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A brief history of mathematics in finance

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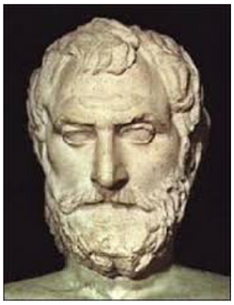
Abstract

In the list of possible scapegoats for the recent financial crises, mathematics, in particular mathematical finance has been ranked, without a doubt, as the first among many and quants, as mathematicians are known in the industry, have been blamed for developing and using esoteric models which are believed to have caused the deepening of the financial crisis. However, as Lo and Mueller (2010) state "Blaming quantitative models for the crisis seems particularly perverse, and akin to blaming arithmetic and the real number system for accounting fraud." Throughout the history, mathematics and finance have always been in a close relationship. Starting from Babylonians, through Thales, and then Fibonacci, Pascal, Fermat, Bernoulli, Bachelier, Wiener, Kolmogorov, Ito, Markowitz, Black, Scholes, Merton and many others made huge contributions to the development of mathematics while trying to solve finance problems. In this paper, we present a brief historical perspective on how the development of finance theory has influenced and in turn been influenced by the development of mathematical finance theory.



Thales (624 − 547 BC)

One of the earliest examples of financial engineering can be traced back to the philosopher Thales (624–547 BC) of Miletus in ancient Greece. Following is from Politics, Book 1 (part XI) by Aristotle: *"According to the story, he knew by his skill in the stars while it was yet winter that there would be a great harvest of olives in the coming year; so having little money, he gave deposits for the use of all olive-presses in Chios and Miletus, which he hired at a low price. When the harvest-time came, and many wanted all at once and of a sudden, he let them out at any rate he pleased. Thus he showed the world that philosophers can easily be rich if they like, but their ambition is another sort."*[1] So 2500 years ago, what Thales traded was nothing but a call option contract on oil presses for the spring olive harvest. As Aristotle mentions, Thales wanted to show that his knowledge as a mathematician (as a philosopher or as an astronomer) was useful for the whole society.

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[1] Aristotle, Politics, Book I, trans. B. Jowett in The Complete Works of Aristotle: the Revised Oxford Translation, ed. Jonathan Barnes, Bollingen Series LXXI:2 (Princeton, N.J.: Princeton University Press, Fourth Printing, 1991), p. 1998, 1259a9-19.

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Leonardo Pisano Bigollo (1170 − 1250)

In 1202, Leonordo of Pisa, commonly called Fibonacci, wrote a very first book on financial engineering, Liber Abaci (The Book of Calculations). His seminal book not only introduced Hindu−Arabic numbers to Europe but also, as Goetzmann (2004) argues, it calculated the present value of alternative cash flows in addition to developing a general method for expressing investment returns, and solving a wide range of complex interest rate problems. Goetzmann and Rouwenhorst (2005) consider the following as one of the most sophisticated interest rate problem from his book Liber Abaci: *"A soldier is granted an annuity by the king of 300 bezants per year, paid in quarterly installments of 75 bezants. The king alters the payment schedule to an annual year-end payment of 300. The soldier is able to earn 2 bezants on 100 per month (over each quarter) on his investment. How much is his effective compensation after the terms of the annuity changed?"*. Clearly, in order to solve this problem you must know the value of money at different points in time. Another problem *"Barter of Merchandise and Similar Things"* from Liber Abaci which is closely related to today's Law of One Price (if two assets offer identical cash flows then they must have the same price) is the following: *"20 arms of cloth are worth 3 Pisan pounds and 42 rolls of cotton are similarly worth 5 Pisan pounds; it is sought how many rolls of cotton will be had for 50 arms of cloth"* (Sigler (2002), p. 180). Leonordo of Pisa has been one of the most famous names in mathematics regarding his contributions to number theory and other related areas. However, he may be considered even more influential in finance because of his contributions to the foundations of credit and banking in Europe through present value computations.

