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Antipodal Distance Transitive Covers of Complete Graphs

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A distance-transitive antipodal cover of a complete graph K_n possesses an automorphism group that acts 2-transitively on the fibres. The classification of finite simple groups implies a classification of finite 2-transitive permutation groups, and this allows us to determine all possibilities for such a graph. Several new infinite families of distance-transitive graphs are constructed.

1. INTRODUCTION

This paper is a contribution towards the determination of all finite distance-transitive graphs. We obtain a classification of all the antipodal distance-transitive graphs having as antipodal quotient a complete graph K_n. Such a graph necessarily has diameter 2 or 3 (see for example [2, Proposition 4.2.2 (ii)]). Those of diameter 2 are simply the complete multipartite graphs $K_{r,\dots,r}$ with n parts of size r, and the heart of the classification lies in finding all the examples with diameter 3. In the diameter 3 case, the original graph Γ and the antipodal quotient have the same valency, and Γ is said to be a *cover* of its antipodal quotient.

We offer the 3-fold cover of K_5 that appears in Figure 1 as a motivating example. The antipodal quotient K_5 is obtained by identifying vertices falling on lines through the centre of radial symmetry of the figure. More formally, this graph is the line graph of the Petersen graph or equivalently, the graph based on the involutions in the alternating group A_5, two involutions being adjacent if their product has order 3.

A 2-fold cover of K_n that is not bipartite is equivalent to a regular 2-graph, see [21] or [2, Theorem 1.5.3], and a result of Gardiner [8, Proposition 4.5] asserts that an $(n-1)$-fold cover of K_n is equivalent to a Moore graph of valency n. Results of Gardiner [8], Taylor [22] and Aschbacher [1] together imply the classification of distance-transitive r-fold covers of K_n unless $3 \le r \le n-2$, and thus we need only deal with r in this range.

The classification of the finite 2-transitive permutation groups is fundamental to our effort. Indeed our Lemma 2.6 shows that any such graph gives rise to two 2-transitive permutation groups and we play these two permutation groups off against each other to obtain our Main Theorem.

MAIN THEOREM. *Suppose G is a distance-transitive automorphism group of the finite graph X. Suppose further that X is antipodal with fibres of size $r \ge 2$ and antipodal quotient the complete graph K_n. Take x to be a vertex of X. Let H be the stabilizer of x in G and let C be the kernel of the action of H on the fibre containing x. Then either X has diameter 2 and is the complete multipartite graph $K_{r,\dots,r}$ with n parts of size r, or X has diameter 3 and one of the following occurs.*

(1) *X is bipartite and equals $K_2 \otimes K_n = (K_{n,n}$ minus a matching$)$, $r = 2$ and $G \le 2 \times S_n$.*
(2) *X appears in [8], $r = n-1$, $n = 7$, and $G \le S_7$.*
(3) *X appears in [22], $r = 2$ and*
    (a) *$n = 2^{2m-1} \pm 2^{m-1}$, $G \le 2 \times Sp(2m, 2)$, for $m \ge 3$.*
    (b) *$n = 3^{2a+1} + 1$, $G \le 2 \times Aut(R(q))$, for $a \ge 1$.*

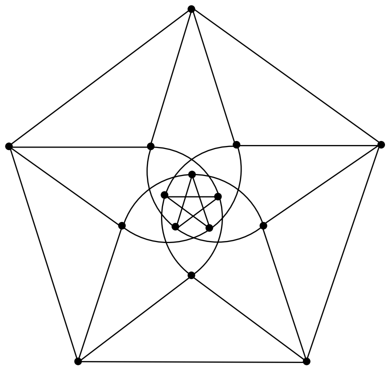

FIGURE 1. The 3-fold cover of K_5 admitting S_5

(c) $n = 176$, $HiS \leq G \leq 2 \times HiS$ *or* $n = 276$, $Co_3 \leq G \leq 2 \times Co_3$.
(d) $n = q^3 + 1$, $G \leq 2 \times P\Gamma U(3, q^2)$, *for* $q > 3$.
(e) $n = q + 1$, $G \leq 2 \times P\Sigma L(2, q)$, *for* $q \equiv 1 (\mathrm{mod}\ 4)$.

(4) X *appears in Example* 3.4*, and one of the following holds, where* $q = p^e$ *for some prime* p*,* r *is an odd prime having* p *as a primitive root, and* G *contains the simple socle of* $Aut X$.

(a) $n = q + 1$, r *divides* $q - 1$, $r - 1$ *divides* e, $Aut X = P\Gamma O(3, q)$*; or*
(b) $n = q^3 + 1$, r *divides* $q^2 - 1$, $r - 1$ *divides* $2e$*, and* $Aut X = P\Gamma U(3, q^2)$.

(5) X *appears in Example* 3.5, $r = 3$, $n = q^3 + 1$ *where* $q = p^e$ *for some prime* p*, and one of the following holds.*

(a) $PSU(3, q^2) \leq G \leq Aut X = P\Gamma U(3, q^2)$*, and* $q - 1 \equiv p + 1 \equiv 0 (\mathrm{mod}\ 3)$*; or*
(b) $PSU(3, q^2) \leq G \leq Aut X = P\Sigma U(3, q^2)$*, and* $q + 1 \equiv 0 (\mathrm{mod}\ 9)$*; or*
(c) $SU(3, q^2) \leq G \leq Aut X = \Sigma U(3, q^2)$*, and* $q + 1 \equiv \pm 3 (\mathrm{mod}\ 9)$.

(6) X *is a graph appearing in Example* 3.6, r *divides* q, $n = q^{2d}$, $C \leq Sp(2d, q)$ *is a transitive linear group and* $A\Gamma L(1, q)$ *involves a* 2*-transitive group of degree* r.

Section 3 consists of a more detailed construction of each instance in this theorem. The graphs in the orthogonal case of Example 3.4 belong to a family of distance regular graphs constructed by Mathon [18] (see also [2, Table 6.10 (A3), and 12.5.3]) and were already known to be distance-transitive (see [2, 12.5.3, Remark (iii)]). To the best of our knowledge, the graphs in the unitary case of Example 3.4 and in Example 3.5 were first constructed in the course of work on this classification. They were first described via a coset graph construction (see Lemma 2.7) and later were given a geometric description by Brouwer *et al.* [3] where a family of distance regular graphs containing this family of distance-transitive graphs was constructed. The graphs in the classical case of Example 3.6 were constructed as distance regular graphs by Thas [23] for q even and by Somma [20] for general q. A generalization of the Thas–Somma construction for distance regular graphs was given by Hensel [11] and Godsil and Hensel [9]. The non-classical case of the construction of Example 3.6 yields graphs which are newly recognized as being distance-transitive.

Although many technical terms are defined in Section 2, our graph theoretic terminology is generally that of [2] and our group theoretic notation follows [7].

Recall that a group G is called *almost simple* if there is a non-Abelian simple subgroup T such that $T \trianglelefteq G \leq Aut T$. In this case T is the unique minimal normal subgroup of G. A permutation group on a set Ω is called *affine* if it has a regular normal subgroup N which is an elementary Abelian p-group, for some prime p. A classical result of Burnside [4, Section 154] asserts that a finite 2-transitive permutation group is either almost simple or affine.

In view of Lemma 2.6, the groups G which act distance-transitively on X must act 2-transitively on the fibres of X. Let K be the kernel of this action. Lemma 4.1 shows that G is almost simple if $K = 1$. This case occupies Section 4 and leads (see Theorem 4.2) to (4) and to (5) (a) and (b) of the Main Theorem. Section 5 treats the case where $K \neq 1$ and G/K is almost simple. It leads to (5) (c) of the Main Theorem, see Theorem 5.1. The remaining possibility, $K \neq 1$, G/K affine, is the subject of Section 6. It leads (see Propositions 6.2 and 6.3) to (6) of the Main Theorem.

2. Preliminaries

The *rth distance graph* X_r of a graph X is the graph with the same vertex set as X, and with two vertices adjacent if and only if they are at distance r in X. If X has diameter d and X_r is connected for all r such that $1 \leq r \leq d$ then X is called *primitive*. Otherwise X is *imprimitive*. If a distance-transitive graph X is imprimitive and X_2 is connected then X_d is not connected [19]. Moreover, if X_i is not connected for some i, then either X_2 is not connected or X_d is not connected. When X_2 is not connected and $d > 2$, X is called *bipartite*, and when X_d is not connected X is said to be *antipodal*.

Suppose now that X is an antipodal distance-regular graph of diameter d. Then we may partition its vertices into sets, called *fibres*, such that any two distinct vertices in the same fibre are at distance d and two vertices in different fibres are at distance less than d. We may therefore define a *quotient* graph which has the fibres of X as its vertices, with two fibres adjacent if and only if there is an edge joining them. This quotient graph is again distance-regular, and is never antipodal. The fibres in an antipodal distance-regular graph all have the same size. The size of a fibre will be referred to as the *index* of the graph.

LEMMA 2.1. *Suppose that X is an antipodal distance-transitive graph of diameter* 2 *with $n \geq 2$ antipodal fibres of size $r \geq 2$. Then $X = K_{r,\ldots,r}$, its antipodal quotient is the complete graph K_n, and its automorphism group is the wreath product $S_r \wr S_n$.*

PROOF. Since X is antipodal of diameter 2, its antipodal quotient is the complete graph K_n, and each vertex in a fibre F is joined to every vertex of $V(X) \setminus F$. Hence $X = K_{r,\ldots,r}$ as claimed. $\square$

Suppose further that X has diameter $d \geq 3$. Then, given two distinct fibres either there are no edges joining them or each vertex in one of the fibres has exactly one neighbour in the other. In this case X is said to be a *cover* of its antipodal quotient, and the natural mapping from X to its antipodal quotient is called a *covering map*. We have the following characterization of antipodal distance-regular graphs of diameter 3 from [9].

LEMMA 2.2 ([10, LEMMA 3.1]). *Suppose that X is a connected antipodal graph of index r, diameter $d \geq 3$, and with antipodal quotient K_n. Then X is distance regular of diameter three if and only if two non-adjacent vertices from different fibres always have the same number of common neighbours.*

We assume from now on that X is a connected antipodal graph of index r, diameter 3, and with antipodal quotient K_n. There are four parameters which are used to describe these graphs. They are the number n of vertices in the complete graph being covered, the index r of the cover, the number a_1 of common neighbours of two adjacent vertices and the number c_2 of common neighbours of two vertices at distance 2. These parameters are related by the identity

$$n - 2 - a_1 = (r - 1)c_2. \tag{1}$$

We will often refer to an antipodal distance-regular graph of diameter 3 as an *antipodal cover of a complete graph with parameters* (n, r, c_2).

NOTATION. The following notation will be used without further reference. X is an antipodal cover of a complete graph with parameters (n, r, c_2) and vertex set $V(X)$, and Σ is the set of fibres of X. If G is a group of automorphisms of X and $F \in \Sigma$, then G_F will be used to denote the subgroup of G fixing F as a set and G_F^F will denote the group of permutations of F induced by G_F. The group of permutations of Σ induced by G will be denoted G^Σ. The kernel of the action of $Aut(X)$ (respectively G) on Σ will be called the *covering group* (respectively *covering group induced by* G). Finally, for a vertex v of X, let $F(v)$ be the fibre of X containing v.

