Discrete Mathematics 310 (2010) 385–389

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Tutte's 5-flow conjecture for highly cyclically connected cubic graphs

Eckhard Steffen

Paderborn Institute for Advanced Studies in Computer Science and Engineering (PACE), Universität Paderborn, Warburger Straße 100, D-33098 Paderborn, Germany

ARTICLE INFO

Article history:
Received 26 September 2006
Accepted 12 March 2009
Available online 8 April 2009

Keywords:
Nowhere-zero flows
Tutte's flow conjectures
5-flows
Cubic graphs
Snarks
Balanced valuations

ABSTRACT

In 1954, Tutte conjectured that every bridgeless graph has a nowhere-zero 5-flow. Let $\omega(G)$ be the minimum number of odd cycles in a 2-factor of a bridgeless cubic graph G. Tutte's conjecture is equivalent to its restriction to cubic graphs with $\omega \geq 2$. We show that if a cubic graph G has no edge cut with fewer than $\frac{5}{2}\omega(G) - 3$ edges that separates two odd cycles of a minimum 2-factor of G, then G has a nowhere-zero 5-flow. This implies that if a cubic graph G is cyclically n-edge connected and $n \geq \frac{5}{2}\omega(G) - 3$, then G has a nowhere-zero 5-flow.

1. Introduction

This paper is about flows on finite graphs. Let $M = (V, E)$ be a graph with vertex set V and edge set E. Each edge is incident to precisely two different vertices, i.e. multiple edges may occur but there are no loops.

An *orientation* D of M is an assignment of a direction to each edge, and for $v \in V$, $D^-(v)$ ($D^+(v)$) is the set of edges whose head (tail) is incident to v. The oriented graph is denoted by $D(M)$, $d^-_{D(M)}(v) = |D^-(v)|$ and $d^+_{D(M)}(v) = |D^+(v)|$ denote the *indegree* and *outdegree* of vertex v in $D(M)$, respectively.

Let $k \geq 2$ be a positive integer and $\varphi : E \longrightarrow \{0, 1, \ldots, k-1\}$ be a function. If for all $v \in V$,

$$\sum_{e \in D^+(v)} \varphi(e) = \sum_{e \in D^-(v)} \varphi(e), \tag{1}$$

then (D, φ) is a *k-flow* on M. If, in addition, $\varphi(e) \neq 0$, for all $e \in E$, then (D, φ) is a *nowhere-zero k*-flow on M. In such a case, we say that M has a nowhere-zero k-flow.

If a graph has a nowhere-zero k-flow, then it has one for every $k' \geq k$. Tutte [7] proved that a graph G has a nowhere-zero k-flow (D, φ) if and only if it has a flow (D', φ') such that for every edge e, $|\varphi'(e)|$ is one of $1, \ldots, k-1$. Thus determining for which number k a graph has a nowhere-zero k-flow is a problem about graphs, not directed graphs.

Tutte [8] raised the problem to determine the smallest number k for which a graph has a nowhere-zero k-flow, and he formulated the 5-Flow Conjecture.

Conjecture 1 ([8]). *Every bridgeless graph has a nowhere-zero* 5*-flow.*

The 5-Flow Conjecture is equivalent to its restriction to cubic graphs, cf. [3]. By Petersen's theorem, every bridgeless cubic graph G has a 2-factor and the *oddness* $\omega(G)$ is the minimum number of odd cycles in a 2-factor of G. Clearly, the oddness must be an even number, and it is well known (cf. [3]) that a cubic graph G has a nowhere-zero 4-flow if and only if it is edge 3-colorable (i.e. $\omega(G) = 0$). Hence the 5-Flow Conjecture is equivalent to its restriction to bridgeless cubic graphs with $\omega \geq 2$.

E-mail address: es@upb.de.

0012-365X/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2009.03.014

Many papers deal with the structure of a possible counterexample to the 5-Flow Conjecture. A connected graph $G = (V, E)$ that contains two disjoint cycles is *cyclically n-edge connected* if there is no edge cut $E' \subset E$ with fewer than n edges such that two components of $G - E'$ contain cycles. The maximum number k so that G is cyclically k-edge connected is the *cyclic connectivity* of G and it is denoted by n^*_G. Kochol [4,5] showed that the length of a shortest cycle in a possible minimum counterexample is at least 9, and that it is cyclically 6-edge connected. This paper proves the following theorems.