Girolamo Cardano (1501 − 1576)

Girolamo Cardano, prominent Italian Renaissance mathematician, in 1565, published his treatise Liber de Ludo Aleae (The Book of Games of Chance) which founded the elementary theory of gambling. His interest in gambling not only enabled him to survive during the poor times of unemployment but also to derive basic rules of the probability. Andrew W. Lo considers the following from his book, Liber de Ludo Aleae, as the footprints of the notion of a fair game which is the essence of a martingale, a precursor to the Random Walk Hypothesis: *" The most fundamental principle of all in gambling is simply equal conditions, e.g., of opponents, of bystanders, of money, of situation, of the dice box, and of the die itself. To the extent to which you depart from that equality, if it is in your opponent's favour, you are a fool, and if in your own, you are unjust."*

Blaise Pascal (1623 − 1662) and Pierre De Fermat (1607 − 1665)

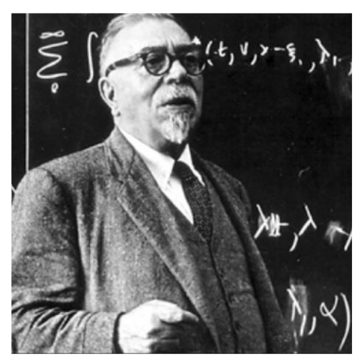
About a century after Cardano, in 1654, two French mathematicians Blaise Pascal and Pierre De Fermat, on the solution of a problem posed by Chevalier de Méré (a French nobleman with an interest in gaming and gambling questions), established the first foundations of the probability theory. The problem originally posed was to decide whether or not to bet even money on the occurrence of at least one "double six" during the 24 throws of a pair of dice. A seemingly well-established gambling rule led de Méré to believe that betting on a double six in 24 throws would be profitable, but his own calculations indicated just the opposite (Apostol (1969)). In a series of letters exchanged, Pascal and Fermat solved this problem and the problem of points (also known as "the unfinished game") which is essentially the same as the

Jacob Bernoulli (1655 − 1705) and Daniel Bernoulli (1700 − 1782) problem of pricing a digital call option on Cox–Ross–Rubinstein tree. Hence, Pascal and Fermat can also be regarded as the first mathematicians to develop a derivative pricing formula.

At the end of 17th and start of the 18th Century, Bernoulli family from Switzerland, made important contributions to the development of probability theory over a couple of generations. Jacob Bernoulli (1655–1705) proved the first version of the law of large numbers (if you perform the same experiment a large number of times, then the observed mean will converge to the expected mean) and core results on expected value in his book Ars Conjectandi (The Art of Conjecturing) on combinatorics and mathematical probability. In 1738, Daniel Bernoulli (1700–1782) took an important step towards a theory of risks by his paper Specimen theoriae novae de mensura sortis (Exposition of a New Theory on the Measurement of Risk) in which he discusses the St. Petersburg paradox. The following passage describing the St. Petersburg paradox is from Daniel Bernoulli's publication (the translation appeared in Econometrica 22 (1954) 123–136): " *My most honorable cousin the celebrated Nicolaus Bernoulli, Professor utriusque iuris at the University of Basle, once submitted five problems to the highly distinguished mathematician Montmort. These problems are reproduced in the work L'analyse sur les jeux de hazard de M. de Montmort, p. 402. The last of these problems runs as follows: Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation."* This game leads to a random variable with infinite expected value and any rational gambler would enter the game with a finite price of entry. However, the game seems to be worth only a very small amount for rational investors compared to the expected value of the game. Daniel Bernoulli solved this paradox by introducing log utility function which has the diminishing marginal utility concept in it. In his own words: " *The determination of the value of an item must not be based on the price, but rather on the utility it yields... There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount."* This seems to be the first time investment decision making is evaluated based on a utility function other than linear utility.

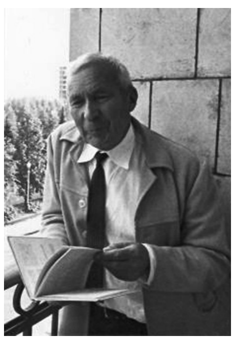
Louis Bachelier (1870 – 1946)
(source: Wikimedia Commons)

At the turn of the 20th Century, March 29, 1900, a French doctoral student Louis Bachelier defended his thesis "Théorie de la Spéculation" (Theory of Speculation) which is today recognized as the birth certificate of the modern mathematical finance. His exceptional work has been published in one of the most influential French scientific journals, Annales Scientiques de lÉcole Normale Supérieure. He is credited with being the first person to derive the mathematics of Brownian motion and to apply its trajectories for modeling stock price dynamics and calculating option prices. Schachermayer and Teichmann (2008) compare the option pricing formulas derived by Louis Bachelier and Black–Merton–Scholes and show that the prices coincide very well. They also present that Bachelier's model yields good short-time approximations of prices and volatilities. His pioneering work regarding the financial markets also led to the development of what is known today as Efficient Market Hypothesis and related theories like capital asset pricing model. He writes in his thesis: " *The influences that determine the movements of the exchange are innumerable; past, current and even anticipated events that often have no obvious connection with its changes ... it is thus impossible to hope for mathematical predictability."* and notes the main idea in a single sentence: *"The mathematical expectation of the speculator is zero."* In honor of his great contributions to the development of stochastic calculus and mathematical finance, a group of prominent financial mathematician formed The Bachelier Finance Society in 1996 to provide academia and practitioners the opportunity to meet and exchange ideas.



