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Almost Periodic Behavior of Nonlinear Waves*

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1. INTRODUCTION

In their very influential paper [3], Fermi, Pasta, and Ulam studied numerically the motion of linearly arranged particles driven by nonlinear forces between nearest neighbors. Contrary to their expectation, the motions were far from being ergodic; on the contrary, each trajectory seemed to occupy only a small portion of phase space; furthermore some of these motions appeared to be almost periodic.

In this talk I shall report briefly on recent theoretical results concerning three nonlinear systems which have a bearing on the questions raised by FPU; here is a brief summary:

Each of the systems discussed has an unusually large number of integrals, i.e., functionals which are conserved during motion; this might explain why some numerically computed trajectories of these systems seem to be confined to such an unexpectedly small portion of phase space. It should be pointed out however, that this cannot be the whole story, since the available part of phase space is still pretty large; in fact other computed trajectories seem to occupy a fairly large portion of phase space. It should be added that it is not known whether the FPU system has any integrals other than total momentum and energy, although the contrary has not been demonstrated either. This shows that there must be an additional mechanism at work; this additional mechanism might very well be the one discovered in low dimensions by Moser, Kolmogoroff, and Arnold.

The first of the systems discussed is Hamiltonian and completely

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integrable; accumulating evidence indicates that so is the second example, with infinitely many degrees of freedom. If so, one might prove the almost periodic behavior of these systems by introducing action and angle variables. Even then it would be desirable to relate the size of almost periods predicted by theory to those observed in calculation.

2. A Method for Constructing Nonlinear Systems with Many Integrals

In [9] a fairly general method was described for constructing nonlinear systems with many integrals. This method has, in the hands of the author and others, led to a number of interesting examples. This section presents very briefly the general method.

Let $L(t)$ be a one-parameter family of operators all of which are similar to each other. That is, we assume that each $L(t)$ can be mapped by a similarity transformation into $L(0)$:

$$U(t)^{-1}L(t)\,U(t) = L(0). \tag{2.1}$$

We assume that both L and U depend differentiably on t, and we introduce the notation

$$U_tU^{-1} = B(t), \tag{2.2}$$

from which we deduce

$$U_t = BU. \tag{2.3}$$

Differentiate (2.1) with respect to t; using

$$\frac{d}{dt}U^{-1} = -U^{-1}U_tU^{-1}$$

and (2.3) we get

$$-U^{-1}BLU + U^{-1}L_tU + U^{-1}LBU = 0$$

which implies

$$L_t = BL - LB. \tag{2.4}$$

Conversely, suppose (2.4) is satisfied and suppose the initial value problem for the differential equation

$$v_t = B(t)v \tag{2.5}$$

can be solved for a sufficiently wide class of initial values $v(0)$. Then the operator

$$U(t): v(0) \to v(t)$$

satisfies (2.1).

Similar operators L have the same spectrum; so it follows from (2.1) that *the eigenvalues* $\{\lambda_j\}$ *of* $L(t)$ *are independent of* t.

In any concrete representation the operator L appears as an integral or differential operator, described in terms of coefficients. Relation (2.4) is a nonlinear differential equation for these coefficients. The eigenvalues of L are functionals of the coefficients; being independent of t, they constitute the sought-after integrals.

If the operators L are symmetric or hermitean symmetric then similarity implies unitary equivalence. In fact if the spectrum of L is simple then the operator U appearing in (2.1) must be unitary.

If $U(t)$ is unitary

$$UU^* = I;$$

differentiating with respect to t we get

$$U_t U^* + U U_t^* = 0.$$

The meaning of this equation is that $U_t U^*$ is antisymmetric. Since U is unitary, $U_t U^* = U_t U^{-1}$, the operator denoted in (2.2) as B. So we conclude:

For L hermitean symmetric, B should be chosen antisymmetric:

$$B^* = -B.$$

3. The Toda Lattice

In his recent interesting paper [4], Flaschka has carried out the following construction:

