Determination of a $GL_2$ automorphic cuspidal representation by twists of critical $L$-values

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Abstract

In this paper, we will generalize the work of Luo and Ramakrishnan on holomorphic modular forms over $\mathbb{Q}$, to the self-contragredient automorphic cuspidal representations of $GL_2$ over any number fields. Our method is double Dirichlet series.
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1. Preliminary

1.1. Introduction

Let $F$ be any number field, and $\pi$ be a self-contragredient cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ with the central character $\chi_\pi$. We choose a finite set of places $S$, including all the archimedean and even ones such that $\pi$ is unramified outside $S$. As in [12], or [10] for function fields, certain quadratic Hecke characters $\chi_D$ are defined over the field $F$, where the $D$ are the integral ideals of $F$ disjoint from $S$ denoted by $D \in I^+(S)$. The $\chi_D$ is a generalization of the usual quadratic residue symbols. We look at the twisted $L$-function $L(s, \pi \otimes \chi_D)$ at the center of the critical strip, $s = 1/2$, and want to know whether the twists of the critical $L$-value can determine the representation. But it is known as in [19] that the quadratic twists do not in general suffice. And there exist cuspidal automorphic representations $\pi_0$ with the property that $L(1/2, \pi_0 \otimes \chi) = 0$ whenever $\chi$ is quadratic or trivial. But we will prove here that, except these bad situations, the twisted critical $L$-values together with the central character will be enough. Precisely, if $L(s, \pi \otimes \chi_D; I - S)$ denotes the partial $L$-functions with the $S$ factors removed, we have

**Theorem 1.1.** Let $\pi$ and $\pi'$ be two self-contragredient cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$ with the same central character. Assume for a fixed finite set of primes $S$, there is a non-zero constant $\kappa$ such that

$$L(1/2, \pi \otimes \chi_D) = \kappa L(1/2, \pi' \otimes \chi_D)$$

for all the $D \in I^+(S)$.

1. If the central character $\chi_\pi$ is trivial, and there is a Hecke character $\chi$ such that $\varepsilon(1/2, \pi \otimes \chi) = 1$, then $\pi \simeq \pi'$.
2. If $\chi_\pi$ is not trivial, then there are infinite many square free $D$ such that $L(1/2, \pi \otimes \chi_D; I - S) \neq 0$, and $\pi \simeq \pi'$.

This generalizes the work of Luo and Ramakrishnan in [16], which concerns holomorphic modular forms. To compare our result to theirs, let us describe their theorem as follows: Let $S_{2k}(N)$ denote the space of holomorphic cusp forms of weight $2k$, level $N$ and trivial character. Let $f \in S_{2k}(N)$, be a normalized holomorphic newform. It is given by a Fourier expansion

$$f(z) = \sum_{n \geq 1} a_n e^{2\pi i nz}$$

with $a_1 = 1$. For every primitive Dirichlet character $\chi$, we have the associated $L$-series

$$L(f, \chi, s) = \sum_{n \geq 1} a_n \chi(n) n^{-s}.$$
Theorem 1.2. Let $f$, $g$ be normalized newforms in $S_{2k}(N)$, $S_{2m}(N')$, respectively. Suppose there is a constant $C$ such that

$$L(f, \chi_d, k) = CL(g, \chi_d, m)$$

for almost all primitive quadratic characters $\chi_d$ of conductor $d$ prime to $NN'$. Then $k = m$, $N = N'$ and $f = g$.

One thing to mention here is that I state and prove our theorem for the complete $L$-functions. But their theorem is for the finite part $L$-series. Since the gamma factor of holomorphic forms just depends on the weight of the forms, so in their case there is no essential difference between using the complete or finite part of the $L$-functions. It is easy to see that our argument works for the $L$-series too. I would like to thank the referee for pointing out to me that the completed or finite-part $L$-functions do not really make any difference, since twisting by a quadratic character unramified at infinity does not change the infinite type of a $GL_2$ automorphic representation.

We will follow the idea of [6], in which they work on certain cuspidal representations of $GL_3$ over $\mathbb{Q}$ whose symmetric square $L$-functions have a simple pole at $s = 1$. We will use the double Dirichlet series method. For good surveys of this, there are [3,4]. Similarly to [16] working on the twisted sum of $L$-series, we will work on the twisted double Dirichlet series. This will be clear later. First, we will review the definition and properties of $\chi_D$ as a generalization of quadratic residue symbols. Next, we will review the constructions and properties of the double Dirichlet series in two variables $s$ and $w$. Note here we stick to the series defined by [9], which is different from those in [4,6,8]. Then, we will sieve out the non-square free terms in the series we have, and continue the sieved series meromorphically to the point $(s, w) = (1/2, 1)$. Finally, by studying the analytic properties of the sieved series at the point, we get the information we need to determine the representation.

1.2. Quadratic symbols

Let $F$ be a number field, and let $O$ denote the integer ring of $F$. As in [5] or [15], we have the quadratic residue symbol $(\frac{a}{D})$ defined for $a \in F^\times$ and a fractional ideal $D$ disjoint from the primes ramified in the field extension $F(\sqrt{a})/F$. In [10,12], the definition of the quadratic symbols is extended to $(\frac{D'}{D})$, where $D'$ and $D$ are ideals, not necessary principal. Let us review this following [7].

First, let us choose $S$, a finite set of places of $F$. Let $S_{\infty}$ be the set of all the archimedean places, and $S_2$ be all the even places above 2. We choose a finite set of odd places denoted as $S_c$, such that the $S_c$ integer ring $O_{S_c}$ has class number 1. Let $S_f = S_2 \cup S_c$, then we have $S = S_f \cup S_{\infty}$.

For each place $v$, let $F_v$ be the completion of $F$ at $v$. For $v$ non-archimedean, let $P_v$ be the corresponding ideal of $O$. Let $C = \prod_{v \in S_f} P_v^{n_v}$ with $n_v \geq 1$ sufficiently large that $a \in F_v$, ord$_v(a - 1) \geq n_v$ implies that $a \in (F_v^\times)^2$. We will fix $n_v = 1$ for the odd places $v$. Let $R_C$ be the narrow ray class group modulo $C$, and let $H_C = R_C \otimes \mathbb{Z}/2\mathbb{Z}$. We use $h$ for the number of elements in $H_C$. For any finite set of places $T \supset S_{\infty}$, let $I(T)$ be denoted the group of fractional ideals of $O$ disjoint from $T$. For its subset of integral ideals, we write as $I^+(T)$. For $C$, let $P_C$ be the subgroup of $I(S)$ consisting of principal ideals $(a)$ with $a \equiv 1 \pmod{C}$ and totally positive. So we have $R_C = I(S)/P_C$ and $H_C = I(S)/I(S)^2P_C$. 
We can consider $H_C$ as a vector space over $F_2$. We choose a basis of $H_C$ and $E_0 \subset \mathcal{I}^+(S)$, a set of representatives of the basis. For each $E_0 \in E_0$, we choose $m_{E_0} \in F^\times$ such that $E_0\mathcal{O}_S = m_{E_0}\mathcal{O}_S$. This is from the assumption on the class number. Then we extend to $E$, a full set of representatives of $H_C$, of the form $\prod_{E_0 \in E_0} E_{n_{E_0}}$. If $E = \prod_{E_0 \in E_0} E_{n_{E_0}}$ is in $E$, we will choose $m_E = \prod_{E_0 \in E_0} m_{E_0}$. It is easy to see that $E\mathcal{O}_S = m_E\mathcal{O}_S$ for all the $E \in E$. Also we have $\mathcal{O} \in E$ and $m_{\mathcal{O}} = 1$.

For any $D \in \mathcal{I}(S)$, write $D = (m)EG^2$ with $E \in E$, $m \in F^\times$, $m \equiv 1 \pmod{C}$, $m$ totally positive and $G \in \mathcal{I}(S)$. We define $(\frac{D}{m}) := (\frac{mE}{mE})$, denoted by $\chi_D$, where $(\frac{mE}{mE})$ is the quadratic symbol. The following lemma shows this is well-defined.

**Lemma 1.3.** If there is another decomposition $D = (m')EG'^2$. Then $E = E'$ and $(\frac{mE}{mE}) = (\frac{m'E}{m'E})$.

We skip the proof, which can be found in [10]. But as in [9], we have the following simple facts on $\chi_D$.

Consistent with class field theory, $\chi_D$ is necessarily not canonical, and it depends on the choices of the representatives.

If $D$ is any ideal of $F$, from now on we will denote its square free part by $D_0$ except when otherwise mentioned. From the definition, we have $\chi_D = \chi_{D_0}$. From the way we chose $E$, we can check that $\chi_{DD'} = \chi_D \chi_{D'}$.

Now let us list some simple observations on the quadratic residue symbol for our better understanding of $\chi_D$.

For any $a \in F^\times$ prime to 2, we use $\chi_a$ to denote $(\frac{a}{\cdot})$. We define a defining module or cycle $M_a$ as follows. Its infinite part has all the real places. For the finite part, the exponent of the prime $P_v$ at each $v$ is given by the $n_v$ defined before if $v$ divides 2; by 1 if ord$_v(a) \neq 0$; and by 0 otherwise. By the reciprocity law of the quadratic residue symbols as in [5] or [15], we can check that $\chi_a$ is a Grössencharakter modulo $M_a$ with the infinite component

$$\chi_a,\infty = \prod_{v \in \mathcal{S}_\infty} (\cdot, a)_v$$

given by the product of local Hilbert norm symbols. If we consider $\chi_a$ as a Hecke character, we have

$$\chi_a = \prod_v (a, \cdot)_v.$$

In other words, locally the quadratic residue symbol is the Hilbert symbol. This can be seen from the multiplicity one on $GL_1$, by which we mean that two Hecke characters are equal if they agree locally for almost all the places.

Locally, for $v$ odd, the ramification of the character $(a, \cdot)_v$ just depends on whether ord$_v(a)$ is odd or even. Precisely, when the order is even, the character is unramified; otherwise ramified, based on the following simple facts:

1. The extension by adding a square root of a unit is non-ramified;
2. A unit is always a norm from a non-ramified extension, while a prime element is never from the non-trivial extensions.
So we have that the primes in the conductor of \( \chi_a \) are from \( S_\infty, S_2 \) and \( (a)_0 \). And the conductor is bounded by \( M_a \). Now we look at \( \chi_D \), from the definition, and we know its conductor consists of some primes from \( S \), bounded by \( C \) at the finite places, plus \( D_0 \), the square free part of \( D \). We also note that the above argument can apply to higher order residue symbols.

**Lemma 1.4. Reciprocity.** Let \( D, D' \in I^+(S) \) be disjoint, and let

\[
\alpha(D, D') := \chi_D(D')\chi_{D'}(D).
\]

Then \( \alpha(D, D') \) depends only on the image of \( D \) and \( D' \) in \( H_C \).

The proof of the lemma can be found at [10]. We can use the reciprocity rule to define a bilinear form:

\[
\alpha : H_C \times H_C \to \{ \pm 1 \},
\]

which will be used later.

For a finite set \( S' \supseteq S \), like what we did before for \( S \), we can define \( C', R_{C'} \) and \( H_{C'} \), choose the representatives, and get the character \( \chi_{D'} \) for all the \( D' \in I(S') \subseteq I(S) \). The following result is useful.

**Proposition 1.5.** We can choose \( E'_0, \mathcal{E}' \) and \( \{ m'_E | E' \in \mathcal{E}' \} \) for \( S' \), such that \( \chi'_{D'} = \chi_{D'} \), where \( \chi'_{D'} \) denotes the character defined with respect to \( S' \), and \( \chi_{D'} \) denotes the characters with respect to \( S \).

**Proof.** The idea is to construct \( \mathcal{E}'_0 \) and \( \{ m'_E | E'_0 \in \mathcal{E}'_0 \} \) from \( \mathcal{E}_0 \) and \( \{ m_{E_0} | E_0 \in \mathcal{E}_0 \} \).

(1) From the surjective homomorphism \( R_{C'} \to R_C \), we get the surjective linear map \( \rho : H_C \to H_C \) over \( \mathbb{F}_2 \). We can write \( H_{C'} = \text{Ker} \rho \oplus N \) for some subspace \( N \), such that \( \rho |_N : N \to H_C \).

(2) We take the preimage of the basis of \( H_C \) in \( N \), and use them as the basis of \( N \). Combining it with a basis of \( \text{Ker} \rho \), we have a basis of \( H_{C'} \).

(3) For the basis of \( \text{Ker} \rho \), let us choose the representatives. First, take any \( E'_0 \) in the class. The class of \( E'_0 \) is in the Ker, so we can write \( E'_0 = (n)G^2 \) with \( n \equiv 1 \pmod{C} \), a totally positive integer and \( G \in I(S) \). Using Approximation theorem, we can assume further that \( G \in I(S') \), and \( n \) is disjoint from \( S' \). Now we change \( E'_0 = (n)E_0 \), and choose \( m'_{E_0} = n \).

(4) For the basis of \( N \), take any \( E'_0 \) in the class. Assume that \( E'_0 \) is in the class of \( E_0 \in H_C \) with \( E_0 \in \mathcal{E}_0 \). So we can write \( E'_0 = (n)E_0G^2 \) with \( n \equiv 1 \pmod{C} \), a totally positive integer, and \( G \in I(S) \). Similarly, we can assume that \( G \in I(S') \). Then we change \( E'_0 = (n)E_0 \), and take \( m'_{E_0} = m_{E_0}n \).