Norbert Wiener (1894 – 1964)
(source: Research Laboratory of Electronics at MIT)

Most of the exciting innovations in the modern history of mathematical finance are rooted in the discovery of the Brownian motion by Scottish botanist Robert Brown. In 1827, he observed rapid oscillatory motion of microscopic particles in a fluid resulting from their collision with atoms or molecules in the fluid. However, as mentioned above, Bachelier was the first to define Brownian motion mathematically and used one dimensional version $t \mapsto B_t, t \geq 0$ to model stock price dynamics. Unaware of Bachelier's work, Albert Einstein also derived the equations for Brownian motion and applied it on the kinetic theory of heat in thermodynamics. However, Norbert Wiener is the first to provide the rigorous mathematical construction of Brownian motion therefore it is also called as Wiener process. He proved the existence of Brownian motion(BM) and constructed the Wiener measure which describes the probability distribution of BM. It has been used to describe many physical phenomena because of its many interesting properties: It is continuous everywhere but nowhere differentiable. It is self-similar in law i.e. if one zooms in or zooms out on a Brownian motion, it is still a Brownian motion. It is one of the best known Lévy processes (càdlàg stochastic processes with stationary independent increments) and also it is a martingale.



Andrey N. Kolmogorov (1903 − 1987)
(source: Oberwolfach Photo Collection)

It was until the publication of Russian mathematician Andrey Nikolaevich Kolmogorov's seminal book *"Foundations of the Theory of Probability"* in 1933 that probability was seen something related to mathematics but somehow different from it. However, Kolmogorov, similar to Euclid's construction of geometry, created a new formulation of probability theory from fundamental axioms and therefore fully integrated integrated probability into mathematics. He relied on measure theory, that was developed by Émile Borel, Henri Lebesgue and many others in the beginning of 20th Century, to set the following axioms: 1) The probability of an event is a non-negative real number, 2) the probability that some elementary event in the entire sample space will occur is 1, 3) The probability of the union of mutually exclusive events is the sum of the probability of the individual events. In his 1933 book, he also introduced the idea of conditional expectation and equivalent measures which enabled financial mathematicians to produce formulas for derivative prices. Kolmogorov today is considered as one of the most brilliant mathematicians that the world has ever known and it is not possible to summarize his mathematical heritage in a paragraph here but his fundamental contributions not only in probability theory but also in statistical mechanics, stochastic processes, information theory, nonlinear dynamics, mathematical statistics have found many interesting applications in finance and economics.

Kiyoshi Ito (1915 − 2008)
(source: Oberwolfach Photo Collection)

One of the most widely used mathematical formula by financial engineers today, Ito's Lemma, was derived by Japanese mathematician Kiyoshi Ito in his spectacular paper: "On stochastic differential equations (1951)". In his attempts to model Markov processes, Itô (in his famous 1942 paper "On stochastic processes") constructed stochastic differential equations of the form

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t,$$

where W is a standard Wiener process and later, in his 1951 paper, he showed that for any twice differentiable function f the following holds

