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Five-dimensional superfield supergravity

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Abstract

We present a projective superspace formulation for matter-coupled simple supergravity in five dimensions. Our starting point is the superspace realization for the minimal supergravity multiplet proposed by Howe in 1981. We introduce various off-shell supermultiplets (i.e. hypermultiplets, tensor and vector multiplets) that describe matter fields coupled to supergravity. A projective-invariant action principle is given, and specific dynamical systems are constructed including supersymmetric nonlinear sigma-models. We believe that this approach can be extended to other supergravity theories with eight supercharges in $D \leqslant 6$ space–time dimensions, including the important case of 4D $\mathcal{N}=2$ supergravity.

1. Introduction

Projective superspace [1,2] is a powerful formalism for building off-shell rigid supersymmetric theories with eight supercharges in $D \leqslant 6$ space–time dimensions. It is ideal for the explicit construction of hyper-Kähler metrics [3]. For more than two decades, it has been an open problem to extend this approach to supergravity. A partial success has been achieved in our recent paper [4] where, in particular, the relevant projective formulation was developed for supersymmetric theories in five-dimensional $\mathcal{N}=1$ anti-de Sitter superspace $\mathrm{AdS}^{5|8} = \mathrm{SU}(2,2|1)/\mathrm{SO}(4,1) \times \mathrm{U}(1)$ which is a maximally symmetric curved background. In this Letter we briefly describe a solution to the problem in the case of 5D simple supergravity. A more detailed presentation will be given elsewhere [5].

For 5D $\mathcal{N}=1$ supergravity[1] [6], off-shell superspace formulations were only sketched by Breitenlohner and Kabelschacht [7] and independently by Howe [8] (who built on the 5D supercurrent constructed in [9]). Later, general matter couplings in 5D simple supergravity were constructed within on-shell components approaches [10–12]. More recently, off-shell component formulations for 5D supergravity-matter systems were developed in [13] and independently, within the superconformal tensor calculus, in [14,15]. Since the approaches elaborated in [13–15] are intrinsically component (i.e. they make use of off-shell hypermultiplets with finitely many auxiliary fields and an intrinsic central charge), they do not allow us to construct the most general sigma-model couplings, similar to the four-dimensional $\mathcal{N}=2$ case, and thus a superspace description is still desirable. Such a formulation is given below.

Before turning to the description of our superspace approach, we should emphasize once more that it is the presence of the intrinsic central charge that hypermultiplets possess, within the component formulations of [13,14] which makes it impossible to cast general quaternionic Kähler couplings in terms of such off-shell hypermultiplets[2] (see, e.g., [16] for a similar discussion in the case of 4D $\mathcal{N}=2$ supergravity). On the other hand, the projective superspace approach offers nice off-shell formulations without central charge. Specifically, there are infinitely many off-shell realizations with *finitely many auxiliary fields* for a neutral hypermultiplet (they are the called $O(2n)$ multiplets, where $n = 2, 3, \dots$, following the terminology of [20]), and a unique formulation

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[1] On historical grounds, 5D simple ($\mathcal{N}=1$) supersymmetry and supergravity are often labeled $\mathcal{N}=2$.

[2] Ref. [15] deal with on-shell hypermultiplets only.

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for a charged hypermultiplet with *infinitely many auxiliary fields* (the so-called polar hypermultiplet). Using covariant polar hypermultiplets introduced below, one can construct sigma-model couplings that cannot be derived within the off-shell component approach. An example is given by the supergravity-matter system (4.17), for a generic choice of the real function $K(\Phi,\bar{\Phi})$ obeying the homogeneity condition (4.14). We hope to discuss this point in more detail elsewhere.

This Letter is organized as follows. In Section 2, we describe the superspace geometry of the minimal multiplet for 5D $\mathcal{N}=1$ supergravity outlined in [8]. Various off-shell supermultiplets are introduced in Section 3. In Section 4, we present a supersymmetric projective-invariant action principle in a Wess–Zumino gauge. This action is ready for applications, in particular for a reduction from superfields to components. We also introduce several families of supergravity-matter systems. Finally, in Section 5 we describe a locally supersymmetric action which we expect to reduce, in the Wess–Zumino gauge, to the action given in Section 4.

2. Superspace geometry of the minimal supergravity multiplet

Let $z^{\hat{M}}=(x^{\hat{m}},\theta^{\hat{\mu}}_i)$ be local bosonic (x) and fermionic (θ) coordinates parametrizing a curved five-dimensional $\mathcal{N}=1$ superspace $\mathcal{M}^{5|8}$, where $\hat{m}=0,1,\dots,4$, $\hat{\mu}=1,\dots,4$, and $i=\underline{1},\underline{2}$. The Grassmann variables $\theta^{\hat{\mu}}_i$ are assumed to obey a standard pseudo-Majorana reality condition. Following [8], the tangent-space group is chosen to be SO(4,1) × SU(2), and the superspace covariant derivatives $\mathcal{D}_{\hat{A}}=(\mathcal{D}_{\hat{a}},\mathcal{D}^i_{\hat{\alpha}})$ have the form

$$\mathcal{D}_{\hat{A}}=E_{\hat{A}}+\Omega_{\hat{A}}+\Phi_{\hat{A}}+V_{\hat{A}}Z. \tag{2.1}$$

Here $E_{\hat{A}}=E_{\hat{A}}{}^{\hat{M}}(z)\,\partial_{\hat{M}}$ is the supervielbein, with $\partial_{\hat{M}}=\partial/\partial z^{\hat{M}}$,

$$\Omega_{\hat{A}}=\frac{1}{2}\Omega_{\hat{A}}{}^{\hat{b}\hat{c}}M_{\hat{b}\hat{c}}=\Omega_{\hat{A}}{}^{\hat{\beta}\hat{\gamma}}M_{\hat{\beta}\hat{\gamma}},\qquad M_{\hat{a}\hat{b}}=-M_{\hat{b}\hat{a}},\qquad M_{\hat{\alpha}\hat{\beta}}=M_{\hat{\beta}\hat{\alpha}} \tag{2.2}$$

is the Lorentz connection,

$$\Phi_{\hat{A}}=\Phi^{kl}_{\hat{A}}J_{kl},\quad J_{kl}=J_{lk} \tag{2.3}$$

is the SU(2)-connection, and Z the central-charge generator. The Lorentz generators with vector indices ($M_{\hat{a}\hat{b}}$) and spinor indices ($M_{\hat{\alpha}\hat{\beta}}$) are related to each other by the rule: $M_{\hat{a}\hat{b}}=(\Sigma_{\hat{a}\hat{b}})^{\hat{\alpha}\hat{\beta}}M_{\hat{\alpha}\hat{\beta}}$ (for more details regarding our 5D notation and conventions, see Appendix in [4]). The generators of SO(4,1) × SU(2) act on the covariant derivatives as follows:

$$\left[J^{kl},\mathcal{D}^i_{\hat{\alpha}}\right]=\varepsilon^{i(k}\mathcal{D}^{l)}_{\hat{\alpha}},\qquad \left[M_{\hat{\alpha}\hat{\beta}},\mathcal{D}^i_{\hat{\gamma}}\right]=\varepsilon_{\hat{\gamma}(\hat{\alpha}}\mathcal{D}^i_{\hat{\beta})}, \tag{2.4}$$

where $J^{kl}=\varepsilon^{ki}\varepsilon^{lj}J_{ij}$, and the symmetrization of n indices involves a factor of $(n!)^{-1}$. The covariant derivatives obey (anti)commutation relations of the general form

$$[\mathcal{D}_{\hat{A}},\mathcal{D}_{\hat{B}}\}=T_{\hat{A}\hat{B}}{}^{\hat{C}}\mathcal{D}_{\hat{C}}+\frac{1}{2}R_{\hat{A}\hat{B}}{}^{\hat{c}\hat{d}}M_{\hat{c}\hat{d}}+R_{\hat{A}\hat{B}}{}^{kl}J_{kl}+F_{\hat{A}\hat{B}}Z, \tag{2.5}$$

where $T_{\hat{A}\hat{B}}{}^{\hat{C}}$ is the torsion, $R_{\hat{A}\hat{B}}{}^{kl}$ and $R_{\hat{A}\hat{B}}{}^{\hat{c}\hat{d}}$ the SU(2)- and SO(4,1)-curvature tensors, and $F_{\hat{A}\hat{B}}$ the central charge field strength.

