Small resolutions of nodal cubic threefolds

by Hans Finkelnberg and Jürgen Werner

Department of Mathematics, Niels Bohrweg 1, P.O. Box 9504, 2300 RA Leiden, the Netherlands

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INTRODUCTION

Let $V$ be a nodal hypersurface in $\mathbb{P}^4$. This means that $V$ has only isolated ordinary double points as singularities. Let $s = \#(V_{\text{Sing}})$. In local affine coordinates $z_1, z_2, z_3$ and $z_4$ on a small neighbourhood $U$ of $P \in V_{\text{Sing}}$ the threefold is given by the equation

$$\sum_{i=1}^{4} z_i^2 = 0.$$ 

Using other affine coordinates this can be written as

$$\varphi_1 \varphi_2 = \varphi_3 \varphi_4.$$

We shall consider two different ways to get a resolution of $P$:

I) The big resolution. We can blow up $P$ in the usual way: $\tilde{V} = B(V, P)$ is the blow-up of $V$ in $P$ (cf. [Ha]) and $\pi_1 : \tilde{V} \rightarrow V$ is the blow-up map. $\pi_1^{-1}(P) \cong \mathbb{P}^1 \times \mathbb{P}^1$. This kind of resolution is called "big" because $P$ is replaced by a surface.

II) The small resolution. This can be described in two ways:

a) In $\tilde{V}$, the big resolution, we can blow down the exceptional quadric $\pi_1^{-1}(P) \cong \mathbb{P}^1 \times \mathbb{P}^1$ in one of the two natural directions (both directions will do). The result is a resolution $\tilde{V}$ of $V$ in which $P$ is replaced by a curve isomorphic to $\mathbb{P}^1$. Therefore it is called a "small" resolution (cf. [Fi1] and [We]).
b) We can also use the local meromorphic function:

\[ \frac{\varphi_1}{\varphi_3} = \frac{\varphi_2}{\varphi_4}. \]

Outside the locus where \( \varphi_1 = \varphi_3 = 0 \), we have a map:

\[ \sigma: U \to \mathbb{P}^1 \]

\( \tilde{V} \) is defined as the closure of the graph of \( \sigma \) in \( U \times \mathbb{P}^1 \). In fact \( \tilde{V} \) is the strict transform of \( V \) in the blowing up of \( U \) in the surface \( \varphi_1 = \varphi_3 = 0 \). \( \tilde{V} \) is smooth and \( P \) is replaced by a \( \mathbb{P}^1 \) (cf. [Br]).

The other small resolution is obtained by using the other meromorphic function

\[ \frac{\varphi_1}{\varphi_4} = \frac{\varphi_3}{\varphi_2}. \]

So globally there are \( 2^5 \) small resolutions but only one big resolution.

It is the aim of this paper to investigate the small resolutions of nodal cubic hypersurfaces in \( \mathbb{P}^4 \). In particular we shall be interested in their projectivity.

This has been done extensively in [Fi1] for the Segre cubic. I.e. the unique cubic in \( \mathbb{P}^4 \) with the maximal number of 10 ordinary double points.

In § 1 we shall state some results from [We] without proofs.

In § 2 we shall consider cubic hypersurfaces in general, we classify them in § 3 where we also consider the projectivity of their small resolutions.

This paper has been written while the second author stayed at the Max-Planck-Institut für Mathematik, Bonn. The mathematical environment in Bonn and in Leiden has been very stimulating.

§ 1. SMALL RESOLUTIONS OF NODAL HYPERSURFACES IN PROJECTIVE SPACE OF DIMENSION 4

Let \( V \subset \mathbb{P}^4 \) be a nodal hypersurface. The defect \( d \) of \( V \) is defined as:

\[ d = \beta_4(V) - \beta_2(V). \]

\( \beta_2(V) = 1 \), so for any small resolution \( \tilde{V} \) we find:

\[ \beta_2(\tilde{V}) = \beta_4(\tilde{V}) = \beta_4(V) = \beta_2(V) + d = 1 + d. \]

Since in general there exist non-trivial relations among the exceptional lines we have \( d \leq s := \# V_{\text{Sing}} \).