(5) Now we can extend the \( \mathcal{E}'_0 \) we have chosen to define \( E' \in \mathcal{E}' \) and \( m'_E \) for \( E' \in \mathcal{E}' \) as what we did for \( S \). It is easy to check that for any \( E' \in \mathcal{E}' \), if we assume that \( E' \) corresponds to the class of \( E \in \mathcal{E} \) in \( H_C \), then \( E' = (n)E \) with \( n \equiv 1 \pmod{C} \) and totally positive. And also \( m'_{E'} = m_{E}n \).

(6) Now we can prove that we get the same characters from the two settings. For any \( D' \in I(S') \), assume that \( D' \) is in the class of \( E' \in H_{C'} \) and in the class of \( E \in H_C \). So we have
$D' = (m') E' G'^2$ with $m' \equiv 1 \pmod{C'}$ and totally positive, and $G' \in I(S')$. By $E' = (n) E$, we have $D' = (n)(m') E G'^2$. So we have

$$\chi'_{D'} = \left( \frac{m'E m'}{E} \right)$$

and

$$\chi_{D'} = \left( \frac{mE nm'}{E} \right).$$

Since $m'E = mEn$, they are equal. □

1.3. Double Dirichlet series

We have chosen the set $S$. For any $r \in I_+(S)$, a square free integral ideal, we let $S_r$ be the set of primes (or places), either in $S$, or dividing $r$. For $S_r$, we have $C_r$, $R_C$, $H_C$, $E_r$, $\{mE\}$ and the quadratic characters $\chi_D$ for $D \in I(S)$. Similarly for $S_r$, we can define $C_r$, $R_{C_r}$ and $H_{C_r}$. For the set of representatives $E_r$ and $\{mE\}$, we will choose them as in Proposition 1.5, so the characters defined over $S_r$ will be identical with those defined over $S$.

We will review the definition and some properties of the double Dirichlet series following [9]. Instead of just $S$, we will generally work on $S_r$. First, let us introduce some notations. For any $D \in I_+(S_\infty)$, let $|D|$ be the norm of the ideal and we extend it multiplicatively to $I(S_\infty)$. We will consider $\pi$, a cuspidal automorphic representation of $GL_2(\mathbb{A})$. And $\beta$, an automorphic representation of $GL_1(\mathbb{A})$. Unless stated otherwise, in this section, we always assume that for any $\nu$ not in $S_r$, $\pi_\nu$ is unramified principal series with Satake parameters $\alpha_1$ and $\alpha_2$ and $\beta$ is unramified outside of $S_r$. For the Satake parameter $\alpha_i$, we extend them as a homomorphism of $I(S_r)$ to $\mathbb{C}^\times$.

We will write $\alpha_i(D)$ or $\alpha_{D,i}$ for the value at $D$. Similarly, we have $\tilde{\alpha}_i$ for $\tilde{\pi}$, the contragredient of $\pi$. For all the Hecke characters, considered as Grössencharakter, will be extended by 0 to ideals not prime to the conductor. We have the double Dirichlet series defined over $S_r$.

$$Z(s, w; \pi, \beta; S_r) := \sum_{D_1, D_2 \in I_+(S_r)} \frac{\beta(D)}{|D|^w} \prod_j \frac{\alpha_j(D_j)}{|D_j|^s} \left( \frac{D/e_j^2}{D_j/e_j^2} \right) |e_j|$$

(1)

and

$$Z_1(s, w; \pi, \beta, S_r) := \sum_{D_1, D_2 \in I_+(S_r)} \frac{\beta(D)}{|D|^w} \prod_j \frac{\alpha_j(D_j)}{|D_j|^s} \left( \frac{D_j/e_j^2}{D/e_j^2} \right) |e_j|.$$ 

(2)

Let $\hat{H}_C$ denote the character group of $H_C$. For $\rho_1, \rho_2 \in \hat{H}_C$, we have $Z(s, w; \pi \otimes \rho_1, \beta \otimes \rho_2; S_r)$ and $Z_1(s, w; \pi \otimes \rho_1, \beta \otimes \rho_2; S_r)$. If we let $V(\pi, \beta; S_r)$ be the vector space spanned by the series $Z(s, w; \pi \otimes \rho_1, \beta \otimes \rho_2; S_r)$ for $\rho_1, \rho_2 \in \hat{H}_C$, we have the following
Proposition 1.6. $V(\pi, \beta; S_r)$ is also the span of the series $Z_1(s, w; \pi \otimes \rho_1, \beta \otimes \rho_2; S_r)$ for $\rho_i \in \hat{H}_C$. And it is a finite-dimensional space and independent of choices of the representatives $E_r$ and the field elements $m_E$ with $E \in E_r$.

We will not use this, but the following statement.

Lemma 1.7. Let $\pi$ and $\beta$ be as before. We have

$$Z(s, w; \pi, \beta; S_r) = h^{-2} \sum_{E, E' \in \hat{H}_C} \alpha(E, E') \times \sum_{\rho_1, \rho_2 \in \hat{H}_C} \rho_1^{-1}(E)\rho_2^{-1}(E')Z_1(s, w; \pi \otimes \rho_2, \beta \otimes \rho_1; S_r)$$

and

$$Z_1(s, w; \pi, \beta; S_r) = h^{-2} \sum_{E, E' \in \hat{H}_C} \alpha(E, E') \times \sum_{\rho_1, \rho_2 \in \hat{H}_C} \rho_1^{-1}(E)\rho_2^{-1}(E')Z(s, w; \pi \otimes \rho_1, \beta \otimes \rho_2; S_r).$$

We will not give the details of the proof, since the idea is the same as in [9]. As we will do again and again, first we divide the summation into the sum over the specific classes in $\hat{H}_C$, then we can flip the quadratic symbols by the reciprocity law, and finally we use the orthogonality relations of $\hat{H}_C$ to recover the formation of $Z$ or $Z_1$. From the lemma, we can see that $Z(s, w; \pi, \beta; S_r)$ can be expressed as linear combination of $Z_1(s, w; \pi \otimes \rho, \beta \otimes \rho; S_r)$ with $\rho_1, \rho_2 \in \hat{H}_C$. Similarly, $Z_1(s, w; \pi, \beta; S_r)$ is a linear combination of $Z(s, w; \pi \otimes \rho_1, \beta \otimes \rho_2; S_r)$ with $\rho_1, \rho_2$, characters of $H_C$. And all the coefficients are constants related to $S$, independent of $r$.

We will use the above lemma later in the estimate. It basically means that the change from $Z(s, w; \pi, \beta; S_r)$ to $Z_1$, or $Z_1(s, w; \pi, \beta; S_r)$ to $Z$, will not change the sizes of the conductors of the representations defining the double Dirichlet series with respect to $|r|$. It just adds something from the set $S$ which is bounded absolutely. So when we later estimate the $Z$ and $Z_1$ over $S_r$ with respect to $|r|$, we can ignore the difference between $Z$ and $Z_1$.

For $T$, a finite set of places of $F$ including all the archimedean, the partial $L$-series of $\pi$ with places in $T$ removed is given by $\prod_{v \notin T} L_v(s, \pi)$. We denote it by $L_T(s, \pi)$ or $L(s, \pi; I - T)$ alternatively. And let $L^T(s, \pi) := \prod_{v \in T} L_v(s, \pi)$. Especially when $T = S_\infty$, we will use $L_\infty(s, \pi)$ and $L^\infty(s, \pi)$.

Following [9], we will express the double Dirichlet series as a sum of twisted $L$-functions times correction factors. First let us introduce these factors. For any homomorphism $\eta : I(S_r) \to \mathbb{C}^\times$, and $D \in I^+(S_r)$, we define the $GL_1$ correction factor:

$$a(s, \eta, D; S_r) := \sum_{d, e \in I^+(S_r)} \mu(d)\chi_D(d)e^{-s}\eta(de^2)|d|^{-s}|e|^{1-2s}.$$
where \( \mu \) is the Möbius function of \( I^+(S_r) \). And for \( \pi \) on \( \text{GL}_2(\mathbb{A}) \) with Satake parameters \( \alpha_1 \) and \( \alpha_2 \), we have the \( \text{GL}_2 \) correction factor \( a(s, \pi, D; S_r) := a(s, \alpha_1, D; S_r)a(s, \alpha_2, D; S_r) \). We have the following.

**Proposition 1.8.** The double Dirichlet series (1) is a weighted sum of quadratic twisted automorphic \( L \)-functions in variable \( s \). More precisely, in the region of absolutely convergence,

\[
Z(s, w; \pi, \beta; S_r) = \sum_{D \in I^+(S_r)} \frac{L(s, \pi \otimes \chi_D; I - S_r)a(s, \pi, D; S_r)\beta(D)}{|D|^w}. \tag{3}
\]

And,

**Proposition 1.9.** In the region of absolute convergence, the double Dirichlet series (2) is also a weighted sum of quadratic twisted \( L \)-functions in \( w \):

\[
Z_1(s, w; \pi, \beta; S_r) = L(4s + 2w - 2, \chi_\pi \beta^2; I - S_r) \times \sum_{D_1, D_2} \frac{\alpha_1(D_1) \alpha_2(D_2)}{|D_1|^s |D_2|^s}
\times L(w, \beta \chi_{D_1D_2}; I - S_r)a(w, \beta, D_1D_2; I - S_r). \tag{4}
\]

Now let us introduce some notations as in [9]. If \( \delta \) is any complex-valued function on \( H_{C_r} \), we can write it as a sum of characters

\[
\delta = \sum_{\rho \in \hat{H}_{C_r}} c_\rho \cdot \rho,
\]

we define

\[
Z(s, w; \pi, \beta \otimes \delta; S_r) = \sum_{\rho \in \hat{H}_{C_r}} c_\rho \cdot Z(s, w; \pi, \beta \rho; S_r).
\]

Similarly, we define

\[
Z_1(s, w; \pi \otimes \delta, \beta; S_r) = \sum_{\rho \in \hat{H}_{C_r}} c_\rho \cdot Z_1(s, w; \pi \otimes \rho, \beta; S_r).
\]

We want to talk about the functional equations of the double Dirichlet series, and we need the following

**Lemma 1.10.** Let \( D, E \in I^+(S) \) be square free ideals, and suppose that they are in the same class in \( H_C \). In another word, \( \chi_D = \chi_E \chi_m \) with some \( m \in F^\times \), totally positive, and \( m \equiv 1 \pmod{C} \). Let \( \pi \) be a unitary automorphic representation of \( GL_n(\mathbb{A}) \) for \( n = 1 \) or \( 2 \) (\( \pi \) is not necessarily cuspidal if \( n = 2 \)), which is unramified outside \( S \). Then

\[
\varepsilon(s, \pi \otimes \chi_D) = \chi_\pi(D/E)|D/E|^{n(1/2-s)}\varepsilon(s, \pi \otimes \chi_E).
\]
The proof of the lemma can be found at [7] or [9], which uses the fact that the root number for a Hecke L-function twisted by quadratic character is 1. We have the following simple observation from the lemma.

**Corollary 1.11.** Let \( \ell \in I^+(S) \) square free and let \( \ell_1 | \ell \). Suppose \( D, E \in I^+(S\ell) \) are in the same class of \( HC \), and \( \pi \) is as in the above lemma. Then

\[
\varepsilon(s, (\pi \otimes \chi_{\ell_1}) \otimes \chi_D) = \chi_\pi(D/E)|D/E|^{\nu(1/2-s)} \cdot \varepsilon(s, (\pi \otimes \chi_{\ell_1}) \otimes \chi_E).
\]

**Proof.** Since

\[
(\pi \otimes \chi_{\ell_1}) \otimes \chi_D = \pi \otimes \chi_{\ell_1} \chi_D = \pi \otimes \chi_{\ell_1} D,
\]

and

\[
(\pi \otimes \chi_{\ell_1}) \otimes \chi_E = \pi \otimes \chi_{\ell_1} \chi_E = \pi \otimes \chi_{\ell_1} E,
\]

we can just apply the lemma. \( \square \)

As in [9], we have the following functional equations:

**Proposition 1.12.** Let \( \pi \) and \( \beta \) be as at the beginning of the section. Let \( E \in I^+(S_r) \), and let \( \delta_E \) be the characteristic function of the class of \( E \) in \( HC \). Then the double Dirichlet series satisfy the following equations:

\[
Z(s, w; \pi, \beta \otimes \delta_E, S_r) = Z(1-s, w+2s-1; \tilde{\pi}, \chi_\pi \beta \otimes \delta_E; S_r)
\]

\[
\times \frac{\varepsilon(s, \pi \otimes \chi_E)}{\chi_\pi(E_0)|E_0|^{1-2s}} \prod_{v \in S_r} \frac{L_v(1-s, \tilde{\pi} \otimes \chi_E)}{L_v(s, \pi \otimes \chi_E)}
\]

(5)

and

\[
Z_1(s, w; \pi \otimes \delta_E, \beta; S_r) \cdot L(2s + 2w - 1, \chi_\pi \beta^2; I - S_r)
\]

\[
= Z_1\left(s + w - \frac{1}{2}, 1 - w; \pi \otimes \beta \otimes \delta_E, \beta^{-1}; S_r\right) \cdot L(2s, \chi_\pi; I - S_r)
\]

\[
\times \frac{\varepsilon(w, \beta \chi_E)}{\beta(E_0)|E_0|^{1/2-w}} \prod_{v \in S_r} \frac{L_v(1-w, \beta^{-1} \chi_E)}{L_v(w, \beta \chi_E)}
\]

(6)

**Proof.** For any place, archimedean or not in the set \( S_r \), the local representation is twisted by a quadratic character only dependent on the class of \( E \), so is the local L factor. This contributes to the last term in the above functional equations. For the \( \epsilon \) factors, we can just use the above lemma, which is true for any global field, instead of doing computation again. Now we can just adapt the proof of [9]. \( \square \)

We now work on the meromorphic continuation of the double Dirichlet series. We need the following result from complex analysis.
Lemma 1.13. If $\Omega$ is a connected tube in $\mathbb{C}^n$, then any holomorphic function in $\Omega$ can be extended to a holomorphic function in its convex hull.