The supergravity gauge group is generated by local transformations of the form

$$\mathcal{D}_{\hat{A}}\to\mathcal{D}'_{\hat{A}}=\mathrm{e}^{K}\mathcal{D}_{\hat{A}}\mathrm{e}^{-K},\quad K=K^{\hat{C}}(z)\mathcal{D}_{\hat{C}}+\frac{1}{2}K^{\hat{c}\hat{d}}(z)M_{\hat{c}\hat{d}}+K^{kl}(z)J_{kl}+\tau(z)Z, \tag{2.6}$$

with all the gauge parameters being neutral with respect to the central charge Z, obeying natural reality conditions, and otherwise arbitrary. Given a tensor superfield $U(z)$, it transforms as follows:

$$U\to U'=\mathrm{e}^{K}U. \tag{2.7}$$

In accordance with [8], in order to realize the so-called minimal supergravity multiplet in the above framework, one has to impose special covariant constraints on various components of the torsion[3] of dimensions 0, 1/2 and 1. They are:

$$T^{i\,j\,\hat{c}}_{\hat{\alpha}\hat{\beta}}=-2\mathrm{i}\varepsilon^{ij}\left(\Gamma^{\hat{c}}\right)_{\hat{\alpha}\hat{\beta}},\qquad F^{i\,j}_{\hat{\alpha}\hat{\beta}}=-2\mathrm{i}\varepsilon^{ij}\varepsilon_{\hat{\alpha}\hat{\beta}}\quad\text{(dimension-0)}, \tag{2.8a}$$

$$T^{i\,j\,\hat{\gamma}}_{\hat{\alpha}\hat{\beta}k}=T^{i}_{\hat{\alpha}\hat{b}}{}^{\hat{c}}=F^{i}_{\hat{\alpha}\hat{b}}=0\quad\text{(dimension-1/2)}, \tag{2.8b}$$

$$T_{\hat{a}\hat{b}}{}^{\hat{c}}=T_{\hat{a}\hat{\beta}(j}{}^{\hat{\beta}}{}_{k)}=0\quad\text{(dimension-1)}. \tag{2.8c}$$

[3] As demonstrated by Dragon [17], the curvature is completely determined in terms of the torsion in supergravity theories formulated in superspace.

Under these constraints, the algebra (2.5) can be shown to take the form[4] (its derivation will be given in [5]):

$$\begin{aligned}\big\{\mathcal{D}^i_{\hat\alpha},\mathcal{D}^j_{\hat\beta}\big\} &= -2\mathrm{i}\varepsilon^{ij}\mathcal{D}_{\hat\alpha\hat\beta} - 2\mathrm{i}\varepsilon^{ij}\varepsilon_{\hat\alpha\hat\beta}Z + 3\mathrm{i}\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij}S^{kl}J_{kl} - 2\mathrm{i}\big(\Sigma^{\hat a\hat b}\big)_{\hat\alpha\hat\beta}(F_{\hat a\hat b}+N_{\hat a\hat b})J^{ij}\\ &\quad - \mathrm{i}\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij}F^{\hat c\hat d}M_{\hat c\hat d} + \frac{\mathrm{i}}{4}\varepsilon^{ij}\varepsilon^{\hat a\hat b\hat c\hat d\hat e}N_{\hat a\hat b}(\Gamma_{\hat c})_{\hat\alpha\hat\beta}M_{\hat d\hat e} + 4\mathrm{i}S^{ij}M_{\hat\alpha\hat\beta},\end{aligned} \tag{2.9a}$$

$$\begin{aligned}\big[\mathcal{D}_{\hat a},\mathcal{D}^j_{\hat\beta}\big] &= \frac12(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\gamma}S^j{}_k\mathcal{D}^k_{\hat\gamma} - \frac12F_{\hat a\hat b}\big(\Gamma^{\hat b}\big)_{\hat\beta}{}^{\hat\gamma}\mathcal{D}^j_{\hat\gamma} - \frac18\varepsilon_{\hat a\hat b\hat c\hat d\hat e}N^{\hat d\hat e}\big(\Sigma^{\hat b\hat c}\big)_{\hat\beta}{}^{\hat\gamma}\mathcal{D}^j_{\hat\gamma}\\ &\quad + \left(-3\varepsilon^{jk}\Xi_{\hat a\hat\beta}{}^l + \frac54(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\alpha}\varepsilon^{jk}\mathcal{F}_{\hat\alpha}{}^l - \frac14(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\alpha}\varepsilon^{jk}\mathcal{N}_{\hat\alpha}{}^l\right)J_{kl}\\ &\quad + \left(\frac12(\Gamma_{\hat a})^{\hat\alpha\hat\gamma}\mathcal{D}^{\hat\delta j}F_{\hat\alpha\hat\beta} - \frac12(\Gamma_{\hat a})_{\hat\beta\hat\alpha}\mathcal{D}^{\hat\gamma j}F^{\hat\delta\hat\alpha} - \frac12(\Gamma_{\hat a})^{\hat\alpha\hat\gamma}\delta^{\hat\delta}_{\hat\beta}\mathcal{D}^{\hat\rho j}F_{\hat\alpha\hat\rho}\right.\\ &\quad \left. + \frac12(\Gamma_{\hat a})_{\hat\rho\hat\alpha}\mathcal{D}^{\hat\rho j}F^{\hat\alpha\hat\gamma}\delta^{\hat\delta}_{\hat\beta} - \frac12(\Gamma_{\hat a})_{\hat\beta\hat\rho}\mathcal{D}^{\hat\rho j}F^{\hat\gamma\hat\delta}\right)M_{\hat\gamma\hat\delta},\end{aligned} \tag{2.9b}$$

$$\begin{aligned}[\mathcal{D}_{\hat a},\mathcal{D}_{\hat b}] &= \frac{\mathrm{i}}{2}\big(\mathcal{D}^{\hat\gamma}_k F_{\hat a\hat b}\big)\mathcal{D}^k_{\hat\gamma} - \frac{\mathrm{i}}{8}\big(\mathcal{D}^{\hat\gamma(k}\mathcal{D}^{l)}_{\hat\gamma}F_{\hat a\hat b}\big)J_{kl} + F_{\hat a\hat b}Z\\ &\quad + \left(\frac14\varepsilon^{\hat c\hat d}{}_{\hat m\hat n[\hat a}\mathcal{D}_{\hat b]}N^{\hat m\hat n} + \frac12\delta^{\hat c}_{[\hat a}N_{\hat b]\hat m}N^{\hat d\hat m} - \frac14N_{\hat a}{}^{\hat c}N_{\hat b}{}^{\hat d} - \frac18\delta^{\hat c}_{\hat a}\delta^{\hat d}_{\hat b}N^{\hat m\hat n}N_{\hat m\hat n}\right.\\ &\quad \left. + \frac{\mathrm{i}}{8}\big(\Sigma^{\hat c\hat d}\big)^{\hat\gamma\hat\delta}\mathcal{D}^k_{\hat\gamma}\mathcal{D}_{\hat\delta k}F_{\hat a\hat b} - F_{\hat a}{}^{\hat c}F_{\hat b}{}^{\hat d} + \frac12\delta^{\hat c}_{\hat a}\delta^{\hat d}_{\hat b}S^{ij}S_{ij}\right)M_{\hat c\hat d}.\end{aligned} \tag{2.9c}$$