Let \( V_{\text{Sing}} = \{ P_1, \ldots, P_s \} \) and let \( L_i \) be the exceptional line over \( P_i \) in some small resolution \( \tilde{V} \) of \( V \). All \( s \)-tuples \( (n_1, \ldots, n_s) \) satisfying \( \sum n_iL_i = 0 \) in \( H_2(\tilde{V}, \mathbb{Z}) \) generate a vectorspace \( \mathcal{B} \subset \mathbb{Q}^s \) of dimension \( s - d \).

Let \( \gamma \in H_4(V, \mathbb{Z}) \) and \( \tilde{\gamma} \) be the strict transform of \( \gamma \) in \( \tilde{V} \). We define: \( m_i = \tilde{\gamma} \cdot L_i \). The \( s \)-tuples \( (m_1, \ldots, m_s) \) obtained by letting \( \gamma \) pass through \( H_4(V, \mathbb{Z}) \) generate a vectorspace \( \mathcal{A} \subset \mathbb{Q}^s \) of dimension \( d \). We find:

\[ \mathcal{Q}^s = \mathcal{A} \cup \mathcal{B}. \]
with respect to the standard Euclidean inner product. The exact decomposition of $Q^3$ into $\mathcal{A}$ and $\mathcal{B}$ depends on the chosen small resolution.

If we choose the other small resolution above $P_j$, then this induces:

$$m_j \mapsto -m_j, \text{ for all } (m_1, \ldots, m_s) \in \mathcal{A}$$

and so:

$$n_j \mapsto -n_j, \text{ for all } (n_1, \ldots, n_s) \in \mathcal{B}.$$ 

The projective small resolutions can be characterised as follows:

**Theorem 1.1.** Let $V$ and $\hat{V}$ be as above, then the following 3 statements are equivalent:

a) $\hat{V}$ is projective.

b) There is a divisor $D$ on $\hat{V}$ such that $D \cdot L_i > 0$ for all $i$.

c) If $\sum n_i L_i = 0$ in $H_2(\hat{V}, \mathbb{Z})$ and $n_i \geq 0 \forall i$ then $n_i = 0$ for all $i$.

Using theorem 1.1 we can characterise the nodal hypersurfaces in $\mathbb{P}^4$ that admit a projective small resolution:

**Lemma 1.2.** Let $V$ be as above, then:

There is a projective small resolution $\Rightarrow L_i \neq 0$ in $H_2(\hat{V}, \mathbb{Q}) \forall i$.

(This last statement is independent of the special choice of $\hat{V}$.)

In terms of $\mathcal{A}$ and $\mathcal{B}$ these results can be formulated as follows:

$\Rightarrow$ A small resolution is projective $\Rightarrow (m_1, \ldots, m_s) \in \mathcal{A}$ s.t. $m_i > 0 \forall i$.

$\Rightarrow$ There exists a projective small resolution of $V \Rightarrow e_i \in \mathcal{B} \forall i$, where $e_i$ is the $i$'th unit vector.

For proofs and further details we refer to [We].

§ 2. NODAL CUBIC THREEFOLDS

Let $K$ be a nodal cubic in $\mathbb{P}^4$ (i.e. $K$ has only isolated ordinary double points). Let $s$ be the number of singularities of $K$, $s \neq 0$. We can choose homogeneous coordinates $(\xi_0 : \ldots : \xi_4)$ on $\mathbb{P}^4$ such that $K$ is given by:

$$\xi_4 Q + R = 0,$$

where $Q$ (resp. $R$) is a homogeneous polynomial of degree 2 (resp. 3) in $\xi_0$, $\xi_1$, $\xi_2$ and $\xi_3$. $P = (0 : 0 : 0 : 1) \in K_{\text{Sing}}$ is called "the watchtower".

