Modules with $n$-acc and the acc on certain types of annihilators

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Abstract

A unitary module $M$ over a commutative ring $R$ with unity satisfies “acc on d-colons,” if for every submodule $N$ of $M$ and every sequence $(a_n)_n$ of elements of $R$ the ascending chain $N:a_1 ⊆ N:a_1a_2 ⊆ ···$ stabilizes. In this paper we study the acc on d-colons and show that it implies the acc on $n$-generated submodules for every $n$ (“pan-acc”), generalizing a result of W. Heinzer and D. Lantz. The method involves a “generalized Nakayama’s Lemma” for these modules. Further it is shown that every $R$-module $R^I$ for a noetherian ring $R$ satisfies pan-acc, and we give a sufficient condition for the acc on principal ideals to rise to the polynomial ring.

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1. Introduction

Several authors [1–3] studied unitary modules over commutative rings with unity that satisfy the ascending chain condition on $n$-generated submodules (“$n$-acc”) for a positive integer $n$. In [3], G. Renault considered also noncommutative rings, but in this paper we assume all rings to be commutative. Renault showed that every free module over a noetherian ring satisfies pan-acc (i.e. $n$-acc for every $n$). Further he proved that for a reduced noetherian ring $R$ and an arbitrary set $I$ the $R$-module $R^I$ has pan-acc. W. Heinzer and D. Lantz pointed
out that a strongly laskerian module also satisfies pan-acc [1]. In this paper, we generalize these results: In Section 2, we consider modules satisfying the acc on d-colons, which were introduced in [4] as “modules satisfying (C)” and build a larger class than the strongly laskerian modules. We will show that these modules have pan-acc and that every free module over a ring satisfying acc on d-colons also has pan-acc. In Section 3, we will show that for a noetherian ring $R$ and an arbitrary set $I$ the $R$-module $R^I$ satisfies pan-acc, and that 1-acc in $R$ rises to the $R$-module $R^I$ if $R$ has the ascending chain condition on annihilators.

Further it is known from [5] that the ascending chain condition on principal ideals (accp) does not rise to the polynomial ring in general, but there are also given several sufficient conditions in [1]. We will show in Section 4 that another sufficient condition is that the polynomial ring satisfies the acc on annihilators. In particular, this holds for reduced rings with only finitely many minimal primes and for subrings of noetherian rings. We further show that localizations at maximal ideals of polynomial and power series rings over strongly laskerian rings have accp.

Throughout this paper, we use the symbol $\subset$ for proper inclusion.

2. Modules satisfying acc on d-colons

In this section, we study the class of modules defined by S. Visweswaran in [4], and we will use the following terminology:

**Definition.** Let $R$ be a ring and $M$ an $R$-module. If, for every sequence $(a_n)_n$ of elements of $R$, the ascending chain $\text{Ann}(a_1) \subseteq \text{Ann}(a_1a_2) \subseteq \cdots$ of submodules of $M$ stabilizes, we say that $M$ satisfies **acc on d-annihilators** (ascending chain condition on annihilators of descending chains of principal ideals). We say that $M$ satisfies **acc on d-colons**, if for every submodule $N$ of $M$ the module $M/N$ satisfies acc on d-annihilators (this is the condition (C) in [4]).

It follows from [6, Exercise 28(a), p. 298] that strongly laskerian modules satisfy acc on d-colons. The following example shows that a ring with acc on d-colons need not be laskerian.

**Example 2.1.** Let $K$ be any field and $x_n (n \geq 1)$ indeterminates over $K$. Let $I$ be the ideal of $K[x_n: n \geq 1]$ generated by all $x_ix_j$, where $i \neq j$, and set $R := K[x_n: n \geq 1]/I$. If now $J$ is an ideal containing $I$ and $f$ is a nonunit in $K[x_n: n \geq 1]$, we have $f \in K[x_1, \ldots, x_m]$ for some $m$, so $J : f$ contains $(x_n: n > m)$. Thus $K[x_n: n \geq 1]/(J : f)$ is noetherian and hence $R$ has acc on d-colons. But if we denote by $P_i$ the prime of $R$ generated by the images of all $x_j$ for $j \neq i$, it is checked easily that the $P_i$’s are infinitely many minimal primes of $R$, so $R$ is not laskerian.
Proof. (a) Let Ann\((A)\) We recall that a prime ideal \(P\) of \(R\) is the only prime ideal of \(Ann(R(a)\) such that \(Ann\(x)\) for some \(x \in M\), and it is weakly associated, if \(P\) is minimal over some \(Ann\(x)\).

**Lemma 2.2.** Let \(R\) be a ring and \(M \neq \{0\}\) an \(R\)-module satisfying acc on d-annihilators. Then we have:

(a) If \(S \subseteq R\) is multiplicatively closed, there is \(s \in S\) such that \(\text{Ker}(M \to M_s) = Ann(s)\), and \(M_s\) also satisfies acc on d-annihilators.

(b) For every \(A \subseteq R\), the factor module \(M/Ann(A)\) has acc on d-annihilators.

(c) For every submodule \(N\) of \(M\) and every sequence \((a_n)\) of elements of \(R\), the ascending chain \(Ann_R(a_1N) \subseteq Ann_R(a_1a_2N) \subseteq \cdots\) of ideals of \(R\) stabilizes.

This means, if \(N \neq \{0\}\), the ring \(R/Ann_R(N)\) has acc on d-annihilators.

(d) If \(P\) is a minimal prime of \(R\) such that \(MP \neq \{0\}\), then \(P\) is an associated prime of \(M\).

(e) Every weakly associated prime of \(M\) is an associated prime of \(M\).

(f) Every submodule of \(M\) has acc on d-annihilators.

Proof. (a) Let \(x_1 \in N := \text{Ker}(M \to M_s)\). Then there is \(s_1 \in S\) such that \(s_1x_1 = 0\). If \(Ann(s_1) = N\), we are finished. Otherwise, it exists \(x_2 \in N \setminus Ann(s_1)\) and \(s_2 \in S\) such that \(s_2x_2 = 0\) and hence \(Ann(s_1) \subseteq Ann(s_1s_2)\). By the hypothesis, this process stops, so we have \(N = Ann(s)\) for some \(s \in S\). If we consider now an ascending chain \(Ann\left(\frac{a_1}{1}\right) \subseteq Ann\left(\frac{a_1a_2}{1}\right) \subseteq \cdots\) in \(M_S\), we may without loss of generality assume that the chain \(Ann(sa_1) \subseteq Ann(sa_1a_2) \subseteq \cdots\) in \(M\) is stationary at the first step. But \(Ann(sa_1) = Ann(sa_1a_2) = \cdots\) implies \(Ann\left(\frac{a_1}{1}\right) = Ann\left(\frac{a_1a_2}{1}\right) = \cdots\) in \(M_S\).

