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Matrix Convexity: Operator Analogues of the Bipolar and Hahn–Banach Theorems*

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Several basic results of convexity theory are generalized to the "quantized" matrix convex sets of Wittstock. These include the Bipolar theorem, a gauge version of the Hahn–Banach theorem, and the existence theorem for support functionals. © 1997 Academic Press

1. INTRODUCTION

Perhaps the most seductive feature of operator algebra theory is its claim to provide a natural framework for "quantizing" mathematics. Ever since von Neumann codified Heisenberg's matrix mechanics [15, 29], and then in collaboration with Murray, formulated non-commutative integration theory [21], the first principle of quantization has remained the same: One begins by replacing functions with operators (the "quantum observables"), and probability measures with density matrices (the "quantum states"). In the last fifty years this program has been successfully implemented in a number of areas. There now exist robust operator algebraic approaches to portions of analysis, geometry, and probability theory.

Operator algebraists have recently turned their attention to functional analysis itself. The notion that one might profitably study "quantized linear spaces" stemmed from Arveson's discovery of an operator-theoretic analogue of the Hahn–Banach Theorem [1]. Generally speaking, the "objects" of functional analysis are infinite dimensional vector spaces, supplemented by order and norm-theoretic structures. Choi and the first author showed [5] that one can use *matrix ordered spaces* (or more precisely "operator systems") to quantize ordered vector space theory (the "function systems" of Kadison [19]). Subsequently Ruan used *matrix normed spaces* (or "operator spaces") to quantize normed spaces [25]. Both of these theories have had significant applications in operator algebra theory (see, e.g., [3, 6–8, 18, 20, 23, 26]).

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Since elementary ideas of convexity underlie both norm and order-theoretic constructions, it would seem only natural to seek an operator version of convexity. In fact, matrix convexity arguments have been implicitly used by operator algebraists in various "averaging" arguments from the very beginnings of their subject. An elegant axiomatization was suggested by Wittstock more than a decade ago [31, 32], (see also [14, 17, 22]). He began by defining the notion of a *matrix convex set*. In classical geometry, a *convex combination* of points in a convex set is associated with a discrete probability measure on the set. In Wittstock's theory, one considers instead *matrix convex combinations* associated with finite density matrices. By introducing semi-linear *set valued* mappings Wittstock went on to prove a difficult analogue of the algebraic form of the Hahn–Banach theorem. He used this to generalize Arveson's theorem to arbitrary operator spaces (see also [22]).

It has long been thought that various more sophisticated aspects of convexity, such as Choquet theory, might have non-commutative analogues (see, e.g., [9]). In order to explore these questions, we have found it necessary to extend Wittstock's theory. Further results in this direction may be found in [30].

In Section 2 we explain our notation. In Section 3 we consider some elementary properties and examples of matrix convex sets, and in Section 4 we discuss the general notion of "matrix convex combinations". In Section 5 we define the matrix analogue of the polar of a convex set, and we prove that as in the scalar case, the matrix bipolar of a matrix convex set containing the origin is the weak closure of that matrix convex set. We use this to show that there are natural *maximal and minimal functors* from the category of convex sets to the category of matrix convex sets. In Section 6 we use systems of Minkowski gauges to prove a straightforward analogue of the "algebraic" Hahn–Banach theorem, without resorting to set-valued functions. We use possibly infinite valued gauges in order to avoid the difficulties associated with non-positive matrix sublinear functionals. We apply this result to prove both metric and order-theoretic versions of Arveson's Hahn–Banach theorem. In Section 7 we prove an operator analogue of the classical "supporting functional" theorem. We conclude in Section 8 with a discussion of continuous gauges.

2. SOME MATRIX CONVENTIONS

We begin by briefly recalling some standard notions of functional analysis. Unless we indicate otherwise, we shall be dealing with complex vector spaces. Given vector spaces V and W, we let $L(V, W)$ be the vector $M_{I,J} = M_{I,J}(\mathbb{C})$, and $M_I = M_{I,I}$, and we identify $M_{I,J}$ with the linear be the corresponding algebraic dual (when appropriate, we will also use this notation for more general dual spaces).

A *-*vector* space is a complex vector space V together with a distinguished *-*operation*, i.e., a conjugate linear mapping $v \mapsto v^*$ such that $v^{**} = v$. We write V_{sa} for the real space of *self-adjoint* elements in V, i.e., the $v \in V$ such that $v^* = v$. Given an arbitrary element $v \in V$, we have that

$$v = \operatorname{Re} v + i \operatorname{Im} v,$$

where $\operatorname{Re} v = \frac{1}{2}(v + v^*)$, and $\operatorname{Im} v = (1/2i)(v - v^*)$. If V and W are *-vector spaces, we have a corresponding *-operation on $L(V, W)$ defined by

$$\varphi^*(v) = \varphi(v^*)^*,$$

and in particular, V^d is a *-vector space. It is easy to see that $F \in V^d$ is self-adjoint if and only if F is real valued on V_{sa}, or equivalently, we have that $F(\operatorname{Re} v) = \operatorname{Re} F(v)$ for all $v \in V$.

A *-vector space V is *partially ordered* if it has a distinguished convex cone $V^+ \subset V_{sa}$. This determines a linear partial ordering $\geqslant$ on V_{sa} in the usual manner. Given two partially ordered *-vector spaces V and W, we have a partial ordering on $L(V, W)$ determined by the cone

$$L(V, W)^+ = \{\varphi\colon V \to W\colon \varphi = \varphi^* \text{ and } \varphi(V^+) \subseteq W^+\}.$$

Given $n \in \mathbb{N}$ we let $\mathbb{C}^n$ have the usual Hilbert space structure, and we let e_i $(i = 1, ..., n)$ be the usual basis vectors, which we shall also regard as *column matrices*. We identify the linear space $M_{m,n}$ of rectangular m by n complex matrices $\alpha = [\alpha_{i,j}]$ with the linear mappings $\alpha\colon \mathbb{C}^n \to \mathbb{C}^m$. We use the standard matrix multiplication for (compatible) scalar matrices, the *-operation

$$M_{m,n} \to M_{n,m}\colon \alpha \mapsto \alpha^* = [\bar{\alpha}_{j,i}],$$

(this is conjugate linear) and the operator norm $\|\alpha\|$. In particular we have that $M_n = M_{n,n}$ is a *-algebra with multiplicative identity I_n. We write $e_{i,j}$ for the usual matrix units in $M_{m,n}$. More generally, given Hilbert spaces H and K, we let $B(H, K)$ denote the Banach space of bounded linear operators $T\colon H \to K$, and we let $B(H) = B(H, H)$ be the corresponding C^*-algebra of operators with identity I_H. We shall always indicate a sesquilinear form on a Hilbert space with the symbol $\langle \cdot | \cdot \rangle$.

Turning to matrices with non-scalar entries, it will be convenient to allow more general indices. Given a finite index set I, we let $\mathbb{C}^I$ denote the Hilbert space of I-tuples $(\alpha_i)_{i \in I}$. Given a vector space V and finite sets I and J, we let $M_{I,J}(V)$ denote the *matrix space* of I by J matrices $[v_{i,j}]$ with $v_{i,j} \in V$, $i \in I$, $j \in J$. We use the abbreviations $M_I(V) = M_{I,I}(V)$,

$M_{I,J}=M_{I,J}(\mathbb{C})$, and $M_I=M_{I,I}$, and we identify $M_{I,J}$ with the linear mappings $\mathbb{C}^J\to\mathbb{C}^I$. We let an integer $n\in\mathbb{N}$ also stand for the finite set $I_n=\{1, ..., n\}$, and thus $j\in n$ just means that $1\leqslant j\leqslant n$. Given $m, n\in\mathbb{N}$, $M_{m,n}(V)$ is the usual vector space of m by n matrices $[v_{i,j}]$ $(v_{i,j}\in V,\ i\in m,\ j\in n)$. On the other hand, letting $m\times n$ stand for the Cartesian product of these index sets,

$$M_{m\times n}(V)=M_{m\times n,\, m\times n}(V)$$

consists of matrices of the form $[v_{(i,k)(j,l)}]$, $(i,k), (j,l)\in m\times n$. The reader must take care not to confuse $M_{m,n}(V)$ with $M_{m\times n}(V)$.

There are natural matrix operations on the matrix spaces. Given $v\in M_m(V)$, $w\in M_n(V)$, and $\alpha\in M_{n,m}$, $\beta\in M_{m,n}$, we have the corresponding *matrix product*

$$\alpha v\beta=\left[\sum_{j,k}\alpha_{i,j}v_{j,k}\beta_{k,l}\right]\in M_n(V).$$

and the *direct sum*

$$v\oplus w=\begin{bmatrix} v & 0\\ 0 & w\end{bmatrix}\in M_{m+n}(V).$$

If V is a *-vector space, there is a natural *-operation on $M_n(V)$ defined by

$$[v_{i,j}]^*=[v_{j,i}^*].$$

Forming matrix spaces is functorial. Given a linear mapping of vector spaces $\varphi\colon V\to W$, we have a corresponding linear mapping

$$\varphi_n\colon M_n(V)\to M_n(W)\colon [v_{i,j}]\mapsto[\varphi(v_{i,j})].$$

where $\varphi_n(v)=[\varphi(v_{i,j})]$. Identifying $M_n(V)$ with the tensor product $M_n\otimes V$, we have that $\varphi_n=id\otimes\varphi$.

A *pairing* of vector spaces V and W is a bilinear function

$$F=\langle\cdot,\cdot\rangle\colon V\times W\to\mathbb{C}$$

such that $\langle v, w\rangle=0$ for all $w\in W$ implies that $v=0$, and similarly $\langle v, w\rangle=0$ for all $v\in V$ implies that $w=0$. For each $n\in\mathbb{N}$ this determines a pairing

$$M_n(V)\times M_n(W)\to\mathbb{C}\colon (v, w)\to\langle v, w\rangle=\sum\langle v_{i,j}, w_{i,j}\rangle. \tag{1}$$

On the other hand, for each $m, n\in\mathbb{N}$ we also have the *matrix pairing*

$$M_m(V)\times M_n(W)\to M_{m\times n}\colon (v, w)\to\langle\langle v, w\rangle\rangle=[\langle v_{i,j}, w_{k,l}\rangle]. \tag{2}$$

Given vector spaces V and W with a distinguished pairing, we say that V and W are *in duality*, or each is the *dual* of the other, and we write $V' = W$ and $W' = V$. We have, for example, that if V is an arbitrary vector space, then V and its algebraic dual $V' = V^d$ are dual vector spaces. A space determines a corresponding *weak topology* on its dual, and one can identify the weakly continuous linear functionals on the space with the elements of the dual space (see [24]). It is easy to see that given dual spaces V and V', the weak topology on $M_n(V)$ determined by (1) coincides with that determined in the same manner by the matrix pairing (2). Thus a net $v^\nu \in M_n(V)$ $(\nu \in N)$ converges to an element v if and only if for each $i, j \in n$,

$$f(v^\nu_{i,j}) = \langle v^\nu, e_{i,j} \otimes f \rangle \to \langle v, e_{i,j} \otimes f \rangle = f(v_{i,j}) \tag{3}$$

for all $f \in V'$, i.e., if and only if $v^\nu_{i,j} \to v_{i,j}$ weakly for all i, j. It follows from this that if V_0 is closed in V, then $M_n(V_0)$ is closed in $M_n(V)$.