Because X is distance-regular with diameter 3, its adjacency matrix has minimal polynomial of degree 4. Because X is a connected cover of K_n two of its eigenvalues are immediate. The valency $n-1$ occurs with multiplicity 1, and -1 occurs with multiplicity $n-1$. The remaining two eigenvalues θ and τ are the solutions of

$$x^2 - (a_1 - c_2)x - (n - 1) = 0. \tag{2}$$

The multiplicity of θ as an eigenvalue is

$$\frac{n(r-1)\tau}{\tau - \theta}. \tag{3}$$

The fact that this quantity is an integer is a non-trivial constraint which must be satisfied by the parameter set (n, r, c_2). It will be referred to as the *multiplicity condition*. These considerations allow us to identify the bipartite examples.

COROLLARY 2.3. *Let X be an antipodal cover of a complete graph with parameters (n, r, c_2). If X is bipartite, then $r = 2$ and $X = K_2 \otimes K_n$ (the complete bipartite graph $K_{n,n}$ minus the edges of a matching).*

PROOF. A bipartite graph of valency $n - 1$ has $1 - n$ as a simple eigenvalue. Thus $\theta = 1$, $\tau = 1 - n$. Further, for X bipartite the parameter a_1 is zero, and it follows from (2) that $c_2 = n - 2$. It is easy to deduce the remaining parameters of X and to see that $X = K_2 \otimes K_n$ as claimed. □

The importance of being able to characterize the bipartite examples is highlighted by the following result due to Smith [19]. Note that each fibre $F \in \Sigma$ is a block for G in its action on the vertices of X and the set of fibres forms a system of imprimitivity for G.

LEMMA 2.4 ([19]). *Let X be a distance-transitive graph with diameter d. Then X is imprimitive if and only if $Aut\,X$ is imprimitive. More precisely, a subset B of $V(X)$ is a block for $Aut\,X$ if and only if it is the vertex set of a connected component of one the graphs X_t, with $t = 2$ or $t = d$.*

In the case where there is a large Abelian covering group it turns out that each prime divisor of the index r also divides n.

THEOREM 2.5 ([9]). *Suppose that X is an antipodal distance-regular graph of diameter* 3 *which is not bipartite, and that X has covering group K. Assume further that K is Abelian and acts transitively on each fibre. If p is a prime divisor of r then p divides n.*

PROOF. Partition the rows and columns of the adjacency matrix A of X according to the fibres Σ. Since X is a covering graph, each non-diagonal $r \times r$ block is a permutation matrix. Label the vertices within a fibre by elements of K. Since K is an Abelian automorphism group of X, the permutation appearing in an $r \times r$ block must centralize K, and hence be in K, [25, Proposition 4.4]. Thus A may be regarded as an $n \times n$ matrix with entries in $\{0\} \cup K$ and only the diagonal entries are 0.

Let $1 \neq \chi$ be a character of K taking values in $R := \mathbb{Z}[\zeta_p]$ the ring of integers with a primitive pth root of unity adjoined. Let $A(\chi) \in Mat_n(R)$ be the image of A under χ. In [16] the matrix $A(\chi)$ is called the generalized intersection array associated with χ. The eigenvalues τ, θ of A are also eigenvalues of $A(\chi)$, since $\chi \neq 1$. (Both actually occur since $A(\chi)$ is not a scalar matrix.) Let $\pi = (1 - \zeta_p)$ be the prime of R over (p) in $\mathbb{Z}$. Then

$$A(\chi) \equiv J - I \pmod{\pi}.$$

Suppose $\theta \not\equiv \tau$ modulo π. Then just one of these eigenvalues is congruent to $n - 1$ modulo π and its multiplicity must equal $r - 1$, the number of choices for χ. The multiplicity condition (3) now implies that $(n - 1)\tau = -\theta$, which leads to $\theta = 1 - n$, $\tau = -1$. It follows that X is bipartite as in the proof of Corollary 2.3.

Thus $\theta \equiv \tau \equiv n - 1 \equiv -1$ modulo π and the multiplicity condition (3) implies $n(r - 1)\tau \in (\pi)$. Since $\tau \notin (\pi)$ and $r - 1 \notin (p)$ it follows that $n \in (\pi) \cap \mathbb{Z} = (p)$. □

We now make explicit the two 2-transitive actions involved in a distance-transitive subgroup of automorphisms of X.

LEMMA 2.6. *Suppose G is a distance-transitive subgroup of $\mathrm{Aut}\, X$. Then*

(1) G acts 2-transitively on Σ and G_F acts 2-transitively on F for $F \in \Sigma$.
(2) Let x, y be adjacent vertices of X. Then G_{xy} acts transitively on $F(y) \setminus \{y\}$.
(3) The G-character afforded by X has constituents of degrees: $1, n - 1, m_\theta, m_\tau$, where m_θ, m_τ are the non-trivial eigenvalue multiplicities of X given by condition (3).

PROOF. First, G acts transitively on the 1-arcs of X, i.e., on the set of ordered pairs of adjacent vertices. Since the fibres are blocks, and since any two fibres contain a pair of adjacent vertices, we deduce that G is 2-transitive on Σ. Since G also acts transitively on the set of ordered pairs of vertices at distance 3, and since any two distinct vertices in a fibre F are at distance 3, G_F acts 2-transitively on the set of vertices in F. Suppose x, y are adjacent vertices of X. Because y is the only vertex in $F(y)$ that is adjacent to x, $G_{xy} = G_{x,F(y)}$. Moreover, the vertices in $F(y) \setminus \{y\}$ are all at distance 2 from x, so they fall in a single G_x-orbit. The second part follows from $F(y)$ being a set of imprimitivity for the action of G_x.

The assertion in (3) follows from the discussion above. □

Graphs admitting 1-arc transitive groups can be constructed from group coset spaces in the following fundamental way.

LEMMA 2.7. *Suppose a non-normal subgroup H of a group G and an element $g \in G$ are given. Let $\Gamma(G, H, HgH)$ denote the graph with vertex set $[G\colon H] := \{Hx | x \in G\}$ and edges the pairs $\{Hx, Hy\}$ such that $xy^{-1} \in HgH$.*

(1) *Assume that G acts faithfully on $[G\colon H]$ and $g^2 \in H$ and $G = \langle H, g\rangle$. Then $\Gamma(G, H, HgH)$ is a simple, undirected, connected graph which admits the group G acting (by right multiplication) faithfully and transitively both on vertices and on arcs.*

(2) *Suppose G acts arc-transitively on a connected graph X, H is the stabilizer of a vertex v, and g is a 2-element interchanging v with some vertex adjacent to v in X. Then $X \cong \Gamma(G, H, HgH)$, and also $g^2 \in H$ and $G = \langle H, g\rangle$.*

(3) *Let $X_1 := \Gamma(G, H, Hg_1H)$ and $X_2 := \Gamma(G, H, Hg_2H)$ be two such graphs with G acting faithfully on $[G\colon H]$. If there exists an element $\phi \in N_{AutG}(H)$ such that $g_1^{\phi} \in Hg_2H$, then $X_1 \cong X_2$. Moreover, the converse is also true provided that $G = Aut\, X_1$.*

(4) *Let M be a proper subgroup of G that properly contains H such that*

   (a) *$G = M \cup MgH$, and*
   (b) *$H^g \cap M \leq H$, and*
   (c) *$H \cap H^g$ acts transitively on $\Gamma_3 := \{Hm \mid m \in M \setminus H\}$.*

   Then $\Gamma(G, H, HgH)$ is a cover of the complete graph on $[G\colon M]$. Moreover, G is a distance-transitive subgroup of automorphisms of $\Gamma(G, H, HgH)$.

PROOF. Proofs of the assertions in (1) and (2) can be found, for example, in [17, Section 1]. Next we prove (3). It is straightforward to verify that, for $\phi \in N_{AutG}(H)$ such that $g_1^{\phi} \in Hg_2H$, the map $Hx \mapsto Hx^{\phi}$, for $x \in G$, defines a graph isomorphism from X_1 to X_2. For the second assertion, assume that $G = Aut\, X_1$ and that $\tau\colon X_1 \to X_2$ is an isomorphism. Then $\tau \in Sym([G\colon H])$ and τ normalizes the common automorphism group G. Further, since G is transitive on vertices we may assume that τ fixes the vertex H, and hence that τ also normalizes the stabilizer of this vertex in G, that is τ normalizes H. It follows that the image of X_1 under τ is the graph $\Gamma(G, H, Hg_1^{\tau}H)$, and hence $Hg_1^{\tau}H = Hg_2H$. Let ϕ be the automorphism of G induced by the conjugation action of τ on G. Then $\phi \in N_{AutG}(H)$ and $g_1^{\phi} \in Hg_2H$.

Finally we prove (4). Since g and H generate all of G, $g \notin M$, so the assumption that $H \cap H^g$ acts transitively on Γ_3, implies that $[G\colon H]$ is the union of the four H-invariant sets:

$$\Gamma_0 := \{H\}, \quad \Gamma_1 := \{Hgh \mid h \in H\}, \quad \Gamma_2 := \{Hx \mid x \notin gH \cup M\} \quad \text{and} \quad \Gamma_3,$$

and only Γ_2 might possibly not be an H-orbit.

Since $g^2 \in H$, $HH^g = (Hg^2)g^{-1}Hg = HgHg$. Now $H \leq M$, so the inclusion (4)(b) can be multiplied on the left and right by H and rewritten $HgHgH \cap M \leq H$. But the right H-cosets at distance at most 2 from H in $\Gamma(G, H, HgH)$ are precisely those in $HgHgH \cup HgH$. It follows that the vertices of Γ_3 are at distance at least three from H in $\Gamma(G, H, HgH)$.

The inclusion $H \supseteq HgHg \cap M$ can also be multiplied on the right by g to obtain $Hg \supseteq (HgHg \cap M)g = HgH \cap Mg$, since $g^2 \in H$. Therefore, $\Gamma(G, H, HgH)$ has but one edge from H to Mg, namely $\{H, Hg\}$. Thus $\Gamma(G, H, HgH)$ gives a matching between $[M\colon H]$ and $[M\colon H]g$ that is $(M \cap M^g)$-invariant. The assumption that $H \cap H^g$ acts transitively on Γ_3 thus implies that $H \cap H^g$ also acts transitively on $\Gamma_3 g$. From this and (4)(a) it follows that Γ_2 is an H-orbit, and it consists of all vertices of $\Gamma(G, H, HgH)$ at distance 2 from H.

Since H has but four orbits on the vertices of $\Gamma(G, H, HgH)$, and $[G\colon M]$ is an antipodal system of imprimitivity, $\Gamma(G, H, HgH)$ is a distance-transitive antipodal graph of diameter 3. In particular $\Gamma(G, H, HgH)$ is a cover of the complete graph on $[G\colon M]$. □

Suppose the distance-transitive cover X has a non-trivial covering group K. Then the orbits of any subgroup $N \leq K$ form an *equitable partition* π of the vertices and the quotient X/π is a distance-regular cover with index $r/|N|$, see [9]. Our next result gives a sufficient condition under which such a quotient of a distance-transitive graph is again distance-transitive.

LEMMA 2.8. *Take G, M, H, g to satisfy the conditions of Lemma 2.7. (4), so $\Gamma(G, H, HgH)$ is a distance-transitive cover of K_n. Assume that the covering group K is non-trivial, and is properly contained in a group $P \trianglelefteq G$, where P acts regularly on the vertices of $\Gamma(G, H, HgH)$. Let C be the centralizer of K in H and assume that*

$$g \in PC \text{ and } G = M \cup MgC.$$

Suppose that $C \leq H_1 \leq H$, and that KH_1 has a homomorphic image KH_1/N which is a 2-transitive permutation group with $K/(K \cap N)$ as regular normal subgroup. Let $\overline{}$ denote reduction modulo $K \cap N$. Then $\Gamma(\overline{PH_1}, \overline{H_1}, \overline{H_1}\overline{g}\overline{H_1})$ is a distance-transitive cover of K_n of index $|K{:}\,K \cap N|$.