(b) This follows by the fact that, if \(Ann(b) = Ann(b)\) \((b, c \in R)\), then \(Ann(A) : b = Ann(b) : A = Ann(bc) : A = Ann(A) : bc\).

(c) We may assume that \(Ann(a_1) = Ann(a_1a_2) = \cdots\) in \(M\). Now pick \(b \in Ann_R(a_1a_2N)\). Then \(bN \subseteq Ann(a_1a_2) = Ann(a_1)\); so \(b \in Ann_R(a_1N)\).

(d) If \(P\) is any prime of \(R\) and \(P_P\) is an associated prime of \(M_P\), then \(P\) is an associated prime of \(M\). For if \(P_P = Ann_R^P\) \((\frac{x}{1}) = Ann_R(x)_P\) and \(\text{Ker}(M \to M_P) = Ann(s)\), it follows \(P = Ann_R(sx)\). So let \(P\) be a minimal prime of \(R\) such that \(MP \neq \{0\}\). By localizing at \(P\) we may assume that \(P\) is the only prime ideal of \(R\). For \(y \in M \setminus \{0\}\) it is \(Ann_R(y) \subseteq P\), and by the acc on d-annihilators, we get \(x \in M\) such that \(Ann_R(x) \subseteq P\) and \(Ann_R(x) = Ann_R(ax)\) for every \(a \in R \setminus Ann_R(x)\) Now \(Ann_R(x)\) is a prime ideal, so it is equal to \(P\) and hence it is an associated prime of \(M\).

(e) If \(P\) is a minimal prime over \(Ann_R(x)\) for some \(x \in M \setminus \{0\}\), (c) and (d) show that \(P = Ann(ax)\) for some \(a \in R\).

(f) This follows from the fact that \(Ann_N(a) = Ann_M(a) \cap N\) for every submodule \(N\) of \(M\). \(\square\)
S. Visweswaran in [4] showed that the zero-dimensional rings with acc on d-colons are exactly the perfect rings. Actually, his proof shows that the acc on d-annihilators is already sufficient. Perfect rings are characterized by being semiquasilocal and having T-nilpotent Jacobson radical [7]. So the concepts of T-nilpotence and the acc on d-annihilators are similar, and the next theorem, which was inspired by the proof of Theorem P in [7], reflects this observation.

**Theorem 2.3.** (a) Let R be a ring and A a T-nilpotent ideal of R. Then, for every sequence \((A_n)_n\) of finitely generated ideals contained in A, there is an n such that \(A_1 \cdots A_n = 0\).

(b) Let \(M\) be an R-module satisfying acc on d-annihilators. Then, for every sequence \((A_n)_n\) of finitely generated ideals of R the ascending chain Ann\((A_1) \subseteq \text{Ann}(A_1 A_2) \subseteq \cdots\) in M stabilizes.

**Proof.** (a) By transfinite induction, we define ideals \(B_\alpha\) of R for every ordinal \(\alpha\). Let \(B_0 := (0)\), \(B_{\alpha+1} := B_\alpha : A\) and, for limit ordinals, \(B_\alpha := \bigcup_{\beta < \alpha} B_\beta\). If \(A \not\subseteq B_\alpha\), pick \(a_1 \in A \setminus B_\alpha\). If \(a_1 A \subseteq B_\alpha\), we have \(a_1 \in B_{\alpha+1}\). Otherwise, we can pick \(a_2 \in a_1 A \setminus B_\alpha\). By the T-nilpotency of A this process terminates, so it is \(B_\alpha \subseteq B_{\alpha+1}\). Hence there is an ordinal \(\gamma\) such that \(A \subseteq B_\gamma\), and, for every finitely generated ideal \(C \subseteq A\), we can define \(h(C) := \min\{\alpha: C \subseteq B_\alpha\}\), which never is a limit ordinal. But if \(h(C) = \beta + 1\) and \(D \subseteq A\) is also finitely generated, we have \(h(CD) \leq \beta < \beta + 1 = h(C)\). So, for any sequence \((A_n)_n\) of finitely generated ideals contained in A, there is an n such that \(h(A_1 \cdots A_n) = (0)\); i.e. \(A_1 \cdots A_n = (0)\).

(b) Again we define sets \(S_\alpha\) by transfinite induction. Let

\[
S_0 := \{M\} = \{\text{Ann}(0)\},
\]

\[
S_{\alpha+1} := \{\text{Ann}(a): a \in R, \text{ for every } b \in R \text{ such that } \text{Ann}(a) \subset \text{Ann}(ab) \text{ it is } \text{Ann}(ab) \in S_\alpha\}, \quad \text{and}
\]

\[
S_\alpha := \bigcup_{\beta < \alpha} S_\beta \quad \text{for a limit ordinals } \alpha.
\]

Then we have \(S_\alpha \subset S_{\alpha+1}\), and similar to (a) \(S_\alpha \subset S_{\alpha+1}\) if \(S_\alpha \subset S := \{\text{Ann}(a): a \in R\}\). Hence there is an ordinal \(\gamma\) such that \(S = S_\gamma\), and, for every finite set \(C = \{c_1, \ldots, c_n\} \subseteq R\), we define \(h(C) := \min\{\alpha: \{\text{Ann}(c_1), \ldots, \text{Ann}(c_n)\} \subseteq S_\alpha\}\), which never is a limit ordinal. If \(h(C) = \beta + 1\) and \(D \subseteq R\) is also a finite set, we have \(\beta + 1 = h(C) \leq h(CD)\) by definition of \(S_{\beta+1}\). It suffices to prove the theorem for sequences of finite sets \(A_n\), and we always have \(h(A_1 \cdots A_k) = h(A_1 \cdots A_j)\) for some \(k\) and every \(l \geq k\), so we always may assume \(k = 1\) for such sequences. Now suppose that there are strictly ascending chains Ann\((A_1) \subset \text{Ann}(A_1 A_2) \subset \cdots\) in \(M\), and out of those with minimal \(\beta + 1 = h(A_1) = h(A_1 A_2) = \cdots\) we chose one with a minimal number of elements in \(A_1\). For every \(m\) there is an element \(b_m \in A_1 \cdots A_m\) such that \(\text{Ann}(b_m) \in S_{\beta+1} \setminus S_{\beta}\).
Since $A_1$ is finite, there is $a \in A_1$ such that for every $m \geq 2$, there exists $c_m \in A_2 \cdots A_m$ such that $\text{Ann}(ac_m) \in S_{\beta+1} \setminus S_\beta$, i.e. $\text{Ann}(a) = \text{Ann}(ac_m)$. Let $A_1 = \{a\} \cup B$. Since $\text{Ann}(A_1 \cdots A_m) = \text{Ann}(a) \cap \text{Ann}(BA_2 \cdots A_m)$, we get the strictly ascending chain $\text{Ann}(B) \subset \text{Ann}(BA_2) \subset \cdots$ in $M$, where $\beta + 1 = h(B) = h(BA_2) = \cdots$ because of the minimality of $h(A_1)$. But $B$ has less elements than $A_1$, a contradiction. $\square$

As a corollary, we can show that the acc on d-annihilators rises to the polynomial ring in some special cases. (But until now, we have no answer for the general case.)