Given vector spaces V and W with duals V' and W', we let $L^w(V, W)$ denote the vector space of weakly continuous linear mappings $\varphi\colon V \to W$. A matrix of linear mappings

$$\varphi = [\varphi_{i,j}] \in M_n(L(V, W))$$

determines a linear mapping $\varphi\colon V \to M_n(W)$ by

$$\varphi(v) = [\varphi_{i,j}(v)]$$

and this in turn determines a natural isomorphism

$$M_n(L^w(V, W)) \cong L^w(V, M_n(W)).$$

In particular, we have that

$$M_n(V') = L^w(V, M_n).$$

Given $v \in M_r(V)$ and $\varphi \in M_n(V')$, we may regard φ as an element of $L^w(V, M_n)$, and v as an element of $L^w(V', M_r)$. With the above conventions, we have that

$$\langle\langle v, \varphi \rangle\rangle = \varphi_r(v) = v_n(\varphi) \in M_{n \times r}. \tag{4}$$

3. MATRIX CONVEX SETS

Following Wittstock [32], we define a *matrix convex set* $K = (K_n)$ in a vector space V to be a collection of non-empty convex sets $K_n \subseteq M_n(V)$ such that

- MC1. for any $\alpha \in M_{r,n}$ with $\alpha^*\alpha = 1$ we have that $\alpha^* K_r \alpha \subseteq K_n$, and
- MC2. for any m and $n \in \mathbb{N}$, $K_m \oplus K_n \subseteq K_{m+n}$.

Perhaps the simplest matrix convex sets are the *closed matrix intervals* in the one-dimensional space $V = \mathbb{C}$. Given α, $\beta \in \mathbb{R}$, we have a corresponding *interval* in M_n defined by

$$[\alpha I_n, \beta I_n] = \{\gamma \in M_n : \alpha I_n \leqslant \gamma \leqslant \beta I_n\}.$$

We define the *matrix interval* $[\alpha I, \beta I]$ to be the collection $([\alpha I_n, \beta I_n])$, $(n \in \mathbb{N})$. We define the *matrix intervals* $[\alpha I, \infty)$ and $(-\infty, \beta I]$ in a similar manner, and we say that a matrix convex set is a closed matrix interval if it has one of these forms or it is equal to the collection $((M_n)_{sa})$.

Lemma 3.1. *Suppose that K is a matrix convex subset of $\mathbb{C}$ with K_1 a bounded closed subset of $\mathbb{R}$. Then K must be a closed matrix interval.*

Proof. Since K_1 is bounded, closed, and convex, it must be a closed interval in $\mathbb{R}$. Let us suppose that $K_1 = [\alpha, \beta]$. If $\gamma \in K_n$, then for any $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$, with $\|\xi\| = 1$, we have that $\xi^*\xi = 1$, and thus

$$\langle \gamma\xi \mid \xi \rangle = \xi^* \gamma \xi \in K_1 = [\alpha, \beta].$$

It follows that $\alpha I_n \leqslant \gamma \leqslant \beta I_n$. Conversely, if $\gamma \in M_n$ satisfies the latter inequality, then we may choose a unitary v and scalars $\lambda_i \in [\alpha, \beta]$ with

$$\gamma = v^*(\lambda_1 \oplus \cdots \oplus \lambda_n)\, v.$$

Since $\lambda_i \in K_1$, it follows that $\gamma \in K_n$. Similar arguments apply if $K_1 = [\alpha, \infty)$ or $K_1 = (-\infty, \beta]$. ∎

By contrast, a convex set D in $\mathbb{C}$ will generally have many "quantizations", i.e., there will exist a multitude of matrix convex sets K with $K_1 = D$. Arveson showed that any bounded closed matrix convex set K in $\mathbb{C}$ is in fact the set of "matrix numerical ranges" of a Hilbert space operator T (see [2], p. 301, and [27]). He went on to show that in certain cases, the matrix numerical ranges provide complete unitary invariants for the operator T.

Matrix convex sets frequently arise in the context of operator spaces, operator systems, and their mapping spaces. We recall that an *operator space* V on a Hilbert space H is a linear subspace of $B(H)$. If in addition, V is self-adjoint and contains the identity operator I, V is said to be an *operator system*. In either case we let $M_n(V)$ have the norm determined by the natural inclusion

$$M_n(V) \subseteq M_n(B(H)) = B(H^n).$$

If V is an operator system, the relative *-operation on $M_n(V)$ coincides with that defined by (2), and we may provide $M_n(V)$ with the relative partial ordering. A linear mapping between operator spaces $\varphi\colon V \to W$ is said to be *completely bounded* if the corresponding *completely bounded norm*

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\}$$

is finite. A mapping φ between operator systems is said to be *completely positive* if $\varphi_n \geqslant 0$ for all n, and if this is the case, we write $\varphi \geqslant_{cp} 0$. If φ is completely positive it follows that $\|\varphi\|_{cb} = \|\varphi(I)\|$. We say that φ is a *morphism* if $\varphi \geqslant_{cp} 0$ and $\varphi(I) = I$.

If V is an operator space, then the balls

$$K_n(V) = \{v \in M_n(V) : \|v\| \leqslant 1\}$$

form a matrix convex set, and if V is an operator system, the same is true for the cones

$$P_n(V) = \{v \in M_n(V) : v \geqslant 0\}.$$

Given operator spaces V and W, we let $CB(V, W)$ be the completely bounded mappings $\varphi\colon V \to W$, with the norm $\|\cdot\|_{cb}$, and we use the identification

$$M_n(CB(V, W)) = CB(V, M_n(W)),$$

to impose a norm on the matrices over $CB(V, W)$. It can be shown that $CB(V, W)$ is again an operator space [11], and thus the sets of complete contractions

$$\mathscr{CC}_n(V, W) = \{\varphi \in CB(V, M_n(W)) : \|\varphi\|_{cb} \leqslant 1\}$$

comprise a matrix convex set $\mathscr{CC}(V, W)$ in $CB(V, W)$. If V and W are operator systems, then it is easy to see that the completely positive maps

$$\mathscr{CP}_n(V, W) = \{\varphi \in CB(V, M_n(W)) : \varphi \geqslant_{cp} 0\},$$

and the morphisms

$$\mathscr{M}_n(V, W) = \{\varphi \in CB(V, M_n(W)) : \varphi \geqslant_{cp} 0, \varphi(I) = I\}$$

for $n \in \mathbb{N}$ determine matrix convex sets $\mathscr{CP}(V, W)$, and $\mathscr{M}(V, W)$ in $L(V, W)$.

Returning to the general theory, we note that if K is a matrix convex set in V and $0 \in K_1$ then $0 = 0 \oplus \cdots \oplus 0 \in K_n$. It follows that $\alpha^* K_r \alpha \subseteq K_n$ for any $\alpha \in M_{r,n}$ with $\|\alpha\| \leqslant 1$, since if we let $\beta = [1 - \alpha^*\alpha]^{1/2}$, then for any $v \in K_r$,

$$\alpha^* v \alpha = [\alpha^* \beta^*](v \oplus 0) \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

where

$$[\alpha^* \beta^*] \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = I.$$

4. MATRIX CONVEXITY

As stressed by Wittstock, the parallel between scalar and matrix convexity becomes more apparent if one uses density matrices, and their higher dimensional variants, the *matrix states*.

We recall that a matrix $\tau = [\tau_{i,j}] \in M_n$ is a *density matrix* if it satisfies $\tau \geqslant 0$ and *trace* $\tau = \sum \tau_{i,i} = 1$. If we use the scalar duality

$$M_n \times M_n \to \mathbb{C}\colon (\alpha, \beta) \mapsto \sum_{i,j} \alpha_{i,j} \beta_{i,j} = \text{trace } \alpha\beta^{tr}, \tag{5}$$

these matrices correspond to the *states* on M_n, i.e., to the positive (or equivalently, completely positive) mappings $\tau\colon M_n \to \mathbb{C}$ such that $\tau(I) = 1$. More generally, we define a *matrix state* on M_n to be a morphism $\sigma\colon M_n \to M_p$, and we let $S(M_n, M_p)$ be the set of all such mappings. Choi [4] (or see [22], Proposition 4.7) showed that these mappings have the form

$$\sigma(\alpha) = \sum_{i=1}^{np} \gamma_i^* \alpha \gamma_i,$$

where $\gamma_i \in M_{n,p}$ $(1 \leqslant i \leqslant np)$ satisfy $\sum_i \gamma_i^* \gamma_i = I$. We may rewrite this

$$\sigma(\alpha) = \gamma^* (\alpha \oplus \cdots \oplus \alpha)\, \gamma,$$

where

$$\gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_{np} \end{bmatrix}.$$

If we are given a vector space V, the corresponding mapping

$$\sigma \otimes id\colon M_n(V) \to M_p(V)$$

is given by

$$w = \sigma \otimes id(v) = \gamma^*(v \oplus \cdots \oplus v)\,\gamma. \tag{6}$$

Returning to matrix convex sets $K = (K_n)$, we see from (6) that conditions MC1 and MC2 imply that,

$$\sigma \otimes id(K_n) \subseteq K_p \tag{7}$$

for any $\sigma \in S(M_n, M_p)$. If K_1 contains 0, then it is easy to see that we have (7) for any completely positive mapping $\sigma\colon M_n \to M_p$ such that $\sigma(I_n) \leqslant I_p$. We call such mappings *matrix quasistates*, and we indicate the collection of such mappings by $QS(M_n, M_p)$.

We interpret (6) as a representation of w in terms of a "non-commutative" convex combination of the entries $v_{i,j}$ of v. To understand this viewpoint, let us reformulate the usual scalar notion of convex combination in a real vector space E. We begin by replacing the matrix space $M_n(V)$ over a vector space V with space of *n-tuples* $E^n = E \otimes \mathbb{R}^n$, and in particular, M_n by $\mathbb{R}^n$. Letting $(\mathbb{R}^n)^+$ be the non-negative n-tuples, we may regard $\mathbb{R}^n$ as an ordered vector space with the distinguished "order unit"

$$1_n = \overbrace{(1, ..., 1)}^{n}.$$

The *vector states* $\sigma\colon \mathbb{R}^m \to \mathbb{R}^n$ (or simply "classical states" if $n = 1$) are by definition the positive linear mappings for which $\sigma(1_m) = 1_n$. They are just the mappings of the form

$$\sigma(\alpha) = \left(\sum_{i=1}^{m} \sigma_i^1 \alpha_i, ..., \sum_{i=1}^{m} \sigma_i^n \alpha_i \right)$$

where $\sigma_i^j \geqslant 0$ and $\sum_i \sigma_i^j = 1$, i.e., σ may be described as n-tuple of "probability measures" σ^j on the set

$$J_m = \overbrace{\{1, ..., 1\}}^{m}.$$

A vector state determines a mapping

$$\sigma \otimes id\colon E^m \to E^m\colon x \mapsto y = \left(\sum_{i=1}^{m} \sigma_i^1 x_i, ..., \sum_{i=1}^{m} \sigma_i^n x_i \right).$$

In particular we see that if $n = 1$, the relation $y = \sigma \otimes id(x)$ is just the statement that $y \in V$ is a convex combination of the x_i.

