Journal of Multivariate Analysis **64**, 196–205 (1998)
Article No. MV971715

A Unified and Broadened Class of Admissible Minimax Estimators of a Multivariate Normal Mean

Yuzo Maruyama*

University of Tokyo, Tokyo, Japan
E-mail: maruyama@e.u-tokyo.ac.jp

Received January 8, 1997; revised August 12, 1997

The problem of estimating the mean of a multivariate normal distribution is considered. A class of admissible minimax estimators is constructed. This class includes two well-known classes of estimators, Strawderman's and Alam's. Further,

generalized Bayes, quadratic loss.

1. INTRODUCTION

Let X be a random variable having p-variate normal distribution $N_p(\theta, I_p)$. Then we consider the problem of estimating the mean vector θ by $\delta(X)$ relative to the quadratic loss function $\|\delta(X)-\theta\|^2$. The usual estimator X, which has constant risk, is minimax. Stein (1956) showed that equivariant estimators relative to an orthogonal transformation group are of the forms $\delta_\phi(X)=(1-\phi(\|X\|^2)/\|X\|^2)X$ and that there exists an estimator dominating X among these when $p \geqslant 3$. James and Stein (1961) explicitly constructed the improved estimator $\delta^{JS}(X)=(1-(p-2)/\|X\|^2)X$. Moreover it turns out that $\delta^{JS}(X)$ is inadmissible since its positive-part estimator is superior to $\delta^{JS}(X)$ as shown in Baranchik (1964). In view of frequentist decision theory, the construction of an admissible estimator is very important. Therefore we are interested in characterizing a class of estimators satisfying two optimalities, that is, minimaxity and admissibility. It is noted that, in this problem, Brown (1971) showed any admissible estimators must be generalized Bayes. For the minimaxity of $\delta_\phi(X)$, Stein (1973) derived a general condition:

$$4\phi'(w)+\phi(w)\{2(p-2)-\phi(w)\}/w \geqslant 0, \tag{1.1}$$

* Research supported by research fellowships of the Japan Society for the Promotion of Science for Young Scientists.

0047-259X/98 $25.00

for every $w \geqslant 0$. This condition obviously includes Baranchik (1970)'s condition: $\phi(w)$ is monotone nondecreasing, and $0 \leqslant \phi(w) \leqslant 2(p-2)$. For many minimax estimators that have been proposed so far, $\phi(w)$ is monotone nondecreasing. We note, however, that Stein's condition allows $\phi(w)$ to decrease with increasing w. In fact, some such classes of minimax estimators have been obtained in Alam (1973), Efron-Morris (1976) and DasGupta and Strawderman (1997). We believe that the investigation of estimators that $\phi(w)$ is not monotone nondecreasing is more and more important because such estimators are considered as candidates for the solution of the more difficult problem in this field, that is, the problem of finding admissible estimators dominating the James-Stein positive-part rule. See also Efron and Morris (1976).

In this paper, we only deal with an well-known class of the generalized Bayes estimators $\delta^*_{a,b}(X) = (1 - \phi^*_{a,b}(\|X\|^2)/\|X\|^2)\, X$, where

$$\phi^*_{a,b}(w) = w\,\frac{\int_0^1 \lambda^{p/2-a+1}(1-\lambda)^b \exp(-\frac{1}{2}w\lambda)\, d\lambda}{\int_0^1 \lambda^{p/2-a}(1-\lambda)^b \exp(-\frac{1}{2}w\lambda)\, d\lambda}.$$

Strawderman (1971) and Berger (1976) showed that $\delta^*_{a,b}(X)$ for $3 - p/2 \leqslant a \leqslant 2$ and $b = 0$ is admissible and minimax. Alam (1973) also proposed the class of minimax admissible estimators, which has been known as a class different from Strawderman's. We verify, however, that $\delta^*_{a,b}(X)$ for $a = b+2$ and some value of $-1 < b < 0$ coincides with Alam's class. Further we show that $\delta^*_{a,b}(X)$ for $3 - p/2 \leqslant a \leqslant 2$ and $b \geqslant -(a - 3 + p/2)/p$ is the admissible minimax estimator and that $\phi^*_{a,b}(w)$ for $-1 < b < 0$ is not monotone nondecreasing. Off course, Strawderman's and Alam's class are both in this class of admissible minimax estimators.