**Corollary 2.4.** Let $R$ be a ring satisfying acc on d-annihilators. If $R$ has one of the following properties (i)–(iii), then the polynomial ring $R[X]$ for any set $X$ of indeterminates also has acc on d-annihilators.

(i) $R$ contains an uncountable field.

(ii) The total quotient ring $Q$ of $R$ is zero-dimensional.

(iii) $R$ is reduced.

**Proof.** (i) It is easy to see that a ring $T$ satisfies acc on d-annihilators if and only if every countable subring of $T$ does, so the assertion follows from a theorem of Camillo and Guralnick in [8].

(ii) Since $R[X]$ is a subring of $Q[X]$ and $Q$ also satisfies acc on d-annihilators by Lemma 2.2(a), we may assume $R = Q$; so $R$ is perfect by the statement made preceding Theorem 2.3. It is well known that a perfect ring is a direct sum of a finite number of quasilocal rings, each with T-nilpotent maximal ideal (see, e.g., [7]). So it suffices to prove the case of a quasilocal ring $(R, P)$ with $P$ T-nilpotent. Now it is an easy consequence of Theorem 2.3(a) that $P[X]$ is also T-nilpotent: For if $f_i \in P[X]$, consider the content ideals $A_{f_i}$ in $R$, and since $A_{f_1} \cdots A_{f_n} = (0)$ for some $n$, we have $f_1 \cdots f_n = 0$. The assertion now follows by the fact that $P[X]$ is the set of zero divisors of $R[X]$.

(iii) Since $R$ is reduced, it is $\text{Ann}(A) = \text{Ann}(A^m)$ for every ideal $A \subseteq R$ and every $m \geq 1$. If $f, g \in R[X]$, by [9, Corollary 28.3] there exists a positive integer $m$ such that $\text{Ann}(A_{f A_g}) = \text{Ann}(A_{f}^{m+1} A_g) = \text{Ann}(A_{f}^mA_{f g}) \supseteq \text{Ann}(A_{f g})$, so it is $\text{Ann}(A_{f A_g}) = \text{Ann}(A_{f g})$. So by Theorem 2.3(b), it suffices to prove that $\text{Ann}(f) = \text{Ann}(f g)$ in $R[X]$ whenever $\text{Ann}(f g) \subseteq \text{Ann}(A_{f g})$ in $R$. For this, it is enough to show $\text{Ann}(f) \cap R[Y] = \text{Ann}(f g) \cap R[Y]$ for every finite subset $Y$ of $X$, so we may assume that $X$ is finite. So let us suppose that $X = \{x\}$ is a single indeterminate and $h = \sum_{i=0}^n a_i x^i \in \text{Ann}(f g)$. If $n = 0$, we have $h \in \text{Ann}(f)$; so assume that $n > 0$ and set $f g = \sum c_i x^i$. We show $a_0 c_i = 0$ for every $i$ by induction on $i$: $a_0 c_0 = 0$ holds trivially, and multiplication of $\sum_{j+k=i} a_j c_k = 0$ by $a_0$ yields $a_0^2 c_i = 0$, and hence $a_0 c_i = 0$. Now it is
Theorem 2.5. If $R$ is a ring and $M$ an $R$-module satisfying acc on $d$-annihilators, then the set of zero divisors on $M$ is a finite union of prime ideals.

Proof. By Lemma 2.2(a), we can localize at $R \setminus Z(M)$, the set of nonzero divisors on $M$. So we assume that every nonunit in $R$ is a zero divisor on $M$, and we have to show that $R$ is semiquasilocal. We denote by $J(R)$ the Jacobson radical of $R$, and throughout this proof, we will make frequent use of the following:

\((\ast)\) For every $b \in R \setminus J(R)$ there is $r \in R$ such that $Ann(b) \subset Ann(b(1 - rb))$ in $M$.

For, if $r \in R$ is such that $1 - rb$ is a nonunit, we have $(1 - rb)x = 0$ for some $x \in M \setminus \{0\}$; so $bx \neq 0$. We denote by $\bar{a}$ the image of any $a \in R$ in $\overline{R} := R/J(R)$ and pick $a \in R$ such that $\bar{a}$ is a nonzero idempotent in $\overline{R}$. Now we show that we can find $a' \in aR$ such that $Ann(a) \subset Ann(a')$ in $M$. $1 - \bar{a}, \bar{a} - a', a'$ is a complete set of orthogonal idempotents in $\overline{R}$, and $(\bar{a} - a')\overline{R}$ is a field.

The acc on $d$-annihilators allows us to find $ab \in aR \setminus J(R)$ such that $Ann(ab) = Ann(abc)$ in $M$ for every $c \in R$ satisfying $abc \notin J(R)$. By \((\ast)\), we have $Ann(ab) \subset Ann(ab(1 - rb))$ for some $r \in R$. Thus, for $a' := a(1 - rb)$ it is $Ann(a) \subset Ann(a')$. Further we have $ab(1 - rab) \in J(R)$ by the choice of $ab$, so $\overline{rab}$ is idempotent in $\overline{R}$. Since $\bar{a}'$ is idempotent (as a product of two idempotents) and $\bar{a} - \bar{a}' = \overline{rab}$, we conclude that $1 - \bar{a}, \bar{a} - a', a'$ is a complete set of orthogonal idempotents. To show that $(\bar{a} - a')\overline{R}$ is a field, observe that $\bar{a} - a' = \overline{rab}$ and $\bar{a}b = \overline{rabb'}$; so $(\bar{a} - a')\overline{R} = \overline{abR} \neq (0)$. If we pick now $\bar{a}b\bar{d} \in abR \setminus (0)$, we find $s \in R$ such that $Ann(ab) \subset Ann(ab(1 - sabcd))$ by \((\ast)\); so the choice of $ab$ implies $ab(1 - sabd) \in J(R)$. Hence $\bar{a}b = \bar{a}bs\bar{abcd}$; i.e. $\overline{rabd}$ is a unit in $\overline{rabR}$. 

\[
0 = hfg = a_0fg + x\left(\sum_{i=0}^{n-1} a_{i+1}x^i\right)fg = x\left(\sum_{i=0}^{n-1} a_{i+1}x^i\right)fg; \text{ so }
\]

\[
h' := \sum_{i=0}^{n-1} a_{i+1}x^i \in Ann(fg).
\]

By induction on $n$, we have $a_0, h' \in Ann(f)$. Hence $h = a_0 + xh' \in Ann(f)$. \qed
Now we apply the above construction to $a_0 := 1$ and get an ascending chain $	ext{Ann}(a_0) \subset \text{Ann}(a_1) \subset \cdots$ in $M$. Since $a_{i+1} \in a_i R$, the acc on d-annihilators implies $\overline{a_{n+1}} = \bar{0}$ for some $n$. Thus $\overline{a_0} - \overline{a_1}, \overline{a_1} - \overline{a_2}, \ldots, \overline{a_n} - \overline{a_{n+1}}$ is a complete set of orthogonal idempotents in $\overline{R}$, and each $(\overline{a_i} - \overline{a_{i+1}}) \overline{R}$ is a field. So $\overline{R}$ is a finite product of fields; i.e. $R$ is semiquasilocal. \hfill \Box

Example 2.1 shows that, although there are only finitely many maximal primes (i.e. prime ideals maximal in the set of zero divisors), a ring satisfying acc on d-annihilators may have infinitely many associated primes.