We may associate with any convex set K a system of convex sets $K^n \subseteq E^n$ by letting

$$K^n = \overbrace{K \oplus \cdots \oplus K}^{n}.$$

The fact that K is convex is equivalent to the assumption that $\sigma \otimes id(K^n) \subseteq K$ for all probability measures σ on J_n with $n \in \mathbb{N}$ arbitrary, i.e., to the hypothesis that K is closed under convex combinations. In contrast to the matrix situation, the assumption that $\sigma \otimes id(K^n) \subseteq K^p$ for vector states $\sigma\colon \mathbb{R}^n \to \mathbb{R}^p$ is redundant.

In Section 6 we shall use matrices $\tau \in M_{m \times n}$ to define linear mappings from M_n into M_m. We have natural vector space isomorphisms

$$M'_{m \times n} = L(M_{m \times n}, \mathbb{C}) \cong L(M_n, M_m) \overset{\theta}{\cong} M_{m \times n} \tag{8}$$

where we identify the first and fourth spaces with the usual scalar duality (5) (replacing n by $m \times n$) and we define $\theta_\tau \in L(M_n, M_m)$ by

$$\theta_\tau(\alpha)_{i,j} = \sum_{k,l \in n} \tau_{(i,k),(j,l)} \alpha_{k,l}.$$

We shall also use the notation

$$\theta_\tau(\alpha) = \tau \cdot_n \alpha = \tau \cdot \alpha, \tag{9}$$

where the subscript n identifies the summation variables k, l.

Letting $M'_{m \times n}$ have the dual ordering, and $L(M_n, M_m)$ the completely positive ordering, we have that these vector space identifications are in fact order isomorphisms [5], and in particular, we have that

$$\tau \geqslant 0 \Leftrightarrow \theta_\tau \geqslant_{cp} 0.$$

On the other hand, the mappings in (8) are not isometric. The dual norm on $M'_{m \times n}$ corresponds to the trace class norm on $M_{m \times n}$ given by

$$\|\tau\|_1 = \text{trace } |\tau|,$$

and in general,

$$\|\tau\| \lneqq \|\theta_\tau\|_{cb} \lneqq \|\tau\|_1. \tag{10}$$

Given any vector space V and a matrix $\tau \in M_{m \times n}$, the corresponding linear mapping

$$\theta_\tau \otimes id_V\colon M_n(V) \to M_m(V)$$

is given by

$$\theta_\tau \otimes id_V(v) = \tau \cdot_n v =_{def} \sum_{k,\, l \in n} \tau_{(i,\,k),\,(j,\,l)} v_{k,\,l}. \tag{11}$$

5. THE BIPOLAR THEOREM

In this section we will assume that V and V' are dual vector spaces. We say that a matrix convex set K in V is *weakly closed* if each K_n is weakly closed in $M_n(V)$ in the weak topology determined by (1).

Given a convex set K in V, we define the *polar* K^0 of K, to be the set

$$\begin{aligned} K^0 &= \{ f \in V' : \operatorname{Re}\langle v, f \rangle \leqslant 1 \text{ for all } v \in K \} \\ &= \{ f \in V' : \operatorname{Re} f|_K \leqslant 1 \}. \end{aligned}$$

This is a weakly closed convex subset of V' which contains 0. The classical Bipolar theorem states that conversely, if K is a weakly closed convex set containing 0, then

$$K^{00} = K. \tag{12}$$

Given a matrix convex set K, we define the *matrix polar* K^π of K by letting

$$\begin{aligned} K_n^\pi &= \{ \varphi \in M_n(V') : \operatorname{Re}\langle\!\langle v, \varphi \rangle\!\rangle \leqslant I_{r \times n} \text{ for all } v \in K_r,\ r \in \mathbb{N} \} \\ &= \{ \varphi \in M_n(V') = L^w(V, M_n) : \operatorname{Re} \varphi_r|_{K_r} \leqslant I_{r \times n} \text{ for all } r \in \mathbb{N} \}. \end{aligned}$$

It is easy to see that this is a weakly closed matrix convex set. We will prove below that if K is a weakly closed convex set containing 0, then $K^{\pi\pi} = K$. The argument is related to that given in [12] to prove Ruan's representation theorem for operator spaces.

The following shows that K_n^π is determined by K_n and the matrix pairing (2) (with $m = n$).

LEMMA 5.1. *Suppose that V and V' are dual vector spaces, and that K is a matrix convex set in V. Then $\varphi \in M_n(V')$ lies in K^π if and only if*

$$\operatorname{Re} \varphi_n|_{K_n} \leqslant I_{n \times n}. \tag{13}$$

Proof. Let us suppose that we have (13) and $r \leqslant n$. Given $v \in K_r$ and $w_0 \in K_{n-r}$, we have that $w = v \oplus w_0 \in K_n$. Letting $e = [I_r Q] \in M_{r,n}$ and using the identification $M_n(M_n) = M_{n \times n}$ we have

$$\operatorname{Re} \varphi_r(v) = e \operatorname{Re} \varphi_n(w)\, e^* \leqslant e I_{n \times n} e^* = I_{r \times n}.$$

On the other hand let us suppose that $v \in K_r$ where $r > n$. We have (14) if and only if

$$\langle \operatorname{Re} \varphi_r(v)\, \xi \mid \xi \rangle \leqslant 1$$

for all unit vectors $\xi \in \mathbb{C}^{r \times n}$. But from [28] (or [10]) we have that there exists an isometry $\beta\colon \mathbb{C}^n \to \mathbb{C}^r$ and a unit vector $\bar{\xi} \in \mathbb{C}^{n \times n}$ for which $\beta \otimes id(\bar{\xi}) = \xi$. It follows that since $\beta^* K_r \beta \subseteq K_n$,

$$\operatorname{Re}\langle \varphi_r(v)\, \xi \mid \xi \rangle = \operatorname{Re}\langle \varphi_n(\beta^* v \beta)\, \bar{\xi} \mid \bar{\xi} \rangle \leqslant 1. \quad \blacksquare$$

We are indebted to Erik Alfsen for the proof of the following result.

LEMMA 5.2. *Suppose that $\mathscr{E}$ is a cone of real continuous affine functions on a compact convex subset S of a topological vector space E, and that for each $e \in \mathscr{E}$, there is a corresponding point $p_e \in K$ with $e(p_e) \geqslant 0$. Then there is a point $p_0 \in S$ for which $e(p_0) \geqslant 0$ for all $e \in \mathscr{E}$.*

Proof. We must show that the sets

$$\{e \geqslant 0\} = \{p \in S\colon e(p) \geqslant 0\}$$

have non-zero intersection. By assumption these sets are non-empty, and they are compact. Thus it suffices to show that they have the finite intersection property. If this is not the case, then there exist $e_1, ..., e_n \in \mathscr{E}$ for which

$$\{e_1 \geqslant 0\} \cap \cdots \cap \{e_n \geqslant 0\} = \varnothing.$$

The mapping $\theta\colon S \to \mathbb{R}^n$ defined by $\theta(p) = (e_1(p), ..., e_n(p))$ is continuous and affine, and thus $\theta(S)$ is a compact convex set in $\mathbb{R}^n$. By assumption we have that

$$\theta(S) \cap (\mathbb{R}^n)^+ = \varnothing.$$

It follows that there is a linear function f on $\mathbb{R}^n$ such that $f((\mathbb{R}^n)^+) \geqslant 0$ and $f(\theta(S)) < 0$. This follows from the usual strict separation lemma for convex sets (see, e.g., [16], Section I.5). But we must have that

$$f(x_1, ..., x_n) = c_1 x_1 + \cdots + c_n x_n$$

for constants $c_j \geqslant 0$. It follows that $e = f \circ \theta = c_1 e_1 + \cdots + c_n e_n$ is an element of $\mathscr{E}$ for which $\{e \geqslant 0\} = \varnothing$, contradicting the hypotheses of the lemma. $\blacksquare$

LEMMA 5.3. *Suppose that V is a vector space, and that $K = (K_n)$ is a matrix convex set containing the origin in V. If F is a linear function on*

$M_n(V)$ which satisfies $\operatorname{Re} F|_{K_n} \leqslant 1$ *then there exists a state p on M_n such that for all $v \in K_r$ and $\alpha \in M_{r,n}$ we have*

$$\operatorname{Re} F(\alpha^* v \alpha) \leqslant p(\alpha^* \alpha) \tag{15}$$

Proof. Let $S = S(M_n)$ be the state space of M_n and let $\mathscr{E}$ be the set of continuous affine real functions on $S(M_n)$ of the form

$$e_{v,\alpha}(p) = p(\alpha^*\alpha) - \operatorname{Re} F(\alpha^* v \alpha).$$

with $\alpha \in M_{r,n}$ and $v \in K_r$. This is a cone of functions since $ce_{v,\alpha} = e_{v,c^{1/2}\alpha}$ for $c \geqslant 0$, and

$$e_{v,\alpha} + e_{w,\beta} = e_{x,\gamma}$$

where $\gamma^* = [\alpha \;\; \beta]$ and $x = v \oplus w$. Furthermore, for each function $e = e_{v,\alpha} \in \mathscr{E}$, there is a point $p_e \in S$ for which $e(p_e) \geqslant 0$. To see this, we may assume that $\alpha \neq 0$, and we then let p_e be a state on M_n with $p_e(\alpha^*\alpha) = \|\alpha\|^2$. Then defining $\beta = \alpha/\|\alpha\|$ we have that $\beta^* v \beta \in K_n$, and thus

$$\begin{aligned} \operatorname{Re} F(\alpha^* v \alpha) &= \|\alpha\|^2 \operatorname{Re} f(\beta^* v \beta) \\ &\leqslant p_e(\alpha^*\alpha). \end{aligned}$$

From Lemma 5.2 there exists such a p for which $e(p) \geqslant 0$ for all $e \in \mathscr{E}$. ▮

THEOREM 5.4. *Suppose that V is a vector space with a distinguished dual space V', and that $K \subseteq V$ is a weakly closed matrix convex set with $0 \in K_1$. Then for any $v_0 \notin K_n$ there exists a weakly continuous linear function $\varphi\colon V \to M_n$ such that* $\operatorname{Re} \varphi_r | K_r \leqslant I_{n \times r}$ *for all $r \in \mathbb{N}$, and for which* $\operatorname{Re} \varphi_n(v_0) \nleqq I_{n \times n}$.