The torsion components obey a number of Bianchi identities some of which can be conveniently expressed in terms of the three irreducible components of $\mathcal{D}^k_{\hat\gamma}F_{\hat\alpha\hat\beta}$: a completely symmetric third-rank tensor $W_{\hat\alpha\hat\beta\hat\gamma}{}^k$, a gamma-traceless spin-vector $\Xi_{\hat a\hat\gamma}{}^k$ and a spinor $\mathcal{F}_{\hat\gamma}{}^k$. These components originate as follows:

$$\begin{aligned}&\mathcal{D}^k_{\hat\gamma}F_{\hat\alpha\hat\beta} = W_{\hat\alpha\hat\beta\hat\gamma}{}^k + (\Gamma_{\hat a})_{\hat\gamma(\hat\alpha}\Xi^{\hat a}{}_{\hat\beta)}{}^k + \varepsilon_{\hat\gamma(\hat\alpha}\mathcal{F}_{\hat\beta)}{}^k,\\ &W_{\hat\alpha\hat\beta\hat\gamma}{}^k = W_{(\hat\alpha\hat\beta\hat\gamma)}{}^k, \qquad \big(\Gamma^{\hat a}\big)_{\hat\alpha}{}^{\hat\beta}\,\Xi_{\hat a\hat\beta}{}^i = 0.\end{aligned} \tag{2.10}$$

The dimension-3/2 Bianchi identities are as follows:

$$\mathcal{D}^k_{\hat\gamma}N_{\hat\alpha\hat\beta} = -W_{\hat\alpha\hat\beta\hat\gamma}{}^k + 2(\Gamma_{\hat a})_{\hat\gamma(\hat\alpha}\Xi^{\hat a}{}_{\hat\beta)}{}^k + \varepsilon_{\hat\gamma(\hat\alpha}\mathcal{N}_{\hat\beta)}{}^k, \tag{2.11a}$$

$$\mathcal{D}^k_{\hat\beta}S^{jl} = -\frac12\varepsilon^{k(j}\big(3\mathcal{F}_{\hat\beta}{}^{l)} + \mathcal{N}_{\hat\beta}{}^{l)}\big). \tag{2.11b}$$

The Bianchi identities of dimension 2 will be described in [5]. A simple consequence of (2.11b) is

$$\mathcal{D}^{(i}_{\hat\alpha}S^{jk)} = 0. \tag{2.12}$$

This result will be important in what follows.

3. Projective supermultiplets

To introduce an important class of off-shell supermultiplets, it is convenient to make use of an isotwistor $u^+_i \in \mathbb{C}^2 \setminus \{0\}$ defined to be inert with respect to the local SU(2) group (in complete analogy with [4,18]). Then, in accordance with (2.8a), the operators $\mathcal{D}^+_{\hat\alpha} := u^+_i\mathcal{D}^i_{\hat\alpha}$ obey the following algebra:

$$\big\{\mathcal{D}^+_{\hat\alpha},\mathcal{D}^+_{\hat\beta}\big\} = -4\mathrm{i}(F_{\hat\alpha\hat\beta} + N_{\hat\alpha\hat\beta})J^{++} + 4\mathrm{i}S^{++}M_{\hat\alpha\hat\beta}, \tag{3.1}$$

where $J^{++} := u^+_iu^+_jJ^{ij}$ and $S^{++} := u^+_iu^+_jS^{ij}$. Relation (3.1) naturally hints at the possibility of introducing covariant superfields $Q(z,u^+)$ obeying the chiral-like condition $\mathcal{D}^+_{\hat\alpha}Q = 0$ (which is a generalization of the so-called analyticity condition in 4D $\mathcal{N}=2$ rigid supersymmetric [1,19] and 5D $\mathcal{N}=1$ anti-de Sitter [4] cases). For this constraint to be consistent, however, such superfields must be scalar with respect to the Lorentz group, $M_{\hat\alpha\hat\beta}Q = 0$, and also possess special properties with respect to the group SU(2), that is, $J^{++}Q = 0$. Now we define such multiplets.

A *covariantly analytic* multiplet of weight n, $Q^{(n)}(z,u^+)$, is a scalar superfield that lives on $\mathcal{M}^{5|8}$, is holomorphic with respect to the isotwistor variables u^+_i on an open domain of $\mathbb{C}^2\setminus\{0\}$, and is characterized by the following conditions:

[4] In [8], the solution to the constraints was not given in detail. In particular, the algebra of covariant derivatives (2.9a)–(2.9c) was not included.

(i) it obeys the analyticity constraint

$$\mathcal{D}^{+}_{\hat{\alpha}} Q^{(n)} = 0; \tag{3.2}$$

(ii) it is a homogeneous function of u^+ of degree n, that is,

$$Q^{(n)}\big(z, cu^+\big) = c^n Q^{(n)}\big(z, u^+\big), \quad c \in \mathbb{C}^*; \tag{3.3}$$

(iii) infinitesimal gauge transformations (2.6) act on $Q^{(n)}$ as follows:

$$\begin{aligned} &\delta Q^{(n)} = \big(K^{\hat{C}}\mathcal{D}_{\hat{C}} + K^{ij}J_{ij} + \tau Z\big)Q^{(n)}, \\ &K^{ij}J_{ij}Q^{(n)} = -\frac{1}{(u^+u^-)}\big(K^{++}D^{--} - nK^{+-}\big)Q^{(n)}, \quad K^{\pm\pm} = K^{ij}u^{\pm}_i u^{\pm}_j, \end{aligned} \tag{3.4}$$

where

$$D^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}, \qquad D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}. \tag{3.5}$$

Transformation law (3.4) involves an additional isotwistor, u^-_i, which is subject to the only condition $(u^+u^-) = u^{+i}u^-_i \neq 0$, and is otherwise completely arbitrary. By construction, $Q^{(n)}$ is independent of u^-, i.e. $\partial Q^{(n)}/\partial u^{-i} = 0$, and hence $D^{++}Q^{(n)} = 0$. It is easy to see that $\delta Q^{(n)}$ is also independent of the isotwistor u^-, $\partial(\delta Q^{(n)})/\partial u^{-i} = 0$, as a consequence of (3.3). It follows from (3.4)

$$J^{++}Q^{(n)} = 0, \qquad J^{++} \propto D^{++}, \tag{3.6}$$

and therefore the constraint (3.2) is indeed consistent. It is important to point out that Eq. (3.6) is purely algebraic.

In what follows, our consideration will be restricted to those supermultiplets that are inert with respect to the central charge, $ZQ^{(n)} = 0$.

Given a covariantly analytic superfield $Q^{(n)}$, its complex conjugate is not analytic. However, similarly to the flat four-dimensional case [1,19] (see also [4]), one can introduce a generalized, analyticity-preserving conjugation, $Q^{(n)} \to \tilde{Q}^{(n)}$, defined as

$$\tilde{Q}^{(n)}\big(u^+\big) \equiv \bar{Q}^{(n)}\big(\tilde{u}^+\big), \quad \tilde{u}^+ = \mathrm{i}\sigma_2 u^+, \tag{3.7}$$

with $\bar{Q}^{(n)}$ the complex conjugate of $Q^{(n)}$. Its fundamental property is

$$\widetilde{\mathcal{D}^{+}_{\hat{\alpha}} Q^{(n)}} = (-1)^{\epsilon(Q^{(n)})}\mathcal{D}^{+\hat{\alpha}}\tilde{Q}^{(n)}. \tag{3.8}$$

One can see that $\tilde{\tilde{Q}}^{(n)} = (-1)^n Q^{(n)}$, and therefore real supermultiplets can be consistently defined when n is even. In what follows, $\tilde{Q}^{(n)}$ will be called the smile-conjugate of $Q^{(n)}$.