Since $P$ is an ordinary double point, $Q = 0$ defines a smooth quadric in $\mathbb{P}^3$, which is also denoted by $Q$. $R = 0$ defines a cubic surface in $\mathbb{P}^3$, not containing $Q$. This cubic surface is denoted by $R$.

The curve $S(K, P) \subset Q$ is defined to be the complete intersection of $Q$ and $R$. $S(K, P)$ is called "the associated curve" of $K$ with respect to $P$ (cf. [Ka], [Fi1], [Fi2]). Let $B(K, P)$ [resp. $B(\mathbb{P}^3, S(K, P))$] denote the blow-up of $K$ [resp. $\mathbb{P}^3$] in the complete intersection $P$ [resp. $S(K, P)$] (cf. [Ha]).
THEOREM 2.1. The natural projection $\pi : K \setminus P \to \mathbb{P}^3$ induces an isomorphism:

$$B(K, P) \cong B(\mathbb{P}^3, S(K, P)).$$

The proof of this theorem can be found in [Fi2].

From this it follows that:
\begin{itemize}
  \item $S(K, P)$ is reduced and has only ordinary double points (cf. [Fi2]).
  \item $\#(S(K, P)_{\text{Sing}}) = s - 1$.
  \item $S(K, P)_{\text{Sing}}$ corresponds bijectively to $K_{\text{Sing}} \setminus P$ via $\pi$.
  \item $\#(\text{irr. components of } S(K, P)) = 1 + d$. Let $S_0, \ldots, S_d$ denote the irreducible components of $S(K, P)$.
  \item $H_4(K, \mathbb{Q}) (\cong H_4(K, \mathbb{Q}))$, for all $K$ is generated by the $\pi^{-1}(S_j)$.
\end{itemize}

The associated curve which is of type $(3, 3)$ contains all information about the intersection numbers of the exceptional lines and divisors on any small resolution $\tilde{K}$. Let $\{P = P_1, \ldots, P_s\} = K_{\text{Sing}}$ and let $L_i$ denote the exceptional line over $P_1$.

LEMMA 2.2.
\begin{itemize}
  \item[a)] $\pi^{-1}(S_j) \cdot L_0 \in \{a - b, b - a\}$ if $S_j$ is of type $(a, b)$ on $Q$.
  \item[b)] If we assume that $s \geq 2$ and that $\pi PP_i) \in S_j$, for some $i \neq 1$, then:
  \item[c)] $\pi^{-1}(S_j) \cdot L_i = 0$ if and only if $S_j$ is singular in $\pi(P_i)$.
\end{itemize}

This lemma is a direct corollary of the following lemma:

LEMMA 2.3. Let $X \subset \mathbb{C}^4$ be defined by $x_1x_2 - x_3x_4 = 0$ and $\tilde{X}$ (resp. $\check{X}$) be a small (resp. big) resolution of $X$.

Let $L$ (resp. $E$) be the exceptional line (resp. quadric) on $\tilde{X}$ (resp. $\check{X}$).

Let $\pi_i$, $i = 1, 2, 3$, be the blow-up maps as indicated in the figure:

Assume that the line $(1, 0)$ on $E$ is blown down by $\pi_3$ and that the line $(0, 1)$ is not.

Let $\Delta$ be a divisor on $\check{X}$ such that $\Delta \cdot E = (a, b)$, then:

$$[(\pi_3)_*(\Delta)] \cdot L = a - b.$$
Since \( \mathcal{N}_{L,X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \) we find that \( c_1((\pi_3)^*(\mathcal{N}_{L,X})) = (-2,0) \) (on \( E \)). So for \((\pi_3)^*(L)\) we have:

\[
[(\pi_3)^*(\Delta)] \cdot L = [(\pi_3)^*(\pi_3)^*(\Delta)] \cdot [(\pi_3)^*(L)]
\]

\[
= [\Delta + aE] \cdot [(-2, 0) - (-1, -1)] \quad \text{(for some } a \in \mathbb{Z})
\]

\[
= [(\Delta + aE) \cdot E] \cdot [( -1, 1)]
\]

\[
= [(a - a, b - a)] \cdot [( -1, 1)]
\]

\[
= (a - b) + (a - a) = a - b.
\]

**LEMMA 2.4.** Let \( K, P_i, L_i, S_j \) and \( S(K, P) \) be as above, then:

There is a projective small resolution \( \tilde{K} \) of \( K \) if and only if every irreducible component \( S_j \) of \( S(K, P) \) is smooth and there is at least one \( S_j \) of type \((a, b)\) with \( a \neq b \).