Proof. From (12), we may choose a continuous linear functional F on $M_n(V)$ such that

$$\operatorname{Re} F|_{K_n} \leqslant 1 < \operatorname{Re} F(v_0).$$

From Lemma 5.3, there is a state p on M_n satisfying (15). Given $0 < \varepsilon < 1$, and letting τ be the normalized trace on M_n, we have that $q = (1-\varepsilon)\, p + \varepsilon\tau$ is a faithful state on M_n. It follows that if $\varepsilon > 0$ is sufficiently small, then $G = (1-\varepsilon)\, F$ satisfies

$$\operatorname{Re} G(\alpha^* v \alpha) \leqslant q(\alpha^*\alpha)$$

for all $v \in M_r(V)$ and $\alpha \in M_{r,n}$, and

$$\operatorname{Re} G|_{K_n} \leqslant 1 < \operatorname{Re} G(v_0).$$

The state q determines a representation π of M_n on a finite dimensional Hilbert space H with a separating and cyclic vector ξ, for which

$$q(\alpha) = \langle \pi(\alpha)\, \xi \mid \xi \rangle.$$

Given a row matrix $\alpha = [\alpha_1, ..., \alpha_n]$, we define the $n \times n$ matrix $\tilde{\alpha} \in M_n$ by

$$\tilde{\alpha} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix},$$

and we let $\tilde{M}_{1,n}$ be the linear space of all such matrices. Owing to the fact that ξ is separating, the corresponding subspace

$$H_0 = \pi(\tilde{M}_{1,n})\, \xi$$

is n-dimensional. Again since ξ is separating, we have a sesquilinear form B_v on H_0 unambiguously defined by

$$B_v(\pi(\tilde{\beta})\, \xi, \pi(\tilde{\alpha})\, \xi) = G(\alpha^* v \beta).$$

Since H_0 is finite dimensional, this determines a unique linear mapping $\varphi(v) \colon H_0 \to H_0$ with

$$G(\alpha^* v \beta) = \langle \varphi(v)\, \pi(\tilde{\beta})\, \xi \mid \pi(\tilde{\alpha})\, \xi \rangle.$$

It is a simple matter to verify that the corresponding mapping $\varphi \colon V \to B(H_0)$ is linear and weakly continuous. Fixing a basis in H_0, we may identify H_0 with $\mathbb{C}^n$, and $B(H_0)$ with M_n. If $v \in M_n(V)$, then using the column matrices e_i determined by the usual basis in $\mathbb{C}^n$ (see Section 2), we have

$$v = [v_{i,j}] = \sum_{i,j} e_i v_{i,j} e_j^*, \tag{16}$$

and thus letting f_j be the row matrix e_j^*,

$$G(v) = \sum_{i,j} \langle \varphi(v_{i,j})\, \pi(\tilde{f}_j)\, \xi \mid \pi(\tilde{f}_i)\, \xi \rangle = \langle \varphi_n(v)\, \eta_0 \mid \eta_0 \rangle,$$

where

$$\eta_0 = \begin{pmatrix} \pi(\tilde{f}_1)\, \xi \\ \vdots \\ \pi(\tilde{f}_n)\, \xi \end{pmatrix}$$

satisfies

$$\|\eta_0\|^2 = \sum \|\pi(\tilde{f}_j)\,\xi\|^2 = \sum q(f_j^* f_j) = q(I) = 1.$$

Given $v = [v_{i,j}] \in K_r$, we claim that $\operatorname{Re} \varphi_r(v) \leqslant I_{r \times n}$, or equivalently,

$$\operatorname{Re}\langle \varphi_r(v)\,\eta \mid \eta \rangle = \langle \operatorname{Re} \varphi_r(v)\,\eta \mid \eta \rangle \leqslant \langle \eta \mid \eta \rangle$$

for any vector $\eta \in (\mathbb{C}^n)^r$. Letting

$$\eta = \begin{pmatrix} \pi(\tilde{\alpha}_1)\,\xi \\ \vdots \\ \pi(\tilde{\alpha}_r)\,\xi \end{pmatrix},$$

where $\alpha_j \in M_{1,n}$, we have that

$$\|\eta\|^2 = \sum \|\pi(\tilde{\alpha}_i)\,\xi\|^2 = \sum q(\alpha_i^* \alpha_i) = q(\alpha^* \alpha)$$

where $\alpha \in M_{r,n}$ is given by

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}.$$

It follows that

$$\begin{aligned} \langle \operatorname{Re} \varphi_r(v)\,\eta \mid \eta \rangle &= \operatorname{Re} \left\langle \sum \varphi(v_{i,j})\,\pi(\tilde{\alpha}_j)\,\xi \mid \pi(\tilde{\alpha}_i)\,\xi \right\rangle \\ &= \operatorname{Re} \sum G(\alpha_i^* v_{i,j} \alpha_j) \\ &= \operatorname{Re} G(\alpha^* v \alpha) \\ &\leqslant q(\alpha^* \alpha) \\ &= \|\eta\|^2. \end{aligned}$$

On the other hand we have that $\operatorname{Re}\langle \varphi_n(v_0)\,\eta_0 \mid \eta_0 \rangle = \operatorname{Re} G(v_0) > 1$, and thus $\operatorname{Re} \varphi_n(v_0) \nleqslant I_{n \times n}$. ∎

Corollary 5.5. *If V, V' and K are as above, then $K^{\pi\pi} = K$.*

Perhaps the simplest application of matrix polarities is to the construction of the "extremal quantizations" of a convex set. Let us suppose that V and V' are vector spaces in duality and that K is a weakly closed convex subset of V which contains the origin. We define the *projective matrix envelope* $\hat{K}$ of K to be the collection $(\hat{K}_n)$ where $\hat{K}_n$ is the weak closure of the set of all elements $v \in M_n(V)$ of the form

$$v = \gamma^*(\overbrace{w \oplus \cdots \oplus w}^{r})\,\gamma,$$

where $w \in K$, $r \in \mathbb{N}$ are arbitrary, and $\gamma \in M_{r,n}$ satisfies $\gamma^*\gamma = 1$. We define the *injective envelope* $\check{K}$ by

$$\check{K} = [(K^0)^\wedge]^\pi.$$

Using the bipolar theorem it is evident that if L is a weakly closed matrix convex set with $L_1 = K$, then

$$\hat{K} \subseteq L \subseteq \check{K}$$

We may generalize this to weakly closed convex sets which do not contain the origin. Given such a weakly closed convex set D and fixing a point d_0, we let $K = D - \{d_0\}$, and we define

$$\hat{D}_n = \hat{K}_n + \overbrace{d_0 \oplus \cdots \oplus d_0}^{n}$$

and

$$\check{D}_n = \check{K}_n + \overbrace{d_0 \oplus \cdots \oplus d_0}^{n},$$

and we again find that if E is any weakly closed matrix convex set with $E_1 = D$, we have that

$$\hat{D} \subseteq E \subseteq \check{D}.$$

6. MATRIX GAUGES

Convex sets and matrix convex sets are closely linked with the theory of gauges. We let $[0, \infty]$ denote the usual one-point compactification of $\mathbb{R}^+$. We use the standard algebraic conventions regarding ∞, namely, we assume that

1. $a + \infty = \infty + a = \infty$ for all $a \in \mathbb{R}$ (we do not define $\infty + (-\infty)$ or $(-\infty) + \infty$).

2. $a \cdot \infty = \infty \cdot a = \infty$ for $a \in (0, \infty]$, and $0 \cdot \infty = \infty \cdot 0 = 0$ (we will not need to define $a \cdot \infty$ or $\infty \cdot a$ for $a < 0$).

A *gauge* on a vector space V is a mapping $\rho\colon V \to [0, \infty]$ such that

- G1. $\rho(v+w) \leqslant \rho(v) + \rho(w)$ for all $v, w \in V$,
- G2. $\rho(\alpha v) = \alpha\rho(v)$ for $\alpha \geqslant 0$ and $v \in V$.

There is a natural correspondence between gauges and convex sets. On the one hand each gauge determines the *unit set*

$$K(\rho) = \{v \in V\colon \rho(v) \leqslant 1\}.$$

Conversely if we are given a convex set $K \subseteq V$, the *Minkowski gauge of* K is the function $\rho^K\colon V \to [0, \infty]$ defined by

$$\rho^K(v) = \inf\{t \in [0, \infty)\colon v \in tK\},$$

where we let $\rho^K(v) = \infty$ if the set on the right is empty. If one introduces topological considerations, one can arrange this to be a bijection between suitable gauges and closed convex sets containing the origin. In this section we shall restrict our attention to gauges in a purely algebraic setting.

We define the *domain* of a gauge to be the set

$$\mathscr{D}(\rho) = \{v \in V\colon \rho(v) < \infty\}.$$

This is a (generally improper) cone in V, and we have that $\mathscr{D}(\rho) = V$ if and only if ρ is *finite*, i.e., $\rho(V) \subseteq [0, \infty)$. It is evident that ρ^K is finite if and only if K is absorbing, i.e., each $v \in V$ lies in tK for some $t > 0$. Our reason for considering infinite gauges is two-fold: a finite gauge need not have a finite dual, and gauges associated with cones are generally infinite. We begin by proving a simple variation of the algebraic Hahn–Banach theorem which permits infinite gauges.

Given a vector space W with a gauge ρ, we say that a subspace $V \subseteq W$ is *cofinal* in W if for each $w \in W$ we have that there exist elements v_+ and v_- in V for which

$$\rho(v_+ + w),\ \rho(v_- - w) < \infty. \tag{17}$$

THEOREM 6.1 (Algebraic Hahn–Banach). *Suppose that W is a real vector space with a gauge ρ and that V is a cofinal subspace of W. Then any linear functional F on V for which $F \leqslant \rho$ has an extension $\bar{F}$ on W for which $\bar{F} \leqslant \rho$.*

Proof. Let us suppose that $W = V + \mathbb{R}w_0$. In order to extend F, it suffices to find a scalar α such that

$$F(v) + \alpha \leqslant \rho(v + w_0) \tag{18}$$

and

$$F(v) - \alpha \leqslant \rho(v - w_0) \tag{19}$$

for all $v \in V$, since we may then let $\bar{F}(v + cw_0) = F(v) + c\alpha$. We define subsets S_- and S_+ of $\mathbb{R}$ by

$$S_- = \{F(v) - \rho(v - w_0) : v - w_0 \in \mathcal{D}(\rho)\}$$

and

$$S_+ = \{-F(v) + \rho(v + w_0) : v + w_0 \in \mathcal{D}(\rho)\}.$$

By assumption, we may choose v_- and v_+ satisfying (17) for $w = w_0$. It follows that

$$F(v_-) - \rho(v_- - w_0) \in S_-$$

and

$$-F(v_+) + \rho(v_+ + w_0) \in S_+$$

hence these sets are non-empty. Given $v_1 - w_0, v_2 + w_0 \in \mathcal{D}(\rho)$ we have that

$$F(v_1) - \rho(v_1 - w_0) \leqslant -F(v_2) + \rho(v_2 + w_0)$$

since

$$F(v_1 + v_2) \leqslant \rho(v_1 + v_2) \leqslant \rho(v_1 - w_0) + \rho(v_2 + w_0),$$

and thus $S_- \leqslant S_+$. We may therefore choose α with $S_- \leqslant \alpha \leqslant S_+$. Given any $v \in V$, we have that (18) is valid since it is trivial if $v + w_0 \notin \mathcal{D}(\rho)$, and it follows from $\alpha \leqslant S_+$ if $v + w_0 \in \mathcal{D}(\rho)$. A similar argument applies to (19).