With respect to the natural projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$, the isotwistor u^+_i plays the role of homogeneous global coordinates for $\mathbb{C}P^1$, and the covariantly analytic superfields $Q^{(n)}(z, u^+)$ introduced describe special supermultiplets living in $\mathcal{M}^{5|8} \times \mathbb{C}P^1$. As is well known, instead of the homogeneous coordinates u^+_i, it is often useful to work with an inhomogeneous local complex variable ζ that is invariant under arbitrary projective rescalings $u^+_i \to cu^+_i$, with $c \in \mathbb{C}^*$. In such an approach, one should replace $Q^{(n)}(z, u^+)$ with a new superfield $Q^{[n]}(z, \zeta) \propto Q^{(n)}(z, u^+)$, where $Q^{[n]}(z, \zeta)$ is holomorphic with respect to ζ, and its explicit definition depends on the supermultiplet under consideration. The space $\mathbb{C}P^1$ can naturally be covered by two open charts in which ζ can be defined, and the simplest choice is: (i) the north chart characterized by $u^{+\underline{1}} \neq 0$; (ii) the south chart with $u^{+\underline{2}} \neq 0$. In discussing various supermultiplets, our consideration below will be restricted to the north chart.

In the north chart $u^{+\underline{1}} \neq 0$, and the projective-invariant variable $\zeta \in \mathbb{C}$ can be defined in the simplest way:

$$u^{+i} = u^{+\underline{1}}(1, \zeta) = u^{+\underline{1}}\zeta^i, \quad \zeta^i = (1, \zeta), \qquad \zeta_i = \varepsilon_{ij}\zeta^j = (-\zeta, 1). \tag{3.9}$$

Since any projective multiplet $Q^{(n)}$ and its transformation (3.4) do not depend on u^-, we can make a convenient choice for the latter. In the north chart, it is

$$u^-_i = (1, 0), \qquad u^{-i} = \varepsilon^{ij}u^-_j = (0, -1). \tag{3.10}$$

The transformation parameters K^{++} and K^{+-} in (3.4) can be represented as $K^{++} = (u^{+\underline{1}})^2 K^{++}(\zeta)$ and $K^{+-} = u^{+\underline{1}}K(\zeta)$, where

$$K^{++}(\zeta) = K^{ij}\zeta_i\zeta_j = K^{\underline{11}}\zeta^2 - 2K^{\underline{12}}\zeta + K^{\underline{22}}, \qquad K(\zeta) = K^{\underline{1}i}\zeta_i = -K^{\underline{11}}\zeta + K^{\underline{12}}. \tag{3.11}$$

If the projective supermultiplet $Q^{(n)}(z, u^+)$ is described by $Q^{[n]}(z, \zeta) \propto Q^{(n)}(z, u^+)$ in the north chart, then the analyticity condition (3.2) turns into

$$\mathcal{D}^{+}_{\hat{\alpha}}(\zeta)Q^{[n]}(\zeta) = 0, \qquad \mathcal{D}^{+}_{\hat{\alpha}}(\zeta) = \mathcal{D}^i_{\hat{\alpha}}\zeta_i = -\zeta\mathcal{D}^{\underline{1}}_{\hat{\alpha}} + \mathcal{D}^{\underline{2}}_{\hat{\alpha}}. \tag{3.12}$$

Let us give several important examples of projective supermultiplets.

An arctic multiplet[5] of weight n is defined to be holomorphic on the north chart. It can be represented as

$$\Upsilon^{(n)}(z,u) = \big(u^{+\underline{1}}\big)^n \Upsilon^{[n]}(z,\zeta), \qquad \Upsilon^{[n]}(z,\zeta) = \sum_{k=0}^{\infty} \Upsilon_k(z)\zeta^k. \tag{3.13}$$

The transformation law of $\Upsilon^{[n]}$ can be read off from Eq. (3.4) by noting (see [18,21] for more details)

$$K^{ij}J_{ij}\Upsilon^{[n]}(\zeta) = \big(K^{++}(\zeta)\partial_\zeta + nK(\zeta)\big)\Upsilon^{[n]}(\zeta), \tag{3.14}$$

or equivalently

$$\begin{aligned}
&J_{\underline{11}}\Upsilon_0 = 0, \qquad J_{\underline{11}}\Upsilon_k = (k-1-n)\Upsilon_{k-1}, \quad k>0,\\
&J_{\underline{22}}\Upsilon_k = (k+1)\Upsilon_{k+1},\\
&J_{\underline{12}}\Upsilon_k = \left(\frac{n}{2}-k\right)\Upsilon_k.
\end{aligned} \tag{3.15}$$

It is important to emphasize that the transformation law of $\Upsilon^{[n]}$ preserves the functional structure of $\Upsilon^{[n]}$ defined in (3.13).

The analyticity condition (3.12) implies

$$\mathcal{D}^{\underline{2}}_{\hat\alpha}\Upsilon_0 = 0, \qquad \mathcal{D}^{\underline{2}}_{\hat\alpha}\Upsilon_1 = \mathcal{D}^{\underline{1}}_{\hat\alpha}\Upsilon_0. \tag{3.16}$$

The integrability conditions for these constraints can be shown to be $J_{\underline{11}}\Upsilon_0 = 0$ and $J_{\underline{11}}\Upsilon_1 = -2J_{\underline{12}}\Upsilon_0$, and they hold identically due to (3.15). Using the algebra of covariant derivatives, Eq. (2.9a), one can deduce from (3.16)

$$\big(\mathcal{D}^{\underline{2}}_{[\hat\alpha}\mathcal{D}^{\underline{2}}_{\hat\beta]} + 3\mathrm{i}\varepsilon_{\hat\alpha\hat\beta}S^{\underline{22}}\big)\Upsilon_1 = 2\mathrm{i}\left(\mathcal{D}_{\hat\alpha\hat\beta} - \frac{3}{2}n\varepsilon_{\hat\alpha\hat\beta}S^{\underline{12}}\right)\Upsilon_0. \tag{3.17}$$

The smile-conjugate of $\Upsilon^{(n)}$ is said to be an antarctic multiplet of weight n. It proves to be holomorphic on the south chart, while in the north chart it has the form

$$\breve{\Upsilon}^{(n)}(z,u) = \big(u^{+\underline{2}}\big)^n \breve{\Upsilon}^{[n]}(z,\zeta), \qquad \breve{\Upsilon}^{[n]}(z,\zeta) = \sum_{k=0}^{\infty}(-1)^k\bar{\Upsilon}_k(z)\frac{1}{\zeta^k}, \tag{3.18}$$

with $\bar\Upsilon_k$ the complex conjugate of U_k. Its transformation follows from (3.4) by noting

$$K^{ij}J_{ij}\breve{\Upsilon}^{[n]}(\zeta) = \frac{1}{\zeta^n}\big(K^{++}(\zeta)\partial_\zeta + nK(\zeta)\big)\big(\zeta^n\breve{\Upsilon}^{(n)}(\zeta)\big). \tag{3.19}$$

The arctic multiplet $\Upsilon^{[n]}$ and its smile-conjugate $\breve{\Upsilon}^{(n)}$ constitute a polar multiplet.