Theorem 1. *Every cubic graph G with cyclic connectivity* $n^*_G \geq \frac{5}{2}\omega(G) - 3$ *has a nowhere-zero 5-flow.*

A *minimum* 2-factor of a cubic graph $G = (V, E)$ has precisely $\omega(G)$ odd cycles. Let $\omega(G) \geq 2$, $\mathcal{F}_2$ be a minimum 2-factor, and let $m_G(\mathcal{F}_2)$ be the maximum number k such that there is no edge cut $E' \subset E$ with fewer than k edges such that two components of $G - E'$ contain odd cycles of $\mathcal{F}_2$. We define $m^*_G = \max\{m_G(\mathcal{F}_2) | \mathcal{F}_2$ is a minimum 2-factor of $G\}$ to be the *cyclic factor connectivity* of G. For graphs G with $\omega(G) = 0$ define $m^*_G = \infty$.

Since $n^*_G \leq m^*_G$ Theorem 1 is a direct consequence of the following theorem.

Theorem 2. *Let G be a bridgeless cubic graph. If* $m^*_G \geq \frac{5}{2}\omega(G) - 3$, *then G has a nowhere-zero* 5*-flow.*

2. Balanced valuations and flow partitions

Bondy [1] and Jaeger [2] introduced the concept of balanced valuations. A *balanced valuation* of a graph $M = (V, E)$ is a function w from the vertex set V into the real numbers such that for all $X \subseteq V$: $|\sum_{v \in X} w(v)| \leq |\partial_M(X)|$, where $\partial_M(X)$ is the set of edges with precisely one end in X. For $v \in V$ let $d_M(v)$ be the degree of v in the undirected graph M. The following theorem relates integer flows to balanced valuations.

Theorem 3 ([2]). *Let* $M = (V, E)$ *be a graph with orientation D and* $k \geq 3$. *Then M has a nowhere-zero k-flow* (D, φ) *if and only if there is a balanced valuation w of M with* $w(v) = \frac{k}{k-2}(2d^+_{D(M)}(v) - d_M(v))$, *for all* $v \in V$.

In particular, Theorem 3 says that a cubic graph G has a nowhere-zero 4-flow (nowhere-zero 5-flow) if and only if there is a balanced valuation of G with values in $\{\pm 2\}$ ($\{\pm \frac{5}{3}\}$).

Let $M = (V, E)$ be a multigraph. If $X \subseteq E$, then $M[X]$ denotes the graph whose vertex set consists of all vertices of edges of X and whose edge set is X. Likewise if $X \subseteq V$, then $M[X]$ is the graph whose vertex set is X and whose edge set consists of those edges incident to two vertices of X. In both instances the subgraph $M[X]$ is called the subgraph of M *induced* by X.

Let $E_i \subseteq E$, and (D_i, φ_i) be flows on $M[E_i]$, $i = 1, 2$. The *sum* $(D_1, \varphi_1) + (D_2, \varphi_2)$ is the flow (D, φ) on $M[E_1 \cup E_2]$ with orientation

$$D := D_1|_{\{e|\varphi_1(e) \geq \varphi_2(e)\}} \cup D_2|_{\{e|\varphi_2(e) > \varphi_1(e)\}}, \quad \text{and}$$

$$\varphi(e) := \begin{cases} \varphi_1(e) + \varphi_2(e) & \text{if } e \text{ received the same direction in } D_1 \text{ and } D_2 \\ |\varphi_1(e) - \varphi_2(e)| & \text{otherwise,} \end{cases}$$

for $e \in E_1 \cup E_2$.

Let $G = (V, E)$ be a bridgeless cubic graph, and $\mathcal{F}_2$ be a 2-factor of G with odd cycles $C_1, C_2, \ldots, C_{2t}$, and even cycles $C_{2t+1}, \ldots, C_{2t+\ell}$ $(t \geq 0, \ell \geq 0)$, and let $\mathcal{F}_1$ be the complementary 1-factor.

A *canonical* 4-coloring of G (with respect to $\mathcal{F}_2$) colors the edges of $\mathcal{F}_1$ with color 1, the edges of the even cycles with 2 and 3, alternately, and the edges of the odd cycles with colors 2 and 3 alternately, except one edge which is colored 0. Then, there are precisely $2t$ vertices $z_1, z_2, \ldots, z_{2t}$ where color 2 is missing.