The usual application of Zorn's lemma provides us with the desired extension $\bar{F}$ (see, e.g., [24]). ∎

COROLLARY 6.2. *Suppose that W is a real or complex vector space with a gauge ρ and that V is a subspace of W which is cofinal in W. If F is a linear functional on V for which*

$$\operatorname{Re} F(v) \leqslant \rho(v)$$

for all $v \in V$, then F has a linear extension $\bar{F}$ on W for which

$$\operatorname{Re} \bar{F}(w) \leqslant \rho(w) \tag{20}$$

for all $w \in W$.

Proof. It suffices to find a real linear extension $\bar{G}$ of $G = \operatorname{Re} F$ with

$$\bar{G}(w) \leqslant \rho(w)$$

since then the function

$$\bar{F}(w) = \bar{G}(w) - i\bar{G}(iw)$$

will be a complex linear function satisfying (20). Thus we may apply the real form of the Hahn–Banach theorem to G. ▮

It is instructive to see how we may use Corollary 6.2 to prove the classical extension theorem for positive linear functionals. Given a partially ordered *-vector space W (see Section 2) we define the order gauge ρ of W to be the Minkowski gauge of the set

$$\{w \in W \colon \operatorname{Re} w \geqslant 0\},$$

i.e., we let

$$\rho(w) = \begin{cases} 0 & \text{if} \quad \operatorname{Re} w \geqslant 0 \\ \infty & \text{otherwise.} \end{cases}$$

LEMMA 6.3. *Suppose that W is a partially ordered *-vector space, and that ρ is the corresponding order gauge. Then*

(a) *a linear functional $F \in W'_{sa}$ satisfies $F \geqslant 0$ if and only if $-\operatorname{Re} F \leqslant \rho$,*

(b) *a*-subspace V of W is cofinal with respect to ρ if and only if for each $w \in W_{sa}$, there exist $v_1, v_2, \in V_{sa}$ such that $v_1 \leqslant w \leqslant v_2$.*

Proof. If $F \geqslant 0$, and $\operatorname{Re} w \geqslant 0$, then since $F = F^*$,

$$-\operatorname{Re} F(w) = -F(\operatorname{Re} w) \leqslant 0 = \rho(w),$$

and thus $-\operatorname{Re} F(w) \leqslant \rho(w)$ for all $w \in W$. Conversely, given the latter condition, we have that $F = F^*$, since if $w = w^*$, $\pm \operatorname{Re} iw = 0$ implies that $\rho(\pm iw) = 0$, and

$$\pm \operatorname{Im} F(w) = -\operatorname{Re} F(\pm iw) \leqslant 0,$$

i.e., $\operatorname{Im} F(w) = 0$. If $w \geqslant 0$, then

$$-F(w) = -\operatorname{Re} F(w) \leqslant \rho(w) = 0.$$

i.e., $F \geqslant 0$.

Given W and ρ as above, we have that a subspace V of W is cofinal with respect to the gauge ρ if and only if there exist v^+ and $v^- \in V$ for which

$$\operatorname{Re} \rho(v^{\pm} \pm w) < \infty,$$

i.e.,

$$\operatorname{Re} v^{\pm} \pm \operatorname{Re} w \geqslant 0.$$

It follows that if $w \in W_{sa}$, then $-\operatorname{Re} v^+ \leqslant w \leqslant \operatorname{Re} v^-$. ∎

We note that if V is cofinal in W and V^+ is *generating for* V, i.e., $V_{sa} = V^+ - V^+$, then we may assume that $v_1 \leqslant 0$ and $v_2 \geqslant 0$ in (b).

COROLLARY 6.4. *Suppose that W is a partially ordered *-vector space and that V is a cofinal *-subspace of W. Then any positive functional F on V has an extension to a positive linear functional $\bar{F}$ on W.*

Proof. From the above remarks, it suffices to apply Corollary 6.2 to the functional $-F$. ∎

We define a *matrix gauge* on a complex vector space V to be a collection $\rho = (\rho_n)$ of gauges $\rho_n = M_n(V) \to [0, \infty]$ such that

- MG1. $\rho_{m+n}(v \oplus w) = \max\{\rho_m(v), \rho_n(w)\}$,
- MG2. $\rho_n(\alpha^* v \alpha) \leqslant \|\alpha\|^2 \rho_m(v)$,

for all $v \in M_m(V)$, $w \in M_n(V)$ and $\alpha \in M_{m,n}$. Each gauge ρ_n determines a corresponding convex set $K_n \subseteq M_n(V)$, and it is evident that $K = (K_n)$ is a matrix convex set in V. Conversely a matrix convex set $K = (K_n)$ determines a matrix gauge on V. As in the scalar, case, one obtains a bijection by introducing suitable topological notions.

It follows from the discussion in Section 4 that if ρ is a matrix gauge, then for any completely positive linear mapping $\theta: M_n \to M_p$ we have that

$$\rho_p((\theta \otimes id)\, v) \leqslant \|\theta\|_{cb}\, \rho_n(v). \tag{21}$$

LEMMA 6.5. *Given a matrix gauge ρ on a complex vector space V, we have that*

$$\rho_n(v) \leqslant n^2 \max_{i,\, j \in n} \{\rho(\pm v_{i,j}), \rho(\pm i v_{i,j})\}$$

for any $v \in M_n(V)$.

Proof. Let us first assume that $n = 2$. We have that for any $v \in V$,

$$\begin{bmatrix} 0 & v \\ v & 0 \end{bmatrix} = \alpha^* \begin{bmatrix} v & 0 \\ 0 & -v \end{bmatrix} \alpha$$

and

$$\begin{bmatrix} 0 & v \\ -v & 0 \end{bmatrix} = \beta^* \begin{bmatrix} iv & 0 \\ 0 & -iv \end{bmatrix} \beta,$$

where

$$\alpha = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{22}$$

and

$$\beta = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

are unitary matrices. It follows that

$$\begin{aligned} \begin{bmatrix} 0 & v_{12} \\ v_{21} & 0 \end{bmatrix} &= \begin{bmatrix} 0 & \tfrac{1}{2}(v_{12}+v_{21}) \\ \tfrac{1}{2}(v_{12}+v_{21}) & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & \tfrac{1}{2}(v_{12}-v_{21}) \\ -\tfrac{1}{2}(v_{12}-v_{21}) & 0 \end{bmatrix} \\ &= A^*(w_1 \oplus w_2) \oplus (w_3 \oplus w_4)\, A, \end{aligned} \tag{23}$$

where

$$w_1 = \frac{1}{2}(v_{12}+v_{21}) \qquad w_2 = -\frac{1}{2}(v_{12}+v_{21})$$

$$w_3 = \frac{i}{2}(v_{12}-v_{21}) \qquad w_4 = -\left(\frac{i}{2}\right)(v_{12}-v_{21}),$$

and

$$A = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Noting that

$$\|A\|^2 = \|A^+A\| = \|2I\| = 2$$

we conclude that

$$\rho_2\left(\begin{bmatrix} 0 & v_{12} \\ v_{21} & 0 \end{bmatrix}\right) \leqslant \|A\|^2 \max \rho(w_i)$$
$$\leqslant 2 \max\{\rho(\pm v_{12}), \rho(\pm iv_{12}), \rho(\pm v_{21}), \rho(\pm iv_{21})\}.$$

Given an $n \times n$ matrix $v = [v_{i,j}]$, we have that

$$v = v_{11} \oplus \cdots \oplus v_{nn} + \sum_{i<j} E_{i,j} \begin{bmatrix} 0 & v_{i,j} \\ v_{j,i} & 0 \end{bmatrix} E_{i,j}^*,$$

where for $i < j$, the scalar matrix

$$E_{i,j} = [e_i \quad e_j].$$

satisfies

$$\|E_{i,j}\|^2 = \left\| \begin{bmatrix} e_i^* \\ e_j^* \end{bmatrix} [e_i \quad e_j] \right\| = \left\| \begin{bmatrix} e_i^* e_i & 0 \\ 0 & e_j^* e_j \end{bmatrix} \right\| = 1.$$

It follows that

$$\rho_n(v) \leqslant \max_i \{\rho(v_{i,i})\} + \frac{1}{2} n(n-1) \max_{i<j} \rho_2\left(\begin{bmatrix} 0 & v_{i,j} \\ v_{j,i} & 0 \end{bmatrix}\right)$$
$$\leqslant \max_i \{\rho(v_{i,i})\} + n(n-1) \max_{i \neq j} \{\rho(\pm v_{i,j}), \rho(\pm iv_{i,j})\}$$
$$\leqslant n^2 \max_{i,j} \{\rho(\pm v_{i,j}), \rho(\pm iv_{i,j})\}. \quad \blacksquare$$

COROLLARY 6.6. *If ρ is a matrix gauge on a vector space V, and ρ_1 a finite gauge on V, then for each $n \in \mathbb{N}$, ρ_n is a finite gauge on $M_n(V)$.*

COROLLARY 6.7. *If ρ is a matrix gauge on a vector space W, and V is a cofinal subspace of W with respect to ρ_1, then $M_n(V)$ is cofinal in $M_n(W)$ with respect to ρ_n for each $n \in \mathbb{N}$.*

Proof. Given a $w = [w_{i,j}] \in M_n(W)$, we claim that there exists a $p \in \mathbb{N}$, $w_k \in W$, $(k \in p)$ and a scalar matrix $\gamma \in M_{p,n}$ with

$$w = \gamma^*(w_1 \oplus \cdots \oplus w_p)\, \gamma. \tag{24}$$

To see this we first note that if w' also has this form, then so does $w + w'$ since if

$$w' = \delta^*(w_1' \oplus \cdots \oplus w_q')\, \delta,$$

then

$$w + w' = [\gamma^* \quad \delta^*](w_1 \oplus \cdots \oplus w'_q) \begin{bmatrix} \gamma \\ \delta \end{bmatrix}.$$

From the proof of Lemma 6.5 we have that

$$w = w_{11} \oplus \cdots \oplus w_{nn} + \sum_{i<j} E_{i,j} \begin{bmatrix} 0 & w_{i,j} \\ w_{j,i} & 0 \end{bmatrix} E^*_{i,j}.$$

On the other hand, from (23), we have that for any $w', w'' \in W$, there exists a scalar matrix A and $w_1, w_2, w_3, w_4 \in W$ with

$$\begin{bmatrix} 0 & w' \\ w'' & 0 \end{bmatrix} = A^*(w_1 \oplus w_2 \oplus w_3 \oplus w_4)\, A,$$

and thus we have (24).

By assumption, we may choose $v_k^\pm \in V$ with $\rho_1(v_k^\pm \pm w_k) < \infty$. It follows that if we let

$$v^\pm = \gamma^*(v_1^\pm \oplus \cdots \oplus v_p^\pm)\, \gamma,$$

we have that

$$\rho_n(v^\pm \pm w) < \|\gamma\|^2 \max \rho_1(v_k^\pm \pm w_k) < \infty. \quad \blacksquare$$

We say that a matrix gauge ρ on a vector W is *finite* if ρ_1, and thus all of the ρ_n are finite. Similarly we say that a subspace V of W is *cofinal with respect to* ρ, if V is cofinal in W with respect to ρ_1, and thus each matrix space $M_n(V)$ is cofinal in $M_n(W)$ with respect to ρ_n.