2. THE MAIN RESULT

2.1. *The Unified Class of the Generalized Bayes Estimators*

First of all, we derive the class of generalized Bayes estimators $\delta^*_{a,b}(X)$. Let the conditional distribution of θ given $\lambda, 0 < \lambda < 1$, be normal with mean 0 and covariance matrix $\lambda^{-1}(1-\lambda)\, I_p$ and a density function of λ is proportional to $\lambda^{-a}(1-\lambda)^b\, I_{(0,1)}(\lambda)$. We assume that $b > -1$. Therefore it is noted that the above prior distribution is proper for $a < 1$ and is improper for $a \geqslant 1$. This idea of the hierarchical prior distribution is from Strawderman (1971) and is broadened by Faith (1978). The generalized Bayes estimator with respect to the above distribution of θ is given by

$$\delta(X) = E(\theta \mid X) = [1 - E(\theta \mid X)]\, X.$$

The joint distribution of λ and X is given by

$$g(\lambda, x) \propto \int \exp\left(-\frac{\|x-\theta\|^2}{2}\right) \cdot \left(\frac{\lambda}{1-\lambda}\right)^{p/2} \cdot \exp\left(-\frac{\lambda}{1-\lambda}\frac{\|\theta\|^2}{2}\right) \cdot \lambda^{-a}(1-\lambda)^b \, d\theta$$
$$\propto \int \exp\left(-\frac{\|\theta-(1-\lambda)x\|^2}{2(1-\lambda)} - \frac{\|x\|^2\lambda}{2}\right) \cdot \left(\frac{\lambda}{1-\lambda}\right)^{p/2} \lambda^{-a}(1-\lambda)^b \, d\theta$$
$$\propto \exp\left(-\frac{1}{2}\|x\|^2 \lambda\right) \cdot \lambda^{p/2-a}(1-\lambda)^b.$$

Therefore if $a < p/2+1$ and $b > -1$, the marginal density of X given by

$$m(x) \propto \int_0^1 \lambda^{p/2-a}(1-\lambda)^b \exp(-\tfrac{1}{2}\|x\|^2 \lambda) \, d\lambda \tag{2.1}$$

exists for all x and the posterior density is well defined. Then, we can get

$$E(\lambda \mid X) = \frac{\int_0^1 \lambda^{p/2-a+1}(1-\lambda)^b \exp(-\frac{1}{2}\|X\|^2 \lambda) \, d\lambda}{\int_0^1 \lambda^{p/2-a}(1-\lambda)^b \exp(-\frac{1}{2}\|X\|^2 \lambda) \, d\lambda},$$

which yields the generalized Bayes estimator $\delta^*_{a,b}(X) = (1-\phi^*_{a,b}(\|X\|^2)/\|X\|^2) X$, where

$$\phi^*_{a,b}(w) = w \frac{\int_0^1 \lambda^{p/2-a+1}(1-\lambda)^b \exp(-\frac{1}{2}w\lambda) \, d\lambda}{\int_0^1 \lambda^{p/2-a}(1-\lambda)^b \exp(-\frac{1}{2}w\lambda) \, d\lambda}.$$