$$\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)\mathrm{d}[X,X]_t.$$

Clearly, Ito's lemma presents a way to construct new SDE's from the given ones. It can be considered as the stochastic calculus counterpart of the chain rule in Newtonian calculus. Ito's formula has been applied not only in different branches of mathematics but also in conformal field theory in physics, stochastic control theory in engineering, population genetics in biology, and in many other various fields. Another extremely useful theorem in mathematical finance, Ito's representation theorem, states that any square integrable martingale of a Brownian filtration has a continuous version. Ito is considered as the father of stochastic integration and stochastic differential equations which lay the foundations of stochastic calculus. In 2006, because of his extraordinary work and outstanding contributions, Carl Friedrich Gauss Prize for Applications of Mathematics was awarded for the first time to Kiyoshi Ito. In a speech given marking his Kyoto prize in 1998, Ito gives a wonderful description of mathematical beauty: *"In precisely built mathematical structures, mathematicians find the same sort of beauty others find in enchanting pieces of music, or in magnificent architecture. There is, however, one great difference between the beauty of mathematical structures and that of great art. Music by Mozart, for instance, impresses greatly even those who do not know musical theory; the cathedral in Cologne overwhelms spectators even if they know nothing about Christianity. The beauty in mathematical structures, however, cannot be appreciated without understanding of a group of numerical formulae that express laws of logic. Only mathematicians can read "musical* *scores" containing many numerical formulae, and play that "music" in their hearts. Accordingly, I once believed that without numerical formulae, I could never communicate the sweet melody played in my heart. Stochastic differential equations, called "to Formula," are currently in wide use for describing phenomena of random fluctuations over time. When I first set forth stochastic differential equations, however, my paper did not attract attention. It was over ten years after my paper that other mathematicians began reading my "musical scores" and playing my "music" with their "instruments." By developing my "original musical scores" into more elaborate "music," these researchers have contributed greatly to developing "Ito Formula"."*

Around the same time Kiyoshi Itō was constructing the foundations of stochastic calculus, Harry Markowitz published his paper "Portfolio Selection" which is considered as the first influential work of mathematical finance capturing the immediate attention outside academia. His 1952 the Journal of Finance paper "Portfolio Selection" together with his 1959 book "Portfolio Selection: Efficient Diversification of Investments" laid the groundwork for what is today referred to as MPT, "modern portfolio theory". Prior to Markowitz's work, investors formed portfolios by evaluating the risks and returns of individual stocks. Hence, this lead to construction of portfolios of securities with the same risk and return characteristics. However, Markowitz argued that investors should hold portfolios based on their overall risk-return characteristics by showing how to compute the mean return and variance for a given portfolio. Markowitz introduced the concept of efficient frontier which is a graphical illustration of the set of portfolios yielding the highest level of expected return at different levels of risk. These concepts also opened the gate for James Tobin's super-efficient portfolio and the capital market line and also William Sharpe's formalization of the capital asset pricing model (CAPM). In the area of linear programming, Harry Markowitz developed "sparse matrix" techniques for solving very large mathematical optimization problems. In simulation, he created a computer language SIMSCRIPT together with Bernard Hauser and Herb Karr which has been widely used to program computer simulations of manufacturing, transportation, and computer systems as well as war games (Wikipedia). In 1989, Markowitz received The John von Neumann Award from the Operations Research Society of America for his work in portfolio theory, sparse matrix techniques and SIMSCRIPT. In 1990, he shared Nobel Prize for Economics for his work in portfolio theory.

Myron Scholes (1941 –)
(source: Wikimedia Commons)

A major breakthrough came in 1973 when Fischer Black and Myron Scholes published the paper "The Pricing of Options and Corporate Liabilities" in the Journal of Political Economy and Robert Merton published the paper "On the pricing of corporate debt: the risk structure of interest rates" in the Bell Journal of Economics and Management Science". These papers introduced a new methodology for the valuation of financial instruments and in particular developed the Black–Scholes model for pricing European call and put options. At the same time another breakthrough on the industry side was the foundation of the Chicago Board Options Exchange to become the first marketplace for trading listed options. Even beyond the imagination of the celebrated authors above, the market was so quick to adapt these models. By 1975, almost all traders were valuing and hedging option portfolios by using the Black–Scholes model built in their hand calculators. From a tiny market trading only 16 option contracts in 1973, the derivatives market has grown enormously in notional amount to trillions of dollars. In addition to huge explosion in the derivatives market, Black–Scholes–Merton work also played a significant role in the expansion of financial mathematics literature. Financial engineers today mainly use two approaches for the calculation of option prices. In the first approach, option price can be found as the risk neutral expected value of the discounted option pay-off. In the second approach the option price is the solution of the famous Black–Scholes PDE

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{1}$$

where V is the option value, S is the underlying asset, σ is the volatility and r is the risk free interest rate. Feynman–Kac Theorem establishes the connection between these two approaches by showing that a classical solution to a linear parabolic PDE has a stochastic representation in terms of an expected value. However, their contributions are not restricted to option pricing formulas. For instance, Fischer Black is also famous for the development of Black–Derman–Toy, Black–Karasinski, Black–Litterman, and Treynor–Black financial models among many other models together with his co-authors. Robert Merton is also well known for Merton model, Jarrow–Turnbull model, ICAPM, and Merton's portfolio problem. In 1997 the Royal Swedish Academy of Sciences awarded the Nobel Prize for Economics to Merton and Scholes, Fischer Black having died a couple of years earlier. The best summary for their significant contribution is given by Rubinstein (1992): "the Black & Scholes model is widely viewed as one of the most successful in the social sciences and perhaps, including its binomial extension, the most widely used formula, with embedded probabilities, in human history".