Our next example is a real $O(2n)$-multiplet, $\breve{H}^{(2n)} = H^{(2n)}$.

$$\begin{aligned}
&H^{(2n)}\big(z,u^+\big) = u^+_{i_1}\cdots u^+_{i_{2n}}H^{i_1\cdots i_{2n}}(z) = \big(\mathrm{i}u^{+\underline{1}}u^{+\underline{2}}\big)^n H^{[2n]}(z,\zeta),\\
&H^{[2n]}(z,\zeta) = \sum_{k=-n}^{n} H_k(z)\zeta^k, \qquad \bar{H}_k = (-1)^k H_{-k}.
\end{aligned} \tag{3.20}$$

The transformation of $H^{[2n]}$ follows from (3.4) by noting

$$K^{ij}J_{ij}H^{[2n]} = \frac{1}{\zeta^n}\big(K^{++}(\zeta)\partial_\zeta + 2nK(\zeta)\big)\big(\zeta^n H^{[2n]}\big). \tag{3.21}$$

This can be seen to be equivalent to

$$\begin{aligned}
&J_{\underline{11}}H_{-n} = 0, \qquad J_{\underline{11}}H_k = (k-1-n)H_{k-1}, \quad -n<k\leqslant n,\\
&J_{\underline{22}}H_n = 0, \qquad J_{\underline{22}}H_k = (k+1+n)H_{k+1}, \quad -n\leqslant k<n,\\
&J_{\underline{12}}H_k = -kH_k.
\end{aligned} \tag{3.22}$$

The analyticity condition (3.12) implies, in particular, the constraints: $\mathcal{D}^{\underline{2}}_{\hat\alpha}H_{-n} = 0$ and $\mathcal{D}^{\underline{2}}_{\hat\alpha}H_{-n+1} = \mathcal{D}^{\underline{1}}_{\hat\alpha}H_{-n}$. The corresponding integrability conditions can be shown to hold due to (3.22). The case $n=1$ corresponds to an off-shell tensor multiplet.

[5] For covariantly analytic multiplets, we adopt the same terminology which was first introduced in [20] in the super-Poincaré case and which is standard nowadays.

Our last example is a real tropical multiplet of weight $2n$:

$$U^{(2n)}\big(z,u^+\big)=\big(\mathrm{i}u^{+\underline{1}}u^{+\underline{2}}\big)^n U^{[2n]}(z,\zeta)=\big(u^{+\underline{1}}\big)^{2n}(\mathrm{i}\zeta)^n U^{[2n]}(z,\zeta),$$
$$U^{[2n]}(z,\zeta)=\sum_{k=-\infty}^{\infty}U_k(z)\zeta^k,\qquad \bar U_k=(-1)^k U_{-k}. \tag{3.23}$$

The SU(2)-transformation law of $U^{[2n]}(z,\zeta)$ copies (3.21). To describe a massless vector multiplet prepotential, one should choose $n=0$. Supersymmetric real Lagrangians correspond to the choice $n=1$, see below.

4. Supersymmetric action in the Wess–Zumino gauge

In our previous paper [4], we formulated the supersymmetric action principle in five-dimensional $\mathcal{N}=1$ anti-de Sitter superspace $\mathrm{AdS}^{5|8}$. From the supergravity point of view, the geometry of this superspace is singled out by setting

$$\mathcal{D}^i_{\hat\alpha}S^{jk}=0,\qquad F_{\hat a\hat b}=N_{\hat a\hat b}=0 \tag{4.1}$$

in the (anti)commutation relations (2.9a)–(2.9c), and the central charge decouples.[6] In a Wess–Zumino gauge, the action functional constructed in [4] is as follows[7]:

$$S=-\frac{1}{2\pi}\oint\frac{u_i^+\,\mathrm{d}u^{+i}}{(u^+u^-)^4}\int\mathrm{d}^5x\,e\bigg[(\mathcal{D}^-)^4-\frac{25}{24}\mathrm{i}S^{--}(\mathcal{D}^-)^2+18S^{--}S^{--}\bigg]\mathcal{L}^{++}\bigg|. \tag{4.2}$$

Here $\mathcal{L}^{++}$ is a real covariantly analytic superfield of weight $+2$, $\mathcal{D}^-_{\hat\alpha}=u^-_i\mathcal{D}^i_{\hat\alpha}$, $S^{--}=u^-_iu^-_jS^{ij}$, and the line integral is carried out over a closed contour in the space of u^+ variables. In the flat-superspace limit, $S^{ij}\to 0$, the action reduces to the 5D version of the projective-superspace action which was originally constructed in [1] and then reformulated in a projective-invariant form in [22].

As demonstrated in [4], the action (4.2) is uniquely fixed by either of the following two conditions: (i) supersymmetry; (ii) projective invariance. The latter means the invariance of S under arbitrary projective transformations of the form

$$\big(u_i^-,u_i^+\big)\to\big(u_i^-,u_i^+\big)R,\qquad R=\begin{pmatrix}a&0\\ b&c\end{pmatrix}\in\mathrm{GL}(2,\mathbb{C}). \tag{4.3}$$

Thus, although the conditions (i) and (ii) seem to be unrelated at first sight, they actually appear to be equivalent. Below, we will put forward the principle of projective invariance in order to construct a supergravity extension of the above action.

4.1. Wess–Zumino gauge

In supergravity theories, reduction from superfields to component fields is conveniently performed by choosing a Wess–Zumino gauge [23,24]. Here we follow a streamlined procedure [25] of introducing the supergravity Wess–Zumino gauge [23] (originally given in [23] for the old minimal formulation of 4D $\mathcal{N}=1$ supergravity[8]). The advantage of this approach is its universality and independence from the dimension of space–time and the number of supersymmetries.

Given a superfield $U(z)=U(x,\theta)$, it is standard to denote as $U|$ its θ-independent component, $U|:=U(x,\theta=0)$. The Wess–Zumino gauge for 5D $\mathcal{N}=1$ supergravity is defined by

$$\mathcal{D}_{\hat a}|=\nabla_{\hat a}+\Psi_{\hat a}{}^{\hat\gamma}_k(x)\mathcal{D}^k_{\hat\gamma}\big|+\phi_{\hat a}{}^{kl}(x)J_{kl}+\mathcal{V}_{\hat a}(x)Z,\qquad \mathcal{D}^i_{\hat\alpha}\big|=\frac{\partial}{\partial\theta^{\hat\alpha}_i}. \tag{4.4}$$

Here $\nabla_{\hat a}$ are space–time covariant derivatives,

$$\nabla_{\hat a}=e_{\hat a}+\omega_{\hat a},\quad e_{\hat a}=e_{\hat a}{}^{\hat m}(x)\partial_{\hat m},\quad \omega_{\hat a}=\frac12\omega_{\hat a}{}^{\hat b\hat c}(x)M_{\hat b\hat c}=\omega_{\hat a}{}^{\hat\beta\hat\gamma}(x)M_{\hat\beta\hat\gamma}, \tag{4.5}$$

with $e_{\hat a}{}^{\hat m}$ the component inverse vielbein, and $\omega_{\hat a}{}^{\hat b\hat c}$ the Lorentz connection. Furthermore, $\Psi_{\hat a}{}^{\hat\gamma}_k$ is the component gravitino, while $\phi_{\hat a}{}^{kl}=\Phi_{\hat a}{}^{kl}|$ and $\mathcal{V}_{\hat a}=V_{\hat a}|$ are the component SU(2) and central-charge gauge fields, respectively. The space–time covariant derivatives obey the commutation relations

$$[\nabla_{\hat a},\nabla_{\hat b}]=\mathcal{T}_{\hat a\hat b}{}^{\hat c}\nabla_{\hat c}+\frac12\mathcal{R}_{\hat a\hat b}{}^{\hat c\hat d}M_{\hat c\hat d}. \tag{4.6}$$

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6 To make contact with the notation used in [4], one should represent $S^{ij}=\mathrm{i}\omega J^{ij}$.

7 We use the following definitions: $(\mathcal{D}^-)^4=-\frac{1}{96}\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\delta}\mathcal{D}^-_{\hat\alpha}\mathcal{D}^-_{\hat\beta}\mathcal{D}^-_{\hat\gamma}\mathcal{D}^-_{\hat\delta}$ and $(\mathcal{D}^\pm)^2=\mathcal{D}^{\pm\hat\alpha}\mathcal{D}^\pm_{\hat\alpha}$.