Now we present $\phi^*_{a,b}(w)$ through the confluent hypergeometric function

$$M(a, b, x) = 1 + ax/b + \cdots + (a)_n x^n/(b)_n n! + \cdots,$$

where $(a)_n = a \cdot (a+1) \cdots (a+n-1)$, $n \geqslant 1$ and $(a)_0 = 1$. Using the following relations due to Abramowitz and Stegun (1964),

$$\Gamma(b-a)\,\Gamma(a)(\Gamma(b))^{-1} M(a, b, x) = \int_0^1 e^{xt} t^{a-1}(1-t)^{b-a-1} \, dt, \qquad \text{for} \quad b > a > 0, \tag{2.2}$$

$$M(a, b, x) = e^x M(b-a, b-x) \tag{2.3}$$

and

$$bM(a, b, x) - bM(a-1, b, x) - xM(a, b+1, x) = 0, \tag{2.4}$$

we obtain

$$\phi^*_{a,b}(w) = w\,\frac{p/2-a+1}{p/2-a+b+2}\,\frac{M(b+1,\, p/2-a+b+3,\, w/2)}{M(b+1,\, p/2-a+b+2,\, w/2)}$$

$$= (p-2a+2)(1-g(w)),$$

where $g(w) = M(b, c, w/2)/M(b+1, c, w/2)$ and $c = p/2 - a + b + 2$. This representation of $\phi^*_{a,b}(w)$ shows that $\delta^*_{a,b}(X)$ for $a = b+2$ and some value of $-1 < b < 0$ coincides with Alam (1973)'s class of generalized Bayes, admissible minimax estimators. Thus we can deal with Strawderman's class and Alam's class simultaneously. For the investigation in Section 2.2, we need to get the properties of the behavior of $\phi^*_{a,b}(w)$. The next results are a generalization of Alam (1973)'s and Takada (1979)'s.

Theorem 2.1.

(A) $\lim_{w\to\infty} \phi^*_{a,b}(w) = p - 2a + 2.$

(B) $\lim_{w\to\infty} w \cdot d/dw\, \phi^*_{a,b}(w) = 0.$

(C) *For* $b \geqslant 0$, $\phi^*_{a,b}(w)$ *is monotone increasing.*

(D) *For* $-1 < b < 0$, $\phi^*_{a,b}(w)$ *is increasing from the origin to a certain point and is decreasing from the point.*

Proof. The following two formula due to Abramowitz and Stegun (1964) and the lemma are useful for our proof:

$$M(a, b, x) = \Gamma(b)(\Gamma(a))^{-1}\, e^x x^{a-b}(1 + O(|x|^{-1})), \qquad \text{for large} \quad x, \tag{2.5}$$

$$\frac{d}{dx} M(a, b, x) = \frac{a}{b} M(a+1, b+1, x) \tag{2.6}$$

and

Lemma 2.1. *Let* $h(y) = (\sum_{i=0}^{\infty} d_i y^i) \,/\, (\sum_{i=0}^{\infty} c_i y^i)$ *where* d_i, c_i *are non-negative, and* $\sum_{i=0}^{\infty} d_i y^i$ *and* $\sum_{i=0}^{\infty} c_i y^i$ *converges for all* $y > 0$. *If the sequence* $\{d_i/c_i\}$ *is monotone increasing (decreasing), then* $h(y)$ *is monotone increasing (decreasing) in* y. *(See Lehmann (1986) p. 428).*

To prove (A), it is sufficient to show that $\lim_{w\to\infty} g(w) = 0$, which is easily verified by (2.5).

To prove (B), observe that a calculation using (2.4) and (2.6) gives

$$w \frac{d}{dw} \phi^*_{a,b}(w) = (p - 2a + 2) \left[(b+1)\, g(w) \frac{M(b+2, c, w/2)}{M(b+1, c, w/2)} - b - g(w) \right].$$

By (2.5), we have the conclusion.

To prove (C), we have only to show $g(w)$ is monotone decreasing in w, which is guaranteed by Lemma 2.1.