PROOF. Note that $G = PH$ and $M = KH$, and set $G_1 := PH_1$, $M_1 := KH_1$. The assumption $G = M \cup MgC$ implies that C acts transitively on $[G{:}\,M] \setminus \{M\}$ and hence also on $[P{:}\,M] \setminus \{K\}$. This in turn implies that $\overline{G_1} = \overline{M_1} \cup \overline{M_1}\overline{g}\overline{H_1}$.

Also, $H_1^g \cap M_1 \leq H \cap M_1 = H_1$ by Lemma 2.7.4, so $\overline{H_1}^g \cap \overline{M_1} \leq \overline{H_1}$. Since $C \leq N$ acts transitively on the fibres $\neq \{Hm{:}\, m \in M\}$, H_1 induces the same group on Γ_3 as $H_1 \cap H_1^g$ and so the transitivity of H_1 on $K/(K \cap N) \setminus \{K \cap N\}$ follows. This implies the third condition of Lemma 2.7.4. □

The classification of the finite 2-transitive permutation groups is fundamental. Of course, this important consequence of the classification of finite simple groups is the result of the work of many people but we only refer explicitly to the lists given in [5, 15]. Recall that, by a result of Burnside [4, Section 154], a finite 2-transitive permutation group is either almost simple or affine.

THEOREM 2.9. *Suppose G is a finite 2-transitive permutation group.*

(1) *Suppose G is almost simple and $T \trianglelefteq G \leq \mathrm{Aut}T$ with T a non-Abelian simple group. Then T is one of the following:*
   - (a) *alternating: T is A_n, of degree $n \geq 5$.*
   - (b) *linear: $T = PSL(d, q)$, of degree $\frac{q^d-1}{q-1}$, with $d \geq 2$ and $(d, q) \neq (2, 2)$ or $(2, 3)$.*
   - (c) *unitary: T is $PSU(3, q^2)$, of degree $q^3 + 1$, with $q \geq 3$.*
   - (d) *symplectic: $T = Sp(2m, 2)$, of degree $2^{2m-1} \pm 2^{m-1}$, with $m \geq 3$.*
   - (e) *Ree: $T = R(q)'$, of degree $q^3 + 1$, with $q = 3^{2a+1} \geq 3$.*
   - (f) *Suzuki: $T = Sz(q)$ of degree $q^2 + 1$, with $q = 2^{2a+1} \geq 8$.*
   - (g) *sporadic:*
     - (i) *$T = M_n$ of degree n, with $n = 11, 12, 22, 23, 24$;*
     - (ii) *$T = PSL(2, 11)$ of degree 11; $T = M_{11}$ of degree 12; $T = A_7$ of degree 15; $T = HiS$ of degree 176; $T = Co_3$ of degree 276.*

(2) *Suppose $G \cong Z_p^n \cdot H$ is an affine 2-transitive permutation group of degree p^n and let the symbol $\circ$ denote a central product. Then H satisfies one of the following:*
   - (a) *linear: $n = cd$, $d \geq 2$, and $SL(d, p^c) \trianglelefteq H \leq \Gamma L(d, p^c)$.*
   - (b) *symplectic: $n = cd$, d even, $d \geq 4$, and $Sp(d, p^c) \trianglelefteq H \leq Z_{p^c-1} \circ \Gamma Sp(d, p^c)$.*
   - (c) *G_2 type: $n = 6c$, $p = 2$ and $G_2(2^c)' \trianglelefteq H \leq Z_{2^c-1} \circ$ Aut $G_2(2^c)$.*
   - (d) *one-dimensional: $H \leq \Gamma L(1, p^n)$.*
   - (e) *exceptional:*
     - (i) *$p^n = 9^2, 11^2, 19^2, 29^2$ or 59^2, and $SL(2, 5) \trianglelefteq H$, or*
     - (ii) *$p^n = 2^4$, and A_6 or $A_7 \trianglelefteq H$, or $p^n = 3^6$ and $SL(2, 13) \trianglelefteq H$.*

(f) extra-special:

(i) $p^n = 5^2, 7^2, 11^2$ *or* 23^2 *and* $Q_8 \trianglelefteq H$*, or*

(ii) $p^n = 3^4$, $R = D_8 \circ Q_8 \trianglelefteq H$, $H/R \leq S_5$*, and* 5 *divides* $|H|$.

We note some useful consequences of this remarkable theorem.

LEMMA 2.10. *Let G be a* 2*-transitive permutation group. Suppose that G is almost simple, with minimal normal subgroup T. Then either T is* 2*-transitive, or T is* $R(3)' \cong PSL(2, 8)$ *acting primitively and with rank four on* 28 *points. (The non-trivial orbits of a point stabilizer have length nine.)* □

LEMMA 2.11 ([10, CHAPTER 2.6]). *Let G be the socle of an almost simple* 2*-transitive permutation group of degree n, and suppose that the Schur multiplier* $M(G)$ *of G has order at least* 3*. Then* $G, n, M(G)$ *are as in one of the columns of the following table.*

G	A_6	A_7	$Sz(8)$	$PSL(3,4)$	M_{22}	$PSU(3,q^2)$	$PSL(m,q)$ and $(q-1,m) \geq 3$
n	6 or 10	7 or 15	6	21	22	q^3+1	$(q^m-1)/(q-1)$
$M(G)$	6	6	$2 \cdot 2$	$4 \cdot 4 \cdot 3$	12	$(q+1,3)$	$(q-1,m)$

The next somewhat technical consequence of Theorem 2.9 is crucial for our analysis in Section 4.

PROPOSITION 2.12. *Suppose that G is a* 2*-transitive permutation group on a set V of size n, and that G is an almost simple group with socle T. Suppose further that, for* $v \in V$, G_v *has a (not necessarily faithful) permutation representation onto a* 2*-transitive permutation group L of degree r, where* $3 \leq r \leq n-2$ *and* $r(n-1)$ *divides* $|G_v|$*. Then* T, n, r *and L are given in a line of the following table, where p is a prime,* $q = p^e$*, and in lines 2 and 3 of the table r is a prime and* $o_r(p)$ *is the order of p* mod r.

	T	n	r	L	Comments
1	$PSL(d,q)$	$(q^d-1)/(q-1)$	q^{d-1}	$\geq ASL(d-1,q)$	$d \geq 3$
2	$PSL(d,q)$	$(q^d-1)/(q-1)$	r	$Frob(r(r-1))$	$r\|(q-1), o_r(p) = r-1$
3	$PSU(3,q^2)$	q^3+1	r	$Frob(r(r-1))$	$r\|(q^2-1), o_r(p) = r-1$
4	$PSL(3,3)$	13	3 or 4	S_3 or S_4	
5	$PSL(3,5)$	31	5	S_5	
6	$PSL(3,8)$	73	28	$R(3)$	
7	$PSL(5,2)$	31	8	A_8	
8	$PSL(7,2)$	127	63	$PSL(6,2)$	

PROOF. If $T = A_n$ then G_v has no permutation representation of degree r such that $3 \leq r < n$ except in the case where $n = 5$, and the conditions in line 2 hold with $d = 2$, $q = 4$ and $r = 3$.

Suppose next that $T = PSL(d, q)$ with $n = (q^d - 1)/(q - 1)$, $q = p^e$ with p prime, and $d \geq 2$. Then G/T and G_v/T_v are both either cyclic or metacyclic with order dividing $(d, q-1)e$ and T_v is the semi-direct product of an elementary Abelian subgroup M of order q^{d-1} by H, where $SL(d-1, q) \leq H \leq GL(d-1, q)$. Further M is a minimal normal subgroup of G_v. There are three cases to consider.

Suppose first that M acts non-trivially in the permutation representation of G_v of degree r. Then, since it is a minimal normal subgroup, M acts faithfully and regularly, so $r = q^{d-1}$.

Since $r \leq n-2$ we must have $d \geq 3$, and the conditions of line 1 hold. Thus we may assume that M acts trivially in the permutation representation of degree r.

If $d = 2$ then G_v/M is metacyclic of order dividing $(q-1)e$, and is a subgroup of the metacyclic group $\langle\alpha\rangle \cdot \langle\sigma\rangle$, where α has order $q-1$, σ has order e, and $\alpha^{\sigma} = \alpha^{p}$. It follows that the socle of the 2-transitive group L must be cyclic and hence regular, whence r is a prime dividing $q-1$. Further, for G_v/M to have a 2-transitive image L, σ must permute transitively by conjugation the non-trivial cosets of $\langle\alpha^{r}\rangle$ in $\langle\alpha\rangle$. A small calculation shows that this is the case if and only if p is a primitive root modulo r, that is $o_r(p) = r-1$. This is line 2 of the table (with $d = 2$).

Suppose now that $d \geq 3$. If $SL(d-1, q)$ is soluble, then $d = 3$ and $q \leq 3$; since $(n-1)r$ divides $|G_v|$, we must have $q = 3$ and $r = 3$ or 4, as in line 4. Now assume that $SL(d-1, q)$ is insoluble. If $SL(d-1, q)$ acts trivially in the permutation representation of degree r, then L is metacyclic of order dividing $(q-1)e$, and arguing as above we see that the conditions of line 2 hold. Finally suppose that $SL(d-1, q)$ acts non-trivially. Then since $GL(d-1, q)$ is involved in T_v it follows that L has $PGL(d-1, q)$ as a normal subgroup. The natural 2-transitive action of degree $r = (q^{d-1}-1)/(q-1)$ is ruled out by the requirement that $(n-1)r$ divide $|G_v|$ in all cases except the case on line 8 of the table. The only other 2-transitive representations of such a group L of degree $r \leq n-2$ occur for $(d, q) = (3, 5)$, $(3, 8)$, or $(5, 2)$ as in line 5, 6 or 7, respectively.

If T is one of $PSU(3, q^2)$, $R(q)$ or $Sz(q)$, then the socle of T_v has order $n-1$; arguing as in the previous paragraph we see that the socle of T_v must act trivially in the permutation representation of degree $r \leq n-2$. The groups $R(q)$ and $Sz(q)$ are then eliminated since $G_v/soc(T_v)$ has no 2-transitive representations, and in the case of $PSU(3, q^2)$ the only possibility for the group L is a metacyclic group $Frob(r(r-1))$ with r prime as in line 3.

Finally none of the symplectic and sporadic cases give examples because G_v has no 2-transitive representation, or because the degree r of such a permutation representation does not satisfy the necessary conditions. □

In Section 5 we require different information about 2-transitive permutation groups. At one stage we need to know which simple socles of almost simple 2-transitive groups have the property that a point stabilizer has an elementary Abelian quotient of order greater than 2.

PROPOSITION 2.13. *Let T be the socle of an almost simple 2-transitive permutation group of degree n, and suppose that a point stabilizer T_v in this representation has an elementary Abelian quotient of order $r = r_o^a \geq 3$ where r_o is a prime dividing n. Then $r = r_o$ is an odd prime, and either*

(a) $T = PSL(d, q)$, $n = (q^d - 1)/(q-1)$, *and r divides* $(d, q-1)$*; or*
(b) $T = PSU(3, q^2)$, $n = q^3 + 1$, *and r divides* $(q+1)/(3, q+1)$.