**PROOF OF LEMMA 2.4.** Using the lemma’s 2.1, 2.2 and 2.3 it follows immediately that the conditions of lemma 2.4 after “if and only if” are equivalent with the condition that every exceptional \( \mathbb{P}^1 \) is non zero in \( H_2(K, \mathbb{Q}) \). This second condition is equivalent to the fact that there exists a projective small resolution \( \tilde{K} \) (= lemma 1.2).

Since every curve of type \((3,3)\) on a smooth quadric in \( \mathbb{P}^3 \) occurs as an associated curve of some cubic in \( \mathbb{P}^4 \) (cf. [Ka]) we can classify the nodal cubic threefolds in \( \mathbb{P}^4 \) by classifying reduced, nodal curves of type \((3,3)\) on a smooth quadric surface. Since the associated curve depends on the watchtower it may happen that two different curves of (total) type \((3,3)\) belong to the same cubic.

The classification is done according to the defect, the number of singularities and the number of singular points which are resolved by zero-homologous curves (modulo torsion).

§ 3. A CLASSIFICATION

In this paragraph we shall give a list of the nodal cubics in \( \mathbb{P}^4 \) and consider the following matters:

\( \rightarrow \) how many planes lie on them,

\( \rightarrow \) which 4-tuples of singular points are contained in planes,

\( \rightarrow \) how many of the \( 2^5 \) small resolutions are projective,

\( \rightarrow \) which are the projective small resolutions (if there aren’t too many),

\( \rightarrow \) which associated curves belong to the same cubic.

**REMARK 1.** The number of planes on a cubic and their positions can quite easily be derived from the associated curve:

a) every line in \( S(K, P) \) gives us a plane through the watchtower \( P \),

b) every \((1,1)\)-part in \( S(K, P) \) (reducible or irreducible) gives us a plane on \( K \), not passing through \( P \). This plane is found by intersecting \( K \) with the \( \mathbb{P}^3 \) spanned by the quadratic cone over the \((1,1)\)-part in \( S(K, P) \). Conversely, every
plane through $P$ gives a line in $S(K, P)$ and every plane not containing $P$ determines a $(1, 1)$-part of $S(K, P)$ (possibly reducible).

**Remark 2.** Let $\hat{K}$ be a projective small resolution of $K$, and let $S_0, \ldots, S_d$ be the irreducible (and smooth!, cf. lemma 2.4) components of $S(K, P)$. If $\pi^{-1}(S_i)$ meets the exceptional line over one singular point above a point of $S_i \cap S_j$ transversally, then $\pi^{-1}(S_i)$ meets all exceptional lines over the singular points above $S_i \cap S_j$ transversally. ($\pi^{-1}(S_i)$ is considered here as a divisor on $K$.)

**Definition.** A triangle on $S(K, P)$ is a triple $\{T_1, T_2, T_3\}$ of singular points of $S(K, P)$ such that these points are connected to each other by three different irreducible components of $S(K, P)$.

**Remark 3.** It is easily verified that in this case there are at most 6 different simultaneous small resolutions of the 3 double points of $K$ above the $T_i$ which fit in a projective $\hat{K}$.

**Remark 4.** Since $\beta_2(K) = 1$ every isomorphism between two small resolutions is induced by an automorphism of $K$. Every automorphism of $K$ is induced by $\text{Aut}(\mathbb{P}^3)$ (cf. [Ff1]). This may be used to verify whether two small resolutions are isomorphic or not.