Recall that an $R$-module $M$ is a ZD-module, if $Z(M/N)$ is a finite union of primes for every submodule $N$ of $M$. This yields the following corollary:

**Corollary 2.6.** A module satisfying acc on d-colons is ZD.

We now want to show that modules with the acc on d-colons have pan-acc. This was motivated from the fact that strongly laskerian modules satisfy pan-acc and other results from the literature: In [4], it was shown that integral domains with acc on d-colons have 1-acc. Further, the equivalence of

(i) $R$ is a perfect ring;
(ii) $R$ is zero-dimensional and satisfies acc on d-colons;
(iii) every $R$-module satisfies acc on d-colons

is shown there (actually in (ii) only the acc on d-annihilators is needed), and from [3] we know that

(iv) every $R$-module satisfies pan-acc

is also equivalent to (i)–(iii).

It is known from [11, Proposition] that in a module satisfying acc on d-colons every submodule has a (possibly infinite) primary decomposition. Further, the proof of Proposition 3.1 in [12] also works for infinite primary decompositions.

**Proposition 2.7.** Let $M$ be an $R$-module such that every submodule has a (possibly infinite) primary decomposition. Then the intersection of the submodules primary to maximal ideals is zero.

**Proof.** Assume that there is $x \neq 0$ which lies in every submodule primary to a maximal ideal of $R$. Then we can find a maximal ideal $P$ of $R$ such that $\text{Ann}(x) \subseteq P$. This yields $x \notin Px$. If we write $Px = \bigcap_{i \in I} Q_i$ as an intersection of primary submodules, then there is some $j \in I$ such that $x \notin Q_j$. But since $Px \subseteq Q_j$, $Q_j$ is $P$-primary, a contradiction. \hfill \Box
In the following, we say that an $R$-module $M$ satisfies Nakayama’s Lemma, or $M$ satisfies (NL), if the zero submodule is the only submodule $N$ such that $J(R)N = N$ (where $J(R)$ denotes the Jacobson radical of $R$).

**Theorem 2.8.** A module with acc on d-colons satisfies (NL).

**Proof.** Let $M$ be an $R$-module with acc on d-colons and $N$ a submodule of $M$ such that $J(R)N = N$. We have to show $N = \{ 0 \}$, and for this, by Proposition 2.7 it suffices to prove that $N$ is contained in any primary submodule which radical is a maximal ideal. So we assume that there is a maximal ideal $P$ of $R$ and a $P$-primary submodule $Q$ of $M$ such that $N \nsubseteq Q$. Then it is not difficult to check that $Q : N$ is a $P$-primary ideal in $R$. Since $PN = N$, we have $(Q : N) : P = Q : N$, and of Exercise 27(a) [6, p. 298] shows that there is $a_1 \in P$ such that $(Q : N) : a_1 = Q : a_1N$ is a $P$-primary ideal that properly contains $Q : N$. Again we have $(Q : a_1N) : P = Q : a_1N$. This yields an ascending chain $(Q : N) : a_1 \subset (Q : N) : a_1a_2 \subset \cdots$ in $R$. But since $M/Q$ has acc on d-annihilators, Lemma 2.2(c) implies that $R/(Q : N)$ has acc on d-annihilators, a contradiction. □

For a module $M$ over a quasilocal ring $(R, P)$ in which $\bigcap_{n=1}^{\infty} P^n M = \{ 0 \}$, Nakayama’s Lemma obviously holds. But the converse is not true: As remarked in [1, p. 263], there are quasilocal rings with T-nilpotent maximal ideal $P$ such that $\bigcap_{n=1}^{\infty} P^n \neq (0)$. An explicit example can be constructed as follows: For a field $K$ and indeterminates $y, x_1, x_2, \ldots$ over $K$ let $I$ be the ideal of $K[y, x_1, x_2, \ldots]$, generated by the terms $y^2, x_n^2$ and $y - x_nx_{k_1} \cdots x_{k_n}$, where $n < k_1 < \cdots < k_n$, and let $R := K[y, x_1, x_2, \ldots]/I$. Then it is not difficult to show that the maximal ideal $P$ of $R$ (which is generated by the images of $y$ and the $x_n$’s) is T-nilpotent, but the image of $y$ is a nonzero element of $\bigcap_{n=1}^{\infty} P^n$.

So our next proposition is a generalization of Proposition 2.1 of [1], and the proof is very similar.

**Proposition 2.9.** Let $M$ be a module over a quasilocal ring $(R, P)$. If $M/N$ satisfies (NL) for every $(n - 1)$-generated submodule of $M$, then $M$ has $n$-acc.

**Proof.** For $n = 1$, let $Rx_1 \subseteq Rx_2 \subseteq \cdots$ an ascending chain of cyclic submodules of $M$, say $x_i = a_ix_{i+1}$. If the chain would not become stationary, we could assume that all inclusions were proper; i.e. $a_i \in P$ for all $i$. But then the submodule $N$ generated by the $x_i$’s would satisfy $N = PN$ in contradiction to (NL).

Now assume that the assertion is true for $n$ and that $M/N$ satisfies (NL) for every $n$-generated submodule of $M$. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of $(n + 1)$-generated submodules of $M$. Suppose first that $N_1 \nsubseteq PN_k$ for all $k$. Then one of the generators of $N_1$, say $x$, can be chosen as one of a set of $n + 1$ generators of $N_k$ for every $k$. By the induction hypothesis the chain stabilizes in $M/Rx$, so the original chain does also. Hence we may assume without loss of generality that
\[ N_1 \subseteq P N_2 \text{ and similarly } N_k \subseteq P N_{k+1} \text{ for all } k. \text{ But then } N := \bigcup_{k=1}^{\infty} N_k \text{ satisfies } N = P N, \text{ so we conclude by (NL) that all } N_k \text{'s are zero}. \]

Theorem 2.8 and Proposition 2.9 yield our result in the quasilocal case.