For more information and the proof of the lemma, our reference is [13] or [8].

Proposition 1.14. Let $Z(s, w; S_r)$ be any series in the vector space $V(\pi, \beta; S_r)$. Then $(w - 1)(w + 2s - 2)L(2s + 2w - 1, \chi_{\pi}; I - S_r)Z(s, w; S_r)$ has a holomorphic continuation to the whole $\mathbb{C}^2$. 

In the proof, we will just consider $Z(s, w; \pi; \beta; S_r)$ and $Z_1(s, w; \pi; \beta; S_r)$, and ignore the twists by the characters of $H_{C_r}$ for looking simple and convenience.

Proof. First, we look at $Z(s, w; \pi; \beta; S_r)$. From (3), we see that in the region $\sigma = \Re(s) > 1$ and $\tau = \Re(w) > 1$, it is absolutely convergent and holomorphic. Now from (5), we get that the series is holomorphic for the region $\sigma < 0$ and $\tau + 2\sigma > 2$. In between, for $0 \leq \sigma \leq 1$ and $\tau$ very large, by the standard convexity method, we know the series is holomorphic. Applying the above lemma, we have $L(2s + 2w - 1, \chi_{\pi}; I - S_r) = Z(s, w; S_r)$ in the convex area bounded by the lines $\tau + 2\sigma > 2$, $\tau + \sigma > 2$ and $\tau = 1$ in the $\sigma \tau$ plane. We denote the area by $R_1$.

Next, we consider the functional equation (6), which moves the area $R_1$ to $R_2$, convex and bounded by $\tau = 0$, $\sigma = 3/2$ and $\tau + 2\sigma > 2$. So we get $L(2s + 2w - 1, \chi_{\pi}; I - S_r) = Z(s, w; S_r)$ is holomorphic on $R_2$. To close the gap between $R_1$ and $R_2$, we look at (4), different from the $L$-series of $\pi$ we worked on, the Hecke $L$-series will have a simple pole at $w = 1$ when the character is trivial. To use the convexity argument, we need add a factor $w - 1$. And we have that the function $(w - 1)L(2s + 2w - 1, \chi_{\pi}; I - S_r) = Z(s, w; S_r)$ is holomorphic for $0 \leq \tau < 1$ and $\sigma$ very large. Taking the convex hull, the function is holomorphic for $\tau + 2\sigma > 2$.

Finally, use the functional equation (5), which moves the area $\tau + 2\sigma > 2$ to $\tau > 1$, the area denoted by $R_3$. It is easy to see that the function $(w + 2s - 2)L(2s + 2w - 1, \chi_{\pi}; I - S_r) = Z(s, w; S_r)$ is holomorphic there. Now the result follows. $\Box$

For the use to the later estimate, we need to refine the functional equations we had before. As before, let $r \in I^+(S)$ square free. Assume $\pi$ and $\beta$ unramified outside $S_r$. Let $f_r(\pi)$ or $f_r(\beta)$ denote the product of the primes in the conductor of $\pi$ or $\beta$, dividing $r$.

Proposition 1.15. Let $E$ be the representative of a class in $H_C$, and $\delta_E$ be the characteristic function of the class, which we consider as a function on $I^+(S_r)$. Then

$$
\prod_{\nu | r} \left( 1 - \frac{\tilde{\alpha}_{v,1}^2}{|v|^2 - 2s} \right) \left( 1 - \frac{\tilde{\alpha}_{v,2}^2}{|v|^2 - 2s} \right) Z(s, w; \pi, \beta \otimes \delta_E; S_r) = \frac{\epsilon(s, \pi \otimes \chi_E)}{\chi_{\pi}(E_0)|E_0|^{2s}} \prod_{v \in S} \frac{L_v(1 - s, \tilde{\pi} \otimes \chi_E)}{L_v(s, \pi \otimes \chi_E)} \cdot h^{-1} \sum_{E' \in H_C} \alpha(E, E') \times \sum_{\rho \in H_C} \rho^{-1}(E') \cdot \sum_{t_1, t_2, s_1, s_2 | r} \frac{\mu(t_1)\alpha_{t_1,1} \mu(t_2)\alpha_{t_2,2}}{|t_1t_2|^s} \cdot \frac{\tilde{\alpha}_{s_1,1} \tilde{\alpha}_{s_2,2}}{s_1s_2} \cdot Z(1 - s, w + 2s - 1; \tilde{\pi}, \beta \delta_E \chi_{\pi} \chi_{t_1 t_2 s_1 s_2}; S_r),
$$

(7)
and

$$\prod_{v|r/f_r(\beta)} \left(1 - \frac{1}{|v|^{2-w}}\right) \cdot L(2s + 2w - 1, \chi_\pi; I - S_r) \cdot Z_1(s, w; \pi \otimes \delta_E, \beta; S_r)$$

$$= \frac{\epsilon(w, \beta \chi_E)}{\beta(E_0)|E_0|^{1/2-w}} \prod_{v \in S} L_v(1 - w, \beta^{-1} \chi_E) \cdot \frac{h^{-1 \sum \alpha(E, E')}}{E' \in H_C} \cdot \sum_{\rho \in H_C} \rho^{-1}(E') \cdot \sum_{m_1, m_2 \in r/f_r(\beta)} \beta(m_1) \mu(m_1) \beta^{-1}(m_2) \cdot \frac{\rho(m_1 m_2)}{|m_1|^w |m_2|^{1-w}}$$

$$\times L(2s, \chi_\pi; I - S_r) \cdot Z_1(s + w - 1/2, 1 - w; \pi \otimes \delta_E \otimes \beta \chi_{m_1 m_2}, \beta^{-1}; S_r). \quad (8)$$

The proof is just the combination of the arguments in [8,9].

**Proof.** By definition, we have

$$Z(s, w; \pi, \beta \otimes \delta_E; S_r) = \sum_{D \in I^+(S_r)} \frac{L(s, \pi \otimes \chi_D; I - S_r) a(s, \pi, D; S_r) \beta \delta_E(D)}{|D|^w}.$$  

From the functional equation of the \(L\)-functions, we have

$$L(s, \pi \otimes \chi_D; I - S_r) = \epsilon(s, \pi \otimes \chi_D) L(1-s, \tilde{\pi} \otimes \chi_D; I - S_r) \cdot \prod_{v \in S} L_v(1-s, \tilde{\pi} \otimes \chi_D) \frac{L_v(s, \pi \otimes \chi_D)}{L_v(1-s, \tilde{\pi} \otimes \chi_D)}.$$  

As we can see that \(\chi_D\) is locally equal to \(\chi_E\) at all the places in \(S\), so are the local twisted \(L\) factors. As in [9], if we write \(D = D_0 D_1^2\), we also have for the correction factors,

$$a(s, \pi, D; S_r) = \frac{\chi_\pi(D_1^2)}{|D_1^2|^{2s-1}} \cdot a(1-s, \tilde{\pi}, D; S_r).$$

Together with the corollary on the epsilon factors, we have

$$Z(s, w; \pi, \beta \otimes \delta_E; S_r)$$

$$= \frac{\epsilon(s, \pi \otimes \chi_E)}{\chi_\pi(E_0)|E_0|^{1-2s}} \prod_{v \in S} L_v(1-s, \tilde{\pi} \otimes \chi_E) \frac{L_v(s, \pi \otimes \chi_E)}{L_v(1-s, \tilde{\pi} \otimes \chi_E)}$$

$$\times \sum_{D \in I^+(S_r)} \frac{L(1-s, \tilde{\pi} \otimes \chi_D; I - S_r) a(1-s, \tilde{\pi}, D; S_r) \beta \chi_\pi \delta_E(D)}{|D|^w + 2s - 1}$$

$$\times \prod_{v|r/f_r(\pi)} \left(1 - \frac{\alpha_v, 1 \chi_D(v)}{|v|^s} \right) \left(1 - \frac{\alpha_v, 2 \chi_D(v)}{|v|^s} \right) \left(1 - \frac{\tilde{\alpha}_v, 1 \chi_D(v)}{|v|^{1-s}} \right)^{-1} \left(1 - \frac{\tilde{\alpha}_v, 2 \chi_D(v)}{|v|^{1-s}} \right)^{-1}. \quad (9)$$
Now if we multiply
\[
\prod_{v | r/f_{r}(\pi)} \left( 1 - \frac{\tilde{\alpha}_{v,1}^{2}}{|v|^{2-2s}} \right) \left( 1 - \frac{\tilde{\alpha}_{v,2}^{2}}{|v|^{2-2s}} \right)
\]
to the above equation, the last product factor becomes
\[
\prod_{v | r/f_{r}(\pi)} \left( 1 - \frac{\alpha_{v,1} \chi D(v)}{|v|^{s}} \right) \left( 1 - \frac{\alpha_{v,2} \chi D(v)}{|v|^{s}} \right) \left( 1 + \frac{\tilde{\alpha}_{v,1} \chi D(v)}{|v|^{1-s}} \right) \left( 1 + \frac{\tilde{\alpha}_{v,2} \chi D(v)}{|v|^{1-s}} \right)
\]
\[
= \sum_{t_{1} | r/f_{r}(\pi)} \frac{\mu(t_{1}) \alpha_{t_{1},1} \chi D(t_{1})}{|t_{1}|^{s}} \cdot \sum_{t_{2} | r/f_{r}(\pi)} \frac{\mu(t_{2}) \alpha_{t_{2},2} \chi D(t_{2})}{|t_{2}|^{s}} \cdot \sum_{s_{1} | r/f_{r}(\pi)} \frac{\tilde{\alpha}_{s_{1},1} \chi D(s_{1})}{|s_{1}|^{1-s}}
\]
\[
\times \sum_{s_{2} | r/f_{r}(\pi)} \frac{\tilde{\alpha}_{s_{2},2} \chi D(s_{2})}{|s_{2}|^{1-s}}
\]
\[
= \sum_{t_{1},t_{2},s_{1},s_{2} | r/f_{r}(\pi)} \frac{\mu(t_{1}) \mu(t_{2}) \alpha_{t_{1},1} \alpha_{t_{2},2} \tilde{\alpha}_{s_{1},1} \tilde{\alpha}_{s_{2},2}}{|t_{1}t_{2}|^{s} |s_{1}s_{2}|^{1-s}} \cdot \chi D(t_{1}t_{2}s_{1}s_{2})
\]
\[
\times \sum_{t_{1},t_{2},s_{1},s_{2} | r/f_{r}(\pi)} \frac{\mu(t_{1}) \mu(t_{2}) \alpha_{t_{1},1} \alpha_{t_{2},2} \tilde{\alpha}_{s_{1},1} \tilde{\alpha}_{s_{2},2}}{|t_{1}t_{2}|^{s} |s_{1}s_{2}|^{1-s}} \cdot \delta_{E'}(t_{1}t_{2}s_{1}s_{2})
\]
\[
\times \chi_{t_{1}t_{2}s_{1}s_{2}}(D)
\]
\[
= \sum_{E' \in H_{C}} \alpha(E, E') \sum_{t_{1},t_{2},s_{1},s_{2} | r/f_{r}(\pi)} \frac{\mu(t_{1}) \mu(t_{2}) \alpha_{t_{1},1} \alpha_{t_{2},2} \tilde{\alpha}_{s_{1},1} \tilde{\alpha}_{s_{2},2}}{|t_{1}t_{2}|^{s} |s_{1}s_{2}|^{1-s}} \cdot \delta_{E'}(t_{1}t_{2}s_{1}s_{2})
\]
\[
\times \chi_{t_{1}t_{2}s_{1}s_{2}}(D).
\]

(10)

Now we can just plug this back to the equation before and get the first functional equation. For the second one, we need use (4), then do as in [9]. □

From now on we will use \( \Phi \) and \( \Psi \) to denote the two involutions correspondent to the two functional equations. So we have
\[
\Phi(s, w) = (1 - s, w + 2s - 1),
\]
and
\[
\Psi(s, w) = (s + w - 1/2, 1 - w).
\]
2. Meromorphic continuation

2.1. The sieving process

In this section, we assume that $\pi$ and $\beta$ are unramified outside of $S$. Let $r$ be as before. For any $\ell | r$, we introduce a series defined over $S$:

$$Z(\ell)(s, w; \pi, \beta) := \sum_{\substack{D \in I^+(S) \setminus D_0D_1^2 \setminus \{D_1, \ell\} = 1}} \frac{L(s, \pi \otimes \chi_D; I - S)a(s, \pi, D; S)\beta(D)}{|D|^w}.$$

Next we define

$$Z(s, w; \pi, \beta; r) := \sum_{\ell | r} \mu(\ell)Z(\ell)(s, w; \pi, \beta).$$

We have

**Lemma 2.1.** In the region of absolute convergence,

$$Z(s, w; \pi, \beta; r) = \sum_{\substack{D \in I^+(S) \setminus D_0D_1^2 \setminus \{D_1, \ell\} = 1}} \frac{L(s, \pi \otimes \chi_D; I - S)a(s, \pi, D; S)\beta(D)}{|D|^w}.$$