8 For an alternative approach to impose a supergravity Wess–Zumino gauge, see [26].

Here the space–time torsion can be shown to be

$$\mathcal{T}_{\hat{a}\hat{b}}{}^{\hat{c}} = 2\mathrm{i}\varepsilon^{jk}\Psi_{\hat{a}}{}^{\hat{\gamma}}_{j}\big(\Gamma^{\hat{c}}\big)_{\hat{\gamma}\hat{\delta}}\Psi_{\hat{b}k}^{\hat{\delta}}. \tag{4.7}$$

The latter occurs in the integration by parts rule:

$$\int \mathrm{d}^5x\, e\nabla_{\hat{a}}U^{\hat{a}} = \int \mathrm{d}^5x\, e\mathcal{T}_{\hat{a}\hat{b}}{}^{\hat{b}}U^{\hat{a}}, \quad e^{-1} = \det\big(e_{\hat{a}}{}^{\hat{m}}\big). \tag{4.8}$$

Those supergravity gauge transformations (2.6) that survive in the Wess–Zumino gauge are described by the following equations:

$$\begin{aligned}
&\mathcal{D}^i_{\hat{\alpha}}K^{\hat{\beta}}_j\big| = K^{\hat{c}}\big|T_{\hat{c}}{}^i_{\hat{\alpha}}{}^{\hat{\beta}}_j\big| + \delta^i_jK_{\hat{\alpha}}{}^{\hat{\beta}}\big| + \delta^{\hat{\beta}}_{\hat{\alpha}}K^i{}_j\big|, \qquad \mathcal{D}^i_{\hat{\alpha}}K^{\hat{b}}\big| = -2\mathrm{i}\big(\Gamma^{\hat{b}}\big)_{\hat{\alpha}\hat{\gamma}}K^{\hat{\gamma}i}\big|,\\
&\mathcal{D}^i_{\hat{\alpha}}K^{\hat{\beta}\hat{\gamma}}\big| = K^{\hat{C}}\big|R_{\hat{C}}{}^i_{\hat{\alpha}}{}^{\hat{\beta}\hat{\gamma}}\big|, \qquad \mathcal{D}^i_{\hat{\alpha}}K^{jk}\big| = K^{\hat{C}}\big|R_{\hat{C}}{}^i_{\hat{\alpha}}{}^{jk}\big|, \qquad \mathcal{D}^i_{\hat{\alpha}}\tau\big| = -2\mathrm{i}K^i_{\hat{\alpha}}\big|.
\end{aligned} \tag{4.9}$$

4.2. *Action principle*

Let $\mathcal{L}^{++}$ be a covariantly analytic real superfield of weight +2. We assume the existence (to be justified later on) of a locally supersymmetric and projective-invariant action associated with $\mathcal{L}^{++}$. Our main result is that the requirement of projective invariance uniquely determines this action in the Wess–Zumino gauge, provided the term of highest order in derivatives is proportional to

$$\oint \frac{u^+_i\,\mathrm{d}u^{+i}}{(u^+u^-)^4}\int \mathrm{d}^5x\, e(\mathcal{D}^-)^4\mathcal{L}^{++}\big|.$$

Direct and long calculations lead to the following projective-invariant action:

$$\begin{aligned}
S\big(\mathcal{L}^{++}\big) = -\frac{1}{2\pi}\oint \frac{u^+_i\,\mathrm{d}u^{+i}}{(u^+u^-)^4}\int \mathrm{d}^5x\, e\Big[&\big(\mathcal{D}^-\big)^4 + \frac{\mathrm{i}}{4}\Psi^{\hat{\alpha}\hat{\beta}\hat{\gamma}-}\mathcal{D}^-_{\hat{\gamma}}\mathcal{D}^-_{\hat{\alpha}}\mathcal{D}^-_{\hat{\beta}} - \frac{25}{24}\mathrm{i}S^{--}\big(\mathcal{D}^-\big)^2 - 2\big(\Sigma^{\hat{a}\hat{b}}\big)_{\hat{\beta}}{}^{\hat{\gamma}}\Psi_{\hat{a}}{}^{\hat{\beta}-}\Psi_{\hat{b}}{}^{\hat{\delta}-}\mathcal{D}_{[\hat{\gamma}}{}^-\mathcal{D}_{\hat{\delta}]}{}^-\\
&- \frac{\mathrm{i}}{4}\phi^{\hat{\alpha}\hat{\beta}--}\mathcal{D}^-_{\hat{\alpha}}\mathcal{D}^-_{\hat{\beta}} + 4\big(\Sigma^{\hat{a}\hat{b}}\big)^{\hat{\alpha}}{}_{\hat{\gamma}}\phi_{[\hat{a}}{}^{--}\Psi_{\hat{b}]}{}^{\hat{\gamma}-}\mathcal{D}^-_{\hat{\alpha}} - 4\Psi^{\hat{\alpha}\hat{\beta}}{}_{\hat{\beta}}{}^-S^{--}\mathcal{D}^-_{\hat{\alpha}} + 2\mathrm{i}\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{m}\hat{n}}(\Sigma_{\hat{m}\hat{n}})_{\hat{\alpha}\hat{\beta}}\Psi_{\hat{a}}{}^{\hat{\alpha}-}\Psi_{\hat{b}}{}^{\hat{\beta}-}\Psi_{\hat{c}}{}^{\hat{\gamma}-}\mathcal{D}^-_{\hat{\gamma}}\\
&+ 18S^{--}S^{--} - 6\mathrm{i}\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{m}\hat{n}}(\Sigma_{\hat{m}\hat{n}})_{\hat{\alpha}\hat{\beta}}\Psi_{\hat{a}}{}^{\hat{\alpha}-}\Psi_{\hat{b}}{}^{\hat{\beta}-}\phi_{\hat{c}}{}^{--} + 18\mathrm{i}\big(\Sigma^{\hat{a}\hat{b}}\big)_{\hat{\alpha}\hat{\beta}}\Psi_{\hat{a}}{}^{\hat{\alpha}-}\Psi_{\hat{b}}{}^{\hat{\beta}-}S^{--}\Big]\mathcal{L}^{++}\big|.
\end{aligned} \tag{4.10}$$

The projective invariance of (4.10) is a result of miraculous cancellations. The technical details will be given in [5]. In the AdS limit (4.1), the action reduces to (4.2).

The important feature of our action (4.10) is that it is practically ready for applications, that is, for a reduction from superfields to component fields. It is instructive to compare (4.10) with the component actions in 4D $\mathcal{N}=1$ supergravity (see Eq. (5.6.60) in [25] and Eq. (5.8.50) in [26]). It should be pointed out that one could also develop a harmonic-superspace formulation for 5D $\mathcal{N}=1$ supergravity, in complete analogy with the 4D $\mathcal{N}=2$ case [27]. But the action functional for supergravity-matter systems in harmonic superspace, as presented in [27], is given in terms of the supergravity prepotential, and some work is still required to reduce it to a form useful for component reduction.

Without loss of generality, one can assume that the integration contour in (4.10) does not pass through the north pole $u^{+i} \sim (0,1)$. Then, one can introduce the complex variable ζ as in (3.9), and fix the projective invariance (4.3) as in (3.10). If we also represent the Lagrangian in the form

$$\mathcal{L}^{++}\big(z,u^+\big) = \mathrm{i}u^{+\underline{1}}u^{+\underline{2}}\mathcal{L}(z,\zeta) = \mathrm{i}\big(u^{+\underline{1}}\big)^2\zeta\mathcal{L}(z,\zeta), \tag{4.11}$$

the line integral in (4.10) reduces to a complex contour integral of a function that is holomorphic almost everywhere in $\mathbb{C}$ except a few points. Let us introduce several supergravity-matter systems.