**Corollary 2.10.** Let \( R \) be a quasilocal ring and \( M \) an \( R \)-module satisfying acc on \( d \)-colons. Then \( M \) has pan-acc.

Before we globalize this, we note a generalization of Proposition 2.3 of [4]:

**Corollary 2.11.** Let \( R \) be a Prüfer ring (not necessarily a domain) satisfying acc on \( d \)-colons. Then \( R \) is noetherian.

**Proof.** Every localization of \( R \) by a prime ideal is a valuation ring satisfying acc on \( d \)-annihilators. Since a valuation ring with accp is noetherian, every localization of \( R \) by a prime ideal is noetherian by Corollary 2.10. Further, \( R \) is a ZD-ring by Theorem 2.5. So \( R \) is noetherian by [13, Proposition].

Since modules with acc on \( d \)-colons may have infinitely many associated primes, the globalization of the pan-acc property from the quasilocal case via Theorem 3.4 of [1] fails. So we have to use the finiteness of the set of maximal primes, which the next theorem allows us to do.

**Theorem 2.12.** Let \( M \) be an \( R \)-module and \( N_1 \subseteq N_2 \subseteq \cdots \) an ascending chain of finitely generated submodules of \( M \) such that \((N_1)_P \subseteq (N_2)_P \subseteq \cdots \) becomes stationary for every maximal ideal \( P \) of \( R \). Assume further that \( Z(M/M_j) \) is contained in a finite union of maximal ideals for every \( j \). Then the chain stabilizes in \( M \).

**Proof.** Suppose the chain is not stationary. Then we may assume \( N_1 \subset N_2 \subset \cdots \). We construct a subchain \( \{M_n\} \) of \( \{N_n\} \) such that \((M_1)_{S_1} \subset \cdots \subset (M_{n+1})_{S_n} \) is the complement of a finite union of maximal ideals that cover \( Z(M/M_j) \) and \( S_n \) is the intersection of \( S_{n-1} \) and the complement of a finite union of maximal ideals that cover \( Z(M/M_n) \).

For this, set \( M_1 := N_1 \) and let \( k_1 \) be the index where the chain \((N_1)_{S_1} \subset \cdots \subset (N_2)_{S_1} \) becomes stationary. Then \( k_1 > 1 \) by the choice of \( S_1 \), so with \( M_2 := N_{k_1} \) we have \((M_1)_{S_1} \subset (M_2)_{S_1} = (N_{k_1+1})_{S_1} = \cdots \). Similarly, if \( M_n \) has been chosen to be \( N_{k_n-1} \), say the chain \((N_1)_{S_n} \subset (N_2)_{S_n} \subset \cdots \) becomes stationary at \( k_n \) and by the choice of \( S_n \) we have \( k_n > k_{n-1} \). Then we set \( M_{n+1} := N_{k_n} \) and it is \((M_1)_{S_n} \subset \cdots \subset (M_{n+1})_{S_n} = (N_{k_n+1})_{S_n} = \cdots \). Thus the subchain \( \{M_n\} \) with the above properties is constructed.

We claim now that for every \( n \) and every \( k > n \) the equality \( M_n : M_{n+1} = M_n : M_k \) holds. For this, pick \( a \in M_n : M_{n+1} \). Since \((M_{n+1})_{S_n} = (M_k)_{S_n} \), we have
a \in (M_n)_{S_n} : (M_k)_{S_n}, and since $M_k$ is finitely generated, there is $s \in S_n$ such that $saM_k \subseteq M_n$. But $s \notin Z(M/M_n)$ implies $aM_k \subseteq M_n$; i.e. $a \in M_n : M_k$.

From this it follows $M_n : M_{n+1} = M_n : M_{n+2} = \cdots = M_n : N$ for every $n$, where $N := \bigcup_{i=1}^\infty M_i$. Now consider the chain $M_1 : N \subseteq M_2 : N \subseteq \cdots$ and choose a maximal ideal $P$ of $R$ that contains this chain. By the hypothesis, there is an $n$ such that $(M_n)_P = (M_{n+1})_P = \cdots$, and since $M_{n+1}$ is finitely generated, we have $sM_{n+1} \subseteq M_n$ for some $s \in R \setminus P$. So $s \in M_n : M_{n+1} = M_n : N \subseteq P$, a contradiction. \hfill \Box

**Corollary 2.13.** In a ZD-module, an ascending chain of finitely generated submodules stabilizes if and only if it stabilizes in every localization by a maximal ideal.

This corollary yields another proof of the fact that a ZD-ring is noetherian if (and only if) every localization by a maximal ideal is noetherian. It also yields our main result in this section:

**Corollary 2.14.** Every module satisfying acc on d-colons has pan-acc.

Except for the case $n = 1$, it is still unknown if every free module over a ring with $n$-acc also satisfies $n$-acc (for $n = 1$, see [14]). But we can show the following theorem.

**Theorem 2.15.** Every free module over a ring with acc on d-colons satisfies pan-acc.

**Proof.** Let $M := \bigoplus_{i \in I} R$. For any maximal ideal $P$ of $R$, we show that $M_P \cong \bigoplus_{i \in I} R_P$ has pan-acc; so we may assume for this that $(R, P)$ is quasilocal. Let $N$ be a finitely generated submodule of $M$. By Proposition 2.9, we have to show that $M/N$ satisfies (NL). So let $U$ be a submodule of $M$ containing $N$ such that $U/N = P(U/N)$. Let $x \in U$ and $J$ a finite subset of $I$ such that $N + Rx \subseteq \bigoplus_{j \in J} R = : M'$. By [4, p. 170], we know that $M'/N$ satisfies acc on d-colons and hence it satisfies (NL). So, if $p : M \to M'$ is the projection, we conclude that $p(U)/N = P(p(U)/N) = N/N$ and (since $x \in p(U)) x \in N$. Thus every localization of $M$ by a maximal ideal of $R$ satisfies pan-acc.

Further, to complete the proof by Theorem 2.12, we have to show that, for every finitely generated submodule $N$ of $M$, $Z(M/N)$ is contained in a finite union of maximal ideals. For this it suffices to show that $M/N$ has acc on d-annihilators. Let $J$ be a finite subset of $I$ such that $N \subseteq \bigoplus_{j \in J} R = : M'$, $p : M \to M'$ the projection and consider a chain $N : a_1 \subseteq N : a_1a_2 \subseteq \cdots$ in $M$ for some $a_n \in R$. Since $R$ and $M'$ satisfy acc on d-colons, respectively, we may assume that $p(N) : a_1 = p(N) : a_1a_2 = \cdots$ in $M'$ and Ann$_R(a_1) = \cdots \subseteq R$. If we pick now $x \in N : a_1a_2$, we have $p(x) \in p(N) : a_1a_2 = p(N) : a_1 \subseteq
$N : a_1$ and $x_i \in \text{Ann}_R(a_1a_2) = \text{Ann}_R(a_1)$ for every component $x_i$ of $x$ such that $i \notin J$. This implies $x \in N : a_1$. Thus $M/N$ has acc on d-annihilators, and we are finished. \hfill \Box

3. The $R$-module $R^I$

In this section, we show, that for a noetherian ring $R$ and an arbitrary set $I$ the $R$-module $R^I$ satisfies pan-acc. We begin with a lemma similar to Proposition 2.1 in [3].