$d = 0$

For any $K$ with $d = 0$, we have that all $L_i = 0$, so there are no projective small resolutions.

For $d = 0$ we have 5 classes $J_1, \ldots, J_5$. The associated curve is irreducible and contains 0, 1, 2, 3 or 4 ordinary double points. These cubics do not contain planes.

$d = 1$

For $d = 1$ we have 4 classes.

$J_6$: $S_0 = (2, 3), S_1 = (1, 0)$. $S_0$ is smooth.

**figure:**

![Diagram](attachment:diagram.png)
This decomposition is seen from every $P_i$.

$\#K_{\text{Sing}} = 4 \quad \#K = 16$
$\#\{\text{proj. } K\} = 2 \quad \#\{\text{planes on } K\} = 1.$

All 4 singular points of $K$ lie in a plane.

The two projective small resolutions are the ones in which the strict transform of the plane meets all exceptional lines transversally OR contains them all (= remark 2). In both cases all exceptional lines are homologous.

The two projective small resolutions are not isomorphic. There are at least 3 isomorphism classes of non-projective small resolutions.

$J7$: $S_0 = (2, 3), S_1 = (1, 0)$. $S_0$ has one ordinary double point.

figure:

This decomposition is seen from $P_1, \ldots, P_4$.

From $P_5$ we see the following decomposition:

$\hat{S}_0 = (2, 2), \hat{S}_1 = (1, 1)$. $\hat{S}_0$ is smooth.

figure:

$\#K_{\text{Sing}} = 5 \quad \#\hat{K} = 32$
$\#\{\text{proj. } K\} = 0 \quad \#\{\text{planes on } K\} = 1.$

The points $P_1, P_2, P_3$ and $P_4$ lie in the plane. $P_5$ is resolved zero-homologously. There are at least 5 isomorphism classes of small resolutions.

$J8$: $S_0 = (2, 3), S_1 = (1, 0)$. $S_0$ has two ordinary double points.

figure:
This decomposition is seen from every $P_1, \ldots, P_4$.

From $P_5$ or $P_6$ we see the following decomposition:

$\overline{S}_0 = (2, 2)$, $\overline{S}_1 = (1, 1)$. $\overline{S}_0$ has one ordinary double point.

The points $P_1, P_2, P_3$ and $P_4$ lie in the plane. $P_5$ and $P_6$ are resolved by zero-homologous curves.

$j9$: $S_0 = (1, 2)$, $S_1 = (2, 1)$.

The two projective small resolutions are the ones for which:

$\overline{S}_0 \cdot L_i = 1$

$\overline{S}_1 \cdot L_i = -1$ \hspace{1cm}$\forall i$

In both cases all exceptional curves are homologous.

In some examples the two projective small resolutions are isomorphic. We do not know whether this is true in general or not.

$d = 2$

For $d = 2$ we have 3 classes.
$J10$: $S_0 = (1, 0), S_1 = (1, 1), S_2 = (1, 2)$.

This decomposition is seen from every $P_i, i = 1, \ldots, 6$.

From $P_7$ we see:
$S_0 = (1, 0), S_1 = (1, 0), S_2 = (1, 3).

$P_7$ is the only point on $K$ through which there pass two planes.

$\# K_{\text{Sing}} = 7 \quad \# \hat{K} = 128$
$\# \{\text{proj. } \hat{K}\} = 6 \quad \# \{\text{planes on } K\} = 2.$

The two 4-tuples of singular points lying in a plane are:

plane 1: $\{P_1, P_5, P_6, P_7\},$
plane 2: $\{P_2, P_3, P_4, P_7\}.$

The 6 projective small resolutions are the following:

<table>
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<tr>
<th></th>
<th>$P_1, P_5, P_6$</th>
<th>$P_2, P_3, P_4$</th>
<th>$P_7$</th>
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</table>