Given a set of $O(2)$ or tensor multiplets H^{++I}, with $I = 1,\dots,n$, their dynamics can be generated by a Lagrangian $\mathcal{L}^{++} = \mathcal{L}(H^{++I})$ that is a real homogeneous function of first degree in the variables H^{++},

$$H^{++I}\frac{\partial}{\partial H^{++I}}\mathcal{L}\big(H^{++}\big) = \mathcal{L}\big(H^{++}\big). \tag{4.12}$$

This is a generalization of superconformal tensor multiplets [1,28].

Given a system of arctic weight-one multiplets $\Upsilon^+(z,u^+)$ and their smile-conjugates $\tilde{\Upsilon}^+$, their dynamics can be described by the Lagrangian

$$\mathcal{L}^{++} = \mathrm{i}K\big(\Upsilon^+,\tilde{\Upsilon}^+\big), \tag{4.13}$$

with $K(\Phi^I, \bar{\Phi}^{\bar{J}})$ a real analytic function of n complex variables Φ^I, where $I = 1, \dots, n$. For $\mathcal{L}^{++}$ to be a weight-two real projective superfield, it is sufficient to require

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}). \tag{4.14}$$

This is a generalization of superconformal polar multiplets [4,18,21].

Let H^{++} be a tensor multiplet, and λ an arctic weight-zero multiplet. Then, the action generated by $\mathcal{L}^{++} = H^{++}\lambda$ vanishes, $S(H^{++}\lambda) = 0$, since the corresponding integrand in (4.10) can be seen to possess no poles (upon fixing the projective gauge). Thus

$$S\big(H^{++}(\lambda + \tilde{\lambda})\big) = 0. \tag{4.15}$$

A massless vector multiplet $\mathbb{V}(z, u^+)$ is described by a weight-zero tropical multiplet possessing the gauge invariance

$$\delta \mathbb{V} = \lambda + \tilde{\lambda}, \tag{4.16}$$

with λ a weight-zero arctic multiplet. Given a tensor multiplet H^{++}, the Lagrangian $\mathcal{L}^{++} = H^{++}\mathbb{V}$ generates a gauge-invariant action.

The minimal supergravity involves a vector multiplet associated with the central charge. If $\mathbb{V}(z, u^+)$ denotes the corresponding gauge prepotential, then the Lagrangian $S^{++}\mathbb{V}$ leads to gauge-invariant coupling. If supersymmetric matter (including a compensator) is described by weight-one polar multiplets, then the supergravity-matter Lagrangian can be chosen to be

$$\mathcal{L}^{++} = S^{++}\mathbb{V} + \mathrm{i}K\big(\Upsilon^+, \tilde{\Upsilon}^+\big), \tag{4.17}$$

with the real function $K(\Phi, \bar{\Phi})$ obeying the homogeneity condition (4.14). Here we have used the fact that S^{++} is covariantly analytic, as a consequence of (2.12).

As a generalization of the model given in [4], a system of interacting arctic weight-zero multiplets $\boldsymbol{\Upsilon}$ and their smile-conjugates $\tilde{\boldsymbol{\Upsilon}}$ can be described by the Lagrangian

$$\mathcal{L}^{++} = S^{++}\mathbf{K}(\boldsymbol{\Upsilon}, \tilde{\boldsymbol{\Upsilon}}), \tag{4.18}$$

with $\mathbf{K}(\Phi^I, \bar{\Phi}^{\bar{J}})$ a real function which is not required to obey any homogeneity condition. The action is invariant under Kähler transformations of the form

$$\mathbf{K}(\boldsymbol{\Upsilon}, \tilde{\boldsymbol{\Upsilon}}) \to \mathbf{K}(\boldsymbol{\Upsilon}, \tilde{\boldsymbol{\Upsilon}}) + \boldsymbol{\Lambda}(\boldsymbol{\Upsilon}) + \bar{\boldsymbol{\Lambda}}(\tilde{\boldsymbol{\Upsilon}}), \tag{4.19}$$

with $\boldsymbol{\Lambda}(\Phi^I)$ a holomorphic function.

In conclusion, we briefly discuss couplings of vector multiplets to supergravity. A U(1) vector multiplet can be described by its gauge-invariant field strength, $W(z)$, which is a real scalar superfield obeying the Bianchi identity (compare with the AdS case [4]),

$$\mathcal{D}^{(i}_{\hat{\alpha}} \mathcal{D}^{j)}_{\hat{\beta}} W = \frac{1}{4}\varepsilon_{\hat{\alpha}\hat{\beta}} \mathcal{D}^{\hat{\gamma}(i} \mathcal{D}^{j)}_{\hat{\gamma}} W. \tag{4.20}$$

The Bianchi identity implies that

$$G^{++}\big(z, u^+\big) = G^{ij}(z) u^+_i u^+_j := \mathrm{i}\left\{\mathcal{D}^{+\hat{\alpha}} W \mathcal{D}^+_{\hat{\alpha}} W + \frac{1}{2} W \big(\mathcal{D}^+\big)^2 W + 2\mathrm{i} S^{++} W^2\right\} \tag{4.21}$$

is a composite real $O(2)$ multiplet, $\mathcal{D}^+_{\hat{\alpha}} G^{++} = 0$. The coupling of the vector multiplet to supergravity is obtained by adding $G^{++}\mathbb{V}$ to the Lagrangian (4.17). For the central-charge vector multiplet, its $O(2)$-descendant $\mathbb{G}^{++}$ reduces to $(-2)S^{++}$, as a result of a super-Weyl gauge fixing $\mathbb{W} = 1$ implicitly made in Howe's formulation [8].

5. $\boldsymbol{\Lambda}$-group and supersymmetric action

In this section, we formulate a locally supersymmetric action underlying the dynamics of supergravity-matter systems. Our construction has some analogies with the prepotential formulation for 4D $\mathcal{N} = 1$ supergravity reviewed in [25,26], specifically the supergravity gauge or Λ group [29,30]. It is also analogous to the harmonic-superspace approach to the minimal multiplet for 4D $\mathcal{N} = 2$ supergravity [27] as re-formulated in Appendix A of [31].

5.1. $\boldsymbol{\Lambda}$-group

Consider the space of analytic multiplets of weight n, $\hat{Q}^{(n)}(z, u^+)$, in flat superspace $\mathbb{R}^{5|8}$. Such superfields are defined by Eqs. (3.2)–(3.4) in which the curved-superspace covariant derivatives $\mathcal{D}^+_{\hat{\alpha}}$ have to be replaced by flat ones, $D^+_{\hat{\alpha}}$. Introduce a Lie algebra of first-order differential operators acting on this linear functional space. Such an operator Λ generates an infinitesimal variation of $\hat{Q}^{(n)}(z,u^+)$ of the form[9]:

$$\delta\hat{Q}^{(n)} = \Lambda\hat{Q}^{(n)}, \quad \Lambda = \Lambda^{\hat{m}}\partial_{\hat{m}} - \frac{1}{(u^+u^-)}\big(\Lambda^{+\hat{\mu}}D^-_{\hat{\mu}} + \Lambda^{++}D^{--}\big) + n\Sigma. \tag{5.1}$$