Denote by u a vector:

$$u = (u_1, \ldots, u_N),$$
$$\| u \|^2 = \sum u_j^2. \tag{3.1}$$

Denote by T cyclic translation:

$$(Tu)_j = u_{j-1}, \tag{3.2}$$

where we set

$$u_0 = u_N .$$

Clearly T is a unitary operator:

$$T^* = T^{-1}. \tag{3.3}$$

Let a be any vector; it is convenient to introduce the abbreviations

$$Ta = a_+ , \qquad T^{-1}a = a_- . \tag{3.4}$$

The following relations are easy to verify:

$$Tau = a_+Tu, \qquad T^{-1}au = a_-Tu. \tag{3.5}$$

Define the operator L by

$$L = a_-T^{-1} + c + aT. \tag{3.6}$$

Using the relation (3.5) we see that L is symmetric:

$$L^* = L.$$

Define B by

$$B = -a_-T^{-1} + aT. \tag{3.7}$$

Again we see easily that B is antisymmetric:

$$B^* = -B.$$

A simple computation gives

$$BL - LB = a_-(c - c_-)\, T^{-1} + 2(a^2 - a_-^2) + a(c_+ - c)T. \tag{3.8}$$

Differentiating (3.6) we get

$$L_t = a_{-t}T^{-1} + c_t + a_tT. \tag{3.8'}$$

Observe that the commutator of B and L belongs to the same class as L_t. Now set, following (2.4)

$$L_t = BL - LB. \tag{3.9}$$

Equating coefficients we get from (3.8), (3.8′)

$$c_t = 2(a^2 - a_-^2), \tag{3.10c}$$

$$a_t = a(c_+ - c). \tag{3.10a}$$

As we saw in Section 2, it follows from Eq. (3.9) that the operators L are similar to one another, and therefore their eigenvalues don't change with t.

L is a matrix, and its eigenvalues are rather complicated functions of its entries. The elementary symmetric functions of the eigenvalues however, being the coefficients of the characteristic polynomial of L, are polynomials in the entries of L. Since L is tridiagonal, the first few are easily computed:

$$\det(\lambda I - L) = \lambda^N + I_1\lambda^{N-1} + \cdots + I_N .$$

Then

$$\begin{aligned} I_1 &= -\sum c_j , \\ I_2 &= \sum c_j c_k - \sum a_j^2, \\ I_3 &= \text{cubic, etc.} \end{aligned} \tag{3.11}$$

Using the well known relations

$$\sum \lambda_j = -I_1 , \qquad \sum \lambda_j \lambda_k = I_2 , \quad \text{etc.}$$

we get from (3.11)

$$\begin{aligned} \sum \lambda_j &= \sum c_j , \\ \sum \lambda_j^2 &= I_1^2 - 2I_2 = \sum c_j^2 + 2 \sum a_j^2, \\ \sum \lambda_j^3 &= \text{cubic, etc.} \end{aligned} \tag{3.12}$$

We turn now to lattice vibrations; denote by q_j the lateral displacement of the jth particle from equilibrium; each particle is linked to its two neighbors by identical springs. We denote by $f(s)$ the force exerted by the spring when stretched by the amount s; $f(s)$ is in physically meaningful cases an increasing function of s.

We take the arrangement of the particles to be periodic, i.e.

$$q_{j+N} \equiv q_j .$$

Assume that each particle has unit mass; then the equations of motion are

$$\frac{d^2}{dt^2} q_j = f(q_{j+1} - q_j) - f(q_j - q_{j-1}). \tag{3.13}$$

This can be written in Hamiltonian form by setting

$$\frac{d}{dt} q_j = p_j , \tag{3.14}$$

the Hamiltonian being

$$\tfrac{1}{2}\sum p_j^2 + \sum F(q_{j+1} - q_j), \tag{3.15}$$

where

$$\frac{d}{ds} F(s) = f(s). \tag{3.16}$$

If $f(s)$ is a linear function of s, the Eqs. (3.13) are linear and analyzable in terms of normal modes. FPU investigated two nonlinear cases, where f was either of the following two forms:

$$f(s) = s + \alpha s^3, \qquad \text{or } f(s) \text{ piecewise linear,}$$

Toda, [12], has introduced and studied the lattice where the dependence of f on s is exponential:

$$f(s) = -e^{-s}. \tag{3.17}$$

The equations of motion are

$$\begin{aligned} dq_j/dt &= p_j , \\ dp_j/dt &= \exp(q_{j-1} - q_j) - \exp(q_j - q_{j+1}). \end{aligned} \tag{3.18}$$