Let $M_G = (V, E(M_G))$ be the graph obtained from G by adding two edges f_i and f'_i between z_{2i-1} and z_{2i} for $i = 1, \ldots, t$. Extend c to a proper edge coloring of M_G by coloring f'_i with color 2 and f_i with color 4. Let $C'_1, \ldots C'_s$ be the 2-factor of M_G induced by the edges of colors 1 and 2 $(s \geq 1)$, and for $i = 1, \ldots, t$ let C''_i be the 2-cycle induced by the edges f_i and f'_i. We construct a nowhere-zero 4-flow on M_G as follows:

For $1 \leq i \leq 2t + \ell$ let (D_i, φ_i) be a nowhere-zero flow on the directed cycle C_i with $\varphi_i(e) = 2$ for all $e \in E(C_i)$.

For $1 \leq i \leq s$ let (D'_i, φ'_i) be a nowhere-zero flow on the directed cycle C'_i with $\varphi'_i(e) = 1$ for all $e \in E(C'_i)$.

For $1 \leq i \leq t$ let (D''_i, φ''_i) be a nowhere-zero flow on the directed cycle C''_i (choose D''_i such that f'_i receives the same direction as in D'_i) with $\varphi''_i(e) = 1$ for all $e \in \{f_i, f'_i\}$. Then

$$(D, \varphi) = \sum_{i=1}^{2t+\ell}(D_i, \varphi_i) + \sum_{i=1}^{s}(D'_i, \varphi'_i) + \sum_{i=1}^{t}(D''_i, \varphi''_i) \quad (2)$$

is a nowhere-zero 4-flow on M_G.

By Theorem 3, there is a balanced valuation $w'(v) = 2(2d^+_{D(M_G)}(v) - d_{M_G}(v))$ of M_G. It holds that $|2d^+_{D(M_G)}(v) - d_{M_G}(v)| = 1$, and hence $w'(v) \in \{\pm 2\}$ for all $v \in V$. The vertices of M_G (and therefore of G as well) are partitioned into two classes $A = \{v | w'(v) = -2\}$ and $B = \{v | w'(v) = 2\}$. Call the elements of A (B) the white (black) vertices of M_G and of G, respectively.

Let $G = (V, E)$ be a bridgeless cubic graph. A partition of V into two classes A and B constructed as above, and using a 2-factor $\mathcal{F}_2$, a canonical 4-coloring c of G, the 4-flow (D, φ) on M_G and the induced balanced valuation w' of M_G is called a *flow partition of* G, and it is denoted by $P_G(A, B) = P_G(A, B, \mathcal{F}_2, c, (D, \varphi), w')$. If we refer to a special 2-factor $\mathcal{F}_2$, we say $P_G(A, B)$ is a flow partition of G with respect to $\mathcal{F}_2$. For $X \subseteq V$ let $A_X = A \cap X$ ($B_X = B \cap X$) be the set of the white (black) vertices of X, and $a_X = |A_X|$, $b_X = |B_X|$. If we consider the vertex set $V(F)$ of a subgraph F of a graph G we also write a_F instead of $a_{V(F)}$ (b_F instead of $b_{V(F)}$).

We will prove some properties of flow partitions of cubic graphs. The following lemma is a direct consequence of the construction of (D, φ) on M_G.

Lemma 1. *Let $P_G(A, B, \mathcal{F}_2, c, (D, \varphi), w')$ be a flow partition of a bridgeless cubic graph $G = (V, E)$, and $xy = e \in E$. If the canonical 4-coloring c colors e with 1or 2, then x and y belong to different classes, i.e. $x \in A$ if and only if $y \in B$.*

Lemma 2. *Let $G = (V, E)$ be a cubic bridgeless graph and $P_G(A, B)$ be a flow partition with respect to a 2-factor $\mathcal{F}_2$. Let $S \subseteq V$ be a set of vertices such that the induced subgraph $G[S]$ is connected, n be the number of edges which have to be removed from $G[S]$ to obtain a spanning tree of $G[S]$, and let n_o be the number of odd cycles of $\mathcal{F}_2$ which are subgraphs of $G[S]$. Then $b_S \leq 4a_S + 3 - 3n + n_o$.*

Proof. Let $P_G(A, B) = P_G(A, B, \mathcal{F}_2, c, (D, \varphi), w')$ and F be a connected subgraph of $\mathcal{F}_2$. We show:

(1) If F is an even cycle, then $b_F = a_F$.
(2) If F is an odd cycle, then $b_F \leq a_F + 1$.
(3) If F is a path, then $b_F \leq a_F + 3$.