To prove (D), it is sufficient to show $g(w)$ is decreasing from the origin to a certain point and is increasing from the point. Let $f(w) = M(b+1, c, w/2)$, and

$$h(w) = \frac{(d/dw)\, M(b, c, w/2)}{(d/dw)\, M(b+1, c, w/2)} = \frac{b}{b+1} \frac{M(b+1, c+1, w/2)}{M(b+2, c+1, w/2)},$$

where second equality is from (2.6). Hence we have $g'(w) = f'(w)(f(w))^{-1}$ $\{h(w) - g(w)\}$. By Lemma 2.1, we show that $h(w)$ is strictly increasing in w from $h(0) = b/(b+1)$ to $h(\infty) = -0$. Since $f'(w) > 0$ and $f(w) > 0$, at first $g(w)$ is decreasing up to the point w_0 which satisfies $h(w) = g(w)$. We are now in position to show that $g(w)$ is increasing when $w > w_0$, that is $h(w) > g(w)$ for $w > w_0$. Our proof is by contradiction. Suppose that there exists a point $w_1 (> w_0)$ such that $h(w) \leqslant g(w)$. Since the differential coefficient of $g(w)$ at w_1 is non-positive, we have $h(w_1 - \varepsilon_1) < g(w_1 - \varepsilon_1)$ for sufficiently small $\varepsilon_1 > 0$. It is noted that since $h'(w_0) > g'(w_0)$ for sufficiently small $\varepsilon_0 > 0$, we have $h(w_0 + \varepsilon_0) > g(w_0 + \varepsilon_0)$. Hence, by intermediate value theorem and by noting that $h(w)$ is strictly increasing, there exists at least one point w_2 in $[w_0 + \varepsilon_0, w_1 - \varepsilon_1]$, which satisfies both $g(w_2) = h(w_2)$ and $g'(w_2) > 0$. This result contradicts the fact $g'(w) = f'(w)(f(w))^{-1}$ $\{h(w) - g(w)\}$, which completes the proof.

2.2. *Admissibility and Minimaxity*

In this section, we obtain conditions on the admissibility and the minimaxity of the generalized Bayes estimator $\delta^*_{a,b}(X)$. Toward the admissibility of $\delta^*_{a,b}(X)$, Brown (1971)'s result is used as the main tool and this is stated in the following.

THEOREM 2.2. *Suppose that $\delta(X)$ is the generalized Bayes estimator with respect to a spherically symmetric prior $F(\theta)$. Hence the marginal density of X is given by*

$$m(x) = \int \exp\left(-\frac{\|x - \theta\|^2}{2} \right) dF(\theta) = m^*(\|x\|).$$

Then if

$$\int_1^\infty (r^{p-1}m^*(r))^{-1}\,dr < \infty,$$

$\delta(X)$ is inadmissible. If this integral is infinite and $r(x)=\delta(x)-x$ is bounded then $\delta(X)$ is admissible.

Using this theorem, we have the following.

THEOREM 2.3. *$\delta^*_{a,b}(X)$ is admissible if and only if $a \leqslant 2$ and $b > -1$.*

Proof. For the estimator $\delta^*_{a,b}(X)$, we have $r(x) = -\phi^*_{a,b}(\|x\|^2)\{x/\|x\|^2\}$. Since $\phi^*_{a,b}(w)$ is a bounded function by Theorem 2.1 and $r(0)$ is a zero vector, we obtain $\|r(x)\| < M$, for some suitable constant M.

Next we will determine whether or not the value of the integral above is infinity. Applying (2.2) and (2.3) to (2.1), we have

$$\begin{aligned} m^*(r) &= \int_0^1 \lambda^{p/2-a}(1-\lambda)^b \exp(-\tfrac{1}{2}r^2\lambda)\,d\lambda \\ &= Ce^{-(r^2/2)}M(b+1,\, p/2-a+b+2,\, r^2/2). \end{aligned}$$

We note that by (2.5) as $r \to \infty$,

$$M(b+1,\, p/2-a+b+2,\, r^2/2) = C_1 \cdot e^{(r^2/2)}(r^2)^{-p/2+a-1}\,(1+O(r^{-2})).$$