If one introduces new variables

$$c_j = \tfrac{1}{2} p_j , \qquad a_j = \tfrac{1}{2} \exp(q_{j-1} - q_j)/2 \tag{3.19}$$

then we can using (3.18) express the derivatives of the new variables as follows:

$$\frac{d}{dt} c_j = \frac{1}{2} \frac{d}{dt} p_j = 2(a_j^2 - a_{j+1}^2), \tag{3.20c}$$

$$\frac{d}{dt} a_j = \tfrac{1}{2}(p_{j-1} - p_j) = a_j = a_j(c_{j-1} - c_j). \tag{3.20a}$$

Observe that (3.20c) and (3.20a) are the same as (3.10a) and (3.10c). So one can conclude that the quantities (3.11) are conserved functionals for the Hamiltonian system (3.18). These conserved quantities as well as the transformation (3.19) were originally found by Hénon [7]; he and Flaschka have proved that the quantities I_j are in involution, so that the Hamiltonian system (3.18) is completely integrable.

Note that the first two functionals (3.12) are total momentum and total energy.

4. The *KdV* Equation

In this application the underlying Hilbert space consists of periodic L_2 functions on the unit interval of the x-axis, and L is the Schroedinger operator

$$L = \partial^2 + u, \qquad \partial = d/dx. \tag{4.1}$$

This is the selfadjoint operator, with a discrete spectrum $\{\lambda_j\}$.

For L given by (4.1), $L_t = u_t$ is multiplication by u_t; therefore in order to satisfy Eq. (2.4) we need operators B_j whose commutator with L is multiplication. In [9] the author has shown how to construct a sequence B_j of such operators; these operators have these properties:

(i) B_j is a differential operator of order $2j + 1$.

(ii) B_j is antisymmetric.

(iii) $B_jL - LB_j$ is multiplication by $K_j(u)$; $K_j(u)$ depends in a nonlinear fashion on u and its derivatives up to order $2j + 1$.

Following (2.4) we consider the equations

$$u_t = B_jL - LB_j = K_j(u); \tag{4.2}$$

these equations have the property that for their solutions the spectrum of L defined by (4.1) is independent of t.

The first two of these operators are

$$B_0 = \partial, \qquad [B_0, L] = K_0(u) = u_x \tag{4.3_0}$$

and

$$\begin{aligned} B_1 &= \partial^3 + \tfrac{3}{2}u\partial + \tfrac{3}{4}u_x, \\ [B_1, L] &= K_1(u) = \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x. \end{aligned} \tag{4.3_1}$$

The zeroth equation (4.2_0) is

$$u_t = u_x \tag{4.2$_0$}$$

and the first one is

$$u_t = \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x\,. \tag{4.2$_1$}$$

(4.2_0) describes translation along the x-axis; (4.2_1) is the *KdV* equation, with some inessential rescaling. The general equation (4.2_j) is usually called the jth generalized *KdV* equation.

Gardner has shown that the jth *KdV* operator K_j has the following structure:

$$K_j(u) = \partial G_j(u),$$

where G_j is the gradient of a functional $F_j(u)$. That is

$$\frac{d}{d\epsilon}F_j(u + \epsilon v)\,|_{\epsilon=0} = (G_j(u), v).$$

Furthermore Gardner has shown [6] that *each functional $F_m(u)$ is a conserved quantity for each generalized KdV flow* (4.2). From this it is easy to deduce, using the Hamiltonian formalism introduced by Gardner, that *the generalized KdV flows* (4.2_j) *commute with each other.*

Numerical calculations by Kruskal and Zabusky have indicated an almost periodic behavior of those solutions of *KdV* which are periodic in space. In [10] the author has constructed an abundance of solutions of *KdV* which are periodic in x and almost periodic in t. These solutions can be characterized by a variational problem suggested by Kruskal and Zabusky: minimize $F_N(u)$ subject to the constraints that $F_j(u)$ have prescribed values of $j < N$. The set of solutions of this variational problem consist of smooth N-dimensional tori on which the *KdV* flow—in fact all generalized *KdV* flows—are almost periodic. For details the reader is referred to [10].

I suspect, but cannot prove, that as N tends to ∞ these special solutions become dense among all C^∞ solutions.