Items (1) and (2) follow from Lemma 1 directly. We distinguish two cases to prove (3).

Case 1: The edges of F are colored with colors 2 and 3.

If $|E(F)| = 2l + 1$, then at least l edges are colored with color 2. Thus Lemma 1 implies that $a_F \geq l$. Since $|V(F)| = 2l + 2$ and $b_F = 2l + 2 - a_F$ it follows that $b_F \leq a_F + 2$.

If $|E(F)| = 2l$, then l edges are colored with color 2. Thus Lemma 1 implies that $a_F = l$. Since $|V(F)| = 2l + 1$ it follows that $b_F = a_F + 1$.

Case 2: F contains an edge of color 0.

By the definition of the coloring there is precisely one edge of color 0.

If the length of F is odd, say $2l + 1$, the first and the last edge of F are colored differently, and there are l edges of color 2. Thus Lemma 1 implies that $a_F \geq l$. Since $|V(F)| = 2l + 2$ it follows that $b_F \leq a_F + 2$.

If $|E(F)| = 2l$, then at least $l - 1$ edges are colored 2. Thus Lemma 1 implies that $a_F \geq l - 1$. Since $|V(F)| = 2l + 1$ it follows that $b_F \leq a_F + 3$. ◦

Let E_1 be the set of edges of color 1 of $G[S]$. By Lemma 1, $|E_1| \leq a_S$. Let $E_1^- \subset E_1$ be a set of edges so that $G[S] - E_1^-$ is connected and no edge of color 1 (in $G[S] - E_1^-$) is contained in a cycle. Each cycle of $G[S] - E_1^-$ is a cycle of $\mathcal{F}_2$. Remove from each cycle precisely one edge of color 2 to obtain a spanning tree of $G[S]$. Let E_2^- be the set of these removed edges of color 2. With $n_i = |E_i^-|$ $(i = 1, 2)$ it follows that $n = n_1 + n_2$.

Let $Z_0, \dots, Z_{a'}$ be the components of $G[S] - E_1$, and a_i (b_i) be the number of white (black) vertices in Z_i, $i = 0, \dots, a'$. Each component is either a cycle of $\mathcal{F}_2$ or a subpath of a cycle of $\mathcal{F}_2$. The number of components is smaller than or equal to 1 plus the number of edges of color 1 in $G[S] - E_1^-$, therefore $a' \leq a_S - n_1$. Furthermore $\sum_{i=0}^{a'} a_i = a_S$.

For $i \in I_P = \{0, 1, \dots, a' - n_2\}$ let Z_i be a path, for $i \in I_C^o = \{a' - n_2 + 1, \dots, a' - n_2 + n_o\}$ let Z_i be an odd cycle, and for $i \in I_C^e = \{a' - n_2 + n_o + 1, \dots, a'\}$ let Z_i be an even cycle. Then it follows with $a' \leq a_S - n_1$ that

$$\begin{aligned}
b_S &= \sum_{i \in I_P} b_i + \sum_{i \in I_C^o} b_i + \sum_{i \in I_C^e} b_i \\
&\leq \sum_{i \in I_P} (a_i + 3) + \sum_{i \in I_C^o} (a_i + 1) + \sum_{i \in I_C^e} a_i \\
&= 3(a' - n_2 + 1) + n_o + \sum_{i=0}^{a'} a_i \\
&\leq 3(a_S - (n_1 + n_2) + 1) + n_o + a_S \\
&= 4a_S + 3 - 3n + n_o. \quad \square
\end{aligned}$$

We finish this section with the following lemma.