Black–Scholes–Merton work has led to enormous research activity within mathematical finance and it is rapidly

Robert Merton (1944 –)
(source: Wikimedia Commons)

growing every day. It is a futile attempt to give a comprehensive account of this activity here but we want to touch a recent topic in financial economics and mathematics i.e. robust hedging. Robustness may have different meanings in different frameworks but in the most general terms it refers to the quality or state of being sturdy and strong in form, constitution, or construction. In economics, robustness is associated with the sustainability of an economic model under different assumptions, parameters, and initial conditions and also the effectiveness of a financial system under different markets and market conditions. Woodward (2006) classifies four different notions of robustness in economics. Inferential robustness refers to the idea that there are different degrees to which inference from some given data may depend on various auxiliary assumptions, and derivational robustness refers to whether a given theoretical result depends on the different modeling assumptions. Measurement robustness means triangulation of a quantity or a value by (causally) different means of measurement. Causal robustness, on the other hand, concerns causal dependencies in the world (Kuorikoski, Lehtinen, & Marchionni, 2007). This word has gained further popularity and importance thanks to the book "Robustness" written by two leading macroeconomists Lars Peter Hansen (co-winner of the 2013 Nobel Prize in Economics) and Thomas J. Sargent (co-winner of the 2011 Nobel Prize in Economics). They develop robust control techniques with applications to a variety of problems in dynamic macroeconomics. These techniques help macroeconomists and decision makers when economic models are not fully trustable or when there are misspecifications or wrong (or sometimes too strong) assumptions in economic modeling.

As it is clear from the recent developments in the economics history, robustness is growing to be more a prominent concept which also needs some mathematical background. It appeared in Mathematical Finance first in the form of robust pricing and hedging of options and it can be traced back to the seminal paper, Pricing with Smile (1994), by Bruno Dupire. The standard approach to option pricing is to assume that the underlying asset price is the solution of a stochastic differential equation and then risk neutrality arguments yield that the price of a contingent claim is given by the discounted expected value of the contingent payoff under the equivalent martingale measure. To reiterate once again, in the classical option pricing methodology first a model is proposed for the behavior of underlying asset and then a fair price and associated hedging strategy is derived from this model. However, Dupire, in his ground-breaking article, only assumes that the underlying asset price follows a diffusion and that there are call options traded with all possible maturities and all possible strikes. By reverse engineering, he extracts the underlying asset distributions from the observed call option prices and any other financial products with payoff contingent on the final value of the asset can be priced and hedged from these distributions. The intuition behind this approach is quite well-founded: Call options today are so liquid that they can be treated as primary assets to price more sophisticated derivative instruments. In a similar vein, Hobson (1998) obtains the model independent bounds and the associated hedging strategies on the prices of exotic derivatives and in particular the look-back option by inferring the information about the potential distribution of asset prices from the call prices. Robust hedging has been an active area of research over the past decade and we refer the reader to the excellent survey of Hobson (The Skorokhod Embedding Problem and Model-Independent Bounds for Option Prices, ParisPrinceton Lectures on Mathematical Finance, Springer, (2010)) for further information.

As it is explained in detail above, robust hedging in Mathematical Finance refers to using liquidly traded financial instruments to reduce the risk that is under consideration. The Chicago Board Options Exchange (CBOE), Volatility Index (VIX) calculation is a good example of robust hedging. VIX is created to calculate the implied volatility of options on the S&P 500 index (SPX), the core index for U.S. equities, for the next 30 calendar days. A wide range of strike prices for SPX put and call options are utilized to calculate VIX. It is also referred as the fear index by the common media and leading financial publications. As the CBOE states "since volatility often signifies financial turmoil, VIX is often referred to as the 'investor fear gauge' ". Inevitably, this measure is closely observed by both buy and sell sides of the market as it provides crucial information about investor sentiment that can be helpful in evaluating potential market turning points. In 2004, CBOE introduced the first exchange-traded VIX futures contract and two years later, in 2006, CBOE launched VIX options which is considered as the most successful new product in exchange history.

