^{n}\) or \(E_{[-\infty,0]}^{n}\) are also critical points of \(E^n\), we may assume that both \(E_{0,[0,\infty]}^{n}\) and \(E_{[-\infty,0]}^{n}\) have no critical values in \((-\infty, -d + \|\tau_0 u_0\|^p_p/n_0]\). So, we can prove the claim above by replacing \(E_{0,[0,\infty]}\), \(E_{[-\infty,0]}\) and \(\Gamma\) with \(E^{n}_{0,[0,\infty]}\), \(E^{n}_{[-\infty,0]}\) and \(\Gamma^n\) (defined in the following) in the previous proof concerning the cases of (i) and (ii), respectively (cf. [17, Theorem 6]):

\[
\Gamma^n := \{ \gamma \in \Gamma(a,b); \ E^n(\gamma(0)), E^n(\gamma(1)) < M_n - 1 \}.
\]

Thus, let \(u_n \in W_{0}^{1,p}(\Omega)\) satisfy

\[
\| (E^n)'(u_n) \|_{W^{-1,p'}} = 0 \quad \text{and} \quad E^n(u_n) \leq -d + \|\tau_0 u_0\|^p_p/n_0.
\]

Here, we admit that \(\{u_n\}\) is bounded in \(W_{0}^{1,p}(\Omega)\), leaving the verification at the end of the proof. Thus, because \(E\) is bounded on every bounded sets, we may assume that \(E(u_n)\) converges to some \(c\) by taking a subsequence. Furthermore, we obtain

\[
E(u_n) \leq E^n(u_n) \leq -d + \|\tau_0 u_0\|^p_p/n_0,
\]

\[
\| E'(u_n) \|_{W^{-1,p'}} = \| E'(u_n) - (E^n)'(u_n) \|_{W^{-1,p'}} \leq \frac{\| u_n \|^{p-1}}{n\lambda_1}
\]

(26)
for every \( n \geq \max\{n_0, 1/(a - \lambda_1), 1/(b - \lambda_1)\}. \) Therefore, \{\( u_n \)\} is a Cerami sequence of \( E \) at level \( c \leq -d + \|\tau_0 u_0\|_p^p/n_0 < 0. \) This yields that \( E \) has a non-trivial critical value since \( E \) satisfies the Cerami condition by Proposition 18.

Finally, we prove the boundedness of \{\( u_n \)\} by contradiction. So, we may assume that \( \|u_n\| \to \infty \) as \( n \to \infty \) by choosing a subsequence. Set \( z_n := u_n/\|u_n\| \). Then, taking again a subsequence, we may suppose that there exists a \( z_0 \in W_{0}^{1,p}(\Omega) \) such that

\[
z_n \rightharpoonup z_0 \quad \text{in} \quad W_{0}^{1,p}(\Omega), \quad z_n \to z_0 \quad \text{in} \quad L^p(\Omega)
\]

and \( z_n(x) \to z_0(x) \) for a.e. \( x \in \Omega \) as \( n \to \infty \). From (26) and [17, Lemma 13], \( z_n \) strongly converges to \( z_0 \) being a solution of \( -\Delta_p u = a u^{p-1} - b u^{p-1} \) in \( \Omega \). Thus, \( |u_n(x)| \to \infty \) for a.e. \( x \in \Omega' := \{ y \in \Omega; z_0(y) \neq 0 \} \). Next, we note that from \((F^+\,-)\) and \((F^-\,-)\), we have

\[
\text{ess inf}\left\{ f(x, t) t - p F(x, t); \; x \in \Omega, \; t \in \mathbb{R} \right\} > -\infty,
\]

and hence

\[
\liminf_{n \to \infty} \int_{\Omega'} f(x, u_n) u_n - p F(x, u_n) \, dx = +\infty
\]

by Fatou’s lemma.

On the other hand, we obtain a contradiction from the following inequality for \( n \geq n_0 \):

\[
d - \|\tau_0 u_0\|_p^p/n_0 \leq (E^n)'(u_n)/p - E^n(u_n)
\]

\[
\leq - \int_{\Omega'} f(x, u_n) u_n - p F(x, u_n) \, dx
\]

\[
- \text{ess inf}\left\{ f(x, t) t - p F(x, t); \; x \in \Omega, \; t \in \mathbb{R} \right\} |\Omega \setminus \Omega'|.
\]

Thus, we conclude the proof. \( \square \)

**Acknowledgment**

The second author would like to express her sincere thanks to Professor Shizuo Miyajima for helpful comments and encouragement.

**Appendix A**

For readers’ convenience, we show the following fact which we used in the final section.

**Lemma.** Let \( I \) be a \( C^1 \) functional on a Banach space \( X \) and satisfy the Cerami condition. If \( I \) has no critical values in \((-\infty, c) \) and \( I^{-1}((-\infty, c)) \neq \emptyset \) holds for some \( c \in \mathbb{R} \), then \( \inf_X I = -\infty \).

**Proof.** Assume \( d := \inf_X I = -\infty \), and then \( c > d \) holds by \( I^{-1}((-\infty, c)) \neq \emptyset \). Then, by the Cerami condition, there exists a \( c - d > \delta > 0 \) such that

\[
\frac{(1 + \|u\|^2)\|I'(u)\|^2}{1 + (1 + \|u\|^2)\|I'(u)\|^2} \geq \delta \quad \text{for every} \quad u \in I^{-1}([d, d + \delta]).
\]
Let $V$ be a locally Lipschitz continuous pseudo-gradient vector field for $I$ (see [14, Lemma 6.1] for the existence of such $V$), that is, $V$ is locally Lipschitz continuous on $X \setminus K(I)$ and satisfies $\|V(u)\| \leq 2\|I'(u)\|$ and $\langle I'(u), V(u) \rangle \geq \|I'(u)\|^2$ for any $u \in X \setminus K(I)$. Now, we consider the following Cauchy problem with initial value $u_0$ with $I(u_0) \leq d + \delta$,

$$
\frac{d\sigma(t)}{dt} = -W(\sigma(t)), \quad \sigma(0) = u_0,
$$

$$
W(u) := \frac{(1 + \|u\|^2)V(u)}{1 + (1 + \|u\|^2)\|V(u)\|^2},
$$

which has a unique continuous solution $\sigma(t)$ defined on $[0, \infty)$ (cf. [22, Theorem 1.41]) because $\|W(u)\| \leq (1 + \|u\|)/2$ for every $u \in X \setminus K(I)$ and $K(I) \cap I^{-1}([d, d + \delta]) = \emptyset$. Noting

$$
\frac{d}{dt} I(\sigma(t)) \leq -\frac{(1 + \|\sigma(t)\|^2\|I'(\sigma(t))\|^2)}{1 + 4(1 + \|\sigma(t)\|^2\|I'(\sigma(t))\|^2)},
$$

we obtain

$$
I(\sigma(5)) \leq I(u_0) - \frac{\delta}{4} \int_0^5 dt \leq d + \delta - \frac{5\delta}{4} < d,
$$

which contradicts to the definition of $d$. $\square$

References