Lemma 3. *Let $P_G(A, B)$ be a flow partition of a cubic bridgeless graph $G = (V, E)$. Let $S \subseteq V$ be a set of vertices such that the induced subgraph $G[S]$ is connected, and n be the number of edges which have to be removed from $G[S]$ to obtain a spanning tree T of $G[S]$. Assume $a_S \leq b_S$, then $b_S \leq 4a_S + 3 - 3n$ if and only if $\frac{5}{3}(b_S - a_S) \leq |\partial_G(S)|$.*

Proof. Consider a spanning tree $T = (S, E(T))$ of $G[S]$ and let $T_i = \{v | v \in S \text{ and } d_T(v) = i\}$, for $i = 1, 2$. Then $|\partial_G(S)| + 2n = 2|T_1| + |T_2|$ and

$$|S| - 1 = |E(T)| = \frac{1}{2}(3(|S| - (|T_1| + |T_2|)) + 2|T_2| + |T_1|) = \frac{1}{2}(3|S| - |\partial_G(S)| - 2n).$$

Since $|S| = a_S + b_S$ it follows that $|\partial_G(S)| = a_S + b_S + 2 - 2n$, and hence $\frac{5}{3}(b_S - a_S) \leq |\partial_G(S)|$ is equivalent to $b_S \leq 4a_S + 3 - 3n$. $\square$

3. Proof of Theorem 2

Let $G = (V, E)$ be a bridgeless cubic graph with oddness ω. If $\omega \in \{0, 2\}$, then G has a nowhere-zero 5-flow, cf. [3]. Thus we may assume that $\omega \geq 4$.

Let $\mathcal{F}_2$ be a minimum 2-factor of G with $m_G(\mathcal{F}_2) = m^*_G \geq \frac{5}{2}\omega - 3$. Let $P_G(A, B) = P_G(A, B, \mathcal{F}_2, c, (D, \varphi), w')$ be a flow partition of G with respect to $\mathcal{F}_2$. Let $w : V \to \{\pm\frac{5}{3}\}$ be a function with $w(v) = -\frac{5}{3}$ if $v \in A$ and $w(v) = \frac{5}{3}$ if $v \in B$. We will show that w is a balanced valuation of G. Then it follows from Theorem 3 that G has a nowhere-zero 5-flow.

Assume to the contrary that w is not balanced. Then there is $S \subseteq V$ with

$$\left|\sum_{v\in S} w(v)\right| > |\partial_G(S)|. \tag{3}$$

If $S = V$, then $|\sum_{v\in S} w(v)| = 0 = |\partial_G(S)|$, and therefore S is a proper subset of V. Let S be of minimum order, so we may assume that $G[S]$ is connected, and without loss of generality $b_S \geq a_S$. With $k = b_S - a_S$ Eq. (3) becomes

$$\frac{5}{3}k > |\partial_G(S)|. \tag{4}$$

We show

Proposition 1. $|\partial_G(S)| \leq \frac{5}{2}\omega - 4$; *in particular*

(1) $|\partial_G(S)| \leq \frac{5}{2}\omega - 4$, *if* $|\partial_G(S)| \equiv 1 \bmod 5$,
(2) $|\partial_G(S)| \leq \frac{5}{2}\omega - 8$, *if* $|\partial_G(S)| \equiv 2 \bmod 5$,
(3) $|\partial_G(S)| \leq \frac{5}{2}\omega - 7$, *if* $|\partial_G(S)| \equiv 3 \bmod 5$,
(4) $|\partial_G(S)| \leq \frac{5}{2}\omega - 11$, *if* $|\partial_G(S)| \equiv 4 \bmod 5$,
(5) $|\partial_G(S)| \leq \frac{5}{2}\omega - 15$, *if* $|\partial_G(S)| \equiv 0 \bmod 5$.

Proof. For $i = 0, 1, 2, 3$ let $E_i \subset E$ be the set of the edges of color i in G and let $c_i = |\partial_G(S) \cap E_i|$. The edges of color 1 form a 1-factor of G. Thus Lemma 1 implies that $k = c_1$ and hence $c_1 > \frac{3}{5}|\partial_G(S)|$ by Eq. (4).

Let l^a_S (l^b_S) be the number of white (black) vertices of S where color 2 is missing, with respect to c. Let $l = |l^b_S - l^a_S|$. From $0 \leq l^a_S, l^b_S \leq \frac{1}{2}\omega$ it follows that $l \leq \frac{1}{2}\omega$, and Lemma 1 implies that $k \leq c_2 + l$. Hence $c_2 + \frac{1}{2}\omega \geq k > \frac{3}{5}|\partial_G(S)|$.

(1) If $|\partial_G(S)| \equiv 1 \bmod 5$, say $|\partial_G(S)| = 5m + 1$, then it follows that $c_1 \geq 3m + 1$ and therefore $c_2 \leq 2m$. Thus $\frac{1}{2}\omega \geq 3m + 1 - c_2 \geq 3m + 1 - 2m = m + 1$ and hence $\frac{5}{2}\omega - 4 \geq |\partial_G(S)|$.
(2) Can be proved analogously.
(3) If $|\partial_G(S)| \equiv 3 \bmod 5$, say $|\partial_G(S)| = 5m + 3$, then it follows that $c_1 \geq 3m + 2$ and therefore $c_2 \leq 2m + 1$.