Given a vector space V with a matrix gauge ρ, let us fix $n \in \mathbb{N}$. We define a new gauge γ_n on $M_n(V)$ by letting

$$\gamma_n(v) = \inf\{ \|\tau\|_1\, \rho_r(\bar{v}) : v = \tau \cdot \bar{v},\ \bar{v} \in M_r(V),\ \tau \in M^+_{n\times r},\ r \in \mathbb{N} \text{ arbitrary}\}, \tag{25}$$

where we recall that

$$\tau \cdot \bar{v} = (\theta_\tau \otimes id_V)(\bar{v}) = \sum_{k,l \in r} \tau_{(i,k),(j,l)} \bar{v}_{k,l}$$

(see Section 4) and the trace class norm $\|\tau\|_1$ is given by

$$\|\tau\|_1 = \sum_{i,k} \tau_{(i,k),(i,k)}.$$

We have that

$$\rho_n(v) \leqslant \gamma_n(v) \leqslant n\rho_n(v),$$

since if $v = \tau \cdot \bar{v}$, then from (21) and (10),

$$\rho_n(v) \leqslant \|\theta_\tau\|_{cb}\, \rho_r(\bar{v}) \leqslant \|\tau\|_1\, \rho_r(\bar{v}), \tag{26}$$

and conversely from the relation $v = \varepsilon \otimes id_V(v)$, where $\varepsilon = [\varepsilon_{i,j}]: M_n \to M_n$ is the identity mapping, we have that

$$\gamma_n(v) \leqslant \|\varepsilon\|_1\, \rho_n(v) = n\rho_n(v).$$

LEMMA 6.8. *Suppose that V has a matrix gauge ρ and define γ_n as above. Then γ_n is a gauge on $M_n(V)$. Furthermore for any $v \in M_n(V)$, we have that*

$$\gamma_n(v) = \inf\{\, \|\beta\|_2^2\, \rho_n(\bar{v}) : v = \beta\bar{v}\beta\}, \tag{27}$$

where $\bar{v} \in M_n(V)$, $\beta \in M_n^+$ is invertible, and $\|\cdot\|_2$ is the Hilbert Schmidt norm.

Proof. In order to prove G1 for γ_n, it suffices to consider the case that $\gamma_n(v)$ and $\gamma_n(w)$ are finite. Let us suppose $v = \tau \cdot \bar{v}$ and $w = \mu \cdot \bar{w}$, where $\tau \in M_{n\times r}^+$, $\bar{v} \in M_r(V)$, $\mu \in M_{n\times s}^+$, $\bar{w} \in M_s(V)$, and $\rho_n(\bar{v}) = \rho_n(\bar{w}) = 1$. We have that

$$v + w = (\tau \oplus \mu) \cdot_{r+s} (v \oplus w),$$

where $\tau \oplus \mu \in M_{n\times(r+s)}^+$ satisfies

$$\|\tau \oplus \mu\|_1 = \|\tau\|_1 + \|\mu\|_1,$$

and

$$\rho_{r+s}(\bar{v} \oplus \bar{w}) \leqslant \max\{\rho_r(\bar{v}), \rho_s(\bar{w})\} \leqslant 1.$$

It follows that

$$\gamma_n(v + w) \leqslant \gamma_n(v) + \gamma_n(w).$$

It is immediate that γ_n satisfies G2, i.e., it is positively homogeneous.

Turning to (27), we let γ_n' denote the expression on the right side of (27). Let us suppose that $v \in M_n(V)$ satisfies $\gamma_n'(v) < 1$. We may assume that $v = \beta\bar{v}\beta$, where $\rho_n(v) < 1$, and $\beta \in M_n^+$ satisfies $\|\beta\|_2^2 < 1$. We then have

$$v = [v_{i,j}] = \left[\sum_{k,l \leqslant n} \beta_{i,k}\bar{v}_{k,l}\beta_{l,j}\right] = \left[\sum_{k,l \leqslant n} \tau_{(i,k),(j,l)}\bar{v}_{k,l}\right] = \tau \cdot (\overbrace{\bar{v} \oplus \cdots \oplus \bar{v}}^{n}),$$

where $\tau=\tilde{\beta}\tilde{\beta}^* \in M_{n\times n}$ with $\tilde{\beta}\in M_{n\times n,\,1}$ defined by $\tilde{\beta}_{(j,\,l),\,1}=\beta_{j,\,l}$. Since $\|\tilde{\beta}\|_2=\|\beta\|_2$, it follows that

$$\|\tau\|_1 \leqslant \|\beta\|_2^2 < 1,$$

and we conclude that $\gamma_n(v)<1$, and thus $\gamma_n(v)\leqslant\gamma'_n(v)$. Dividing by a suitable constant, it is evident that this inequality is valid for any v with $\gamma'_n(v)<\infty$, and it is obvious if $\gamma'_n(v)=\infty$.

Conversely if $\gamma_n(v)<1$, we may suppose that $v=\tau\cdot\bar{v}$ where

$$\bar{v}=[\bar{v}_{k,\,l}]\in M_r(V)$$

and

$$\tau=[\tau_{(i,\,k),\,(j,\,l)}]\in M^+_{n\times r}$$

satisfy $\|\tau\|_1<1$, and $\rho_n(\bar{v})<1$. Since τ is positive, we have that $\tau=\beta^2$, we have $\beta\geqslant 0$ and

$$\|\beta\|_2^2=\|\tau\|_1<1.$$

It follows that

$$\tau_{(i,\,k),\,(j,\,l)}=\sum_{g\in n\times r}\beta_{(i,\,k),\,g}\beta_{g,\,(j,\,l)},$$

and thus

$$(\tau\cdot\bar{v})_{i,\,j}=\sum_{k,\,l,\,g}\beta_{(i,\,k),\,g}\bar{v}_{k,\,l}\beta_{g,\,(j,\,l)}.$$

Equivalently, letting $p=n\times r\times r$, we have that $v=\tau\cdot\bar{v}=b^*\bar{\bar{v}}b$, where $b\in M_{p,\,n}$, and $\bar{\bar{v}}$ are defined by

$$b_{(g,\,l),\,j}=\beta_{g,\,(j,\,l)},$$

and

$$\bar{\bar{v}}=\overbrace{\bar{v}\oplus\cdots\oplus\bar{v}}^{n\times r}\in M_p(V).$$

We have that $\|b\|_2=\|\beta\|_2$, and $\rho_p(\bar{\bar{v}})=\rho_r(\bar{v})$ are less than 1.

Regarding b as linear mapping $\mathbb{C}^n\to\mathbb{C}^p$, we let $b=u\,|b|$ be the corresponding polar decomposition, with $|b|=(b^*b)^{1/2}$ a positive semidefinite

$n \times n$ matrix and u the isometry of $K = (\ker b)^{\perp}$ onto the range of b. We note that since $|b| = u^*b$, $\| \, |b| \, \|_2 < 1$. Letting P be the projection of $\mathbb{C}^p$ onto K, we also have that for $\varepsilon > 0$, $b_1 = |b| + \varepsilon(I - P)$ is an invertible positive $n \times n$ matrix, and $b = ub_1$. If ε is sufficiently small, we may assume that $\|b_1\|_2 < 1$. It follows that

$$v = b^* \bar{\bar{v}} b = b_1 \bar{v}_1 b_1,$$

where $\bar{v}_1 = u^* \bar{\bar{v}} u$, is the desired decomposition of v. It follows that $\gamma_n'(v) \leqslant \gamma_n(v)$, and as before, we conclude that this is also true for arbitrary $v \in M_n(V)$. ∎

THEOREM 6.9. *Let us suppose that W is a complex vector space with a matrix gauge ρ, and that H is a Hilbert space. Given a cofinal subspace $V \subseteq W$ and a linear function $\varphi\colon V \to B(H)$ such that*

$$\operatorname{Re} \varphi_r(v) \leqslant \rho_r(v) \, I, \tag{28}$$

for all $v \in M_r(V)$ and $r \in \mathbb{N}$, φ has an extension $\bar{\varphi}\colon W \to B(H)$ such that

$$\operatorname{Re} \bar{\varphi}_r(w) \leqslant \rho_r(w) \, I, \tag{29}$$

for all $w \in M_r(W)$ and $r \in \mathbb{N}$.

Proof. We begin by recalling that since V is assumed cofinal in W with respect to ρ_1, it follows from Lemma 6.7 that $M_n(V)$ is cofinal in $M_n(W)$.

First let us consider the case that $H = \mathbb{C}^n$, and thus $B(H) = M_n$. We let γ_n^V and γ_n^W denote the gauges on $M_n(V)$ and $M_n(W)$ determined by (25). Given an element $v \in M_n(V)$, we have that

$$\gamma_n^V(v) = \gamma_n^W(v). \tag{30}$$

Indeed, it is obvious that $\gamma_n^W(v) \leqslant \gamma_n^V(v)$. Conversely, let us suppose that $\gamma_n^W(v) < 1$. Then from Lemma 6.8 we have that $v = \beta \bar{w} \beta$, where $\bar{w} \in M_n(W)$ and $\beta \in M_n^+$ is an invertible matrix with $\|\beta\|_2 < 1$ and $\rho_n(\bar{w}) < 1$. But we have that $\bar{w} = \beta^{-1} v \beta^{-1} \in M_n(V)$, and therefore from Lemma 6.8, $\gamma_n^V(v) < 1$.

We have a one-to-one correspondence between linear mappings $F\colon M_n(V) \to \mathbb{C}$ and linear mappings $\varphi\colon V \to M_n$ given by the relation

$$\langle \varphi(v), \tau \rangle = F([\tau_{i,j} v]),$$

where $\tau \in M_n$ and $v \in V$. It follows from this relation that for any $r \in \mathbb{N}$, $\bar{v} = [\bar{v}_{k,l}] \in M_r(V)$ and $\tau \in M_{n \times r}$,

$$\begin{aligned}\langle \varphi_r(\bar{v}), \tau\rangle &= \sum_{i,\,j,\,k,\,l} \tau_{(i,\,k),\,(j,\,l)}\,\varphi(\bar{v}_{k,\,l})_{i,\,j}\\ &= \sum_{k,\,l\in r} \langle \varphi(\bar{v}_{k,\,l}), \tau_{(\cdot,\,k),\,(\cdot,\,l)}\rangle\\ &= \sum_{k,\,l\in r} F([\tau_{(i,\,k),\,(j,\,l)}\bar{v}_{k,\,l}]_{i,\,j\in n})\\ &= F(\tau \cdot_r \bar{v}).\end{aligned}$$

We have that (28) is valid if and only if we have that for all $\tau \in M^+_{n\times r}$,

$$\langle \operatorname{Re} \varphi_r(\bar{v}), \tau\rangle \leqslant \rho_r(\bar{v})\langle I_{n\times r}, \tau\rangle = \rho_r(\bar{v})\ \|\tau\|_1,$$

i.e., if and only if

$$\operatorname{Re} F(\tau \cdot_r \bar{v}) \leqslant \rho_r(\bar{v})\ \|\tau\|_1.$$

It follows that (28) is equivalent to the hypothesis that $\operatorname{Re} F(v) \leqslant \gamma^V_n(v)$ for all $v \in M_n(V)$. From (30) we have that $\operatorname{Re} F(v) \leqslant \gamma^W_n(v)$. From Corollary 6.2 we have that F has an extension $\bar{F}$ satisfying $\operatorname{Re} \bar{F}(w) \leqslant \gamma^W_n(w)$ for all $w \in W$, and thus reversing the above reasoning, we have an extension $\bar{\varphi}\colon W \to M_n$ satisfying (29).