Therefore there exists an L such that $r > L$ implies

$$C_2 \cdot r^{-p+2a-2} < m^*(r) < C_3 \cdot r^{-p+2a-2},$$

for suitable values C_2 and C_3. Hence, in the case of $a > 2$, we have

$$\int_1^\infty \frac{1}{r^{p-1}m^*(r)}\,dr < \int_1^L \frac{1}{r^{p-1}m^*(r)}\,dr + \frac{1}{C_2}\int_L^\infty r^{-2a+3}\,dr < \infty.$$

Next we have the inequality

$$\int_1^\infty (r^{p-1}m^*(r))^{-1}\,dr > (1/C_3)\int_L^\infty r^{-2a+3}\,dr.$$

If $-2a+3 \geqslant -1$, the r.h.s of above inequality diverges. This completes the proof.

Next for the minimaxity of $\delta^*_{a,b}(X)$, we have the following.

THEOREM 2.4. *$\delta^*_{a,b}(X)$ for $3-p/2 \leqslant a < 1+p/2$ and $b \geqslant -(a+p/2-3)/p$ is minimax.*

Combining Theorem 2.3 and Theorem 2.4 shows that $\delta^*_{a,b}(X)$ for $3-p/2 \leqslant a \leqslant 2$ and $b \geqslant -(a+p/2-3)/p$ is the admissible minimax estimator. Off course, this class includes Strawderman-Berger's class(for $3-p/2 \leqslant a \leqslant 2$ and $b=0$) and Alam's class (for $a=b+2$ and $(2p+1-(4p^2+8p-7)^{1/2})/4 < b < 0$). In particular, the estimator in this class for $b<0$ is admissible minimax estimator and satisfies that $\phi(w)$ is not nondecreasing.

Proof. By (A) and (B) of Theorem 2.1, we have

$$\lim_{w \to \infty} \left[4w \frac{d}{dw} \phi^*_{a,b}(w) + \phi^*_{a,b}(w)\{2(p-2) - \phi^*_{a,b}(w)\} \right]$$
$$= -4(a-p/2-1)(a+p/2-3),$$

which implies that we have only to consider the case for $3-p/2 \leqslant a < 1+p/2$ and $b > -1$. For $b \geqslant 0$ and $3-p/2 \leqslant a < 1+p/2$, by (A) and (C) of Theorem 2.1, $\phi^*_{a,b}(w)$ satisfies Baranchik's condition. Thus, in this case, $\delta^*_{a,b}(X)$ is minimax. It is noted that this result has already been derived in Faith (1978). For $-1<b<0$ and $3-p/2 \leqslant a < 1+p/2$, $\phi^*_{a,b}(w)$ is not monotone nondecreasing as claimed in (D) of Theorem 2.1. Thus, by Stein's condition (1.1), we would like to get conditions that

$$S(a,b,w) = \left[4 \frac{d}{dw} \phi^*_{a,b}(w) + \phi^*_{a,b}(w)\{2(p-2) - \phi^*_{a,b}(w)\}/w \right]$$

is nonnegative for every $w \geqslant 0$. Letting $c = p/2 - a + b + 2$ and $x = w/2$, after some calculation, we obtain

$$\begin{aligned} S(a,b,w) = \frac{2(c-b-1)}{cM^2(b+1,c,x)} & [(p-c-b-1)\, M(b+1,c+1,x)\, M(b+1,c,x) \\ & + (c-b-1)\, M(b+1,c+1,x)\, M(b,c,x) \\ & + 2(b+1)\, M(b+2,c+1,x)\, M(b,c,x)]. \end{aligned} \tag{2.7}$$