The transformation parameters $\Lambda^{\hat{m}}$, $\Lambda^{+\hat{\mu}}$, Λ^{++} and Σ are such that the variation of $\hat{Q}^{(n)}$, $\Lambda\hat{Q}^{(n)}$, is also a flat analytic superfield of weight n. The requirement of analyticity, $D^+_{\hat{\alpha}}\hat{Q}^{(n)} = 0$, can be seen to imply

$$D^+_{\hat{\mu}}\Lambda^{\hat{\nu}\hat{\gamma}} = 8\mathrm{i}\left(\delta^{[\hat{\nu}}_{\hat{\mu}}\Lambda^{+\hat{\gamma}]} + \frac{1}{4}\varepsilon^{\hat{\nu}\hat{\gamma}}\Lambda^{+\hat{\mu}}\right), \qquad D^+_{\hat{\mu}}\Lambda^{+\hat{\nu}} = \delta^{\hat{\nu}}_{\hat{\mu}}\Lambda^{++}, \qquad D^+_{\hat{\mu}}\Lambda^{++} = D^+_{\hat{\mu}}\Sigma = 0. \tag{5.2}$$

The requirement of $\Lambda\hat{Q}^{(n)}$ to be independent of u^-_i can be shown to hold if

$$\frac{\partial}{\partial u^{-i}}\Lambda^{\hat{m}} = \frac{\partial}{\partial u^{-i}}\Lambda^{+\hat{\mu}} = \frac{\partial}{\partial u^{-i}}\Lambda^{++} = 0, \qquad u^{-i}\frac{\partial}{\partial u^{-i}}\Sigma = 0, \qquad D^{++}\Sigma = \frac{\Lambda^{++}}{(u^+u^-)}. \tag{5.3}$$

It is also clear that the requirement of $\Lambda\hat{Q}^{(n)}$ to have weight n holds if $\Lambda^{\hat{m}}$, Σ, $\Lambda^{+\hat{\mu}}$ and Λ^{++} are homogeneous functions of u^+ of degrees 0, 0, 1 and 2, respectively.

A solution to the above constraints is:

$$\Lambda^{\hat{\mu}\hat{\nu}} = \mathrm{i}\left(D^{+\hat{\mu}}D^{+\nu} + \frac{1}{4}\varepsilon^{\hat{\mu}\hat{\nu}}\big(D^+\big)^2\right)\Omega^{--}, \qquad \Lambda^{+\hat{\mu}} = -\frac{1}{2}D^{+\hat{\mu}}\big(D^+\big)^2\Omega^{--}, \tag{5.4a}$$

$$\Lambda^{++} = -\frac{1}{8}\big(D^+\big)^2\big(D^+\big)^2\Omega^{--}, \tag{5.4b}$$

as well as

$$2\Sigma = \partial_{\hat{m}}\Lambda^{\hat{m}} + \frac{1}{(u^+u^-)}\big(D^-_{\hat{\mu}}\Lambda^{+\mu} - D^{--}\Lambda^{++}\big). \tag{5.5}$$

Here the parameter Ω^{--} is required to be (i) independent of u^-; and (ii) homogeneous in u^+ of degree -2.

Let $\hat{\mathcal{L}}^{++}(z,u^+)$ be a real analytic superfield of weight $+2$. Its transformation can be seen to be a total derivative:

$$\Lambda\hat{\mathcal{L}}^{++} = \partial_{\hat{m}}\big(\Lambda^{\hat{m}}\hat{\mathcal{L}}^{++}\big) + \frac{1}{(u^+u^-)}D^-_{\hat{\mu}}\big(\Lambda^{+\mu}\hat{\mathcal{L}}^{++}\big) - \frac{1}{(u^+u^-)}D^{--}\big(\Lambda^{++}\hat{\mathcal{L}}^{++}\big). \tag{5.6}$$

Therefore, the following functional

$$S = -\frac{1}{2\pi}\oint\frac{u^+_i\,\mathrm{d}u^{+i}}{(u^+u^-)^4}\int \mathrm{d}^5x\,\big(D^-\big)^4\hat{\mathcal{L}}^{++} \tag{5.7}$$

is invariant under (5.6).

A crucial element in the above construction is the Lie algebra of first-order operators of the form

$$\mathcal{A} = \big(D^{++}\Sigma\big)D^{--} - n\Sigma, \qquad u^{+i}\frac{\partial}{\partial u^{+i}}\Sigma = u^{-i}\frac{\partial}{\partial u^{-i}}\Sigma = 0, \qquad \big(D^{++}\big)^2\Sigma = 0, \tag{5.8}$$

which act on the space $\mathcal{F}^{(n)}$ of functions $Q^{(n)}(u^+)$ being homogeneous in u^+ of order n. This algebra can be viewed to be a gauging of the algebra $\mathrm{su}(2)\oplus\mathbb{R}$ naturally acting on $\mathcal{F}^{(n)}$, where $\mathbb{R}$ corresponds to infinitesimal scale transformations.

5.2. A partial solution to the constraints

Given a covariantly analytic superfield $Q^{(n)}$, Eq. (3.1) implies that the covariant derivatives $\mathcal{D}^+_{\hat{\alpha}}$ and $Q^{(n)}$ can be represented as follows:

$$\mathcal{D}^+_{\hat{\alpha}} = \mathrm{e}^{\mathcal{H}}\Delta^+_{\hat{\alpha}}\mathrm{e}^{-\mathcal{H}}, \quad \Delta^+_{\hat{\alpha}} = N_{\hat{\alpha}}{}^{\hat{\beta}}D^+_{\hat{\beta}} + \hat{\Omega}^{+\hat{\beta}\hat{\gamma}}_{\hat{\alpha}}M_{\hat{\beta}\hat{\gamma}} + \hat{\Phi}^-_{\hat{\alpha}}J^{++}, \tag{5.9a}$$

$$Q^{(n)} = \mathrm{e}^{\mathcal{H}}\hat{Q}^{(n)}, \qquad D^+_{\hat{\alpha}}\hat{Q}^{(n)} = 0. \tag{5.9b}$$

Here $\mathcal{H}(z,u)$ is some first-order differential operator, and $\hat{Q}^{(n)}(z,u^+)$ a flat analytic superfield of weight n. The covariant derivatives are left invariant under gauge transformations of the prepotentials of the form:

$$\delta\mathrm{e}^{\mathcal{H}} = -\mathrm{e}^{\mathcal{H}}\boldsymbol{\Lambda}, \qquad \delta\Delta^+_{\hat{\alpha}} = \big[\boldsymbol{\Lambda},\Delta^+_{\hat{\alpha}}\big], \quad \boldsymbol{\Lambda} = \Lambda + \rho^{-\hat{\mu}}D^+_{\hat{\mu}} + \rho^{\hat{\beta}\hat{\gamma}}M_{\hat{\beta}\hat{\gamma}} + \rho^{--}J^{++}, \tag{5.10}$$

[9] For simplicity, the analytic multiplets are chosen to be independent of the central charge, $Z\hat{Q}^{(n)} = 0$. It is not difficult to extend out analysis to the general case; compare also with [31].

where Λ is the same as in Eq. (5.1), while the parameters $\rho^{-\hat{\mu}}$, $\rho^{\hat{\beta}\hat{\gamma}}$ and ρ^{--} are arbitrary modulo homogeneity conditions. The analytic superfield $\hat{Q}^{(n)}$ transforms as in (5.1). The variation $\hat{Q}^{(n)}$ involves not only general coordinate and local SO(4, 1) $\times$ SU(2) transformations, but also Weyl transformations.

Let $\mathcal{L}^{++}$ be the covariantly analytic Lagrangian in the action (4.10). It can be represented in the form $\mathcal{L}^{++} = \mathrm{e}^{\mathcal{H}}\hat{\mathcal{L}}^{++}$, for some flat analytic superfield $\hat{\mathcal{L}}^{++}(z, u^+)$ of weight $+2$. Then, the action (5.7) generated by $\hat{\mathcal{L}}^{++}$ is locally supersymmetric and projective-invariant. In the Wess–Zumino gauge, it should turn into (4.10).

In order to extend our construction to the case of 4D $\mathcal{N} = 2$ supergravity, one should build on the superspace formulation for the minimal supergravity multiplet given in [32].

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