**Proof.** Since $r$ is square free, we have

$$Z(s, w; \pi, \beta; r) = \sum_{\ell | r} \mu(\ell) \sum_{\substack{D \in I^+(S) \setminus D_0D_1^2 \setminus \{D_1, \ell\} = 1}} \frac{L(s, \pi \otimes \chi_D; I - S)a(s, \pi, D; S)\beta(D)}{|D|^w}.$$

$$= \sum_{\substack{D \in I^+(S) \setminus D_0D_1^2 \setminus \{D_1, \ell\} = 1}} \frac{L(s, \pi \otimes \chi_D; I - S)a(s, \pi, D; S)\beta(D)}{|D|^w}. $$

We know that $\sum_{\ell | r} \mu(\ell) = 1$ if $\frac{r}{(r, D_1)} = 1$, i.e. $r | D_1$. Otherwise, it is 0. \qed

Next, if we let $\sum'$ stand for sum over the square free terms only, we define

$$Z^*(s, w; \pi, \beta) := \sum' \frac{L(s, \pi \otimes \chi_D; I - S)a(s, \pi, D; S)\beta(D)}{|D|^w},$$

$$= \sum' \frac{L(s, \pi \otimes \chi_D; I - S)\beta(D)}{|D|^w}. $$
Here we use the fact that for the square free $D$, from the definition the correction factor $a(s, \pi, D; S) = 1$. We also have

**Lemma 2.2.** In the region of absolute convergence,

$$Z^b(s, w; \pi, \beta) = \sum_{r \in I^+(S)} \mu(r) Z(s, w; \pi, \beta; r).$$

**Proof.**

$$\sum_{r \in I^+(S)} \mu(r) Z(s, w; \pi, \beta; r) = \sum_{r \in I^+(S)} \mu(r) \sum_{D \in I^+(S)} \frac{L(s, \pi \otimes \chi_D; I - S)a(s, \pi, D; S)\beta(D)}{|D|^w}$$

$$= \sum_{D \in I^+(S)} \left( \sum_{r | D_1} \mu(r) \right) \frac{L(s, \pi \otimes \chi_D; I - S)a(s, \pi, D; S)\beta(D)}{|D|^w}.$$

Now the lemma follows from the property of $\mu$. $\square$

We want to express $Z^{(\ell)}(s, w; \pi, \beta)$ in term of the double Dirichlet series we defined.

**Proposition 2.3.** We have

$$\prod_{\nu | \ell} \left( 1 - \frac{\alpha^{2}_{\nu, 1}}{|\nu|^{2s}} \right) \left( 1 - \frac{\alpha^{2}_{\nu, 2}}{|\nu|^{2s}} \right) Z^{(\ell)}(s, w; \pi, \beta)$$

$$= \sum_{\ell_3 | \ell} \beta(\ell_3) \prod_{\nu | \ell_3} \left( 1 - \frac{\alpha^{2}_{\nu, 1}}{|\nu|^{2s}} \right) \left( 1 - \frac{\alpha^{2}_{\nu, 2}}{|\nu|^{2s}} \right) \cdot h^{-2} \sum_{E, E' \in \mathcal{HC}} \alpha(E, E')$$

$$\times \sum_{\rho, \rho' \in \mathcal{HC}} \rho(E) \rho'(E') \cdot \sum_{g_1, g_2 | \ell/\ell_3} \frac{\alpha_{g_1, 1} \alpha_{g_2, 2} \chi_{\ell_3} (g_1 g_2) \rho(g_1 g_2)}{|g_1 g_2|^\delta}$$

$$\times Z(s, w; \pi \otimes \chi_{\ell_3}, \beta \rho' \chi_{g_1 g_2}; S_\ell).$$

**Proof.**

$$Z^{(\ell)}(s, w; \pi, \beta) = \sum_{D \in I^+(S)} \frac{L(s, \pi \otimes \chi_D; I - S)a(s, \pi, D; S)\beta(D)}{|D|^w}$$

$$= \sum_{\ell_3 | \ell} \sum_{D \in I^+(S)} \frac{L(s, \pi \otimes \chi_D; I - S)a(s, \pi, D; S)\beta(D)}{|D|^w}.$$
\[= \sum_{\ell_3 | \ell} \sum_{D \in I^+(S)} \frac{L(s, \pi \otimes \chi_{\ell_3} D; I - S)a(s, \pi, \ell_3 D; S)\beta(\ell_3 D)}{|\ell_3 D|^w} \]

\[= \sum_{\ell_3 | \ell} \beta(\ell_3) \sum_{D \in I^+(S_\ell)} \frac{L(s, \pi \otimes \chi_{\ell_3} \chi D; I - S)a(s, \pi, \ell_3 D; S)\beta(D)}{|D|^w}. \]

Because \(\pi\) is twisted by \(\chi_{\ell_3}\), the local factors at \(v | \ell_3\) in the partial \(L\)-functions will be 1. We have

\[
L(s, \pi \otimes \chi_{\ell_3} \chi D; I - S) \\
= L(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi D; I - S_\ell) \prod_{v | \ell/\ell_3} L_v(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi D) \\
= L(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi D; I - S_\ell) \\
\times \prod_{v | \ell/\ell_3} \left(1 - \frac{\alpha_v \chi_{\ell_3}(v) \chi D(v)}{|v|^s}\right)^{-1} \left(1 - \frac{\alpha_v 2 \chi_{\ell_3}(v) \chi D(v)}{|v|^s}\right)^{-1}. \tag{12}
\]

For the correction factors,

\[
a(s, \pi, \ell_3 D; S) = a(s, \alpha_1, \ell_3 D; S)a(s, \alpha_2, \ell_3 D; S),
\]

where

\[
a(s, \alpha_i, \ell_3 D; S) = \sum_{d,e \in I^+(S) \atop d^2e^2 \equiv \ell_3 D} \frac{\mu(d)\chi_{\ell_3} \chi D(d)\alpha_i(de^2)}{|d|^s|e|^{2s-1}}.
\]

Since \(d^2e^2 \equiv \ell_3 D\), \(d\) and \(e\) appear in the square part of \(D\). And \(D\) is prime to \(\ell\), hence \(d\) and \(e\) are in \(I^+(S_\ell)\). Also,

\[
\chi_{\ell_3} \chi D(d)\alpha_i(de^2) = \chi D(d)(\alpha_i\chi_{\ell_3})(de^2).
\]

So by the definition,

\[
a(s, \alpha_i, \ell_3 D; S) = a(s, \alpha_i \chi_{\ell_3}, D; S_\ell)
\]

and

\[
a(s, \pi, \ell_3 D; S) = a(s, \pi \otimes \chi_{\ell_3}, D; S_\ell).
\]

Now we compute:
\[
\prod_{v|\ell} \left(1 - \frac{\alpha_{v,1}^2}{|v|^{2s}}\right) \left(1 - \frac{\alpha_{v,2}^2}{|v|^{2s}}\right) \cdot Z^{(l)}(s, w; \pi, \beta)
\]

\[
= \sum_{\ell_3|\ell} \frac{\beta(\ell_3)}{|\ell_3|^w} \cdot \prod_{v|\ell_3} \left(1 - \frac{\alpha_{v,1}^2}{|v|^{2s}}\right) \left(1 - \frac{\alpha_{v,2}^2}{|v|^{2s}}\right) \times \sum_{D \in I^+(S_\ell)} \frac{L(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi_D; I - S_\ell) \cdot a(s, \pi \otimes \chi_{\ell_3}, D; S_\ell) \beta(D)}{|D|^w} \\
\times \prod_{v|\ell/\ell_3} \left(1 + \frac{\alpha_{1,v} \chi_{\ell_3} \chi_D(v)}{|v|^s}\right) \left(1 + \frac{\alpha_{2,v} \chi_{\ell_3} \chi_D(v)}{|v|^s}\right)
\]

\[
= \sum_{\ell_3|\ell} \frac{\beta(\ell_3)}{|\ell_3|^w} \cdot \prod_{v|\ell_3} \left(1 - \frac{\alpha_{v,1}^2}{|v|^{2s}}\right) \left(1 - \frac{\alpha_{v,2}^2}{|v|^{2s}}\right) \times \sum_{D \in I^+(S_\ell)} \frac{L(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi_D; I - S_\ell) a(s, \pi \otimes \chi_{\ell_3}, D; S_\ell) \beta(D)}{|D|^w} \\
\times \left(\sum_{g_1|\ell/\ell_3} \frac{\alpha_{g_1,1} \chi_{\ell_3}(g_1) \chi_D(g_1)}{|g_1|^s} \cdot \sum_{g_2|\ell/\ell_3} \frac{\alpha_{g_2,2} \chi_{\ell_3}(g_2) \chi_D(g_2)}{|g_2|^s}\right)
\]

\[
= \sum_{\ell_3|\ell} \frac{\beta(\ell_3)}{|\ell_3|^w} \prod_{v|\ell_3} \left(1 - \frac{\alpha_{v,1}^2}{|v|^{2s}}\right) \left(1 - \frac{\alpha_{v,2}^2}{|v|^{2s}}\right) \cdot \sum_{g_1,g_2|\ell/\ell_3} \frac{\alpha_{g_1,1} \alpha_{g_2,2} \chi_{\ell_3}(g_1 g_2)}{|g_1 g_2|^s} \\
\times \sum_{D \in I^+(S_\ell)} \frac{L(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi_D; I - S_\ell) a(s, \pi \otimes \chi_{\ell_3}, D; S_\ell) \beta(D) \chi_D(g_1 g_2)}{|D|^w}. \tag{13}
\]

Now we consider

\[
\sum_{g_1,g_2|\ell/\ell_3} \frac{\alpha_{g_1,1} \alpha_{g_2,2} \chi_{\ell_3}(g_1 g_2)}{|g_1 g_2|^s} \\
\times \sum_{D \in I^+(S_\ell)} \frac{L(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi_D; I - S_\ell) a(s, \pi \otimes \chi_{\ell_3}, D; S_\ell) \beta(D) \chi_D(g_1 g_2)}{|D|^w}
\]

\[
= \sum_{E,E' \in H_C} \alpha(E, E') \sum_{g_1,g_2|\ell/\ell_3} \frac{\alpha_{g_1,1} \alpha_{g_2,2} \chi_{\ell_3}(g_1 g_2)}{|g_1 g_2|^s} \\
\times \sum_{D \in I^+(S_\ell)} \frac{L(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi_D; I - S_\ell) a(s, \pi \otimes \chi_{\ell_3}, D; S_\ell) \beta(D) \chi_D(g_1 g_2)}{|D|^w}
\]

\[
= \sum_{E,E' \in H_C} \alpha(E, E') \sum_{g_1,g_2|\ell/\ell_3} \frac{\alpha_{g_1,1} \alpha_{g_2,2} \chi_{\ell_3}(g_1 g_2)}{|g_1 g_2|^s} \cdot \frac{1}{h} \sum_{\rho \in H_C} \frac{\rho(g_1 g_2)}{\rho(E)}
\]
\[
\times \sum_{D \in \mathcal{I}^+(S_\ell)} \frac{L(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi_D; S_\ell) a(s, \pi \otimes \chi_{\ell_3}, D; S_\ell) \beta(D) \chi_{g_{1g_2}}(D)}{|D|^w} \cdot \frac{1}{h} \sum_{\rho', E} \frac{\rho'(D)}{\rho'(E')}
\]

\[
= h^{-2} \sum_{E, E' \in H_C} \alpha(E, E') \sum_{\rho, \rho' \in \hat{H}_C} \rho(E) \rho'(E') \sum_{g_{1g_2} \ell/\ell_3} \frac{\alpha_{g_1, 1} \alpha_{g_2, 2} \chi_{\ell_3}(g_{1g_2}) \rho(g_{1g_2})}{|g_{1g_2}|^s} \\
\times \sum_{D \in \mathcal{I}^+(S_\ell)} \frac{L(s, (\pi \otimes \chi_{\ell_3}) \otimes \chi_D; S_\ell) a(s, \pi \otimes \chi_{\ell_3}, D; S_\ell) (\beta \chi_{g_{1g_2}})(D)}{|D|^w} \\
= h^{-2} \sum_{E, E' \in H_C} \alpha(E, E') \sum_{\rho, \rho' \in \hat{H}_C} \rho(E) \rho'(E') \sum_{g_{1g_2} \ell/\ell_3} \frac{\alpha_{g_1, 1} \alpha_{g_2, 2} \chi_{\ell_3}(g_{1g_2}) \rho(g_{1g_2})}{|g_{1g_2}|^s} \\
\times Z(s, w; \pi \otimes \chi_{\ell_3}, \beta \rho' \chi_{g_{1g_2}}; S_\ell). \quad (14)
\]

2.2. Estimates

We will make the meromorphic continuation of \( Z^*(s, w; \pi, \beta) \) to the point \((s, w) = (1/2, 1)\). To achieve this, we need do some estimates. Note that throughout this section, we will not track those functions of \( \epsilon \) from the estimates. Instead, we will use \( \epsilon \) for all of them for simplicity. Those functions go to 0 as \( \epsilon \) goes to 0. This is the only property we will use. Eventually we will take \( \epsilon \) very small so we can ignore these functions. From this point, it makes no difference if we just write \( \epsilon \). In this section the basic tool we need here is the following Rademacher’s version of Phragmen–Lindelöf as in [18].