Lemma 3.1. Let $R$ be a noetherian ring, $S$ a multiplicatively closed subset of $R$, $I$ an arbitrary set and $M$ a finitely generated submodule of the $R_S$-module $(R^I)_S$. Then there is a finite subset $J$ of $I$ such that the projection $p : (R^I)_S \rightarrow (R^J)_S$ is injective on $M$.

Proof. Say $M$ is generated by $x^{(1)}, \ldots, x^{(n)}$ and $y_i$ ($i \in I$) are the projections of any $y \in (R^I)_S$ to $R_S$. Let $U$ be the submodule of $R_S^n$ which is generated by all $n$-tuples $(x^{(1)}_i, \ldots, x^{(n)}_i)$ ($i \in I$). Since $R_S$ is noetherian, there is a finite subset $J$ of $I$ such that $U = \sum_{j \in J} R_S(x^{(1)}_j, \ldots, x^{(n)}_j)$. Taking $y \in M \cap \text{Ker}(p)$ we have $y = \sum_{k=1}^n r_k x^{(k)}$ for some $r_k \in R_S$ and $y_j = \sum_{k=1}^n r_k x^{(k)}_j = 0$ for every $j \in J$. Further, for a fixed $i \in I$ there are $a_j \in R_S$ such that $(x^{(1)}_i, \ldots, x^{(n)}_i) = \sum_{j \in J} a_j (x^{(1)}_i, \ldots, x^{(n)}_i)$. Hence $y_i = \sum_{k=1}^n r_k x^{(k)}_i = \sum_{k=1}^n r_k \sum_{j \in J} a_j x^{(k)}_j = \sum_{j \in J} a_j \sum_{k=1}^n r_k x^{(k)}_j = 0$, so we have shown $M \cap \text{Ker}(p) = 0$. \hfill \Box

Lemma 3.2. Let $R$, $S$, $I$ and $M$ be as above. Then $(R^I)_S/M$ has only finitely many maximal primes.

Proof. By Lemma 3.1, there is a finite subset $J$ of $I$ such that the projection $p : (R^I)_S \rightarrow (R^J)_S$ is injective on $M$. Since $R_S$ is noetherian, the set $\Delta := \{M : z : z \in (R^I)_S \setminus M\}$ has maximal elements, and each of them is prime. To see that there are only finitely many, pick $M : x$ maximal in $\Delta$. If $(M + R_Sx) \cap \text{Ker}(p) = 0$, we have $M : x = p(M) : p(x)$, hence it is an associated prime of the noetherian module $(R^J)_S/p(M)$, which has only finitely many. So we may assume there is a nonzero $y \in M + R_Sx$ such that $p(y) = 0$. Then $M : x \subseteq M : y$, and since $y \notin M$ and $M : x$ is maximal in $\Delta$, equality holds. Further, $p(y) = 0$ yields $M : y = \text{Ann}(y) = \text{Ann}(y_1) \cap \cdots \cap \text{Ann}(y_n)$ for some $y_k \in R_S$ (since $R_S$ is noetherian). Say $M : x = \text{Ann}(y_1)$; so in this case $M : x$ is one of the finitely many associated primes of $R_S$. \hfill \Box
Now we are able to show the result promised at the beginning of this section. This is another application of Theorem 2.12.

**Theorem 3.3.** Let $R$ be a noetherian ring and $I$ be an arbitrary set. Then the $R$-module $R^I$ satisfies pan-acc.

**Proof.** We first show that every localization of $R^I$ by a maximal ideal $P$ has pan-acc. Since $R$ is noetherian, we can embed $(R^I)_P$ in $R^I_P$. So we may assume that $R$ is local with maximal ideal $P$, and by [1, Proposition 2.1] or our Proposition 2.9, it suffices to show that $\bigcap_{k=1}^{\infty} P^k (R^I/M) = \{0\}$ for every finitely generated submodule $M$ of $R^I$. Let $x + M \in \bigcap_{k=1}^{\infty} P^k (R^I/M)$. By Lemma 3.1, there is a finite subset $J$ of $I$ such that the projection $p: R^I \to R^J$ is injective on $M + Rx$. Then we have $p(x) + p(M) \in \bigcap_{k=1}^{\infty} P^k (R^J/p(M)) = \{0\}$, and $p(x) \in p(M)$ yields $x \in M$. Hence $\bigcap_{k=1}^{\infty} P^k (R^I/M) = \{0\}$.

By Lemma 2.2, $R^I/M$ has only finitely many maximal primes for every finitely generated submodule $M$ of $R^I$. Thus we can apply Theorem 2.12. $\square$

**Remark.** In fact, $R^I/M$ has only finitely many weakly associated primes for every finitely generated submodule $M$ of $R^I$; so Theorem 3.3 also follows from Theorem 3.4 of [1]. To see this, denote by $\text{Ass}(N)$ the set of weakly associated primes of a module $N$, and assume that $\text{Ass}(R^I/M)$ is infinite for some finitely generated submodule $M$ of $R^I$. Now we can argue as in the proof of Proposition 3.7 of [1]: By Lemma 3.2, $R^I/M$ has only finitely many maximal primes. Since weakly associated primes localize, there is a maximal prime $P$ such that $\text{Ass}((R^I)_P/M_P)$ is infinite. If $P_p$ is generated by $x_1, \ldots, x_n$ and we pick an associated prime $Q_P \neq P_P$ of $(R^I)_P/M_P$, we have $Q_{T_k} \in \text{Ass}(((R^I)_P/M_P)_{T_k})$ for some $k$, where $R_{T_k} = (R_P)_{S_k}$ and $S_k$ is the set of powers of $x_k$. Hence we may assume that $\text{Ass}((R^I)_{T_k}/M_{T_k})$ with $T := T_1$ is infinite; so by Lemma 3.2, this process can be iterated. The descending chain of the resulting primes yields a contradiction to the finite rank of $P$.

As another result concerning ascending chain conditions in $R^I$, we prove the following Lemma 3.4. For this, we state some basic facts about the acc on annihilators, e.g., from [15]. For a ring $R$, the following are equivalent:

(i) $R$ satisfies the ascending chain condition on annihilators.
(ii) $R$ satisfies the descending chain condition on annihilators.
(iii) For every subset $A$ of $R$ there is a finite subset $A'$ of $A$ such that $\text{Ann}(A) = \text{Ann}(A')$.