In the general case let us assume that we are given a linear mapping $\varphi\colon V \to B(H)$ satisfying (28). For each finite dimensional projection F on H, we let

$$\varphi_F = F\varphi F\colon V \to B(F(H)).$$

Using the notation $F_r = F \oplus \cdots \oplus F$,

$$\operatorname{Re}(\varphi_F)_r\,(v) = \operatorname{Re} F_r\varphi_r(v)\,F_r \leqslant \rho_r(v)\,I_{F(H)^r}$$

for all $v \in M_r(V)$, and thus we may find an extension $\bar{\varphi}_F\colon W \to B(F(H))$ of φ_F with

$$\operatorname{Re}(\bar{\varphi}_F)_r\,(w) \leqslant \rho_r(w)\,I_{F(H)^r}.$$

Let us fix $w \in W$. Since V is cofinal in W, we may select $v^+_j, v^-_j \in V (j = 1, 2)$ with

$$v^-_1 - w,\ v^+_1 + w,\ v^-_2 - iw,\ v^+_2 + iw \in \mathscr{D}(\rho_1).$$

We then have that

$$\operatorname{Re}(\bar{\varphi}_F(w)+\varphi_F(v_1^+))\leqslant\rho_1(w+v_1^+)\,I_{F(H)},$$
$$\operatorname{Re}(-\bar{\varphi}_F(w)+\varphi_F(v_1^-))\leqslant\rho_1(-w+v_1^-)\,I_{F(H)}$$
$$\operatorname{Re}(i\bar{\varphi}_F(w)+\varphi_F(v_2^+))\leqslant\rho_1(iw+v_2^+)\,I_{F(H)}$$
$$\operatorname{Re}(-i\bar{\varphi}_F(w)+\varphi_F(v_2^-))\leqslant\rho_1(-iw+v_2^-)\,I_{F(H)}$$

and thus for each w, the operators $\bar{\varphi}_F(w)$ are uniformly bounded. Letting $\mathscr{F}$ be the collection of finite dimensional projections on H, ordered in the usual way, the net of operators $\{\bar{\varphi}_F(w): F\in\mathscr{F}\}$ is bounded. Letting $L(W, B(H))$ have the topology of point-weak convergence, it follows form the Tychonoff theorem that any set $S\subseteq L(W, B(H))$ with $\{s(w): s\in S\}$ bounded for each $v\in V$ is precompact. It follows that the net $\{\bar{\varphi}_F: F\in\mathscr{F}\}$ is precompact. Thus it has a subnet which converges to a mapping $\bar{\varphi}: W\to B(H)$. It is a simple exercise to show that this mapping has the desired properties. ∎

In order to illustrate this result, we reprove the operator system and operator space forms of the Hahn–Banach theorem.

Given an operator system W, we may define a matrix gauge ρ on W by letting

$$\rho_n(w)=\begin{cases}0 & \operatorname{Re} w\geqslant 0\\ \infty & \text{otherwise.}\end{cases}$$

As in the ordered case, we have that a linear mapping $\varphi: W\to B(H)$ satisfies $\varphi\geqslant_{cb} 0$ if and only if

$$-\operatorname{Re}_r \varphi_r(w)\leqslant\rho_r(w)\,I_{H^r}.$$

for all $w\in M_r(W)$ and $r\in\mathbb{N}$. Furthermore, a self-adjoint subspace V of W is a cofinal if and only if for each $w\in W_{sa}$ there exists v_1 and $v_2\in V_{sa}$ for which $v_1\leqslant w\leqslant v_2$, and this condition is valid for matrices as well. We conclude from the above theorem, that completely positive mapping φ has a completely positive extension $\bar{\varphi}: V\to B(H)$.

Turning to the non-ordered case, let us suppose that we have an operator space W, a subspace V, and a completely contractive mapping $\psi = V\to B(H)$ for some Hilbert space H. We have that $\rho_n(w)=\|w\|$ determines a finite matrix gauge on W. Following Wittstock [33] we have that the linear mapping

$$\varphi: V\to M_2(B(H))=B(H\oplus H): v\mapsto\begin{bmatrix}0 & \psi(v)\\ 0 & 0\end{bmatrix}$$

satisfies $\operatorname{Re}\varphi_n(v) \leqslant \frac{1}{2}\|v\|$ for $v \in M_n(V)$. It follows that we have an extension $\bar{\varphi}\colon V \to M_2(B(H))$ satisfying $\operatorname{Re}\bar{\varphi}_n(w) \leqslant \frac{1}{2}\|w\|$ for all $w \in M_n(W)$. It is a simple matter to check that

$$\bar{\psi} = [1 \quad 0]\,\bar{\varphi}\begin{bmatrix} 0 \\ 1 \end{bmatrix} : W \to B(H)$$

is a completely contractive extension of ψ.

It should be noted that the construction of the gauge γ from the matrix gauge ρ (see (25)) may be expressed in terms of matrix convexity. Letting $\varepsilon_{i,j}$ be the usual basis in M_n, the relation

$$v = \tau \cdot_r \bar{v} = \left[\sum_{k,\, l \in r} \tau_{(i,k),(j,l)} \bar{v}_{k,l}\right] = \sum_{i,\, j \in n,\, k,\, l \in r} \tau_{(i,k),(j,l)} (\varepsilon_{i,j} \otimes \bar{v}_{k,l})$$

simply provides that v is a matrix (sub)-convex combination of the elementary tensors

$$\varepsilon_{i,j} \otimes \bar{v}_{k,l} \in M_n \otimes V.$$

This definition is motivated by tensor product constructions in the theory of operator spaces. The analogy is discussed further in [30].

7. SUPPORT FUNCTIONS

Let us suppose that V is a complex linear space with a finite gauge ρ. Given a point $v_0 \in V$, we say that a linear functional f on V is a *support function for* v_0 if we have that $\operatorname{Re} f(v) \leqslant \rho(v)$ for all v, and $\operatorname{Re} f(v_0) = \rho(v_0)$.

In order to define the corresponding notion for matrix gauges, it is useful to introduce a special matrix gauge on $\mathbb{C}$. Given $\alpha \in M_n$, we define $\operatorname{Re}_n^+(\alpha)$ to be the least scalar $\lambda \geqslant 0$ with $\operatorname{Re}\alpha \leqslant \lambda I_n$. To put it another way, this is just the Minkowski functional of the set

$$K_n = \{\alpha \in M_n\colon \operatorname{Re}\alpha \leqslant I_n\}.$$

It is easy to check these sets comprise a matrix convex set, and thus that $\operatorname{Re}^+ = (\operatorname{Re}_n^+)$ is a matrix gauge for $\mathbb{C}$. Given a finite matrix gauge $\rho = (\rho_n)$ on a complex vector space V and a point $v_0 \in M_n(V)$, we say that a linear mapping $\varphi = V \to M_n$ *is a support mapping for* v_0 if $\operatorname{Re}^+_{n\times r}\varphi_r(v) \leqslant \rho_r(v)$ and $\operatorname{Re}^+_{n\times n}\varphi_n(v_0) = \rho_n(v_0)$.

In the scalar case, the existence of support functionals is a consequence of the algebraic Hahn–Banach theorem. The situation is more subtle for the matrix case, and we shall instead reduce the problem to the scalar case.

This proof entails a quadratic version of the technique used in Theorem 5.4. We have included the usual proof of the scalar case for the reader's convenience.

LEMMA 7.1. *If V is a vector space with a finite gauge ρ, then given $v_0 \in V$, there exists a function $F \in V'$ such that* $\operatorname{Re} F(v) \leqslant \rho(v)$ *for all $v \in V$, and* $\operatorname{Re} F(v_0) = \rho(v_0)$.

Proof. Let us define a real linear function G_0 on $\mathbb{R}v_0$ as follows. We let

$$G_0(\alpha v_0) = \alpha \rho(v_0)$$

for $\alpha \in \mathbb{R}$. We have that if $\alpha \geqslant 0$,

$$G_0(\alpha v_0) = \alpha \rho(v_0) = \rho(\alpha v_0).$$

If $\alpha < 0$, then

$$G_0(\alpha v_0) = -G_0((-\alpha)\, v_0) = -\rho((-\alpha)\, v_0) \leqslant \rho(\alpha v_0),$$

where we have used the inequality

$$0 = \rho(0) \leqslant \rho(v) + \rho(-v)$$

for arbitrary $v \in V$. Applying Theorem 6.2, we may extend G_0 to a function G on V which satisfies

$$G(v) \leqslant \rho(v)$$

for all $v \in V$. We then define a complex linear function F on V by

$$F(v) = G(v) - iG(iv).$$

We have that

$$\operatorname{Re} F(v) = G(v) \leqslant \rho(v)$$

and

$$\operatorname{Re} F(v_0) = \rho(v_0). \quad \blacksquare$$

THEOREM 7.2. *Suppose that V is a complex vector space with a finite matrix gauge ρ, and that $v_0 \in M_n(V)$. Then there exists a linear function $\varphi: V \to M_n$ such that*

$$\operatorname{Re}^+_{r \times n} \varphi_r(v) \leqslant \rho_r(v)$$

for all $v \in M_r(V)$ *and*

$$\operatorname{Re}^{+}_{n \times n} \varphi_n(v_0) = \rho_n(v_0).$$

Proof. From Lemma 7.1 we may choose a functional $F\colon M_n(V) \to \mathbb{C}$ such that

$$\operatorname{Re} F(v) \leqslant \rho_n(v)$$

and

$$\operatorname{Re} F(v_0) = \rho_n(v_0).$$

Using the positive homogeneity of ρ, we have from Lemma 5.3 that there exists a state p on M_n such that

$$\operatorname{Re} F(\alpha^* v\alpha) \leqslant p(\alpha^*\alpha)\, \rho_r(v)$$

for all $v \in M_r(V)$ and $\alpha \in M_{r,n}$, $r \in \mathbb{N}$ arbitrary. Applying the GNS theorem, we have a corresponding representation π of M_n on a finite dimensional Hilbert space H with a cyclic unit vector $\xi_0 \in H$ satisfying $p(\alpha) = \langle \pi(\alpha)\, \xi_0 \mid \xi_0 \rangle$ for all $\alpha \in M_n$.