**Lemma 2.4.** We denote the strip \( a < \sigma = \text{Re}(s) < b \) by \( S(a, b) \). Let \( f(s) \) be an analytic function in \( S(a, b) \) and satisfy for certain positive constants \( c \) and \( C \)

\[
|f(s)| < Ce^{\sigma |s|^c}
\]

Suppose moreover that

\[
|f(a + it)| \leq A|Q + a + it|^\alpha
\]

and

\[
|f(b + it)| \leq B|Q + b + it|^\beta
\]

with

\[
Q + a > 0.
\]

Then in the strip \( S(a, b) \), we have

\[
|f(s)| \leq M_Q \left( A|Q + s|^\alpha \right)^{\frac{b-a}{b-a}} \left( B|Q + s|^\beta \right)^{\frac{\sigma-a}{b-a}},
\]

where \( M_Q \) is a constant number just depending on \( Q, a, b, \alpha, \) and \( \beta \).
If $\pi$ is a cuspidal automorphic representation on $GL_2(\mathbb{A})$, unramified outside $S$, we have the partial $L$-series $L_S(s, \pi)$. We let $A$ be a constant number such that the Satake parameters are bounded as $|v|^{-A} \leq |\alpha_{i,v}| \leq |v|^A$. We call $A$ an exponential bound of the Satake parameters. The famous Ramanujan conjecture predicts that $A = 0$. It is easy to see that the partial $L$-series converges absolutely for $\sigma = \text{Re}(s) > 1 + A$. In fact there is a better result.

**Lemma 2.5.** Suppose $\pi$ is a unitary cuspidal automorphic representation on $GL_2(\mathbb{A})$, unramified outside $S$. Then the $L$-series $L_S(s, \pi)$ converges absolutely for $\sigma = \text{Re}(s) > 1$.

**Proof.** For any $v \not\in S$, by the unitarity of $\pi_v$, we have for the Satake parameters

$$\{\alpha_{j,v}^{-1} \mid j = 1, 2\} = \{\tilde{\alpha}_{j,v} \mid j = 1, 2\}.$$

So the Satake parameters of $\tilde{\pi}$ are given by the complex conjugates of those of $\pi$. We have, for $\sigma = \text{Re}(s) > 1$, the Euler product

$$L_S(s, \pi \times \tilde{\pi}) = \prod_{v \not\in S} \prod_{j} \left(1 - \frac{\alpha_{i,v} \tilde{\alpha}_{j,v}}{|v|^s}\right)^{-1}.$$

It is known that $L_S(s, \pi \times \tilde{\pi})$ has a simple pole at $s = 1$ with none to the right, and it has positive coefficients in the series expansion. So the series converges absolutely for $\sigma > 1$. This is a well-known fact from analytic number theory. From the first lemma in [1], we have that, for $\sigma$ large, by looking at the Euler factors,

$$\sum_{D \in I^+(S)} \frac{|c(D)|^2}{|D|^s} = L_S(s, \pi \times \tilde{\pi}) \prod_{v \not\in S} (1 - |v|^{-2s}).$$

So it is absolutely convergent for $\sigma > 1$. Now we use the Cauchy–Schwartz inequality,

$$\sum_{D} \frac{|c(D)|}{|D|^{1+\epsilon}} = \sum_{D} \frac{|c(D)|}{|D|^{(1+\epsilon)/2}} \cdot \frac{1}{|D|^{(1+\epsilon)/2}} \leq \sum_{D} \frac{|c(D)|^2}{|D|^{1+\epsilon}} \cdot \sum_{D} \frac{1}{|D|^{1+\epsilon}}. \quad \Box$$

We also want to mention that as in [2], a cuspidal automorphic representation of $GL_n(\mathbb{A})$ are automatically unitary provided that its central character is unitary.

From now on, we will fix the representation $\pi$ and the Hecke character $\beta$. We assume that they are unramified outside the set $S$ we chose before and $\beta$ is quadratic. Let $\pi' = \pi \otimes \chi'$ be a twist of $\pi$ by some $\chi' \in \hat{H}_C$. Now let us look at some simple estimates on the twisted partial $L$-series.

First, for any $\epsilon > 0$, and $\sigma = \text{Re}(s) > 1 + \epsilon$, we have

$$L(s, \pi' \otimes \chi_D; I - S_r) \ll_\epsilon 1.$$
\[ L(s, \pi' \otimes \chi_D; I - S_r) = \varepsilon(s, \pi' \otimes \chi_D) L(1 - s, \tilde{\pi}' \otimes \chi_D; I - S_r) \]
\[ \times \prod_{v \in S} \frac{L_v(1 - s, \tilde{\pi}' \otimes \chi_D)}{L_v(s, \pi' \otimes \chi_D)} \cdot \prod_{v \mid r \atop v \not| f_r(\chi')} \frac{L_v(1 - s, \tilde{\pi}' \otimes \chi_D)}{L_v(s, \pi' \otimes \chi_D)}. \]

Looking on the vertical line \( s = -\epsilon + it \), we can get the estimates for each factor on the right-hand side of the above equation. We have
\[ \varepsilon(s, \pi' \otimes \chi_D) \ll_\epsilon |D_0 f_r(\chi')|^{1+2\epsilon}, \]
and
\[ \prod_{v \in S} \frac{L_v(1 - s, \tilde{\pi}' \otimes \chi_D)}{L_v(s, \pi' \otimes \chi_D)} \ll_\epsilon (|t| + 1)^{b_0}, \]
where \( b_0 \) is some constant number. Also for \( v \mid f_r(\chi') \),
\[ \frac{1}{L_v(s, \pi' \otimes \chi_D)} = \left( 1 - \frac{\alpha_v,1 \chi'(v)\chi_D(v)}{|v|^s} \right) \left( 1 - \frac{\alpha_v,2 \chi'(v)\chi_D(v)}{|v|^s} \right) \ll_\epsilon |v|^{A+\epsilon}. \]

We get, by putting all these together,
\[ L(s, \pi' \otimes \chi_D; I - S_r) \ll_\epsilon |D_0 f_r(\chi')|^{1+2\epsilon} \left( \frac{r}{f_r(\chi')} \right)^{A+\epsilon} (|t| + 1)^{b_0}. \]

Note that the estimates we need later are for \( s \) near \( 1/2 \). But we will just assume that \( s = 1/2 \) for convenience. It will not essentially effect the result. Applying the Phragmen–Lindelöf theorem to above, we have
\[ L \left( \frac{1}{2}, \pi' \otimes \chi_D; I - S_r \right) \ll_\epsilon |D_0|^{1/2+\epsilon} |f_r(\chi')|^{1/2+\epsilon} \left( \frac{r}{f_r(\chi')} \right)^{A/2+\epsilon}. \]

Next we work on \( L(s, \pi' \otimes \chi_D; I - S_r)a(s, \pi', D; S_r) \). Let \( E \) be the representative of the class of \( D \) in \( H_C \), from the proof of (7), we have the functional equations
\[ L(s, \pi' \otimes \chi_D; I - S_r)a(s, \pi', D; S_r) \]
\[ = \frac{\varepsilon(s, \pi' \otimes \chi_E)}{\varepsilon_\pi(E_0)|E_0|^{1-2s}} L(1 - s, \tilde{\pi}' \otimes \chi_E; I - S_r)a(1 - s, \tilde{\pi}', D; S_r) \]
\[ \times \prod_{v \in S} \frac{L_v(1 - s, \tilde{\pi}' \otimes \chi_E)}{L_v(s, \pi' \otimes \chi_E)} \cdot \prod_{v \mid r \atop v \not| f_r(\chi')} \frac{L_v(1 - s, \tilde{\pi}' \otimes \chi_E)}{L_v(s, \pi' \otimes \chi_E)}. \]

So from the estimates on the right-hand side, we have on the vertical line \( \sigma = \text{Re}(s) = -\epsilon \),
\[ L(s, \pi' \otimes \chi_D; I - S_r)a(s, \pi', D; S_r) \ll_\epsilon |f_r(\chi')|^{1+2\epsilon} \left( \frac{r}{f_r(\chi')} \right)^{A+\epsilon} |D|^{1+2\epsilon} (|t| + 1)^{d_0}. \]
Then by the Phragmen–Lindelöf theorem,
\[
L\left(\frac{1}{2}, \pi' \otimes \chi_D; I - S_r\right) a(s, \pi', D; S_r) \ll \varepsilon \left| f_r(\chi') \right|^{1/2+\varepsilon} \left| r \right|^{A/2+\varepsilon} \left| D \right|^{1/2+\varepsilon}.
\]
We have the following

**Corollary 2.6.** Let \( \beta' \) be a Hecke character unramified outside of \( S_r \). Then for \( s = 1/2 \) and \( \tau = \text{Re}(w) = 1 + 1/2 + \varepsilon \), the series \( Z(s, w; \pi', \beta'; S_r) \) converges absolutely, and
\[
Z(1/2, 1 + 1/2 + \varepsilon + it; \pi', \beta'; S_r) \ll \varepsilon \left| f_r(\chi') \right|^{1/2+\varepsilon} \left| r \right|^{A/2+\varepsilon}.
\]

Now we will work on the double Dirichlet series and further assume that \( \pi \) is also self-contragredient. We have
\[
Z^*(s, w; \pi, \beta) = \sum_{D \in I^+(S)} L(s, \pi \otimes \chi_D; I - S) \beta(D) / |D|^w.
\]
Our goal is to get the meromorphic continuation for it near the point \((s, w) = (1/2, 1)\). As we have seen before,
\[
Z^*(s, w; \pi, \beta) = \sum_r \mu(r) Z(s, w; \pi, \beta; r).
\]
What we will do is to show \((w - 1)(w + 2s - 2) \cdot Z(s, w; \pi, \beta; r)\) is holomorphic near the point, and give an estimate of the function there with respect to \( r \), which will imply \((w - 1)(w + 2s - 2) \sum_r \mu(r) Z(s, w; \pi, \beta; r)\) converges absolutely and uniformly. So we have continuation we want. We will work with \((w - 1)(w + 2s - 2) \cdot L(2s + 2w - 1, \chi_\pi; I - S_r) \cdot Z(s, w; \pi, \beta; r)\).

Note that the factor \( L(2s + 2w - 1, \chi_\pi; I - S_r) \) converges absolutely near the point and uniformly on \( r \). For convenience, we will ignore the factor \((w - 1)(w + 2s - 2)\). We can write
\[
L(2s + 2w - 1, \chi_\pi; I - S_r) = L(2s + 2w - 1, \chi_\pi; I - S_\ell) \cdot \prod_{v|\ell/\ell'} L_v(2s + 2w - 1, \chi_\pi)^{-1}
\]
\[
= L(2s + 2w - 1, \chi_\pi; I - S_\ell) \cdot \prod_{v|\ell/\ell'} \left(1 - \frac{\chi_\pi(v)}{|v|^{2s+2w-1}}\right).
\]
This plus Proposition 2.3 gives us
\[
L(2s + 2w - 1, \chi_\pi; I - S_r) \cdot Z(s, w; \pi, \beta; r)
\]
\[
= \sum_{\ell|r} \mu(\ell) \cdot \prod_{v|\ell/\ell'} \left(1 - \frac{\chi_\pi(v)}{|v|^{2s+2w-1}}\right) \cdot h^{-2} \sum_{E, E' \in H_C} \alpha(E, E')
\]
\[
\times \sum_{\rho, \rho' \in H_C} \rho(E) \rho'(E') \cdot \sum_{\ell|\ell_3} \frac{\beta(\ell)}{[\ell_3]^w} \prod_{v|\ell/\ell_3} \left(1 - \frac{\alpha_1(v)}{|v|^{2s}}\right)^{-1} \left(1 - \frac{\alpha_2(v)}{|v|^{2s}}\right)^{-1}
\]
representations are given by functional equations, but we will ignore the changes in the S part, which will not contribute to the estimate to \(|r|\). We will follow the idea of [8] and estimate the function at the two ends. One is \((s, w) = (1/2, 1)\) with respect to \(|r|\). It is easy to see that if we change \(s\) slightly, the estimate will not change much. So we will just assume that \(s = 1/2\) for simplicity, and this will not effect our later argument. We will work in steps to show the process just outlined above. In each step, we will show how the coordinates change, and track how the \(GL_2\) representations are changed by the functional equations. It is easy to compute that \(Z(s, w; \pi \otimes \chi_{\ell_3}, \beta \chi_{g_{1}g_2} \rho'; S_\ell)\).

$$
\times \sum_{g_1g_2|\ell/\ell_3} \frac{\alpha_{g_1,1}\alpha_{g_2,2}\chi_{\ell_3}(g_{1}g_2)\rho(g_{1}g_2)}{|g_{1}g_2|^3} 
\times L(2s + 2w - 1, \chi_\pi; I - S_\ell) \cdot Z(s, w; \pi \otimes \chi_{\ell_3}, \beta \chi_{g_{1}g_2} \rho'; S_\ell) \tag{15}
$$

Because \(s\) is near \(1/2\), from Proposition 1.14, we have that \((w - 1)(w + 2w - 2) \cdot L(2s + 2w - 1, \chi_\pi; I - S_\ell) \cdot Z(s, w; \pi, \beta; r)\) is holomorphic.

We will estimate the above function near the point \((s, w) = (1/2, 1)\) with respect to \(|r|\). It is easy to see that if we change \(s\) slightly, the estimate will not change much. So we will just assume that \(s = 1/2\) for simplicity, and this will not effect our later argument. We will work in steps to show the process just outlined above. In each step, we will show how the coordinates change, and track how the \(GL_2\) and \(GL_1\) representations are changed by the functional equations, but we will ignore the changes in the \(S\) part, which will not contribute to the estimate to \(|\ell|\). We will also list the factors added by the functional equations.