Further, a reduced ring satisfies the acc on annihilators if and only if it has only finitely many minimal primes.
Lemma 3.4. If \( R \) has \( \text{accp} \) and \( \text{acc} \) on annihilators, then the \( R \)-module \( R^I \) has 1-\( \text{acc} \) for every set \( I \).

Proof. Let \( Rx_1 \subseteq Rx_2 \subseteq \cdots \) be an ascending chain of cyclic submodules of \( R^I \), say \( x_n = r_n x_{n+1} \) for some \( r_n \in R \). Let \( x_{n,i} (i \in I) \) be the components of \( x_n \) and \( A_n \) the ideal in \( R \) generated by \( \{ x_{n,i} : i \in I \} \). Since the \( \text{acc} \) on annihilators holds in \( R \), we may assume that \( A := \text{Ann}(A_1) = \text{Ann}(A_n) \) for every \( n \); and further there are \( i_1, \ldots, i_m \in I \) such that \( A = \text{Ann}(x_{n,i_1}, \ldots, x_{n,i_m}) \) for every \( n \). From [14] we know that \( R^J \) has 1-\( \text{acc} \) (where \( J := \{ i_1, \ldots, i_m \} \)); so we have elements \( s_n \in R \) such that \( x_{n+1,ik} = s_n x_{n,ik} \) for \( k = 1, \ldots, m \). Hence \( (1 - r_n s_n)x_{n,ik} = 0 \); i.e. \( 1 - r_n s_n \in A \). It follows \( (1 - r_n s_n)x_{n+1} = 0 \); so \( x_{n+1} = s_n x_n \) and the chain stabilizes. \( \blacksquare \)

Remark. It is easy to see that another sufficient condition for the 1-\( \text{acc} \) property to rise to the \( R \)-module \( R^I \) is that every zero divisor is contained in the Jacobson radical. Note also that Lemma 3.4 fails for general \( \text{accp} \)-rings (cf. [14]).

4. Polynomial rings over rings with \( \text{accp} \)

We begin this section with a proof of the following result: \( \text{accp} \) rises to the polynomial ring \( R[X] \) if the \( \text{acc} \) on annihilators holds in \( R[X] \).

Theorem 4.1. Let \( R \) be an \( \text{accp} \)-ring and \( X \) an arbitrary set of indeterminates. If \( R[X] \) satisfies the \( \text{acc} \) on annihilators, then \( R[X] \) has \( \text{accp} \).

Proof. We assume the contrary. For a polynomial \( f \in R[X] \), denote by \( A_f \) the content ideal of \( f \). Then the set

\[
\Delta := \left\{ \text{Ann}\left( \bigcup_{i=1}^{\infty} A_{f_i} \right) : (f_1) \subseteq (f_2) \subseteq \cdots \text{ is a nonstabilizing chain in } R[X] \right\}
\]

is nonempty and has maximal elements, since \( R \) satisfies the \( \text{acc} \) on annihilators. Let \( \text{Ann}(\bigcup_{i=1}^{\infty} A_{f_i}) \) be maximal in \( \Delta \), where without loss of generality the chain \( (f_1) \subseteq (f_2) \subseteq \cdots \) strictly ascends. The chain \( \text{Ann}(A_{f_1}) \supseteq \text{Ann}(A_{f_2}) \supseteq \cdots \) stabilizes, so we may assume that \( P := \text{Ann}(A_{f_1}) = \text{Ann}(\bigcup_{i=1}^{\infty} A_{f_i}) \).

We show that \( P \) is prime. If not, we find \( a, b \in R \setminus P \), such that \( ab \in P \). This yields \( a \in \text{Ann}(\bigcup_{i=1}^{\infty} A_{f_i}) \setminus P \), so we may assume that \( (0) \neq (b f_1) = (b f_2) = \cdots \). If \( f_i = g_i f_{i+1} \) and \( b f_{i+1} = h_i b f_i \), we get \( b f_{i+1}(1 - h_i g_i) = 0 \) for every \( i \geq 1 \). Now we set \( p_i^{(k)} := f_i(1 - h_k g_k) \) \((i, k \geq 1)\). Then \( b \in \text{Ann}(\bigcup_{i=1}^{\infty} A_{p_i^{(k)}}) \setminus P \) for every \( k \); so the chains \( (p_1^{(k)}) \subseteq (p_2^{(k)}) \subseteq \cdots \) become stationary, say at \( n_k \). Since \( R[X] \) satisfies the \( \text{acc} \) on annihilators, there is \( m \) such that \( A := \text{Ann}(p_i^{(k)}) \):
i, k ≥ 1) = Ann(p_i^{(k)}: 1 ≤ i, k ≤ m) = Ann(p_1^{(1)}, ..., p_m^{(m)}) and hence we have
A = Ann(p_1^{(1)}, ..., p_m^{(m)}) for every l ≥ m. Set l := max{n_1, ..., n_m, m}. Then
(p_i^{(k)}) = (p_{l+1}^{(k)}) and (p_l^{(k)}) = g_l p_{l+1}^{(k)} for k = 1, ..., m; so as in the proof of the
Lemma in [14], there is a common factor q ∈ R[X] such that p_{l+1}^{(k)} = q p_l^{(k)} (k = 1, ..., m). It follows 1 − g_l q ∈ A and so p_{l+1}^{(l)}(1 − h_l g_l) = p_{l+1}^{(l)} = g_l q f_{l+1}(1 − h_l g_l); hence we have the contradiction f_{l+1} ∈ (f_l).

So P is a prime ideal, and without loss of generality every coefficient of g_i (where f_i = g_i, f_{i+1}) is either zero or not in P. Considering the decreasing degrees of the f_i’s in R_P[X], we may further assume that g_i ∈ R for every i. But by Lemma 3.4 (or [14, Theorem]) the chain stabilizes. □

It is shown in [16] that the acc on annihilators does not rise to the polynomial ring in general, so it would be interesting to know if the statement of Theorem 4.1 remains true if we only assume that R satisfies the acc on annihilators.

**Corollary 4.2.** Let X be an arbitrary set of indeterminates over the ring R.

(a) If R is a reduced accp-ring with only finitely many minimal primes, then R[X] satisfies accp.
(b) If R is a subring of a noetherian ring and R has accp, then R[X] satisfies accp.

**Proof.** (a) Since R[X] is also reduced with only finitely many primes, the statement follows from Theorem 4.1 and the remark made preceding Lemma 3.4.

(b) This follows from Lemma 4.3 and the fact that any subring of a ring with acc on annihilators inherits this property. □

**Lemma 4.3.** Let R be a noetherian ring and X be an arbitrary set of indeterminates. Then R[X] and R[[X]] := \bigcup{R[Y]}: Y is a finite subset of X satisfy the acc on annihilators.