The following calculation is mainly based on the method of Alam (1973). Using the series expansion for the confluent hypergeometric function, we have

$$M(b+1, c+1, x)\, M(b+1, c, x)$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\gamma=0}^{n} \binom{n}{\gamma} \frac{(b+1)_\gamma\, (b+1)_{n-\gamma}}{(c+1)_\gamma\, (c)_{n-\gamma}}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\gamma=0}^{n} \binom{n}{\gamma} \frac{(b+1)_\gamma\, (b+1)_{n-\gamma}\, c}{(c)_\gamma\, (c)_{n-\gamma}\, (c+n-\gamma)}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\gamma=0}^{[n/2]} \binom{n}{\gamma} \frac{(b+1)_\gamma\, (b+1)_{n-\gamma}}{(c)_\gamma\, (c)_{n-\gamma}}\, cq_\gamma \left[\frac{1}{c+\gamma} + \frac{1}{c+n-\gamma}\right], \tag{2.8}$$

where $[n/2]$ denotes the largest integer less than or equal to $n/2$, $q_\gamma = 1$ for $\gamma < n/2$ and $q_\gamma = 1/2$ for $\gamma = n/2$. Similarly,

$$M(b+1, c+1, x)\, M(b, c, x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\gamma=0}^{[n/2]} \binom{n}{\gamma} \frac{(b+1)_\gamma\, (b+1)_{n-\gamma}}{(c)_\gamma\, (c)_{n-\gamma}}\, cq_\gamma \times b\left[\frac{1}{(c+\gamma)(b+n-\gamma)} + \frac{1}{(c+n-\gamma)(b+\gamma)}\right], \tag{2.9}$$

$$M(b+2, c+1, x)\, M(b, c, x)$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\gamma=0}^{[n/2]} \binom{n}{\gamma} \frac{(b+1)_\gamma\, (b+1)_{n-\gamma}}{(c)_\gamma\, (c)_{n-\gamma}}\, cq_\gamma \frac{b}{b+1}\left[\frac{b+1+\gamma}{(c+\gamma)(b+n-\gamma)} + \frac{b+1+n-\gamma}{(c+n-\gamma)(b+\gamma)}\right]. \tag{2.10}$$

Combining (2.7), (2.8), (2.9) and (2.10), we can see that $S(a, b, w)$ is nonnegative if

$$T(c, b, n, \gamma) = (c-b-1)\, b\left[\frac{1}{(c+\gamma)(b+n-\gamma)} + \frac{1}{(c+n-\gamma)(b+\gamma)}\right] + 2(b+1)\frac{b}{b+1}\left[\frac{b+1+\gamma}{(c+\gamma)(b+n-\gamma)} + \frac{b+1+n-\gamma}{(c+n-\gamma)(b+\gamma)}\right] + (p-c-b-1)\left[\frac{1}{c+\gamma} + \frac{1}{c+n-\gamma}\right] \geqslant 0,$$

for each $\gamma = 0, 1, ..., [n/2]$ and for each $n = 0, 1, ...$ Now $n = 0$ implies $\gamma = 0$ clearly and we have $T(c, b, 0, 0) = 2p/c > 0$. Thus we deal with the case for $n \geqslant 1$. Note that $n \geqslant 1$ implies $n - \gamma \geqslant 1$. We can arrange $T(c, b, n, \gamma)$ as

$$T(c, b, n, \gamma) = (p - c - 1)\frac{2c+n}{(c+\gamma)(c+n-\gamma)} + b\left[\frac{c+1-n+3\gamma}{(c+\gamma)(b+n-\gamma)} + \frac{c+1+2n-3\gamma}{(c+n-\gamma)(b+\gamma)}\right].$$