First step, as in (15), is at \((s, w) = (1/2, -1/2 - \epsilon + it)\). For those factors, we have

$$
\prod_{v|r/\ell} \left( 1 - \frac{\chi_\pi(v)}{|v|^{2s+2w-1}} \right) \ll \epsilon \left| \frac{r}{\ell} \right|^{1+2\epsilon}, \tag{16}
$$

$$
\frac{\beta(\ell_3)}{|\ell_3|^{w}} \ll \epsilon \left| \ell_3 \right|^{1/2+\epsilon}, \tag{17}
$$

$$
\prod_{v|\ell/\ell_3} \left( 1 - \frac{\alpha_{v,1}^2}{|v|^{2r}} \right)^{-1} \left( 1 - \frac{\alpha_{v,2}^2}{|v|^{2s}} \right)^{-1} \ll \epsilon \left| \frac{\ell}{\ell_3} \right|^{\epsilon}. \tag{18}
$$

If we let the square free part \((g_{1}g_2)_0 = \ell_2\), we have \((\ell_2, \ell_3) = 1\) and

$$
\frac{\alpha_{g_1,1}\alpha_{g_2,2}\chi_{\ell_3}\rho(g_{1}g_2)}{|g_{1}g_2|^3} \ll \epsilon \left| \ell_2 \right|^{4-1/2+\epsilon}. \tag{19}
$$

Second step, applying the functional equation (8), the changes of the coordinates and the representations are given by

$$
\Psi(1/2, -1/2 - \epsilon + it) = (-1/2 - \epsilon + it, 1 + 1/2 + \epsilon - it), \tag{19}
$$

$$
(\pi \otimes \chi_{\ell_3}, \beta \chi_{\ell_2}) \rightarrow (\pi \otimes \chi_{\ell_2 \chi_{(\ell_3m_1m_2)_0}}, \beta \chi_{\ell_2}). \tag{20}
$$

And at \(w = -1/2 - \epsilon + it\), the factors added are
\(\varepsilon(w, \beta \chi_{\ell_2}) \ll \varepsilon|\ell_2|^{1+\varepsilon},\)  
(21)

\[
\sum_{m_1, m_2 \mid \ell_2} \frac{\mu(m_1)\beta(m_1 m_2)}{|m_1|^w |m_2|^{1-w}}. 
\]  
(22)

Third step, we fix a pair of \(m_1\) and \(m_2\) from the step before. Applying the functional equation (7), we have

\[
\Psi(-1/2 - \varepsilon + it, 1 + 1/2 + \varepsilon - it) = (1 + 1/2 + \varepsilon - it, -1/2 - \varepsilon + it)
\]

and

\[
(\pi \otimes \chi_{\ell_2} \chi(\ell_3 m_1 m_2)_0, \beta \chi_{\ell_2}) \rightarrow (\pi \otimes \chi_{\ell_2} \chi(\ell_3 m_1 m_2)_0, \beta \chi_{\ell_2} \chi(t_1 t_2 s_1 s_2)_0).
\]

And at \(s = -1/2 - \varepsilon + it\), the factors added are

\[
\varepsilon(s, \pi \otimes \chi_{\ell_2} \chi(\ell_3 m_1 m_2)_0) \ll \varepsilon|\ell_2(\ell_3 m_1 m_2)_0|^{2+2\varepsilon},
\]

(23)

\[
\sum_{t_1, t_2, s_1, s_2 \mid \ell_2(\ell_3 m_1 m_2)_0} \frac{\alpha_{t_1, 1} \alpha_{t_2, 2} \alpha_{s_1, 1} \alpha_{s_2, 2} \mu(t_1)\mu(t_2)}{|t_1 t_2|^w |s_1 s_2|^{1-w}}.
\]

(24)

Next step, we fix a set of \(t_1, t_2, s_1\) and \(s_2\) from the step before. Applying (8), then the changes are

\[
\Psi(1 + 1/2 + \varepsilon - it, -1/2 - \varepsilon + it) = (1/2, 1 + 1/2 + \varepsilon - it).
\]

And

\[
(\pi \otimes \chi_{\ell_2} \chi(\ell_3 m_1 m_2)_0, \beta \chi_{\ell_2} \chi(t_1 t_2 s_1 s_2)_0) \rightarrow (\pi \otimes \chi(\ell_3 m_1 m_2)_0 \chi(t_1 t_2 s_1 s_2)_0 \chi_1 \chi_2 \chi(t_1 t_2 s_1 s_2)_0, \beta \chi_{\ell_2} \chi(t_1 t_2 s_1 s_2)_0).
\]

(25)

The factors added at \(w = -1/2 - \varepsilon + it\) are given by

\[
\varepsilon(w, \beta \chi_{\ell_2} \chi(t_1 t_2 s_1 s_2)_0) \ll \varepsilon|\ell_2|^{1+\varepsilon}|(t_1 t_2 s_1 s_2)_0|^{1+\varepsilon},
\]

(26)

\[
\sum_{n_1, n_2 \mid \ell_2(t_1 t_2 s_1 s_2)_0} \frac{\mu(n_1)\beta \chi_{\ell_2} \chi(t_1 t_2 s_1 s_2)_0(n_1 n_2)}{|n_1|^w |n_2|^{1-w}}.
\]

(27)

At the final step, we reach the vertical line \((1/2, 1 + 1/2 + \varepsilon - it)\). Since \(L(2s + 2w - 1, \chi_\pi; I - S_\ell)\) is absolutely bounded on the line, we have from Corollary 2.6

\[
Z(s, w; \pi \otimes \chi(\ell_3 m_1 m_2)_0 \chi(t_1 t_2 s_1 s_2)_0 \chi_1 \chi_2 \chi(t_1 t_2 s_1 s_2)_0; S_\ell)
\ll \varepsilon|((\ell_3 m_1 m_2)_0 n_1 n_2)_0 \cdot (t_1 t_2 s_1 s_2)_0|^{1/2 - A/2+\varepsilon} |\ell|^{1/2+\varepsilon}.
\]

(28)

Now we estimate \(L(2s + 2w - 1, \chi_\pi; I - S_\ell) \cdot Z^{(\ell)}(s, w; \pi, \beta)\) at the left vertical line with respect to \(|\ell|\). We gather all the contributions in the above steps.
For the first part, we have lower the exponent as much as possible.

In order to do the estimate, let us regroup the above as follows

1. \( |\ell_3|^{1/2+\epsilon} \cdot |\ell_2|^{1/2+\epsilon} \cdot |\ell_2|^{-1/2} \cdot |m_1|^{1/2+\epsilon} \cdot |m_2|^{-3/2+\epsilon} \);
2. \( |\ell_2|^{1+\epsilon} \cdot |m_1|^{1/2+\epsilon} \cdot |m_2|^{-3/2+\epsilon} \);
3. \( |\ell_2(\ell_3m_1m_2)_0|^{2+2\epsilon} \cdot |t_1t_2|^{1/2+A+\epsilon} \cdot |s_1s_2|^{-3/2+A+\epsilon} \);
4. \( |\ell_2(t_1t_2s_1s_2)_0|^{1+\epsilon} \cdot |n_1|^{1/2+\epsilon} |n_2|^{-3/2+\epsilon} \);
5. \( |((\ell_3m_1m_2)_0n_1n_2)(t_1t_2s_1s_2)_0|^{1/2-A/2+\epsilon} \cdot |\ell|^{1/2+\epsilon} \);
6. \( |\ell_2|^A \cdot |t_1t_2|^A \cdot |s_1s_2|^{A-1/2} \).

The estimate is given by the following

**Lemma 2.7.** The product of the first three groups is bounded by \( |\ell|^{2.5+\epsilon} \), and the product of the last three is bounded by \( |\ell|^{1+\lambda+A/2+\epsilon} \).

The idea for the proof is that we will decompose the (norms) of the ideals into the product of small pieces, and then regroup and combine the disjoint pieces to \( \ell \) in such a way that we can lower the exponent as much as possible.

**Proof.** For the first part, we have \( \ell_3, m_1 \) and \( m_2 \) are all relatively prime to \( \ell_2 \). The decomposition of these three into small pieces is just like considering them as three sets and looking at their Venn diagram. It is easy to see that we can put together the disjoint pieces of the first group above that do not involve \( \ell_2 \) into something dividing \( \ell/\ell_2 \) with exponent at most \( 1/2 + \epsilon \). But there is an extra piece, which is the part of the common divisor of \( \ell_3 \) and \( m_1 \) that is prime to \( m_2 \). So the first group of the factors above is at most \( |\ell|^{1/2+\epsilon} \) times the extra piece. Now we look at the second group of factors, to see whether we can combine the extra piece with \( t_1 \) or \( t_2 \). For each prime divisor, which can be combined with neither \( t_1 \) nor \( t_2 \), we know it must divide both. We look at the term \( (t_1t_2s_1s_2)_0 \) in the third group, and try to combine the divisor with it. Since \( t_1 \) and \( t_2 \) both have the divisor, the square free part of their product will not. If \( (s_1s_2)_0 \) does not have the divisor, then the combination is successful. Otherwise, we know either \( s_1 \) or \( s_2 \) will have the divisor, but look at the factor \( |s_1s_2|^{1+\epsilon} \) we have in the second group. So we have the result for the product of the first three. Similarly we can deduce the rest of the lemma.

So from the lemma, we have that at the left vertical line \( \tau = -1/2 - \epsilon \),

\[
L(2s + 2w - 1, \chi_{\pi}; I - S_{\ell}) \cdot Z^{(\ell)}(s, w; \pi, \beta) \ll \epsilon |\ell|^{3.5 + A + A/2 + \epsilon}.
\]
And

\[ L(2s + 2w - 1, \chi_\pi; I - S_r) \cdot Z(s, w; \pi, \beta; r) \ll \epsilon |r|^{3.5 + A + A/2 + \epsilon}. \]

Now let us look at the estimate at the right vertical line \( \tau = 1 + 1/2 + \epsilon \). Since the \( L \) factor \( L(2s + 2w - 1, \chi_\pi; I - S_r) \) converges absolutely there, we just ignore it.

\[
Z(1/2, w; \pi, \beta; r) = \sum_{\substack{D \in I^+(S) \setminus \{D_0 \} \cap \mathbb{Z} \mid r \mid D_1}} \frac{L(1/2, \pi \otimes \chi_{D_0}; I - S) a(1/2, \pi, S; \beta(D))}{|D_0|^w} \cdot \frac{a(1/2, \pi, D; S) \beta(D)}{|D_1^2|^w}.
\]

From the definition and the fact that the central character \( \chi_\pi \) is unitary, it is easy to deduce that

\[ a(1/2, \pi, D; S) \ll \epsilon |D_1|^{2A}. \]

Let \( D_1 = rD'_1 \), at \( \tau = \Re(w) = 1 + 1/2 + \epsilon \), we have

\[
Z(s, w; \pi, \beta; r) \ll \epsilon |r|^{2A - 3 - 2\epsilon} \cdot \left( \sum_{D_0} \frac{L(1/2, \pi \otimes \chi_{D_0}; I - S)}{|D_0|^{1+1/2}} \right) \times \left( \sum_{D'_1} |D'_1|^{2A - 3} \right) \ll \epsilon |r|^{2A - 3}.
\]

Now we will apply Phragmen–Lindelöf. First, we note that the series we are working with satisfy the conditions needed for Phragmen–Lindelöf. The ideas and arguments on this can be found in [8], and we will not repeat them here. We see that for \( s = 1/2 \) and \( 1/2 - \epsilon \leq \tau \leq 1 + 1/2 + \epsilon \), the function \( L(2s + 2w - 1, \chi_\pi; I - S) \cdot Z(s, w; \pi, \beta; r) \) is bounded by \(|r|\) raised to the following exponent:

\[ \frac{\tau + 1/2 + \epsilon}{2 + 2\epsilon} \cdot (-3 + 2A + \epsilon) + \frac{1 + 1/2 + \epsilon - \tau}{2 + 2\epsilon} \cdot (3.5 + A + A/2 + \epsilon). \]

Let \( \epsilon \) be very small, so that we can ignore it. We set the above expression less than \(-1\), and solve for \( \tau \). We see that, in order to get \( \tau < -1 \), or in other words, the continuation to hold for \( s \) near \( 1/2 \) and \( \tau = \Re(w) < 1 \), we need \( A \), the exponential bound for Satake parameter, less than \( 1/5 \). From the result of Shahidi and Kim [14], the bound for any cuspidal automorphic representation over any number field can be given by \( A = 5/34 \), which is smaller than what we need. So we achieve the continuation for all the cuspidal automorphic representations over \( F \).
3. Local analysis

3.1. A new series $Z^*(s, w; \pi; \mathcal{K}; r)$

From now on, we will specify $r$ to be either $\mathcal{O}$, or a prime not in $S$. We define $\chi_r$ to be the usual quadratic character defined before if $r$ is a prime ideal, and the trivial character if $r = \mathcal{O}$. We also assume that $\pi$ is self-contragredient. For any class $\mathcal{K} \in H_C$, we introduce

$$Z^*(s, w; \pi; \mathcal{K}; r) := \sum'_{D \in \mathcal{K}} \frac{L(s, \pi \otimes \chi_D) \chi_r(D)}{|D|^w}.$$ 

Let $E$ be the representative of the class $\mathcal{K}$. We know, for any $\nu \in S$, the local $L$-factor

$$L_{\nu}(s, \pi \otimes \chi_D) = L_{\nu}(s, \pi \otimes \chi_E),$$

independent of $D$ for $D \in \mathcal{K}$. We have

$$L(s, \pi \otimes \chi_D) = L^S(s, \pi \otimes \chi_E)L(s, \pi \otimes \chi_D; I - S).$$