**Proof.** The zero ideal of R[X] has a primary decomposition, so R[X] is a subring of a finite direct sum of polynomial rings over primary noetherian rings (i.e., (0) is a primary ideal). So we may assume that R is primary with nil radical P; hence R[X] is primary with nil radical P[X]. By Theorem 2.3 of [15], it remains to see that

(a) R[X] has only finitely many primes maximal in the zero divisors (this holds since P[X] is the only one);
(b) Ker(R[X] → R[X]_P[X]) = Ann(s) for some s ∈ P[X] (take s = 1); and
(c) R[X]_P[X] has the acc on annihilators (R[X]_P[X] is noetherian since P[X] is finitely generated).
Remark. Corollary 4.2(a) does not hold for reduced accp-rings in general. To see this, we modify the example of W. Heinzer and D. Lantz in [5]. Let $K$ be a field and $A_1, A_2, \ldots$ indeterminates over $K$. Set $A := \{A_n: n \geq 1\}$, $I := (A_n(A_{k-1} - A_k): 1 < k \leq n)K[A]$ and $S := K[A]/I$. Then it is easy to see that $Q_i := ((A_{n-1} - A_n: 1 < n \leq i) \cup \{A_n: n > i\})K[A]$ $(i \geq 1)$ and $Q := (A_{n-1} - A_n: n > 1)K[A]$ are exactly the primes that are minimal over $I$. By showing $\bigcap_{i=1}^n Q_i = (I, A_{n+1}, A_{n+2}, \ldots)K[A]$ for every $n$ we conclude that $I$ is the intersection of its minimal primes; i.e. $S$ is reduced. So the set of zero divisors of $S$ is the union of the minimal primes of $S$ and thus contained in $P := (a_1, a_2, \ldots)K[A]$, where $a_n$ denotes the image of $A_n$ in $S$. The ring $R := S_p$ is the desired example, for with the same proof as in [5], it can be shown that $R$ is an accp-ring while $R[x]$ is not.

We want to show now that for a strongly laskerian ring $R$ every localization of $R[X]$ by a prime ideal has accp. If principal ideals in $R[X]$ had only finitely many weakly associated primes (a question considered in [1, p. 269]) or at least $Z(R[X] / (f))$ for every $f \in R[X]$ would be contained in a finite union of primes, Theorem 2.12 would show that polynomial rings over strongly laskerian rings have accp.

**Proposition 4.4.** Let $(R, P)$ be a quasilocal ring satisfying $\bigcap_{k=1}^\infty P^k = (0)$, $X$ an arbitrary set of indeterminates, and $Q$ a prime ideal of $R[X]$ such that $Q \cap R = P$. Then $\bigcap_{k=1}^\infty Q^k \subset (0)$ in $R[X]_Q$.

**Proof.** (i) Consider at first the case where $X$ is a single indeterminate. Let $g \notin \bigcap_{k=1}^\infty Q^k$. If $Q = P[x]$, we have $g \in \bigcap_{k=1}^\infty P^k[x] = (0)$ (since $Q^k \subset P^k[x]$ and $P^k[x]$ is primary). Hence we may suppose that $Q = (P, f)$ is a maximal ideal of $R[X]$, where the image of $f$ is irreducible in $(R/P)[x]$. Then $g \in \bigcap_{k=1}^\infty Q^k$; so assume $g \neq 0$. There is an integer $n$ such that $g \notin P^n[x]$, and we pass to the factor ring $\bar{R} := R[x]/P^n[x] \cong (R/P^n)[x]$. Then $Q^k \subset f^{k-n+1}R$ for every $k \geq n$; so it is $g \in \bigcap_{k=1}^\infty f^kR$. Let $g = y_k f^k$ for every $k$. Since $f$ is not a zero-divisor in $\bar{R}$, we get the ascending chain $(u_1) \subset (u_2) \subset \cdots$ in $\bar{R}$. By [1, Corollary 2.2(a) and Proposition 3.8], $\bar{R}$ has accp; so $u_{k+1} = u_k$ for some $k$. Hence $g = wfg$ in the primary ring $\bar{R}$. Since $g \neq 0$ in $\bar{R}$, we have $1 - wf \in P[x] \subset Q$. This yields the contradiction $1 \in Q$.

(ii) The case where $X$ is a finite set of indeterminates follows by induction (using the fact that $R[x_1, \ldots, x_n]_Q$ is isomorphic to a localization of $R'_{Q \cap R'[x_n]}$, where $R' := R[x_1, \ldots, x_{n-1}]$).

(iii) If $X$ is an infinite set of indeterminates, we pick $x \in \bigcap_{k=1}^\infty Q^k_0$ and since only finitely many indeterminates appear in $g$, we may assume $g \in \bar{R}$ (by using similar techniques as in (ii)). So it suffices to show that $Q^{(k)} \cap R \subset P^k$ holds for
every symbolic power $Q^{(k)}$ of $Q$, and this follows as in the proof of Theorem 2.5 of [1]. □

Since a quasilocal ring $(R, P)$ with $\bigcap_{k=1}^{\infty} P^k = (0)$ satisfies accp, in Proposition 4.4 we conclude that $R[X]_Q$ has accp.

**Corollary 4.5.** Let $R$ be a strongly laskerian ring and $X$ a set of indeterminates. Then $R[X]_Q$ has accp for every prime $Q$ of $R[X]$.

**Proof.** By localizing at $Q \cap R$ we may assume that $R$ is quasilocal with only maximal ideal $P = Q \cap R$, and by Proposition 3.1 of [12], we have $\bigcap_{k=1}^{\infty} P^k = (0)$. □

A similar statement for power series rings also holds:

**Lemma 4.6.** Let $R$ be a ring such that for every maximal ideal $P$ of $R$ one has $\text{Ker}(R \to R_P) = \text{Ann}(s)$ for some $s \notin P$, and let $X$ be a set of indeterminates. If $R_P$ has accp for every maximal ideal $P$ of $R$, then $R[[X]]_M$ satisfies accp for every maximal ideal $M$ of $R[[X]]$.

**Proof.** Every maximal ideal $M$ of $R[[X]]$ has the form $M = (P, X)$ with $P$ maximal in $R$. Since $\text{Ker}(R \to R_P) = \text{Ann}(s)$ for some $s \notin P$, it is easy to see that $R[[X]]_M$ is a subring of $T := R_P[[X]]$, and since $R_P$ is a quasilocal accp-ring, so is $T$. The units of $R[[X]]_M$ are exactly the elements of $R[[X]]_M$ that are units in $T$. From this it follows easily that $R[[X]]_M$ has accp. □

**Corollary 4.7.** If $R$ is a ring with acc on d-colons and $X$ an arbitrary set of indeterminates, then $R[[X]]_M$ satisfies accp for every maximal ideal $M$ of $R[[X]]$. So, if in addition $R$ is semiquasilocal, $R[[X]]$ has accp.

**References**