PROOF. For T in Theorem 2.9 (1) (a), (d), or (g), the largest Abelian quotient of T_v has order at most 2. For T in Theorem 2.9 (1) (e) or (f), an elementary Abelian quotient of T_v is cyclic of prime order r dividing $q-1$; however such primes do not divide n. Finally for T in Theorem 2.9 (1) (b) or (c), an elementary Abelian quotient of T_v is cyclic of prime order r, where r divides $q-1$ or $(q^2-1)/(3, q-1)$, respectively. Since r also divides n, and $r \geq 3$, it follows that r divides $(d, q-1)$ or $(q+1)/(3, q+1)$, respectively. □

Recall that the groups H in Theorem 2.9 (2) may be regarded as subgroups of $GL(n, p)$ and as such are called *transitive linear groups*. We need the following proposition which asserts that an almost simple transitive linear group cannot also have a faithful 2-transitive representation of large degree. Note that the lower bound p^a is sharp, since the group $H = GL(a, 2)$ has a faithful 2-transitive action of degree $2^a - 1$.

PROPOSITION 2.14. *Suppose that, for a prime p and integer $a \geq 1$, the group $H \leq GL(a, p)$ is an almost simple transitive linear group. Then any faithful 2-transitive representation of H has degree less than or equal to $p^a - 1$.*

PROOF. By assumption (since H is almost simple) H is one of the groups in Theorem 2.9 (2) (a)–(c), or (e) (ii) with $p^a > 5$, and the non-Abelian simple socle S of H is isomorphic to $PSL(d, p^c)$ $(a = cd \geq 2c)$, $PSp(d, p^c)$ $(a = cd \geq 4c$, d even), $G_2(p^c)$ $(a = 6c, p = 2)$, or A_6 or A_7 $(p^a = 16)$, respectively. In the first case faithful 2-transitive representations exist, but for all of them the degree is at most $p^a - 1$. Similarly in the second case 2-transitive representations exist (only) when $p^c = 2$ but their degree is again less than 2^a. There is a 2-transitive representation in the third case only for $G_2(2)' \cong PSU(3, 3^2)$ but its degree is 28 (less than 2^6), and in the final case the 2-transitive representations of the groups A_6 and A_7 are all of degree less than 16. □

The following result plays an important role in Section 6. In the statement and proof we use the notation from Theorem 2.9.2.

LEMMA 2.15. *Let $H \leq GL(b, p)$ be a transitive linear group. Suppose that $C \trianglelefteq H$ and H/C is isomorphic to a transitive linear subgroup of $GL(a, p)$, where $p^a > 2$ and $a < b$.*

(1) If $H \leq \Gamma L(1, p^b)$, then C has a non-trivial cyclic p'-subgroup C_1 normalized by H.
(2) If $a > 1$ and $H \not\leq \Gamma L(1, p^b)$, then C contains one of the groups $SL(d, p^c)$, $Sp(d, p^c)'$ or $G_2(2^c)$ where $b = cd$ (with $d \geq 2$), cd (with $d \geq 4$), or $6c$ (with $c > 1$), respectively, and $|H/C|$ divides $c(p^c - 1)$.
(3) If $a = 1$ and $H \not\leq \Gamma L(1, p^b)$, p is odd and $Z(C)$ is non-trivial and has order prime to p.

PROOF. Suppose first that $H \leq \Gamma L(1, p^b)$. If $b = 2$, then $a = 1$, p is odd, and $H/C \cong GL(1, p)$. In this case the subgroup $C_1 := C \cap GL(1, p^2)$ has order divisible by $(p+1)/2 > 1$ and is normalized by H. Suppose now that $b > 2$. If there is a p-primitive prime divisor p_1 of $p^b - 1$, take C_1 to be the Sylow p_1-subgroup of H. Otherwise $p^b = 2^6$, a equals 2 or 3, and the subgroup C_1 of order 3 in $H \cap GL(1, 2^6)$ is characteristic in C.

Next suppose that H is one of the groups appearing in Theorem 2.9 (2) (a)–(c) (but $H \not\leq \Gamma L(1, p^b)$) and let T be the normal subgroup of H there indicated. If the smallest degree of a (not necessarily faithful) representation of T over $GF(p)$ is less than b, then $T = SL(2, 3)$ and (3) holds. Otherwise, the assumption that $a < b$ implies that $T \leq C$, from which (2) follows. Note that if $C \geq G_2(2)'$ then a divides $b = 6$ and $2^a > 2$; but this means that $2^a - 1$ does not divide $|H/C|$, so in the case of $G_2(2^c)$ we must have $c > 1$.

There are two more types in Theorem 2.9 (2), namely 2.9 (2) (e) *exceptional* and 2.9 (2) (f) *extra special*. In these cases $p = 2$ only if $H = A_6$ or A_7, and these groups have minimal 2-modular degree 4, so are excluded by the hypothesis $a < b$. Thus p is odd, and either $b = 2$, or $p^b = 3^4$ or 3^6. In each case $a = 1$ and L has centre of order at least 2. □

We close this section with a technical result which plays a central role in the construction of the new affine examples in Example 3.6, the first instance of which occurs for $p = 2, a = 2, b = 3, c = 1$.

LEMMA 2.16. *Let p be a prime, $a < b$ and let $q = p^c$ where $c|(a, b)$. Suppose c divides $s := (p^a - 1)(q - 1, a/c)/(q - 1)$ and s divides b. Then there is a subgroup $G \leq A\Gamma L(1, p^b)$ having a homomorphic image that is a 2-transitive group of degree p^a.*

PROOF. Write V for the additive group of $GF(p^b)$ viewed as a $GF(q)$-vector space. Let σ be an automorphism of $GF(p^b)$ of order s. Since c divides s, σ fixes $GF(q)$. Suppose $x \in GF(p^b)$ generates a normal basis for $GF(p^b)$ over $GF(q)$. Then the cyclic $K\langle\sigma\rangle$-module $S \leq V$ generated by x has dimension s and is free. Since p does not divide s, V is completely reducible and every irreducible $K\langle\sigma\rangle$-module appears in S. Thus V has a submodule W of $GF(q)$-codimension a/c for which σ acts faithfully on V/W. Since the order of the group of $GF(q)$-scalars that σ induces on V/W is $(q-1)/(q-1, a/c)$, the group $H := \langle\sigma, GF(q)^\star\rangle$ acts as a Singer group on V/W. □

3. EXPLICIT GRAPH CONSTRUCTIONS

We now give explicit constructions of the graphs arising in the Main Theorem, beginning with those having the extreme values for the index r. Since the diameter 2 case is fully described in Lemma 2.1, we discuss only the graphs having diameter 3.

EXAMPLE 3.0 (BIPARTITE). If X is an antipodal distance-regular graph of diameter three, and is also bipartite, then Lemma 2.3 implies that $r = 2$ and $X = K_2 \otimes K_n = K_{n,n}$ minus the edges of a matching. Its automorphism group is $S_n \times Z_2$.

EXAMPLE 3.1 (GARDINER). Here (see [8, Proposition 4.6]) the index $r = n - 1$, and $n \in \{3, 7, 57\}$. If $n = 3$, $X = C_6$, a cycle of length 6, with automorphism group D_{12}. If $n = 7$, $X = 6 \cdot K_7$ is the subgraph induced on the vertices at distance two from a chosen vertex in the Hoffman Singleton graph, and X has automorphism group S_7. A result of M. Aschbacher [1] implies that there is no distance-transitive example when $n = 57$. (In fact it follows from a result of G. Higman (which is discussed in [6]) that there is not even a vertex-transitive Moore graph of valency 57.)

EXAMPLE 3.2. The distance-transitive covers of K_n which have index 2 and are not bipartite were classified by Taylor [22], see also [2, Theorem 1.5.3]. The possible values for n and the automorphism group $2 \cdot G$ are as follows.

Type	n	$2 \cdot G$	Comments
symplectic	$2^{2m-1} \pm 2^{m-1}$	$2 \cdot Sp(2m, 2)$	$m \geq 3$
Ree	$q^3 + 1$	$2 \cdot Aut(R(q))$	$q = 3^{2a+1} > 3$
Higman–Sims	176	$2 \cdot HiS$	
Conway	276	$2 \cdot Co_3$	
unitary	$q^3 + 1$	$2 \cdot P\Gamma U(3, q^2)$	$q > 3$
linear	$q + 1$	$2 \cdot P\Sigma L(2, q)$	$4 \mid (q-1)$
affine	2^{2d}	$2 \cdot ASp(2d, 2)$	

The classification of the graphs in Example 3.2 and the structure of G follow from [22, Theorems 1 and 2]. Information about the splitting of the automorphism group $2 \cdot G$ can be deduced from [22, Theorem 2.1], and we are grateful to Taylor for his advice in settling the question of splitting for each of the cases above. In all cases $2 \cdot G$ can be represented as a group of $(n - m) \times (n - m)$ monomial matrices over the real numbers acting on a certain set of n equiangular lines in $\mathbb{R}^{n-m}$, where m is one of the multiplicities given by equation (3), and the centre of the group is $Z = \{I, -I\}$. In line 4, for example, $G = Co_3$ splits over Z because $-I$ is outside the derived group (see also [2, 11.4.H]). Moreover the 176 equiangular lines in 22 dimensions on which $2 \cdot HiS$ acts are a subset of the 276 equiangular lines for Co_3 in 23 dimensions, and since $2\cdot Co_3$ splits we have that $2\cdot HiS$ splits also. In lines 1, 2 and 5 of the table, the group $2\cdot G$ must split since in these cases G has Schur multiplier of odd order. In the linear case the parameters a_1 and c_2 are both equal to $(q-1)/2$, and equation (2) yields that both possibilities for the multiplicity m are equal to $(q+1)/2$. Since in this case $(q+1)/2$ is odd, the group $2\cdot P\Sigma L(2,q)$ splits over Z (see the remark following [22, Theorem 2.1]). Finally in the affine case the group $2\cdot G$ does not split.

The graphs occurring in the last line of this table do indeed appear in the Main Theorem as special cases under case 6. The graphs occurring in lines 5 and 6 of the table could also be described using the construction of Example 3.4 below. (Note that $P\Sigma L(2,q) \cong P\Gamma O(3,q)$.) For the orthogonal case in Example 3.4, if we take $r=2$ (which is not allowed in Example 3.4) and if we require that $q \equiv 1 (\mathrm{mod}\ 4)$, then we obtain the graphs in the linear case above, and their automorphism groups are $2\cdot P\Gamma O(3,q)$ rather than $P\Gamma O(3,q)$. Similarly for the unitary case in Example 3.4, if we take $r=2$ (which again is not allowed in Example 3.4) and if we require that $q>3$, then we obtain the graphs in the unitary case above, and their automorphism groups are $2\cdot P\Gamma U(3,q^2)$ rather than $P\Gamma U(3,q^2)$.

There is a 3-fold cover of K_5 mentioned in the introduction. It appears in Example 3.4 below with $G = P\Gamma O(3,4)$ but we take this opportunity to illustrate the graph coset construction method of Lemma 2.7 for this small example.

EXAMPLE 3.3 (PETERSEN). Let $G = S_5$ the symmetric group on $\{1,\dots,5\}$, let $M = S_4$ be the stabilizer in G of 1 and let H be a Sylow 2-subgroup of M. Take $g = (1,2) \in G$. Then the hypotheses of Lemma 2.7 (4) hold. We comment only on properties (4) (b) and (c). Now $H = \langle (2345), (35)\rangle$, so $H^g = \langle (1345), (35)\rangle$ and $H^g \cap M = H^g \cap H = \langle (35)\rangle$, whence conditions (4) (b) and (c) of Lemma 2.7 hold. Thus $\Gamma(G, H, HgH)$ is an antipodal G-distance-transitive 3-fold cover of K_5; it is isomorphic to the line graph of the Petersen graph.

The next examples possess an almost simple automorphism group that acts faithfully on Σ and are dealt with in Section 4. For this reason they are similar to the unitary, Ree and symplectic examples of Taylor. First we recall the groups involved and their underlying geometry.