So

$$Z^*(s, w; \pi; \mathcal{K}; r) = L^S(s, \pi \otimes \chi_E) \cdot \sum'_{D \in \mathcal{K}} \frac{L(s, \pi \otimes \chi_D; I - S) \chi_r(D)}{|D|^w}.$$ 

If we let

$$Z^*_0(s, w; \pi; \mathcal{K}; r) = \sum'_{D \in \mathcal{K}} \frac{L(s, \pi \otimes \chi_D; I - S) \chi_r(D)}{|D|^w},$$

we get

$$Z^*(s, w; \pi; \mathcal{K}; r) = L^S(s, \pi \otimes \chi_E) \cdot Z^*_0(s, w; \pi; \mathcal{K}; r).$$

Now let us look at $Z^*_0(s, w; \pi; \mathcal{K}; r)$. If $r = \mathcal{O}$, we have

$$Z^*_0(s, w; \pi; \mathcal{K}; \mathcal{O}) = \sum'_{D \in \mathcal{K}} \frac{L(s, \pi \otimes \chi_D; I - S)}{|D|^w}.$$ 

By the orthogonality, it is a linear combination of $Z^*(s, w; \pi, \beta)$ for $\beta \in \hat{H}_C$. In the sieving process, we get the continuation of $Z^*(s, w; \pi, \beta)$, so $Z^*_0(s, w; \pi; \mathcal{K}; \mathcal{O})$ has the continuation too. If $r$ is a prime and if we let $c(n)$ be the coefficient of $n$ in the $L$-series of $\pi$, then
\[ Z_0^*(s, w; \pi; \mathcal{K}; r) = \sum_{D \in \mathcal{K}}' \frac{L(s, \pi \otimes \chi_D; I - S)\chi_r(D)}{|D|^w} \]

\[ = \sum_{D \in \mathcal{K}}' \frac{L(s, \pi \otimes \chi_D; I - S_r)(\sum_{k \geq 0} c(r^k) \chi_D(r^k))\chi_r(D)}{|D|^w}. \quad (34) \]

If we let

\[ L_{1,r}(s) = \sum_{k \geq 0} \frac{c(r^{2k+1})}{|r^{2k+1}|^s} \]

and

\[ L_{2,r}(s) = \sum_{k \geq 0} \frac{c(r^{2k})}{|r^{2k}|^s}, \]

then since \((D, r) = 1\), we have

\[ \sum_{k \geq 0} \frac{c(r^k)}{|r^k|^s} \chi_D(r^k) = L_{1,r}(s)\chi_D(r) + L_{2,r}(s). \]

By the estimate \(c(n) \ll |n|^{1/5+\varepsilon}\) in [17], we see that \(L_{1,r}(s)\) and \(L_{2,r}(s)\) are absolutely convergent for \(\sigma = \text{Re}(s) > 1/5\). Also from reciprocity, \(\chi_D(r) = \alpha(E, r)\chi_r(D)\). We have

\[ Z_0^*(s, w; \pi; \mathcal{K}; r) = \alpha(E, r)L_{1,r}(s) \sum_{D \in \mathcal{K}}' \frac{L(s, \pi \otimes \chi_D; I - S_r)}{|D|^w} \]

\[ + L_{2,r}(s) \sum_{D \in \mathcal{K}}' \frac{L(s, \pi \otimes \chi_D; I - S_r)\chi_r(D)}{|D|^w} \]

\[ = a(E, r)L_{1,r}(s)Z^*(s, w; \pi, \delta; S_r) + L_{2,r}(s)Z^*(s, w; \pi, \chi_r\delta; S_r). \quad (35) \]

So it has a meromorphic continuation near \((s, w) = (1/2, 1)\). More precisely, we have the function \((w - 1)(w + 2s - 2)Z_0^*(s, w; \pi; \mathcal{K}; r)\) is holomorphic near the point.

3.2. The computation of the residues

First, we will compute the following meromorphic function in \(s\),

\[ \lim_{w \to 1} (w - 1)Z_0^*(s, w; \pi; \mathcal{K}; r), \]

which we call the residue at \(w = 1\).
The function \((w - 1)(w + 2s - 2)Z^*_0(s, w; \pi; K; r)\) is a holomorphic function near the point \((s, w) = (1/2, 1)\). If we substitute \(w = 1\), we have a holomorphic function in \(s\), and it is

\[
(2s - 1) \lim_{w \to 1} (w - 1)Z^*_0(s, w; \pi; K; r).
\]

So the residue makes sense and it is holomorphic except for a possible simple pole at \(s = 1/2\).

From the definition of \(Z^*_0(s, w; \pi; K; r)\), it is easy to see that, when \(\sigma = \Re(s) \gg 0\) and \(\tau = \Re(w) > 1\), the series converges absolutely. So we can interchange the order of summation and compute the residue, then make the continuation to \(s = 1/2\).

When \(r\) is a prime,

\[
Z^*_0(s, w; \pi; K; r) = \sum_{D \in \mathcal{K}} \frac{\mu(D \oplus I - S)}{|D|^w} \frac{c(D)\chi_D(n)}{|n|^s}.
\]

For each \(n \in I^+(S)\), we write \(n = n_0n_1^2n_2^2\), such that \(\forall p \mid n_1 \Rightarrow p \mid n_0\). Since \(D\) is square free, if \((D, n_1^2) \neq 1\), we have \(\chi_D(n_2) = \chi_D(n_2) = 0\). So we will assume \((D, n_1^2) = 1\). Both \(n_0\) and \(D\) are square free. If \((D, n_0) = 1\),

\[
\chi_D(n) = \chi_D(n_0) = \alpha(E, n_0)\chi_n(D);
\]

if \((D, n_0) \neq 1\),

\[
\chi_D(n) = \chi_D(n_0) = 0 = \alpha(E, n_0)\chi_n(D).
\]

So we have

\[
Z^*_0(s, w; \pi; K; r) = \sum_{n \in I^+(S)} \frac{c(n)}{|n|^s} \sum_{D \in \mathcal{K}} \frac{\mu(D \oplus I - S)}{|D|^w} \frac{\chi_D(n)}{|n|^s} \frac{\chi_D(n)}{|n|^s}.
\]

\[
= \sum_{n \in I^+(S)} \frac{c(n)}{|n|^s} \alpha(E, n_0) \sum_{D \in \mathcal{K}} \frac{\mu(D \oplus I - S)}{|D|^w} \frac{\chi_D(n)}{|n|^s} \frac{\chi_D(n)}{|n|^s}.
\]

\[
= \sum_{n \in I^+(S)} \frac{c(n)}{|n|^s} \alpha(E, n_0) \frac{1}{h} \sum_{\rho \in \mathcal{H}_C} \rho^{-1}(E) \sum_{D \in I^+(S)} \frac{(\rho \chi_D(n_0)(D)}{|D|^w}.
\]
We let $\psi = \rho \chi_r \chi_{n_0}$, and let $\zeta_F$ be the Dedekind zeta function of the field $F$. We will consider

$$\zeta_F(2w) \cdot \sum'_{D \in I^+(S) \atop (D, rn_2) = 1} \frac{\psi(D)}{|D|^w}.$$ 

We have

$$\zeta_F(2w) = \prod_p \left(1 - \frac{1}{|p|^{2w}}\right)^{-1},$$

and

$$\sum'_{D \in I^+(S) \atop (D, rn_2) = 1} \frac{\psi(D)}{|D|^w} = \prod_{p \notin S \atop p \nmid nr} \left(1 + \frac{\psi(p)}{|p|^w}\right).$$

So we get

$$\zeta_F(2w) \sum'_{D \in I^+(S) \atop (D, rn_2) = 1} \frac{\psi(D)}{|D|^w} = \prod_{p \in S \atop \text{or } p \nmid r} \left(1 - \frac{1}{|p|^{2w}}\right)^{-1} \cdot \prod_{p \notin S \atop p \nmid nr} \left(1 - \frac{\psi(p)}{|p|^w}\right)^{-1}.$$

If $\psi$ is non-trivial, we know the product

$$\prod_{p \notin S \atop \text{or } p \nmid nr} \left(1 - \frac{\psi(p)}{|p|^w}\right)^{-1}$$

is a partial $L$-series of $L(w, \psi)$. So we have for $\tau > 0$, it is holomorphic. And we get

$$\lim_{w \to 1} (w - 1)\zeta_F(2w) \cdot \sum'_{D \in I^+(S) \atop (D, rn_2) = 1} \frac{\psi(D)}{|D|^w} = 0.$$

If $\psi$ is trivial, looking at the primes outside $S$, we know that $n_0$ must be $r$ and $\chi_r \chi_{n_0}$ is trivial. So $\rho$ must be trivial. In this case, we have $n = r^{2k+1}n_2^2$ for some $k \geq 0$ and $n_2 \in I^+(S)$, $(n_2, r) = 1$. We have

$$\zeta_F(2w) \sum'_{D \in I^+(S) \atop (D, rn_2) = 1} \frac{\psi(D)}{|D|^w} = \prod_{p \in S \atop \text{or } p \nmid r} \left(1 + \frac{1}{|p|^w}\right)^{-1} \zeta_F(w)$$

and

$$\lim_{w \to 1} (w - 1)\zeta_F(2w) \sum'_{D \in I^+(S) \atop (D, rn_2) = 1} \frac{\psi(D)}{|D|^w} = \prod_{p \in S \atop \text{or } p \nmid r} \left(1 + \frac{1}{|p|^w}\right)^{-1} \cdot A(F),$$
where \( A(F) = \text{Res}_{w=1} \zeta_F (w) \).

Now we consider

\[
\lim_{w \to 1} \zeta_F (2w)(w - 1)Z^a_0(s, w; \pi; K; r)
\]

\[
= \frac{1}{h} \sum_{\rho \in HC} \rho^{-1}(E) \cdot \lim_{n \to 1} \sum_{n \in I^*(S)} \frac{c(n)}{|n|^s} \alpha(E, n_0)
\]

\[
\times (w - 1)\zeta_F (2w) \sum_{D \in I^*(S) \atop (D, r n_2) = 1} \frac{\chi_r(D) \chi_{n_0}(D) \rho(D)}{|D|^w} .
\]

(38)

We will interchange the order of \( \lim_{w \to 1} \) and \( \sum_{n \in I^*(S)} \). This is valid if we count on the following:

For the part of summation of the trivial characters, the process of taking the limit \( \lim_{w \to 1} (w - 1)\zeta_F (w) \) is independent of \( n \), and also

\[
\prod_{p|n} \left( 1 + \frac{1}{|p|^w} \right)^{-1} \ll |n|^\epsilon
\]

for any \( \epsilon > 0 \) near \( w \to 1 \). When \( \sigma \gg 0 \), the interchange is OK.

For the part of the summation over the non-trivial characters, the value of the partial \( L \)-series near \( w = 1 \) can be estimated by the Phragmen–Lindelöf theorem in term of \( |n| \), which is controlled uniformly by \( \sum_n \frac{c(n)}{|n|^s} \) for \( \sigma \) very large.

So we have

\[
\lim_{w \to 1} \zeta_F (2w)(w - 1)Z^a_0(s, w; \pi; K; r)
\]

\[
= \frac{1}{h} \sum_{\rho \in HC} \rho^{-1}(E) \sum_{n \in I^*(S)} \frac{c(n)}{|n|^s} \alpha(E, n_0) \lim_{w \to 1} (w - 1)\zeta_F (2w) \sum_{D \in I^*(S) \atop (D, r n_2) = 1} \frac{(\rho \chi_r \chi_{n_0})(D)}{|D|^w}
\]

\[
= \frac{1}{h} \cdot \alpha(E, r) \sum_{n_2 \in I^*(S) \atop k \geq 0} \frac{r^{2k+1}n_2^2}{|p|^{2k+1}n_2^2|s|} \prod_{p \in S} \left( 1 + \frac{1}{|p|} \right)^{-1}
\]

\[
\times \left( 1 + \frac{1}{|r|} \right)^{-1} \prod_{p | n_2} \left( 1 + \frac{1}{|p|} \right)^{-1} \cdot A(F)
\]

\[
= \frac{A(F)}{h} \cdot \alpha(E, r) \prod_{p \in S} \left( 1 + \frac{1}{|p|} \right)^{-1} \left( 1 + \frac{1}{|r|} \right)^{-1}
\]

\[
\times L_{1, r}(s) \sum_{n_2 \in I^*(S) \atop (r, n_2) = 1} \frac{c(n_2^2)}{|n_2^2|^s} \prod_{p | n_2} \left( 1 + \frac{1}{|p|} \right)^{-1} .
\]

(39)
We have the following lemma:

**Lemma 3.1.**

\[
(1 + \frac{1}{|r|})^{-1} \cdot \sum_{\substack{n^2 \in \mathcal{I}^+(S) \setminus (n_2, r) = 1}} \frac{c(n^2)}{|n^2|^s} \cdot \prod_{p|n} \left(1 + \frac{1}{|p|}\right)^{-1} = \frac{1}{1/|r| + L_{2, r}(s)} \cdot \sum_{n \in \mathcal{I}^+(S)} \frac{c(n^2)}{|n^2|^s} \cdot \prod_{p|n} \left(1 + \frac{1}{|p|}\right)^{-1}.
\]

Now

**Corollary 3.2.** We have

\[
\text{Res}_{w=1} Z_0^*(s, w; \pi; \mathcal{K}; r) = \frac{A(F)}{h} \cdot \frac{\alpha(E, r)}{\zeta_F(2)} \cdot \prod_{p \in S} \left(1 + \frac{1}{|p|}\right)^{-1} \times \sum_{n \in \mathcal{I}^+(S)} \frac{c(n^2)}{|n^2|^s} \cdot \prod_{p|n} \left(1 + \frac{1}{|p|}\right)^{-1} \cdot \frac{L_{1, r}(s)}{1/|r| + L_{2, r}(s)}.
\]

We will need the following lemma later.