Given a row matrix $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n] \in M_{1,n}$, we define $\tilde{\alpha} \in M_n$ by

$$\tilde{\alpha} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}.$$

We let $\widetilde{M}_{1,n}$ be the linear space of all such $n \times n$ matrices, and we let $H_0 = \pi(\widetilde{M}_{1,n})\, \xi_0 \subseteq H$. Fixing an element $v \in V$, we wish to show we may define a sesquilinear from B_v on $H_0 \times H_0$ by

$$B_v(\pi(\tilde{\beta})\, \xi_0, \pi(\tilde{\alpha})\, \xi_0) = F(\alpha^* v\beta). \tag{31}$$

To begin, we note that

$$\operatorname{Re} F(\alpha^* v\alpha) \leqslant \rho_1(v)\, p(\alpha^*\alpha)$$

and substituting $-v$ for v,

$$-\rho_1(-v)\, p(\alpha^*\alpha) \leqslant \operatorname{Re} F(\alpha^* v\alpha) \leqslant \rho_1(v)\, p(\alpha^*\alpha).$$

Replacing v by $(-i)\, v$, we also have that

$$-\rho_1(iv)\, p(\alpha^*\alpha) \leqslant \operatorname{Im} F(\alpha^* v\alpha) \leqslant \rho_1((-i)\, v)\, p(\alpha^*\alpha).$$

It follows that if we let $\sigma(v) = 2 \max\{\rho_1(\pm v), \rho_1(\pm iv)\}$,

$$|F(\alpha^* v \alpha)| \leqslant \sigma(v)\, p(\alpha^* \alpha).$$

We have that $\Theta_v(\beta, \alpha) = F(\alpha^* v \beta)$ is a sequilinear form, and

$$|\Theta_v(\alpha, \alpha)| \leqslant \sigma(v)\, p(\alpha^* \alpha) = \sigma(v)\, \|\pi(\tilde{\alpha})\, \xi_0\|^2$$

It follows that

$$\begin{aligned} |\Theta_v(\beta, \alpha)| &= \left| \frac{1}{4} \sum_{k=0}^{4} i^k \Theta_v(\beta + i^k \alpha, \beta + i^k \alpha) \right| \\ &\leqslant \frac{\sigma(v)}{4} \{ \|\pi(\tilde{\beta})\, \xi_0 + \pi(\tilde{\alpha})\, \xi_0\|^2 + \|\pi(\tilde{\beta})\, \xi_0 - \pi(\tilde{\alpha})\, \xi_0\|^2 \\ &\quad + \|\pi(\tilde{\beta})\, \xi_0 + i\pi(\tilde{\alpha})\, \xi_0\|^2 + \|\pi(\tilde{\beta})\, \xi_0 - i\pi(\tilde{\alpha})\, \xi_0\|^2 \} \\ &\leqslant \sigma_n(v) \{ \|\pi(\tilde{\alpha})\, \xi_0\|^2 + \|\pi(\tilde{\beta})\, \xi_0\|^2 \}. \end{aligned}$$

This is true for arbitrary row matrices α and β, and replacing α by $t\alpha$ and β by $t^{-1}\beta$ for $t > 0$, the left hand side is unaffected. Taking the infimum over such t, we conclude that

$$|\Theta_v(\beta, \alpha)| \leqslant 2\sigma_n(v)\, \|\pi(\tilde{\alpha})\, \xi_0\|\, \|\pi(\tilde{\beta})\, \xi_0\|.$$

We conclude that the bilinear form (31) is well-defined, and thus there exists a unique linear mapping $\varphi_0(v): H_0 \to H_0$ for which

$$F(\alpha^* v \beta) = \langle \varphi_0(v)\, \pi(\tilde{\beta})\, \xi_0 \mid \pi(\tilde{\alpha})\, \xi_0 \rangle.$$

It is a simple matter to verify that the corresponding map $\varphi_0: V \to \mathscr{B}(H_0)$ is linear. The space H_0 has dimension $h \leqslant n$, and we may thus identify it with the subspace $\mathbb{C}^h \oplus 0_{n-h}$ in $\mathbb{C}^n$. Letting E be the projection of $\mathbb{C}^n$ onto that subspace, and letting $\varphi(v) = E\varphi_0(v)\, E: \mathbb{C}^n \to \mathbb{C}^n$, we obtain a mapping $\varphi: V \to M_n$ satisfying

$$F(\alpha^* v \beta) = \langle \varphi(v)\, \pi(\tilde{\beta})\, \xi_0 \mid \pi(\tilde{\alpha})\, \xi_0 \rangle.$$

Following the argument for Theorem 5.4, if $v \in M_n(V)$, then $v = [v_{i,j}] = \Sigma_{i,j} f_i^* v_{ij} f_j$, where $f_i = e_i^*$, and thus

$$\begin{aligned} F(v) &= \Sigma \langle \varphi(v_{ij})\, \pi(\tilde{f}_f)\, \xi_0 \mid \pi(\tilde{f}_i)\, \xi_0 \rangle \\ &= \langle \varphi_n(v)\, \xi \mid \xi \rangle, \end{aligned}$$

where

$$\xi = \begin{pmatrix} \pi(\tilde{f}_1)\,\xi_0 \\ \vdots \\ \pi(\tilde{f}_n)\,\xi_0 \end{pmatrix} \in \mathbb{C}^n$$

satisfies

$$\|\xi\|^2 = \Sigma\, \|\pi(\tilde{f}_j)\,\xi_0\|^2 = \Sigma p(\tilde{f}_j^{\,*}\,\tilde{f}_j) = p(I) = 1.$$

Given $v = [v_{i,\,j}] \in M_r(V)$, we claim that $\operatorname{Re} \varphi_r(v) \leqslant \rho_r(v)\, I_{r \times n}$. Since we have that $\varphi_r(v) = E_r \varphi_r(v)\, E_r$, where $E_r = E \oplus \cdots \oplus E$, it suffices to show that

$$\operatorname{Re}\langle \varphi_r(x)\,\eta \mid \eta\rangle = \langle [\operatorname{Re} \varphi_r(x)]\,\eta \mid \eta\rangle \leqslant \langle \rho_r(v)\,\eta \mid \eta\rangle.$$

for any vector η of the form

$$\eta = \begin{pmatrix} \pi(\tilde{\alpha}_1)\,\xi_0 \\ \vdots \\ \pi(\tilde{\alpha}_r)\,\xi_0 \end{pmatrix},$$

where $\alpha_j \in M_{1,\,n}$. We have that

$$\|\eta\|^2 = \sum \|\pi(\tilde{\alpha}_i)\,\xi_0\|^2 = \sum p(\alpha_i^* \alpha_i) = p(\alpha^* \alpha)$$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix},$$

and thus

$$\begin{aligned} \operatorname{Re}\langle (\varphi)_r\,(v)\,\eta \mid \eta\rangle &= \operatorname{Re} \sum \langle \varphi(v_{i,\,j})\,\pi(\tilde{\alpha}_j)\,\xi_0 \mid \pi(\tilde{\alpha}_i)\,\xi_0\rangle \\ &= \operatorname{Re} \sum F(\alpha_i^*\, v_{i,\,j}\, \alpha_j) \\ &= \operatorname{Re} F(\alpha^* v \alpha) \\ &\leqslant \rho_r(v)\; p(\alpha^* \alpha) \\ &= \rho_r(v)\; \|\eta\|^2. \end{aligned}$$

Finally we have that

$$\mathrm{Re}^{+}_{n\times n}\, \varphi_n(v) \geqslant \mathrm{Re}\langle \varphi_n(v_0)\, \xi \mid \xi\rangle = \mathrm{Re}\, F(v_0) = \rho_n(v_0),$$

and we are done. ∎

8. GAUGE CONTINUITY

In order to complete our discussion of gauges and matrix gauges, it is necessary to review a few simple facts about their topological properties. Let us suppose that V is a locally convex vector space, and that V' consists of all the continuous functionals on V. Then V and V' are in duality, and the closed convex sets in V are weakly closed (this follows, for example, from the Bipolar theorem).

We say that a gauge $\rho\colon V \to [0, \infty]$ is *lower continuous* (respectively, *continuous*) if for each convergent net $v_\nu \to v$, we have that

$$\rho(v) \leqslant \underline{\lim}\ \rho(v_\nu).$$

(respectively,

$$\rho(v) = \lim \rho(v_\nu).)$$

We have that ρ is lower semicontinuous if and only if the sets $\{v\colon \rho(v) \leqslant c\}$ are closed, or equivalently, if and only if the unit set $K(\rho) = \{v\colon \rho(v) \leqslant 1\}$ is closed. We say that a matrix gauge is lower semicontinuous if that is the case for each ρ_n, i.e., each of the sets $K_n(\rho)$ is closed.

Continuous gauges play an important role in the theory of locally convex spaces. The corresponding notion is similarly of importance in a corresponding theory of locally matrix convex spaces, which will be considered elsewhere. The following simple result is important in this context.

LEMMA 8.1. *Suppose that ρ is a gauge on a vector space V. Then the following are equivalent*:

(1) 0 *is in the interior of the unit set* $K(\rho)$,

(2) ρ *is continuous at* 0,

(3) ρ *is continuous.*

Proof. If $K(\rho)$ contains 0 in its interior, we may assume that $0 \in N \subseteq K(\rho)$, where N is an open symmetric neighborhood of 0. It follows that if $x \in \varepsilon N$, then $\rho(x) < \varepsilon$. Let us suppose that $v_\nu \to v \in V$. Then eventually we have that $v_\nu - v \in \varepsilon N$ and thus $\rho(v_\nu - v) < \varepsilon$. Since N is symmetric we also have that $\rho(v - v_\nu) < \varepsilon$. It follows that eventually,

$$\rho(v) \leqslant \rho(v - v_\nu) + \rho(v_\nu) \leqslant \varepsilon + \rho(v_\nu),$$

and thus ρ is lower semicontinuous. On the other hand, eventually

$$\rho(v_\nu) \leqslant \rho(v_\nu - v) + \rho(v) \leqslant \varepsilon + \rho(v),$$

from which we conclude that ρ is in fact continuous.

It is trivial that $(3) \Rightarrow (2) \Rightarrow (1)$, and thus we are done. ∎

COROLLARY 8.2. *If ρ is a matrix gauge on a locally convex space V and ρ_1 is continuous on V, then ρ_n is continuous on $M_n(V)$.*

Proof. This is immediate from (2) of the above result, and Lemma 6.5. ∎

If ρ is a continuous gauge on a locally convex space V and f is a linear function on V with $\operatorname{Re} f(v) \leqslant \rho(v)$ for all $v \in V$, then in particular, $\operatorname{Re} f(v) \leqslant 1$ on the interior of $K(\rho)$. It follows that $\operatorname{Re} f(x)$ is continuous on V, and thus the same is true for f. From this we also see that if ρ is a continuous matrix gauge on V and $\varphi\colon V \to M_n$ is a linear function on V such that $\operatorname{Re} \varphi_r(v) \leqslant \rho_r(v)$ for all $v \in M_r(V)$, $r \in \mathbb{N}$, then φ is continuous. Similarly, this may be used to show that one obtains continuous extensions in Theorem 7.2.

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