Let p be a prime and set $q = p^e$. The isometry group of an irreducible ternary quadratic form over $GF(q)$ is $O(3,q)$. The set of singular points of the form is an oval in $PG(2,q)$ whose full isometry group is called $P\Gamma O(3,q)$. When the coefficient field is extended to $GF(q^2)$ and the form is extended to a Hermitian form with respect to the involutory field automorphism ρ, the isometry group becomes $SU(3,q)$ and the singular points form a unital in $PG(2,q^2)$ whose full isometry group is $P\Gamma U(3,q)$. We point out that all of this holds in arbitrary characteristic although standard treatments of these groups, for example in Huppert [13, Theorem II.10.12], ignore the even characteristic case.

Following [13], we work with a basis with respect to which the Hermitian form is

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = v_1 w_3^q + v_2 w_2^q + v_3 w_1^q. \tag{4}$$

In addition, we extend the Frobenius field automorphism $\sigma : a \mapsto a^p$ to act on matrices with respect to this basis. The reader is cautioned that there is no simple characteristic-free relation between σ and ρ.

We shall next construct two families of graphs, the first of which embeds in the second like the oval in the unital. Let Ω be the associated oval or unital and let F be $GF(q)$ or $GF(q^2)$, respectively. Then $|\Omega| = q+1$ or q^3+1, respectively. Further, let

$$G(\Omega) = P\Gamma O(3,q) \text{ or } P\Gamma U(3,q^2),$$
$$S(\Omega) = PGO(3,q) \text{ or } PGU(3,q^2) \text{ respectively.}$$

We shall refer to these two cases as the orthogonal case and the unitary case, respectively. Choose α, β to be distinct elements of Ω that are fixed by σ. We may take α, β to be the span of the first basis vector and the span of the last basis vector in Huppert's basis (except when $G(\Omega) = P\Gamma O(3, 2^e)$). Then $G(\Omega)_{\alpha\beta} = \langle S(\Omega)_{\alpha\beta}, \sigma\rangle$ where $S(\Omega)_{\alpha\beta} = G(\Omega)_{\alpha\beta} \cap S(\Omega)$ is the group W modulo scalars and

$$W = \left\langle \begin{pmatrix} h & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & h^{-q} \end{pmatrix} : h, k \in F^\star \right\rangle. \tag{5}$$

In general, $S(\Omega)_{\alpha\beta} = \langle s\rangle$ is cyclic of order $q-1$ or q^2-1, respectively. The Sylow p-subgroup $P(\Omega)$ of $S(\Omega)_\alpha$ has order q or q^3, respectively. Also $S(\Omega)_\alpha = P(\Omega)S(\Omega)_{\alpha\beta}$ and $G(\Omega)_\alpha = P(\Omega)G(\Omega)_{\alpha\beta} = P(\Omega)S(\Omega)_{\alpha\beta}\cdot\langle\sigma\rangle$. The next construction refers to Ω, $G(\Omega)$, $S(\Omega)$, σ, α, β, s and W as introduced in this discussion.

EXAMPLE 3.4 (FAITHFUL). The graphs in this construction are r-fold covers of the complete graph on Ω, so $n = |\Omega|$, where r satisfies:

(1) r is an (odd) prime dividing $|S(\Omega)_{\alpha\beta}|$, and in the unitary case we require that $r \geq 5$,
(2) $r - 1$ divides $|Aut F|$, and
(3) p is a primitive root modulo r.

Let $L(\Omega) = \langle s^r\rangle$ be the unique subgroup of $S(\Omega)_{\alpha\beta}$ of index r, and let $H(\Omega) = P(\Omega)L(\Omega)\cdot\langle\sigma\rangle$. Further let g be any 2-element in $N_{G(\Omega)}(L(\Omega)\cdot\langle\sigma\rangle) \setminus G(\Omega)_{\alpha\beta}$. If in the case where $G(\Omega) = P\Gamma O(3, q)$ we have $(q-1)/r \leq 2$, then we also require that g normalize $G(\Omega)_{\alpha\beta}$; in all other situations this is a consequence of the requirement that $g \in N(L(\Omega)\cdot\langle\sigma\rangle)$. Then the graph $X(\Omega) := \Gamma(G(\Omega), H(\Omega), H(\Omega)gH(\Omega))$ is a distance-transitive cover of K_n of index r. Further (see Theorem 4.3) $X(\Omega)$ is independent of the choice of the 2-element g, and $Aut X(\Omega) = G(\Omega)$.

To prove that $X = X(\Omega)$ is indeed an example we verify the conditions of Lemma 2.7. For typographic simplicity we suppress some references to Ω. First let $M = G_\alpha$. An important consequence of the three conditions in Example 3.4 is that M has a homomorphism onto the 2-transitive Frobenius group *Frob*$(r(r-1))$ of degree r with kernel $PL\cdot\langle\sigma^{r-1}\rangle$, and one of the stabilizers in M in this 2-transitive representation is the subgroup H.

Observe that $G = G(\Omega)$ acts faithfully on $\Omega = [G\colon M]$ since Ω spans the underlying projective space. Now $[M\colon H]$ has odd order and g is a 2-element interchanging α and β so $g^2 \in H$. Also $G = \langle H, g\rangle$ since M is the only maximal subgroup of G containing H. Consequently part (1) of Lemma 2.7 holds.

Lemma 2.7 (4) (a) follows from the facts that $M = PG_{\alpha\beta}$ and H contains P. To see that Lemma 2.7 (4) (b) holds requires more care. First observe that $H^g \cap M \leq M \cap M^g = G_{\alpha\beta} = \langle S_{\alpha\beta}, \sigma\rangle$, and so $H^g \cap M \cap S \leq S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is cyclic, its subgroup $H^g \cap M \cap S$ is characteristic and hence (since g normalizes $G_{\alpha\beta}$) is equal to $(H^g \cap M \cap S)^g$ which equals $H \cap M^g \cap S$. Then since $L \leq H \cap M^g \cap S \leq H \cap S = PL$, it follows that $H^g \cap M \cap S = L$. Further, $\sigma^g \in \langle L, \sigma\rangle$ by the definition of g, and also $\sigma^g \in H^g \cap M$. Since $L \leq H^g \cap M$ it follows that we also have $\sigma \in H^g \cap M$, and hence $H^g \cap M = \langle H^g \cap M \cap S, \sigma\rangle = \langle L, \sigma\rangle \leq H$, as required.

Finally we prove that Lemma 2.7 (4) (c) holds. The automorphism σ acts on $S_{\alpha\beta} = \langle t\rangle$ as the pth-power map. Thus σ^i leaves the coset Ht invariant if and only if $t^{p^i-1} \in H \cap S_{\alpha\beta}$, that is if and only if r divides $p^i - 1$, which is the case if and only if $r - 1$ divides i (since p is a primitive root modulo r). It follows that $\langle\sigma\rangle$, and hence also $H \cap H^g$, act transitively on the

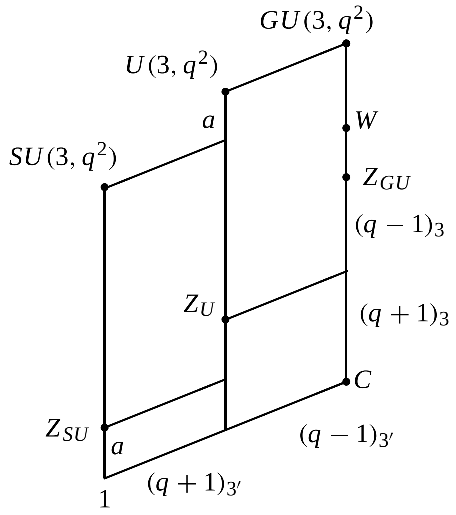

FIGURE 2. Unitary groups and scalars Z, $a := (q+1, 3)$

non-trivial cosets of H in M. This completes the proof that X is indeed a distance transitive cover of K_n of index r.

It turns out that, if conditions (1)–(3) of Example 3.4 hold in the unitary case with $r = 3$, then the construction above yields an example if and only if $q \equiv 1 (\text{mod } 3)$. In that case the proof above goes through unchanged. If on the other hand these conditions hold for $r = 3$ but $q \equiv -1 (\text{mod } 3)$, then the graph X of Example 3.4 is disconnected and is the disjoint union of three copies of the complete graph K_{q^3+1}. However, there are two variants of the construction of Example 3.4 which between them produce examples in this situation. To emphasize the special nature of the case $r = 3$ we have chosen to present all three of these constructions together in Example 3.5 below.

First we explain some of the reasons why the case $r = 3$ is so different from the case of larger primes r, by giving more details about the subgroups of the unitary groups.

Consider a three-dimensional vector space over the field $F = GF(q^2)$, equipped with a Hermitian form with respect to the involutory field automorphism ρ. We may choose a basis with respect to which the Hermitian form is given by equation (4). The group of matrices preserving this form is called the unitary group $U(3, q^2)$. The subgroup Z_U of non-singular scalar matrices in $U(3, q^2)$ has order $q + 1$. Further, the subgroup of determinant 1 matrices in $U(3, q^2)$ has index $q + 1$ and is called the special unitary group $SU(3, q^2)$; and the subgroup Z_{SU} of non-singular scalar matrices in $SU(3, q^2)$ has order $a := (3, q + 1)$. Also the central product $Z_U \circ SU(3, q^2)$ has index a in $U(3, q^2)$. It seems to be a combination of the facts that $Z_{SU} \neq 1$ and $Z_U \circ SU(3, q^2) \neq U(3, q^2)$ when $q + 1$ is divisible by 3 which makes the constructions behave so differently when $r = 3$ and $q \equiv -1 (\text{mod } 3)$. These subgroup inclusions are represented in Figure 1. Figure 1 also shows the general unitary group $GU(3, q^2)$ which is the central product of $U(3, q^2)$ and the full group $Z_{GU} \cong F^*$ of non-singular scalar matrices.

The other groups which appear in the graph construction are $\Sigma U(3, q^2) = SU(3, q^2)\langle\sigma\rangle$ and $\Gamma U(3, q^2) = GU(3, q^2)\langle\sigma\rangle$, where as above σ is the Frobenius field automorphism $\sigma \colon a \mapsto a^p$

extended to act on the matrices in $GU(3,q^2)$. Note that $GU(3,q^2)/Z_{GU} \cong U(3,q^2)/Z_U$ and this group is denoted $PGU(3,q^2)$. Note also that the group $\Gamma U(3,q^2)$ acts on Ω with kernel Z_{GU} and the stabilizer in $GU(3,q^2)$ of the points α and β is the subgroup W defined in equation (5).

We now prepare for the construction. We use the notation introduced before Example 3.4 for the groups S, P, the element σ, and the points α, β of Ω. Let L_0 be the unique subgroup of $S_{\alpha\beta}$ of index 3, and if $q+1 \equiv 0 \pmod 9$ let L_1 be the unique subgroup of $S_{\alpha\beta}$ of index 9. In the case where $q+1 \equiv \pm 3 \pmod 9$, $SU(3,q^2)_{\alpha\beta} = W \cap SU(3,q^2)$ is cyclic of order q^2-1, and in this case we let L_2 be the unique subgroup of $SU(3,q^2)_{\alpha\beta}$ of index 3. In this case also, we use P to denote the unique Sylow p-subgroup of $SU(3,q^2)_\alpha$ (of order q^3).