One closely related problem to the robust hedging problem in mathematics and economics is the Monge–Kantorovich optimal transport problem. This problem was first formulated by Monge (1781) to move the soil during the construction of the building of forts and roads with minimal transport expenses. Mathematically, given two measures ν and μ of equal mass, we search for an optimal transport map S i.e. an optimal bijection of $\mathbb{R}^d$ so that $S_{\#}\nu = \mu$ i.e.

$$\int_{\mathbb{R}^d} \varphi(S(x))\mathrm{d}\nu(x) = \int_{\mathbb{R}^d} \varphi(S(x))\mathrm{d}\mu(x) \tag{2}$$

for all continuous functions φ. Then objective is to minimize

$$\int_{\mathbb{R}^d} c(x, \varphi(S(x)))\mathrm{d}\nu(x) \tag{3}$$

for a given cost function c, over all bijections S.

Kantorovich relaxed this problem by considering

$$\text{minimize} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x, y)\, \mathbb{Q}(\mathrm{d}x, \mathrm{d}y) \tag{4}$$

over all probability measures $\mathbb{Q} \in \mathcal{M}(\nu, \mu)$, i.e.,

$$(Proj_x)\#\mathbb{Q} = \nu, \quad (Proj_y)\#\mathbb{Q} = \mu, \tag{5}$$

or equivalently, for every Borel sets $A, B \subset \mathbb{R}^d$,

$$\mathbb{Q}(A \times \mathbb{R}^d) = \nu(A), \qquad \mathbb{Q}(\mathbb{R}^d \times B) = \mu(B). \tag{6}$$

In this formulation, it is a linear program and easily admits a solution. Moreover, its convex dual is given by,

$$\text{maximize} \left[\int g(x)\, \nu(\mathrm{d}x) + \int h(y)\, \mu(\mathrm{d}y) \right]$$

over all $g \in \mathbb{L}^1(\mathbb{R}^d, \nu)$, $h \in \mathbb{L}^1(\mathbb{R}^d, \mu)$ satisfying

$$g(x) + h(y) \geq c(x, y), \quad \forall\, x, y.$$

The connection between Monge–Kantorovich and robust hedging problems is explained in Dolinsky and Soner (2013) in the following way. An optimal connection need to be constructed between two measures i.e. the initial and final distributions of a stock process in robust hedging problems. In general, however, the cost functional depends on the whole path of this connection and not simply on the final value. Hence, one needs to consider processes instead of simply the maps S. The probability distribution of this process has prescribed marginals at final and initial times. Thus, it is in direct analogy with the Kantorovich measure.

References

Apostol, T. (1969). *Calculus,* volume II (2nd ed.). John Wiley & Sons.

Dolinsky, Y., & Soner, H. M. (2013). Martingale optimal transport and robust hedging in continuous time. *Probability Theory and Related Fields*, 1–37.

Goetzmann, W. N. (2004). *Fibonacci and the financial revolution*. http://www.nber.org/papers/w10352 (Working Paper).

Goetzmann, W. N., & Rouwenhorst, G. (2005). *The origins of value: The financial innovations that created modern capital markets*. Oxford University Press.

Hobson, D. G. (1998). Robust hedging of the lookback option. *Finance and Stochastics, 2–4*, 329–347.

Kuorikoski, J., Lehtinen, A., & Marchionni, C. (2007). *Economics as robustness analysis*. http://philsci-archive.pitt.edu/3550/1/econrobu.pdf (Working paper).

Lo, A., & Mueller, M. (2010). *Warning: Physics envy may be hazardous to your wealth!*. http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1563882 (Working Paper).

Monge, G. (1781). Memoire sur la Theorie des Oéblais et des Remblais Histoire de L'Acad. *des Sciences de Paris*, 666–704.

Rubinstein, M. (1992). "Guiding force". From Black–Scholes to black holes: new frontiers in options. *Risk Magazine*.

Schachermayer, W., & Teichmann, J. (2008). How close are the option pricing formulas of Bachelier and Black–Merton–Scholes? *Mathematical Finance, 18–1*, 155–170.

Sigler, L. E. (2002). *Fibonacci's Liber Abaci*.

Woodward, J. (2006). Some varieties of robustness. *Journal of Economic Methodology, 13–2*, 219–240.