The quantity inside the braces on the right-hand side can be written as

$$\begin{aligned}
&\left[\frac{c-n+3\gamma}{(c+\gamma)(b+n-\gamma)} + \frac{c+2n-3\gamma}{(c+n-\gamma)(b+\gamma)}\right. \\
&\qquad \left. + \frac{1}{(c+\gamma)(b+n-\gamma)} + \frac{1}{(c+n-\gamma)(b+\gamma)}\right] \\
&= \frac{1}{(c+\gamma)(c+n-\gamma)}\left[\frac{c+n-\gamma}{b+n-\gamma}(c-n+3\gamma) + \frac{c+\gamma}{b+\gamma}(c+2n-3\gamma)\right. \\
&\qquad \left. + \frac{c+n-\gamma}{b+n-\gamma} + \frac{c+\gamma}{b+\gamma}\right] \\
&\leqslant \frac{1}{(c+\gamma)(c+n-\gamma)}\left[\frac{c+1}{b+1}(c-n+3\gamma) + \frac{c+1}{b+1}(c+2n-3\gamma)\right. \\
&\qquad \left. + \frac{c+n-\gamma}{b+1} + \frac{c+\gamma}{b+1}\right] \\
&= \frac{2c+n}{(c+\gamma)(c+n-\gamma)}\frac{c+2}{b+1}.
\end{aligned}$$

Since $b < 0$, we have

$$T(c, b, n, \gamma) \geqslant \frac{2c+n}{(c+\gamma)(c+n-\gamma)}\left[p - c - 1 + \frac{b}{b+1}(c+2)\right],$$

which is nonnegative if $p - (p/2 - a + b + 2) - 1 + b(p/2 - a + b + 2 + 2)/(b+1) \geqslant 0$. In this case, $S(a, b, w)$ is also nonnegative, completing the proof.

ACKNOWLEDGMENTS

I thank Professor T. Kubokawa and the referee for many valuable comments and helpful suggestions that led to an improved version of the paper.

REFERENCES

1. Abramowitz, M., and Stegun, I. A. (1964). *Handbook of Mathematical Functions*. Dover, New York.
2. Alam, K. (1973). A family of admissible minimax estimators of the mean of a multivariate normal distribution. *Ann. Statist.* **1** 517–525.
3. Baranchik, A. J. (1964). Multiple regression and estimation of the mean of a multivariate normal distribution, Tech. Rep. 51, Stanford University.
4. Baranchik, A. J. (1970). A family of minimax estimators of the mean of a multivariate normal distribution. *Ann. Math. Statist.* **41** 642–645.
5. Berger, J. (1976). Admissible minimax estimation of multivariate normal mean with arbitrary quadratic loss. *Ann. Statist.* **4** 223–226.
6. Brown, L. D. (1971). Admissible estimators, recurrent diffusions, and insoluble boundary value problems. *Ann. Math. Statist.* **42** 855–903.
7. DasGupta, A., and Strawderman, W. E. (1997). All estimates with a given risk, Riccati differential equations and a new proof of a theorem of Brown. *Ann. Statist.* **25** 1208–1221.
8. Efron, B., and Morris, C. (1976). Families of minimax estimators of the mean of a multivariate normal distribution. *Ann. Statist.* **4** 11–21.
9. Faith, R. E. (1978). Minimax Bayes estimators of a multivariate normal mean. *J. Multivariate Anal.* **8** 372–379.
10. James, W., and Stein, C. (1961). Estimation of quadratic loss. In *Proc. 4th Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, pp. 361–379. Univ. of California Press, Berkeley.
11. Lehmann, E. L. (1986). *Testing Statistical Hypotheses*, 2nd ed. Wiley, New York.
12. Stein, C. (1956). Inadmissibility of the usual estimator for the mean of a multivariate normal distribution. In *Proc. 3rd Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, pp. 197–206. Univ. of California Press, Berkeley.
13. Stein, C. (1973). Estimation of the mean of a multivariate normal distribution. In *Proc. Prague Symposium Asymptotic Statistics*, pp. 345–381.
14. Strawderman, W. E. (1971). Proper Bayes minimax estimators of multivariate normal mean. *Ann. Statist.* **42** 385–388.
15. Takada, Y. (1979). Stein's positive part estimator and Bayes estimators. *Ann. Inst. Statist. Math.* **31** 177–183.