EXAMPLE 3.5 (UNITARY, $r=3$). The graphs in this construction are 3-fold covers of the complete graph on Ω where $|\Omega| = n = q^3+1$. The graph X is defined as $X := \Gamma(G, H, HgH)$ where G is a group given in one of the lines of the table below, H is the subgroup $H = PL \cdot \langle\sigma\rangle$ of G where L is as in the table, and g is a 2-element in $N_G(L \cdot \langle\sigma\rangle) \setminus G_{\alpha\beta}$.

G	L	q
$P\Gamma U(3,q^2)$	L_0	$q-1 \equiv p+1 \equiv 0 (\mathrm{mod}\ 3)$
$P\Sigma U(3,q^2)$	L_1	$q+1 \equiv 0 (\mathrm{mod}\ 9)$
$\Sigma U(3,q^2)$	L_2	$q+1 \equiv \pm 3 (\mathrm{mod}\ 9)$

The graph $X := \Gamma(G, H, HgH)$ is a distance-transitive cover of K_n of index 3. Further (see Theorems 4.3 and 5.2), when $q > 2$, X is independent of the choice of the 2-element g, and $Aut X = G$.

To prove that X is indeed an example we verify the conditions of Lemma 2.7. As we mentioned above, the proof for q, G, L in line 1 is the same as that for the graphs in Example 3.4. The details of proof for q, G, L in line 2 are entirely analogous. Consider the case where q, G, L are as in line 3. Then as 9 does not divide $|SU(3,q^2)_{\alpha\beta}|$, it follows that L, and hence also H, intersect Z_{SU} trivially. Thus G acts faithfully on the coset space $[G\colon H]$. Similar arguments to those in the proof for the graphs in Example 3.4 can then be used to verify all the other conditions of Lemma 2.7.(1) and (4). In fact some aspects of the verification are made easier by the fact that in this case $|PL|$ is not divisible by 3.

Note that in Example 3.4 and in lines 1 and 2 of Example 3.5, it follows from the parameter restrictions that $q \geq 4$. However, in line 3 of Example 3.5, we may have $q = 2$. The example obtained in this case is the same as the graph constructed in Example 3.6 below with $q = 3$ $n = 1$.

The final examples possess an affine 2-transitive group acting on the fibres Σ and the kernel of the action on Σ is non-trivial. For this reason they are similar to the affine examples of Taylor. This case is the subject of Section 6. Although these graphs can be presented as a group coset construction we prefer a more geometric approach.

EXAMPLE 3.6 (AFFINE). Let p be a prime and $q = p^e$. The isometry group of a non-degenerate $(2n+2)$-ary alternate bilinear form over $GF(q)$ is $Sp(2n+2,q)$. Fix a point ∞ in the associated projective geometry $PG(2n+1,q)$ and consider the embedded affine space A having $\infty^\perp$ as hyperplane at infinity. The classical examples have the q^{2n+1} points of A as vertices. The fibres of these graphs are the affine lines in the direction ∞. Two vertices x, y are adjacent if and only if $x \in y^\perp$. The required transitivity follows from Witt's Theorem applied to $Sp(2n+2,q)$. These graphs are uniquely determined by their parameters (see Propositions 6.2 and 6.3).

Note that the unipotent radical P of $Sp(2n+2,q)_\infty$ acts vertex transitively, and that the stabilizer of a vertex contains a group $C \cong Sp(2n,q)$ acting trivially on the associated fibre F. Moreover $Aut(X)_F^F = A\Gamma L(1,q)$.

Whenever a subgroup $G_1 \leq A\Gamma L(1,q)$ has a 2-transitive permutation representation of degree r properly dividing q, as in Lemma 2.16, Lemma 2.8 gives an r-fold distance-transitive cover of $K_{q^{2d}}$ having a symplectic group acting on the fibres. The first instance of such a graph is the 4-fold cover of K_{8^2} having point stabilizer $Sp(2,8)\cdot 3$ where $N = GF(2) \leq GF(8)$ in Lemma 2.8.

Note that some classical examples cover others. By the above construction, each of the groups $G_1 := Sp(2d+2,q^e)$, $G_2 := Sp(2de+2,q)$ yields a covering of $K_{q^{2de}}$. But the covering indices are q^e and q, respectively and the associated covering groups K_1 and K_2 are the additive groups of $GF(q^e)$ and $GF(q)$, respectively. Taking N to be the kernel of the natural map from $A\Gamma L(1,q^e)$ to $A\Gamma L(1,q)$ in Lemma 2.8 we see that the first graph covers the second.

4. Faithful

For the rest of the paper we use the notation introduced in Section 2 after Lemma 2.2. Suppose that the group G acts distance-transitively on X, let $v \in X$ be a vertex in the fibre $F \in \Sigma$, set $H := G_v$, and let K denote the kernel of the action of G on Σ (that is, K is the covering group induced by G). Because the classification has already been done when $r = 2$ or $n-1$, as discussed earlier, we assume further that $3 \leq r \leq n-2$.

Throughout this section suppose that K is trivial so the group G acts faithfully on Σ. We shall show that the only graphs satisfying this hypothesis are those in (4) of the Main Theorem.

Lemma 4.1. *The group G is almost simple.*

Proof. Since $K = 1$, G and G^Σ are isomorphic. Assume by way of contradiction that G is affine. Then there is an elementary Abelian normal subgroup N of G acting regularly on Σ. Therefore, there is a prime p and integer e such that

$$n = |N| = p^e,$$

and (since N is regular) $G = G_F N$. Since N is regular on Σ it follows that N has r orbits of length n on vertices, and since N is normal in G these N-orbits comprise a system of imprimitivity for G on vertices. Since $3 \leq r < n$ these N-orbits are neither the connected components of the graph X_2 nor those of the graph X_3, contradicting Lemma 2.4. □

Theorem 4.2. *The graph X is one of the graphs in Example 3.3, 3.4, or 3.5 (lines 1 or 2).*

Proof. Let T be the socle of G. Then G_F has a permutation representation onto a 2-transitive subgroup L of the symmetric group $Sym(F)$ of degree r, and the hypotheses of Proposition 2.12 hold. Thus T, n, r and L are as in one of the lines of the table in Proposition 2.12. Note that, since T is transitive on Σ and Σ is the unique non-trivial block system by Lemma 2.4, it follows that T is transitive on vertices. Moreover, since G is transitive on the arcs of X, by Lemma 2.7.2 we have $X \cong \Gamma(G, G_v, G_v g G_v)$ where g is a 2-element in $N_G(G_{vw}) \setminus G_{vw}$ and $\{v, w\}$ is an edge.

Suppose next that $T = PSL(d,q)$ with $n = (q^d-1)/(q-1)$. There are several lines in the table of Proposition 2.12 corresponding to these groups and we treat them in turn. First let

$r = q^{d-1}$ with $d \geq 3$ as in line 1 of the table in Proposition 2.12. Then Equation (1) may be written as

$$(n-1) - (r-1)c_2 = a_1 + 1$$

and substituting the given values for r and n on the left here, we deduce that

$$(q^{d-1} - 1)\left(\frac{q}{q-1} - c_2\right) = a_1 + 1 > 0.$$

This implies that $c_2 = 1$. From [9, Theorem 3.4(d)] it follows that $(n-r)^2 \leq n-1$, and therefore

$$q^{d-1} - 1 \leq q(q-1).$$

This is not possible for any values of $q > 1$ and d with $d \geq 3$. Suppose next that r is an odd prime divisor of $q-1$ as in line 2 of the table in Proposition 2.12. If $d = 2$ then $T \cong P\Omega(3, q)$, and we shall see that X is one of the graphs of Example 3.4. We use the notation of Example 3.4 for the subgroups M, H and L (note that M is defined just below Example 3.4). Certainly we may choose F so that G_F is the subgroup $M \cap G$ with M as in Example 3.4, and since T is vertex-transitive, $T_v = G_v \cap T$ is the unique subgroup of $T_F = M \cap T$ of index r. Thus we may choose v so that G_v is $H \cap G$ with H as in Example 3.4, and we may choose w adjacent to v such that T_{vw} is the subgroup $T \cap L$ with L as in Example 3.4. Then $T \cap L \leq G_{vw} \leq \langle L, \sigma\rangle$, and the 2-element $g \in N_G(G_{vw}) \setminus G_{vw}$ satisfies the conditions required in Example 3.4. Thus X is one of the graphs of Example 3.4. (Note here that if $q = 4$ then $T \cong A_5$, $r = 3$, and X is the line graph of the Petersen graph, as in Example 3.3, see [2, p. 222].)

On the other hand if $d \geq 3$ then for $F' \in \Sigma \setminus F$, the subgroup $G_{vF'}$ is transitive on F'. This can be seen easily by considering the action of the preimage $S := SL(d, q)$ of T on the underlying vector space V. The action of T on Σ is permutationally isomorphic to the action of S on the 1-spaces in V. Moreover, for a 1-space U_F corresponding to the fibre F, the action induced by T_F on F is permutationally isomorphic to the action of S_{U_F} on a block system on the non-zero vectors in U_F comprising r blocks of length $(q-1)/r$. Clearly, for distinct 1-spaces U, U', and for a non-zero vector $u \in U$, the stabilizer $S_{uU'}$ is still transitive on the non-zero vectors of U'. It follows from this (choosing $U = U_F$, U' corresponding to the fibre F', and choosing u in the block in U_F corresponding to v) that $T_{vF'}$ is transitive on F'. This contradicts the fact that $G_{vF'}$ fixes the unique point in F' adjacent to v in X. For the values of d, q, n, and r in lines 4–8, Equation (1) yields the possibilities for a_1 and c_2, and from these the eigenvalues τ and θ may be computed to test the multiplicity condition (3). In no case is the expression (3) an integer.

Finally suppose that T is $PSU(3, q^2)$ with $n = q^3 + 1$ ($q \geq 3$), and q and r as in line 3 of the table of Proposition 2.12. As in the previous case, if $r \geq 5$, we may choose the fibre F, the vertex $v \in F$, and a vertex w adjacent to v in such a way that, for M, H and L as in Example 3.4, we have $G_F = G \cap M$, $G_v = G \cap H$, $G_{vw} = G \cap \langle L, \sigma\rangle$, and hence also the 2-element g satisfies the requirements of Example 3.4. It follows that X is one of the graphs of Example 3.4. Similarly, if $r = 3$ and $q \equiv 1 \pmod 3$, arguing in the same way we see that X is one of the graphs of Example 3.5 (for q, G, L in line 1 of the table in Example 3.5). Consider now the case where $r = 3$ and 3 divides $q + 1$. Here T has index 3 in $PGU(3, q^2)$. Again we may choose F so that G_F is the stabilizer of the point α of the unital Ω, using the notation introduced before Example 3.4. Since T is vertex-transitive, G_v must be a subgroup of index 3 in G_F and G_v must intersect T in a subgroup of index 3 in $G_F \cap T$. This is only possible if $G \cap PGU(3, q^2) = PSU(3, q^2)$ and $q + 1$ is divisible by 9. Replacing G by a conjugate in $P\Gamma U(3, q^2)$ if necessary, we may assume that $G \leq P\Sigma U(3, q^2)$. Thus we may choose the vertices $v \in F$, and w, such that $G_v = PL_1{\cdot}\langle\sigma^i\rangle$ and $G_{vw} = L_1{\cdot}\langle\sigma^i\rangle$ for some integer i, where

P, L_1 are as in Example 3.5 (in line 2 of the table). Then also g satisfies the requirements of Example 3.5, and it follows that X is again one of the graphs of Example 3.5. $\square$

We end this section by determining the full automorphism groups of the graphs in Example 3.4 and in Example 3.5 (lines 1 and 2).

THEOREM 4.3.