If $c_2 = 2m + 1$, then $c_1 \leq |\partial_G(S)| - c_2 = 3m + 2$ and hence $c_1 = 3m + 2$ and $c_0 = c_3 = 0$. Let X be the set of vertices of $G[S]$ which are incident (in G) to an edge of $|\partial_G(S) \cap E_2|$, and Y be the set of vertices which are incident to an edge of color 0 in $G[S]$. Color 2 or 3 is missing on each vertex of $X \cup Y$ and $Z = X \cap Y$ consists of those vertices of $G[S]$ where both colors, 2 and 3, are missing. Each vertex z of Z is incident to an edge $e = zz'$ of color 0 in $G[S]$. Furthermore, color 2 is missing and color 3 appears at z'. Therefore, for each vertex of $z \in Z$ there is precisely one vertex z' in $G[S]$ where only color 2 is missing. Since $|X| = c_2$ is odd and $c_0 = c_3 = 0$ it follows that the total number of vertices of $G[S]$ where either color 2 or color 3 is missing is odd. This is a contradiction, since every path induced by edges of colors 2 and 3 in $G[S]$ has precisely two end vertices in $G[S]$.

Therefore $c_2 \leq 2m$ and hence $c_2 + \frac{1}{2}\omega \geq 3m+2$ implies that $\frac{1}{2}\omega \geq 3m+2-2m = m+2$. Thus $\frac{5}{2}\omega - 7 \geq 5m+3 = |\partial_G(S)|$.
Items (4) and (5) can be proved analogously to (3). ∘

Since G has no edge cut with fewer than $\frac{5}{2}\omega - 3$ edges that separates two odd cycles of $\mathcal{F}_2$ it follows with Proposition 1 that $n_o = 0$. Hence $b_S \leq 4a_S + 3 - 3n$ by Lemma 2 and therefore $\frac{5}{3}k \leq |\partial_G(S)|$ by Lemma 3. This contradicts Eq. (4) and completes the proof. □

4. Remarks on r-flows

The notion of nowhere-zero flows can be extended to rational numbers. Let $1 \leq p \leq q$ be integers, and let φ be a function from the edge set E of the directed graph $G = (V, E)$ (with orientation D) into the rational numbers. (D, φ) is a *nowhere-zero* $\frac{q}{p} + 1$-*flow* on $G = (V, E)$ if $1 \leq \varphi(e) \leq \frac{q}{p}$ for all $e \in E$ and Eq. (1) is satisfied for all $v \in V$. The *circular flow number* $F_c(G)$ of G is the minimum number r such that G has a nowhere-zero r-flow.

Seymour [6] proved that every bridgeless graph has a nowhere-zero 6-flow. Some methods of this paper can be extended to the study of nowhere-zero r-flows on graphs. For instance, it can be proved that $F_c(G) < 6$ for all bridgeless cubic graphs G with $m^*_G \geq \frac{3}{2}\omega(G) + 1$.

References

[1] J.A. Bondy, Balanced colourings and graph orientation, Congr. Numer. XIV (1975) 109–114.
[2] F. Jaeger, Balanced valuations and flows in multigraphs, Proc. Amer. Math. Soc. 55 (1975) 237–242.
[3] F. Jaeger, Nowhere-zero flow problems, in: L.W. Beineke, R.J. Wilson (Eds.), in: Topics in Graph Theory, vol. 3, Academic Press, London, 1988, pp. 70–95.
[4] M. Kochol, Reduction of the 5-flow conjecture to cyclically 6-edge-connected snarks, J. Combin. Theory, Ser. B 90 (2004) 139–145.
[5] M. Kochol, Restrictions on smallest counterexamples to the 5-flow conjecture, Combinatorica 26 (2006) 83–89.
[6] P.D. Seymour, Nowhere-zero 6-flows, J. Comb. Theory Ser. B 30 (1981) 130–135.
[7] W.T. Tutte, On the embedding of linear graphs in surfaces, Proc. London Math. Soc. Ser. 2 51 (1949) 474–489.
[8] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1954) 80–91.