A """+""
under $P_i$ indicates that the strict transform of the plane(s) through $P_i$
meet(s) $L_i$ transversally. A """-""
indicates the other small resolution. This list
is easily obtained by using remark 2.
Because \( s = 7 \) and \( d = 2 \), there are 5 relations between the exceptional lines. The vectorspace \( B \) is of dimension 5. On a projective small resolution we have \( L_1 = L_5 = L_6 \) and \( L_2 = L_3 = L_4 \) and one relation \( L_1 \pm L_2 \pm L_2 = 0 \). The 6 projective small resolutions correspond to the 3 possibilities of choosing sign combinations such that not both are "+". For each of these combinations there are two projective small resolutions which are dual to each other.

There are at least 4 isomorphism classes of projective small resolutions.

\[ J11: S_0 = (1, 0), S_1 = (0, 1), S_2 = (2, 2). \]

\[ \begin{array}{c}
S_0 \\
P_1 \\
P_1 \\
P_2 \\
P_2 \\
P_3 \\
P_3 \\
P_4 \\
P_4 \\
P_5 \\
P_5 \\
P_6 \\
P_6 \\
S_1 \\
\end{array} \]

\# \text{Sing} = 6 \\
\# \{\text{proj. } K\} = 6 \\
\# \{\text{planes on } K\} = 3.

The 4-tuples of singular points lying in a plane are:

plane 1 = \{P_1, P_2, P_3, P_4\}
plane 2 = \{P_1, P_2, P_3, P_6\}
plane 3 = \{P_3, P_4, P_5, P_6\}.

We know from remark 2 that for a projective \( \tilde{K} \), \( L_1 \) is determined by \( L_2 \), \( L_3 \) by \( L_4 \) and \( L_5 \) by \( L_6 \) (and reversely). That \( L_1 \) is determined by \( L_2 \) can be seen by choosing \( P_6 \) as watchtower.

Since \( P_2 \), \( P_3 \) and \( P_6 \) form a triangle, there are at most 6 projective \( \tilde{K} \)'s. It turns out that there are exactly 6 of them. They can be characterized by the 6 different orders of the 3 planes on \( K \). Any order of the 3 planes determines a projective \( \tilde{K} \): The four singular points in the first plane are replaced by lines transversally to that plane. The two remaining singular points in the second plane are replaced by lines, transversally to that second plane. Clearly these 6 small resolutions are different and projective. They are isomorphic if and only if \( S_3 \) acts on the plane configuration.

The relations between the exceptional lines on a projective small resolution are

\[ L_1 = L_2, L_3 = L_4, L_5 = L_6 \] and \( L_1 \pm L_3 \pm L_5 = 0 \).

Again the 6 projective small resolutions are those on which we do not have \( L_1 + L_3 + L_5 = 0 \).

\[ J12: S_0 = (1, 0), S_1 = (0, 1), S_2 = (2, 2). \] \( S_2 \) has one node.
This decomposition is seen from $P_1, ..., P_6$.

From $P_7$ we see:

$S_0 = (1, 1), S_1 = (1, 1), S_2 = (1, 1)$.

The 3 planes are:

- Plane 1: $\{P_1, P_2, P_4, P_6\}$
- Plane 2: $\{P_1, P_3, P_5, P_6\}$
- Plane 3: $\{P_2, P_3, P_4, P_5\}$.

$P_7$ is resolved zero-homologously.

For $d = 3$ we have one class.

$J13$: $S_0 = (1, 0), S_1 = (0, 1), S_2 = (1, 1), S_3 = (1, 1)$. 
This decomposition is seen from $P_1, \ldots, P_4$.

From $P_5, \ldots, P_8$ we see:
$S_0 = (0, 1), S_1 = (1, 0), S_2 = (1, 0), S_3 = (1, 2).

From $P_9, \ldots, PH$ we see:
$5_0 = (0, 1), 5_1 = (1, 0), 5_2 = (1, 0), 5_3 = (1, 2).