(a) *Let $X = \Gamma(G, H, HgH)$ be a graph constructed in Example* 3.4*, or in Example* 3.5 *(for the groups in line* 1 *of the table). Then $Aut X$ is $P\Gamma O(3, q)$ in the orthogonal case, and $P\Gamma U(3, q^2)$ in the unitary case.*

(b) *If $X = \Gamma(G, H, HgH)$ is a graph constructed in Example* 3.5*, with the groups as in line* 2 *of the table, then $Aut X$ is $P\Sigma U(3, q^2)$.*

Moreover, in either case the graph X is independent of the choice of the 2*-element g.*

PROOF. Let X be one of the graphs in Example 3.4 or Example 3.5 defined in terms of the group $G = P\Gamma O(3, q)$, $P\Gamma U(3, q^2)$ or $P\Sigma U(3, q^2)$. Let V denote the vertex set of order nr where $n = q + 1$ or $q^3 + 1$, let T denote the socle of G, and set $A = Aut X$. By construction $G \leq A$, and in the unitary case $G = P\Gamma U(3, q^2) \cap Sym(V)$. Since X is antipodal, A preserves the set Σ of n fibres of size r. Let $F \in \Sigma$, and $v \in F$.

Our first step is to show that T has trivial centralizer in A. Let C be the centralizer of T in $Sym(V)$. Then C is semi-regular on vertices and $|C|$ is equal to the number of fixed vertices of T_v (see for example [25, Exercise 4.5′]). Hence $C = \langle c \rangle$ is cyclic of order r. Moreover the C-orbit containing v is the fibre F containing v. Thus the set of C-orbits is Σ and so C acts on V with n orbits of length r. Suppose that $T \times C \leq A$. Then X affords four irreducible characters of both $T \times C$ and T having degrees as in Lemma 2.6.2. Since the irreducible characters of $T \times C$ all have the form $\chi_T \otimes \chi_C$, for irreducible characters χ_T of T and χ_C of C, X affords at most four C-characters and their multiplicities are determined by Lemma 2.6.2. However, since C has n orbits of length r, X affords n copies of the regular representation of C. It follows that $m_\theta = m_\tau = n$ and that $r = 3$. Moreover, the θ-eigenspace of the adjacency matrix $A(X)$ affords a non-trivial representation of C, and consequently is spanned by vectors of the form:

$$x_v = v + \omega v^c + \overline{\omega} v^{c^2}; \qquad v \in V$$

where ω is an appropriate complex cube root of unity. But now, for w adjacent to v, the wth entry of $Ax_v = \theta x_v$ on the one hand equals $1 + 0 + 0$, (as w is not adjacent to v^c or v^{c^2}) and on the other hand equals 0 (as w is not in the support of x_v). This contradiction shows that $C \cap A = 1$.

By [9] the covering group K (the kernel of the action of A on Σ) is semiregular and hence is either trivial or cyclic of order r. Suppose that $K = Z_r$. Then $Aut K \cong Z_{r-1}$, and it follows that the derived subgroup A' of A centralizes K. In particular the socle T of G centralizes K, contradicting the fact we have just proved that $C_A(T) = 1$. Thus $K = 1$ and so A acts faithfully on Σ. Since G is 2-transitive on Σ, A is isomorphic to a 2-transitive permutation group of degree n containing $G^\Sigma \cong G$. If A contains the alternating group A_n, then $A_F = A_{n-1}$ or S_{n-1}, which has no transitive representation of degree r ($3 \leq r \leq n - 2$) unless $n = 5$, $r = 3$; but in this case we have $A = G \cong S_5$ as required. Thus we may assume that A does not contain A_n. It now follows from Theorem 2.9 that $A = G$. (Note that in the orthogonal case $G \cong P\Gamma L(2, q)$, and in the case where $G = P\Sigma U(3, q^2)$ with $r = 3$ and 9 dividing $q + 1$, we know that $A \cap P\Gamma U(3, q^2) = G$.)

From what we have just proved about their automorphism groups, any isomorphism between two of the graphs under consideration must be between graphs $X_1 = \Gamma(G, H, Hg_1H)$ and $X_2 = \Gamma(G, H, Hg_2H)$, for the same group G and subgroup H. Using the notation of Example 3.4 or 3.5, the elements g_1 and g_2 are 2-elements in $N_G(L \cdot \langle\sigma\rangle) \setminus G_{\alpha\beta}$. Note that $N_G(L \cdot \langle\sigma\rangle)$ contains $L \cdot \langle\sigma\rangle$ as a subgroup of index 2. Also $G_{\alpha\beta} = S_{\alpha\beta} \cdot \langle\sigma\rangle$ has a homomorphism onto $Z_r \cdot Z_{r-1}$ and it follows that its subgroup $L \cdot \langle\sigma\rangle$ of index r is self-normalizing in $G_{\alpha\beta}$. Thus, for a given G and H, all choices for the 2-elements g_1, g_2 determine the same double coset $Hg_1H = Hg_2H$, and hence the same graph $X_1 = X_2$. $\square$

5. UNFAITHFUL AND NEARLY SIMPLE

We continue the notation and assumptions introduced in the first paragraph of Section 4. Further, we assume throughout this section that the covering group $K \trianglelefteq G$ is non-trivial and that $G^\Sigma \cong G/K$ is almost simple, and therefore appears in Theorem 2.9.1. Case 4 (b) (iii) of the Main Theorem arises under this hypothesis.

Since the group K^F induced by K on F is a non-trivial normal subgroup of the 2-transitive group $(G_F)^F$, it follows that K is transitive on F (and hence on each of the fibres); and since X is a cover of its antipodal quotient K_n it follows that K acts faithfully and regularly on F. Hence, K^F is a regular normal subgroup of the 2-transitive group $(G_F)^F$, and so $K \cong Z_{r_o}^a$ is an elementary Abelian r_o-group for some prime r_o. It follows that $r = r_o^a$, $(G_F)^F$ is an affine 2-transitive group, and by Theorem 2.5 the prime r_o divides n. Since $(G_F)^F$ is 2-transitive, it follows that G_F, and hence also G, act transitively by conjugation on the $r-1$ non-trivial elements of K. Let C be the kernel of this G-action, that is, $C = C_G(K)$. Since K is Abelian, $K \leq C$. Further, let N be the normal subgroup of G containing K such that N/K is the non-Abelian simple socle of G/K.

THEOREM 5.1. *The subgroup $N = SU(3, q^2)$, with $r = 3$, $n = q^3 + 1$, and $q \equiv \pm 3 \pmod 9$, and the graph X is one of the graphs of Example 3.5.*

PROOF. Suppose first that $C = K$. Then G/K is isomorphic to a subgroup of $GL(a, p)$ with G/K acting transitively on the $r - 1 = r_o^a - 1$ non-zero vectors. Also G/K is an almost simple group which has a faithful 2-transitive action on Σ of degree $n > r$, contradicting Proposition 2.14. Hence $C \neq K$. It follows that C/K contains the non-Abelian simple socle of G/K, that is C contains N.

Suppose next that the derived subgroup N' of N is a proper subgroup of N. Since N/K is a non-Abelian simple group and N/N' is Abelian, it follows that $N = KN'$ and $K \not\leq N'$. Then, since K is a minimal normal subgroup of G we have $K \cap N' = 1$, and hence $N = K \times N'$, and $N' \cong N/K$ is simple. We claim that $(N'_F)^F = K^F \cong K$. (Note that a consequence of this is that the hypotheses of Proposition 2.13 hold for N'.) Since $N \trianglelefteq G$, the set of N'-orbits on vertices forms a system of imprimitivity for G in X. Moreover, since N is transitive on Σ, so also is N'. Thus the N'-orbits are not the antipodal blocks and so by Lemma 2.4 (since X is not bipartite), N' is transitive on vertices. In particular, N'_F is transitive on F and commutes with K. Since K^F is Abelian and regular, it is self-centralizing in the symmetric group on F, (see [25, 4.4]). It follows that $(N'_F)^F = K^F \cong K$, as claimed.

It now follows from Proposition 2.13 that either (a) $N' = PSL(d, q)$, $n = (q^d - 1)/(q - 1)$, and r divides $(d, q - 1)$, or (b) $N' = PSU(3, q^2)$, $n = q^3 + 1$, and r divides $(q + 1)/(3, q + 1)$. In case (a), since $r \geq 3$, we have $d \geq 3$ and so, for $F' \in \Sigma \setminus \{F\}$, the subgroup $G_{vF'}$ is transitive on F', contradicting the fact that this group fixes the unique point of F' adjacent to v in Γ. Thus case (b) holds. Now N' is transitive on vertices and is 2-transitive on Σ. Moreover, N'_v

is the unique subgroup of index r in N'_F (whence if $r = 3$ then 9 divides $q + 1$). This implies that N'_v is transitive on $\Sigma \setminus \{F\}$ and hence on the $n - 1$ vertices adjacent to v. It therefore follows that $X \cong \Gamma(N', N'_v, N'_v g N'_v)$ for some 2-element g as in Lemma 2.7.1, and thence that X is one of the graphs constructed in Example 3.4, or in Example 3.5 with the groups as in line 1 or 2 of the table. However, by Theorem 4.3, the automorphism group of X is then $P\Gamma U(3, q^2)$ or $P\Sigma U(3, q^2)$, and in particular $Aut X$ acts faithfully on Σ. This contradicts the fact that $K \neq 1$.

Thus $N = N'$. Then $K \leq Z(N) \cap N'$, that is, K is contained in the Schur multiplier $M(N/K)$ of the simple group N/K. Since $|K| = r \geq 3$, the group N/K, n, and $M(N/K)$ are as in one of the columns of the table in Lemma 2.11. Since we have that $|K| = r = r_o^a$ divides $|M(N/K)|$, and also that r_o divides n, one of the following must hold:

G	A_6	A_7	$PSL(3,4)$	M_{22}	$PSU(3,q^2)$	$PSL(m,q)$
n	6	15	21	22	q^3+14	$(q^m-1)/(q-1)$
r	3	3	3	4	$r=3, 3 \vert q+1$	$r=r_o, r_o \mid (m, q-1)$

In the cases of A_6, A_7, and M_{22}, we find all possibilities for a_1 and c_2 from equation (1), and for each of these, we find the eigenvalues θ and τ and check the multiplicity condition (3). In no case is the multiplicity condition satisfied. This procedure in the case of $PSL(3, 4)$ leads to the unique possibility: $c_2 = 9$ and $a_1 = 1$. However, by [2, Theorem 1.2.3] no such graph exists. In the case of $PSL(m, q)$, $m \geq r \geq 3$ and we obtain a contradiction by arguing as in the previous paragraph. Thus $N/K = PSU(3, q^2)$, and hence $N = SU(3, q^2)$. Now N_F is an extension of a normal subgroup of order q^3 by a cyclic group of order $q^2 - 1$, and hence N_v is its unique subgroup of index $r = 3$. Since N_v acts faithfully on the vertices of Γ, N_v does not contain the centre K of N, and hence $q + 1 \equiv \pm 3 \pmod 9$. Further, N_v is transitive on $\Sigma \setminus \{F\}$ and hence also on the vertices adjacent to v. By Lemma 2.7, it follows that $X \cong \Gamma(N', N'_v, N'_v g N'_v)$ for some 2-element g as in Lemma 2.7.1, and hence X is one of the graphs constructed in Example 3.5 with the groups as in line 3 of the table there. □

Finally, we determine the automorphism groups of the graphs arising in Theorem 5.1.

THEOREM 5.2. *Let* $X = \Gamma(G, H, HgH)$ *be a graph constructed in Example* 3.5, *with* q, G, H *satisfying line* 3 *of the table, and with* $q > 2$. *Then* $Aut X = G = \Sigma U(3, q^2)$. *Moreover, the graph* X *is independent of the choice of the* 2*-element* g.