5. The Sine-Gordon Equation

In this section we show how to present in the framework of Section 2 a portion of a very interesting theory developed by Ablowitz, Kaup, Newell, and Segur, [1]. AKNS consider the first order matrix operator

$$L = \begin{pmatrix} -\partial & q \\ r & \partial \end{pmatrix}. \tag{5.1}$$

The analysis of AKNS suggests to seek B of the form

$$B = RL^{-1}, \tag{5.2}$$

where

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.3}$$

Setting B as given by (5.2) into (2.4) gives

$$L_t = R - LRL^{-1}.$$

Multiplying by L on the right we get

$$L_tL = RL - LR. \tag{5.4}$$

We proceed now to solve this equation for R of form (5.3) when L is of form (5.1).

A straightforward calculation gives

$$RL = \begin{pmatrix} -a\partial + br & aq + b\partial \\ -c\partial + dr & cq + d\partial \end{pmatrix},$$

$$LR = \begin{pmatrix} -\partial a + qc & -\partial b + qd \\ ra + \partial c & rb + \partial d \end{pmatrix}.$$

So

$$RL - LR = \begin{pmatrix} a_x - qc + br & 2b\partial + b_x + aq - qd \\ -2c\partial - c_x + dr - ra & -d_x + cq - rb \end{pmatrix}. \tag{5.5}$$

Differentiating (5.1) we get

$$L_t = \begin{pmatrix} 0 & q_t \\ r_t & 0 \end{pmatrix}.$$

A straightforward calculation gives

$$L_tL = \begin{pmatrix} q_tr & q_t\partial \\ -r_t\partial & r_tq \end{pmatrix}. \tag{5.6}$$

Substituting (5.5) and (5.6) into (5.4) we get 4 sets of relations from the 4 components:

$$\begin{aligned} &\text{(i)} \quad q_t r = a_x - qc + br, \\ &\text{(ii)} \quad q_t = 2b, \qquad b_x + aq - qd = 0, \\ &\text{(iii)} \quad r_t = 2c, \qquad -c_x + dr - ra = 0, \\ &\text{(iv)} \quad r_t q = -d_x + cq - rb. \end{aligned} \tag{5.7}$$

Substituting the first relation in (5.7ii) into (5.7i) and the first relation in (5.7iii) into (5.7iv) we get

$$br + qc = a_x \tag{5.8$_1$}$$

and

$$qc + rb = -d_x \,. \tag{5.8$_2$}$$

Subtracting these two we get

$$a_x + d_x = 0,$$

which we satisfy by setting $d = -a$. Substituting this into the second relation in (5.7ii) and the second relation in (5.7iii) gives

$$b_x = (d - a)q = -2aq, \tag{5.9$_1$}$$

$$c_x = (d - a)r = -2ar. \tag{5.9$_2$}$$

Multiply (5.9$_1$) by c, (5.9$_2$) by b, and (5.8$_1$) by $2a$, and add; we get

$$cb_x + bc_x + 2aa_x = 0;$$

from this we conclude that

$$cb + a^2 = \text{const.}$$

We take that constant to be 1; so

$$a = (1 - bc)^{1/2}. \tag{5.10}$$

Relations (5.9) and (5.10) constitute a system of differential equations for b and c; if initial values are specified, b and c are uniquely determined in terms of q and r. The first relations in (5.7ii) and (5.7iii):

$$q_t = 2b, \qquad r_t = 2c \tag{5.11}$$

is a system of evolution equations for q and r; the right side is a nonlocal function of q and r.

Equation (5.11) is particularly simple when $q = r$; in this case we choose $b = c$; the resulting system occurs in the theory of self-induced transparency, see [8]. Relation (5.10) suggests the parametrization

$$b = \sin u, \qquad a = \cos u. \tag{5.12}$$

Substituting this into (5.9) gives

$$\cos u u_x = -2 \cos u q$$

from which we deduce

$$q = -\tfrac{1}{2} u_x .$$

Substituting this into (5.11) and using (5.12) we get

$$u_{xt} + 4 \sin u = 0, \tag{5.13}$$

the so-called sine-Gordon equation. For application of these ideas to solutions of the sine-Gordon equation we refer the reader to [1].

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