The 5 planes are:

plane 1 = $\{P_1, P_2, P_6, P_8\}$
plane 2 = $\{P_1, P_2, P_5, P_7\}$
plane 3 = $\{P_5, P_6, P_7, P_8\}$
plane 4 = $\{P_3, P_4, P_5, P_6\}$
plane 5 = $\{P_3, P_4, P_7, P_8\}$.

The 24 projective small resolutions can be found by using remark 2 and 3.

It would take us too far to investigate the isomorphism classes in the cases
where $d = 3$ or 4.

$d = 4$

There is one class for $d = 4$.

$J14$: $S_0 = (1, 1), S_1 = (0, 1), S_2 = (0, 1), S_3 = (1, 0), S_4 = (1, 0)$.
This decomposition is seen from every $P_i$.

# $K_{\text{Sing}} = 9$
# $K = 256$
# $\{\text{proj. } K\} = 102$
# $\{\text{planes on } K\} = 9$.

The 9 planes are:

1. Plane 1 = $\{P_1, P_2, P_5, P_8\}$
2. Plane 2 = $\{P_1, P_3, P_4, P_9\}$
3. Plane 3 = $\{P_6, P_7, P_8, P_9\}$
4. Plane 4 = $\{P_3, P_5, P_6, P_8\}$
5. Plane 5 = $\{P_2, P_4, P_6, P_9\}$
6. Plane 6 = $\{P_3, P_5, P_7, P_9\}$
7. Plane 7 = $\{P_2, P_4, P_7, P_8\}$
8. Plane 8 = $\{P_1, P_2, P_3, P_6\}$
9. Plane 9 = $\{P_1, P_4, P_5, P_7\}$

The calculation of the number of projective small resolutions and the decomposition of $\mathcal{O}^9$ in $\mathcal{A}$ and $\mathcal{B}$ can be found in [We], page 71, 108 and 109.

\[ d = 5 \]

There is one class for $d = 5$.

\[ J15: S_0 = (0, 1), S_1 = (0, 1), S_2 = (0, 1), S_3 = (1, 0), S_4 = (1, 0), S_5 = (1, 0). \]

\[ \text{figure:} \]

\[ 
\begin{array}{ccc}
S_0 & S_1 & S_2 \\
S_3 & P_2 & P_3 & P_4 \\
S_4 & P_5 & P_6 & P_7 \\
S_5 & P_8 & P_9 & P_{10} \\
\end{array} \\
\]

This decomposition is seen from every $P_i$.

The cubic in this class is the unique cubic with the maximal number of 10 double points. It is called the Segre cubic.

# $K_{\text{Sing}} = 10$
# $K = 1024$
# $\{\text{proj. } K\} = 332$
# $\{\text{planes on } K\} = 15.$
The 15 planes are:

plane 1 = \{P_1, P_2, P_3, P_{10}\}  
plane 2 = \{P_1, P_2, P_3, P_4\}
plane 3 = \{P_1, P_4, P_7, P_{10}\}  
plane 4 = \{P_1, P_2, P_3, P_4\}
plane 5 = \{P_1, P_5, P_6, P_7\}  
plane 6 = \{P_1, P_8, P_9, P_{10}\}
plane 7 = \{P_2, P_5, P_9, P_{10}\}  
plane 8 = \{P_2, P_6, P_7, P_8\}
plane 9 = \{P_2, P_4, P_6, P_9\}  
plane 10 = \{P_2, P_3, P_7, P_{10}\}
plane 11 = \{P_3, P_4, P_5, P_8\}  
plane 12 = \{P_3, P_5, P_7, P_9\}
plane 13 = \{P_3, P_6, P_8, P_{10}\}  
plane 14 = \{P_4, P_5, P_6, P_{10}\}
plane 15 = \{P_4, P_7, P_8, P_9\}.

The small resolutions as well as other aspects of the Segre cubic are studied extensively in [Fi1] and [Fi2]. It has been shown in [Fi1] that the 332 projective small resolutions divide into 6 isomorphism classes.

REFERENCES