PROOF. Let X be one of the graphs in Example 3.5 defined in terms of the group $G = \Sigma U(3, q^2)$, where $r = 3$ and $q \equiv \pm 3 \pmod 9$. Let V denote the vertex set of order nr where $n = q^3 + 1$, let S denote the normal subgroup $SU(3, q^2)$ of G, and set $A = Aut X$. By construction $G \leq A$. Since X is antipodal, A preserves the set Σ of n fibres of size $r = 3$. Let $F \in \Sigma$, and $v \in F$. Since the kernel of the action of A on fibres contains $K \cong Z_3$ and is semiregular on vertices, it follows that the kernel is equal to K. Thus the permutation group $A^\Sigma = A/K$ induced by A on Σ is a 2-transitive subgroup of S_n containing $G/K = P\Sigma U(3, q^2)$. In particular, since $q^3 + 1$ is not a prime power when $q > 2$, A^Σ is almost simple. Let $soc(A^\Sigma)$ denote its socle. Note that the extension of K by $soc(A^\Sigma)$ does not split since N does not split over K. It follows that $K \subseteq A' \cap Z(A')$ (where A' is the derived subgroup of A), and hence that K is contained in the Schur multiplier of $soc(A^\Sigma)$. Using the facts that $n - 1$ is a cube, and $|K| = 3$, it follows from Lemma 2.11 that $A^\Sigma \leq P\Gamma U(3, q^2)$, and hence that $|A : G| = 1$ or 3.

Consider the group $\Gamma U(3, q^2)$, and recall our discussion just before Example 3.5 of its subgroups, and in particular Figure 1. We use some of the notation introduced there. Let T

be the quotient of $\Gamma U(3, q^2)$ obtained by factoring out the $3'$-Hall subgroup C of the group Z_{GU} of non-singular scalar matrices. Then T is an extension of a cyclic group of order 3 by $P\Gamma U(3, q^2)$. We may identify A with a subgroup of T and G with the image of $\Sigma U(3, q^2)$ under the quotient map (since $\Sigma U(3, q^2) \cap Z_{GU} \cong Z_3$). With this identification, K is the cyclic normal subgroup of T of order 3. Let $\bar{W}$ denote the image of the subgroup W (defined in equation (5)) under the quotient map, that is $\bar{W} = W/(W \cap C)$. Then $\bar{W}$ has order $3(q^2 - 1)$, and $T = G\bar{W}$.

Suppose for a contradiction that $A = T$. Let v be the vertex of X such that $G_v = H = PL_2 \cdot \langle\sigma\rangle$, with P, L_2 as in Example 3.5. Then $T_v = P\bar{W} \cdot \langle\sigma\rangle$ (identifying P, L_2 and σ with their images in T). We may therefore now identify the vertex set V with the set $[T : T_v]$ of right cosets of T_v, with T acting by right multiplication. There is a fibre F' such that $G_{vF'} = L_2 \cdot \langle\sigma\rangle$ and we have that $G_{vF'} = G_{vw}$ where w is the unique vertex of F' adjacent to v in X. Then $T_{vF'} = \bar{W} \cdot \langle\sigma\rangle$. However, the vertices in F' are the three cosets of T_v with coset representatives in $\bar{W}$, and $T_{vF'}$ therefore acts transitively on these three vertices. This contradicts the fact that, as a subgroup of automorphisms of X, $T_{vF'}$ should fix w. Hence $A = G$ as claimed.

To see that the graph X is independent of the choice of 2-element g, suppose that $X_2 = \Gamma(G, H, Hg_2H)$ is a second graph constructed as in Example 3.5, with a second 2-element $g_2 \in N(L_2 \cdot \langle\sigma\rangle)$. An isomorphism φ from X to X_2 is an element of the symmetric group $Sym(V)$ which normalizes the common automorphism group G. However it follows from the discussion above that G has trivial centralizer in $Sym(V)$, that $N_{Sym(V)}(G) \leq N_{Sym(V)}(S) = T$, and that $N_T(G) = G$, whence G is self-normalizing in $Sym(V)$. Hence $\varphi \in G$, and so $X_2 = X^{\varphi} = X$. $\square$

6. Affine

We continue to use the notation and assumptions introduced in the first paragraph of Section 4, but we put $H := G_F$. Further, we assume that the covering group $K \trianglelefteq G$ is non-trivial and that $G^{\Sigma} \cong G/K$ is an affine 2-transitive group. Case (6) of the Main Theorem arises under this hypothesis.

Recall that a p-group is called *special* if it is non-Abelian and has but one proper non-trivial characteristic subgroup and it is called *extra special* if that subgroup has order p.

LEMMA 6.1. *There is a prime p such that $r = p^a$, $n = p^b$ (where $b > a \geq 1$) and the maximal normal p-subgroup P of G acts regularly on the vertices of X. The group P is either elementary Abelian or special of exponent p. Moreover K is the only proper non-trivial H-invariant subgroup of P.*

PROOF. Arguing as in the second paragraph of Section 5 we see that $K = Z_p^a$ is an elementary Abelian p-group, for some prime p and some $a \geq 1$, and that K acts faithfully and regularly on each fibre. Moreover, p divides n by Theorem 2.5. Therefore, since G^{Σ} is affine, we have $n = p^b$ for some $b > a$. The maximal normal p-subgroup P of G acts regularly on Σ so $P_x \leq K \cap H = 1$. The group H acts on P by conjugation and the H-conjugacy classes of elements in P reflect the four H-orbits on X. Since any characteristic subgroup of P must be a union of H-conjugacy classes the last sentence follows from Lemma 2.4. In particular this means that $K \leq Z(P)$.

Suppose that P does not have exponent p. Take $s \in P$ of order p^2. Then $s \notin K$ and $Q = \langle K, s\rangle$ is Abelian. Also the group $\langle q^p : q \in Q\rangle \leq K$ is generated by s^p. But now the stabilizer in H of the block $F^s \neq F$ must centralize $1 \neq s^p \in K$, and therefore fix both x and x^{s^p} in F, contrary to Lemma 2.6.

Thus, P has exponent p and if not Abelian, all of its proper non-trivial characteristic groups coincide with K, in particular, $K = P' = \Phi(P) = Z(P)$ and so P is special. □

Recall from the notation introduced at the beginning of Section 4 that $v \in X$ is in the fibre F. Identify vertices of X with elements of P, by identifying $v^g \in X$ with $g \in P$. The fibres are labelled by (elements of) cosets of the covering group K and form the unique non-trivial system of imprimitivity for the action of G on X.

Let $C = C_H(K)$ denote the kernel of the action of H on F. Then the transitive linear group $H^\Sigma \cong H$ has the transitive linear group $H^F \cong H/C$ as a homomorphic image. Note that the hypotheses of Lemma 2.15 hold for H and C, and hence C satisfies one of the three alternative conclusions of that lemma. In particular $C \neq 1$.

PROPOSITION 6.2. *Suppose P is elementary Abelian. Then $p = 2$, $C \leq Sp(2d, 2^c)$ where $b = 2cd$, and C is a transitive linear group. If $a = c$, then X is a classical example in Example 3.6.*

PROOF. Suppose that C has a non-trivial Abelian characteristic p'-subgroup C_1. Then C_1 is a normal subgroup of G_F, and since G_F is transitive on $\Sigma \setminus \{F\}$, it follows that the only vertices fixed by C_1 are the r vertices of F, and hence that $C_P(C_1) = K$. Hence $P_1 := \langle [x, z]: x \in P, z \in C_1\rangle$ is non-trivial. Since C_1 is a p'-group, P_1 intersects K trivially and so P_1 acts faithfully on Σ. Since C_1 is H-invariant, P_1 is an H-invariant complement to K in P contrary to Lemma 6.1. Therefore Lemma 2.15 (2) holds.

Regard H as a $GF(p)$-linear group by way of its action on P. Then H leaves invariant the $GF(p)$-subspace K and acts as a transitive linear group on $P/K(\cong \Sigma)$ and on $K(\cong F)$. Moreover, H acts indecomposably on P, since the orbits of an H-invariant complement to K in P would provide a second system of imprimitivity for the action of G contrary to Lemma 6.1. By definition, C centralizes the a dimensional $GF(p)$-subspace K. Thus there is a $GF(p)$-basis of P and group homomorphism $\phi: C \to GL(b, p)$ with respect to which the elements of $C \leq Aut(P) = GL(b + a, p)$ have the form:

$$\begin{pmatrix} \phi(\ell) & h(\ell) \\ 0 & I_a \end{pmatrix}, \qquad \ell \in C. \tag{6}$$

It follows that the first cohomology group $H^1(C, P/K)$ is non-trivial. Results of Higman [12, Lemma 4], Jones and Parshall [14] imply that $p = 2$ and $C \leq Sp(2d, 2^c)$, $b = 2cd$. By Theorem 2.9, C has $Sp(2d, 2^c)'$ or $G_2(2^c)$ as a normal subgroup, and the first cohomology groups $H^1(C, P/K)$ for these groups C are known to be one-dimensional over $GF(2^c)$, see [14]. This implies that the set of possible functions $h(\ell)$ appearing in (6) are all equivalent under conjugation by matrices of the form $diag(I_b, f I_{a/c})$, $f \in GF(2^c)$ (where we interpret $f I_{a/c}$ as an element of $GL(a, 2)$). When $a = c$ it follows that any two such groups C_1 and C_2 are conjugate under these matrices. This implies that the associated graphs are isomorphic. □

PROPOSITION 6.3. *If P is special, then p is odd, $C \leq Sp(2d, p^c)$ where $b = 2cd$ and $d \geq 1$, and C is a transitive linear group. Moreover if $a = c$, then the graph is a classical example in Example 3.6.*

PROOF. Since groups of exponent 2 are elementary Abelian, Lemma 6.1 implies that p is odd. If $a = 1$, then P is *extra-special*. The extra-special groups are classified (cf. [13, Theorem 13.7]) and the only ones of exponent p which have automorphism groups transitive on P/K are such that $Aut P/O_p(Aut P) = Sp(2d, p)$ with $b = 2d$. Thus $C \leq H \leq Sp(2d, p)$, and H is a transitive linear group. In particular, $H \not\leq \Gamma L(1, p^{2d})$, and hence part 2 or 3 of Lemma 2.15 holds. In either case C is a transitive linear group and Proposition 6.3 holds with $c = 1$. note that the *exceptional* groups in Theorem 2.9 (2) (e) (i) and the *extra special* groups in Theorem 2.9 (2) (f) (i) arise here.

Now suppose that $a > 1$. Since C normalizes P and acts trivially on K it acts on P/K in such a way as to leave invariant the non-trivial bi-additive form afforded by the commutator map of P. This commutator form appears as an H-invariant skew symmetric element of the $GF(p)$-vector space $P/K \otimes P/K$. By Schur's lemma, the centralizer in $Aut(P/K)$ of the full isometry group of this form is $GF(p^c)$ for some c dividing a. Again when $a = c$ the form and therefore the isomorphism type of P is unique. The inner automorphism group of P is isomorphic to P/K and so its full automorphism group has the structure of $A\Gamma Sp(2d, p^c)$ where $b = 2cd$. But since p is odd, $H^1(H, P/K) = 0$ by [14], for each possible transitive linear group H. This implies that there is only one conjugacy class of possible groups $H \leq A\Gamma Sp(d, q)$ and uniqueness follows. As in the previous paragraph $H \not\leq \Gamma L(1, p^{2d})$, and so since $a > 1$, part 2 of Lemma 2.15 holds and hence C is as claimed. □

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