**Lemma 3.3.** There is a $T(s)$, an absolutely convergent Euler product for $\sigma > 1/5$, such that for $\sigma \gg 0$, we have

\[
\sum_{n \in \mathcal{I}^+(S)} \frac{c(n^2)}{|n^2|^s} \cdot \prod_{p|n} \left(1 + \frac{1}{|p|}\right)^{-1} = L_S(2s, \pi, \text{sym}^2) T(s),
\]

where $L_S(s, \pi, \text{sym}^2)$ is the partial symmetric square $L$-function.

**Proof.** We have

\[
\sum_{n \in \mathcal{I}^+(S)} \frac{c(n^2)}{|n^2|^s} \cdot \prod_{p|n} \left(1 + \frac{1}{|p|}\right)^{-1} = \prod_{p \notin S} \left(1 + \frac{|p|}{1 + |p|} \sum_{k \geq 1} \frac{c(p^{2k})}{|p^{2k}|^s}\right)
\]

\[
= \prod_{p \notin S} \left(L_{2, p}(s) - \frac{1}{1 + |p|} \sum_{k \geq 1} \frac{c(p^{2k})}{|p^{2k}|^s}\right)
\]

\[
= \prod_{p \notin S} (L_{2, p} + O(|p|^{-(2s+3/5+\epsilon)})).
\]

And it is easy to compute that

\[
L_{1, p}(s) = \frac{c(p)}{|p|^s (1 - \alpha_{p, 1}^2/|p|^{2s})(1 - \alpha_{p, 2}^2/|p|^{2s})}.
\]
and
\begin{align*}
L_{2,p}(s) &= \frac{1 + \chi_{\pi}(p)/|p|^{2s}}{(1 - \alpha_{p,1}^2/|p|^{2s})(1 - \alpha_{p,2}^2/|p|^{2s})} \\
&= \frac{1 - 1/|p|^{4s}}{(1 - \alpha_{p,1}^2/|p|^{2s})(1 - \alpha_{p,2}^2/|p|^{2s})(1 - \chi_{\pi}(p)/|p|^{2s})}.
\end{align*}

So we have
\[\zeta_{F,S}(4s) \cdot \prod_{p \notin S} L_{2,p}(s) = L_S(2s, \pi, \text{sym}^2).\]

The lemma now follows from the comparison of Euler factors. \qed

As before, we have
\[Z^*(s, w; \pi; \mathcal{K}; r) = L^S(s, \pi \otimes \chi_E)Z_0^*(s, w; \pi; \mathcal{K}; r).\]

So
\[
\text{Res}_{w=1}Z^*(s, w; \pi; \mathcal{K}; r) = L^S(s, \pi \otimes \chi_E) \cdot \text{Res}_{w=1}Z_0^*(s, w; \pi; \mathcal{K}; r)
\]
\[
= \frac{L^S(s, \pi \otimes \chi_E)}{h} \cdot \frac{\alpha(E, r)A(F)}{\zeta_F(2)} \cdot \prod_{p \in S} \left(1 + \frac{1}{|p|}\right)^{-1}
\times \sum_{n \in I^+(S)} \frac{c(n^2)}{|n^2|^s} \cdot \prod_{p \mid n} \left(1 + \frac{1}{|p|}\right)^{-1} \cdot \frac{L_{1,r}(s)}{1/|r| + L_{2,r}(s)}.
\]

Let
\[
R_1(s, \pi, \mathcal{K})
\]
\[
= \frac{L^S(s, \pi \otimes \chi_E)}{h} \cdot \frac{A(F)}{\zeta_F(2)} \cdot \prod_{p \in S} \left(1 + \frac{1}{|p|}\right)^{-1} \cdot \sum_{n \in I^+(S)} \frac{c(n^2)}{|n^2|^s} \cdot \prod_{p \mid n} \left(1 + \frac{1}{|p|}\right)^{-1}
\]

and let
\[
R_r(s, \pi, \mathcal{K}) = \alpha(E, r) \cdot \frac{L_{1,r}(s)}{1/|r| + L_{2,r}(s)}.
\]

So
\[
\text{Res}_{w=1}Z^*(s, w; \pi; \mathcal{K}; r) = R_1(s, \pi, \mathcal{K})R_r(s, \pi, \mathcal{K}).
\]

For the case \(r = \mathcal{O}\), by the same argument, we have
\[
\text{Res}_{w=1}Z^*(s, w; \pi; \mathcal{K}; r) = R_1(s, \pi, \mathcal{K}).
\]
3.3. Proof of the theorem

First, we have the following expansion of $Z^*_0(s, w; \pi; K; r)$ near $(1/2, 1)$. Let

$$A(s) = \lim_{w \to 1} (w - 1)Z^*_0(s, w; \pi; K; r)$$

and

$$B(s) = \lim_{w \to 2 - 2s} (w + 2s - 2)Z^*_0(s, w; \pi; K; r).$$

We have

$$Z^*_0(s, w; \pi; K; r) = \frac{A(s)}{w - 1} + \frac{B(s)}{w + 2s - 2} + H(s, w),$$

where $H(s, w)$ is holomorphic near the point.

Next we will consider the first case of the theorem:

Suppose $\pi$ is self-contragredient with trivial central character. Also, there is a quadratic Hecke character $\chi$ such that $\varepsilon(1/2, \pi \otimes \chi) = 1$.

Let us look at the functional equation of $Z^*_0(s, w; \pi; K; r)$. We have

$$Z^*_0(s, w; \pi; K; r) = \sum_{D \in K} L(s, \pi \otimes \chi_D; I - S) \chi(D) |D|^w.$$ 

Use the equations

$$L(s, \pi \otimes \chi_D; I - S) = \varepsilon(s, \pi \otimes \chi_D) L(1 - s, \pi \otimes \chi_D; I - S) \prod_{v \in S} L_v(1 - s, \pi \otimes \chi_E) L_v(s, \pi \otimes \chi_E)$$

and

$$\varepsilon(s, \pi \otimes \chi_D) = |D/E_0|^{1 - 2s} \varepsilon(s, \pi \otimes \chi_E).$$

Then we have

$$Z^*_0(s, w; \pi; K; r) = \frac{\varepsilon(s, \pi \otimes \chi_E)}{|E_0|^{1 - 2s}} \prod_{v \in S} L_v(1 - s, \pi \otimes \chi_E) L_v(s, \pi \otimes \chi_E)$$

$$\times \sum_{D \in K} L(1 - s, \pi \otimes \chi_D; I - S) \chi(D) |D|^{w + 2s - 1}$$

$$= M(s, E) Z^*(\Phi(s, w); \pi; K; r),$$

where $\Phi(s, w) = (1 - s, w + 2s - 1)$, and

$$M(s, E) = \frac{\varepsilon(s, \pi \otimes \chi_E)}{|E_0|^{1 - 2s}} \prod_{v \in S} L_v(1 - s, \pi \otimes \chi_E) L_v(s, \pi \otimes \chi_E).$$
It is holomorphic at $s = 1/2$. And the product factor is either 1 or $-1$ at $s = 1/2$, depending on whether the number of poles of the local factors is even or odd.

Now we use the results from [11]. From the assumption that $\varepsilon(1/2, \pi \otimes \chi) = 1$ for some $\chi$, and [11], there exist infinitely many $D \in I^+(S)$ such that $L(1/2, \pi \otimes \chi_D) \neq 0$.

Also, in [11] it is proved that there are infinitely many partial twisted $L$-series with archimedean factors removed that do not vanish at $s = 1/2$. So we may choose the class $\mathcal{K}$ such that there is a $D$ in the class and the partial $L$-series twisted by $\chi_D$ is not 0 at $s = 1/2$. And it is easy to see that $M(1/2, E) = 1$.

Applying the functional equation to the expansion, we have

$$Z^*_0(s, w; \pi; \mathcal{K}; r) = A(s) + \frac{B(s)}{w + 2s - 2} + H(s, w)$$

$$= M(s, E) \left[ \frac{A(1-s)}{w + 2s - 2} + \frac{B(1-s)}{w - 1} + H(1-s, w + 2s - 1) \right]. \quad (44)$$

So

$$B(s) = M(s, E)A(1-s).$$

Since $\pi$ is self-contragredient, and $\chi_\pi$ is trivial, we can see that

$$L_S(s, \pi \times \pi) = L_S(s, \pi, \text{sym}^2) \cdot \zeta_{F,S}(s).$$

Both $L_S(s, \pi \times \pi)$ and $\zeta_{F,S}(s)$ have simple poles at $s = 1$, so $L_S(1, \pi, \text{sym}^2)$ is defined and non-zero, hence so is $A(1/2)$. Also $L_S(s, \pi \otimes \chi_\pi)$ has no pole at $s = 1/2$. In fact, for the archimedean factors, this follows from [11]; and for the finite factors, it follows from the result of Gelbart and Jacquet as in [19].

We substitute $s = 1/2$, and deduce that

$$Z^*(1/2, w; \pi; \mathcal{K}; r) = \frac{2R_1(1/2, \pi, \mathcal{K})R_r(1/2, \pi, \mathcal{K})}{w - 1} + H(1/2, w)$$

has a simple pole at $w = 1$ with residue $2R_1(1/2, \pi, \mathcal{K})R_r(1/2, \pi, \mathcal{K})$.

Now we consider the second case of the theorem. In this case we have

$$L_S(s, \pi \times \pi) = L_S(s, \pi, \text{sym}^2) \cdot L_S(s, \chi_\pi).$$

The simple pole of $L(s, \pi \times \pi)$ at $s = 1$ implies that $L_S(s, \pi, \text{sym}^2)$ has a simple pole at $s = 1$. So $R_1(s, \pi, \mathcal{K})$ has a simple pole at $s = 1/2$. We have the $A(s)$ in the expansion of $Z^*(s, w; \pi; \mathcal{K}; r)$ has a simple pole at $s = 1/2$. From the functional equation, $B(s)$ has a simple pole too. So we can refine the expansion

$$Z^*_0(s, w; \pi; \mathcal{K}; r) = \frac{A_0}{(w - 1)(s - 1/2)} + \frac{A_1(s)}{w - 1} + \frac{A'_0}{(w + 2s - 2)(s - 1/2)}$$

$$+ \frac{A'_1(s)}{w + 2s - 2} + H(s, w), \quad (45)$$
where we have
\[ A_0 = \text{Res}_{s=1/2} R_1(s, \pi, \mathcal{K}) \cdot R_r(1/2, \pi, \mathcal{K}). \]

Since \((w - 1)(w + 2s - 2)Z^*_0(s, w; \pi; \mathcal{K}; r)\) is holomorphic, we have \(A_0 = -A'_0\). We substitute \(s = 1/2\), and deduce that
\[ Z^*_0(1/2, w; \pi; \mathcal{K}; r) = \frac{2A_0}{(w - 1)^2} + \frac{A_2}{w - 1} + H(1/2, w) \]
has a double pole at \(w = 1\).

Now we shall give the final proof of our theorem.

**Proof.** Let \(\pi'\) be another self-contragredient representation with the same central character as \(\pi\). We can extend the set \(S\) so that it works for both representations. Let \(\kappa\) be a non-zero constant such that for all \(D \in I^+(S)\),
\[ L(1/2, \pi \otimes \chi_D) = \kappa L(1/2, \pi' \otimes \chi_D). \]
Then we get
\[ Z^*(1/2, w; \pi; \mathcal{K}; r) = \kappa Z^*(1/2, w; \pi'; \mathcal{K}; r). \]

In the first case, if we take \(r = 0\) and compute the residue at \(w = 1\), we have
\[ R_1(1/2, \pi, \mathcal{K}) = \kappa R_1(1/2, \pi', \mathcal{K}). \]
Taking \(r\) a prime, we have
\[ R_1(1/2, \pi, \mathcal{K}) R_r(1/2, \pi, \mathcal{K}) = \kappa R_1(1/2, \pi', \mathcal{K}) R_r(1/2, \pi', \mathcal{K}). \]
So we get
\[ R_r(1/2, \pi, \mathcal{K}) = R_r(1/2, \pi', \mathcal{K}). \]

In the second case, we compute
\[ \lim_{w \to 1} (w - 1)^2 Z^*(1/2, w; \pi; \mathcal{K}; r). \]
We will also get
\[ R_r(1/2, \pi, \mathcal{K}) = R_r(1/2, \pi', \mathcal{K}). \]

By definition, we have
\[ R_r(1/2, \pi, \mathcal{K}) = \alpha(E, r) \cdot \frac{L_{1,r}(1/2)}{1/|r| + L_{2,r}(1/2)} \]
\[ = \alpha(E, r) \cdot \frac{|r|^{2.5} c(r)}{|r|^3 + (1 + \chi_{\pi}(r))|r|^2 + (2\chi_{\pi}(r) - c(r))^2|r| + 1}. \quad (46) \]
Now we fix $r$, and change $c(r)$ to a variable $t$, and we have a function

$$H_r(t) = \frac{t}{|r|^3 + (1 + \chi\pi(r))|r|^2 + (2\chi\pi(r) - t^2)|r| + 1}.$$ 

By taking the derivative, we see that $H_r(t)$ is monotone. Together with the same central character, we get $\pi_r \cong \pi'_r$. So by the strong multiplicity one theorem, we have

$$\pi \cong \pi'.$$

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