General Theorems on Rates of Convergence in Distribution
of Random Variables
II. Applications to the Stable Limit Laws and
Weak Law of Large Numbers

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This part is concerned with the applications of the general limit theorems
with rates of Part I, achieved by specializing the limiting r.v. X. This leads to
new convergence theorems with higher order rates in the one- and multi-dimen-
sional case for the stable limit law, for the central limit theorem, and the weak
law of large numbers.

This is Part II of the preceding paper [9]. The contents of Part I are assumed
to be known.

References are in alphabetical order in each part, some of the basic papers
of Part I being recalled here. The sections are numbered consecutively.

Whereas Part I is concerned with several general limit theorems on con-
vergence in distribution with rates of sequences of r.v., the limit being the
d.f. of an arbitrary r.v. X, the purpose of Part II is to deal with the applications
of these theorems for special choices of the limiting r.v. and a normalisation
function \( \varphi(n) \) (tending to zero for \( n \to \infty \)). These are the stable r.v., the normally
distributed r.v. \( X^* \), and the r.v. \( X_\delta \) that vanishes almost surely.

6. Stable Limit Law with Rates

The first applications of our general approximation theorems lead to con-
vergence theorems with rates for which the d.f. of the limiting r.v. is the
Let \((X_i)_{i \in \mathbb{N}}\) be a sequence of real independent r.v. and \(r \in \mathbb{N}\), \(0 < \beta \leq 1, 0 < \alpha \leq r\).

(a) Assume that

\[
\nu_{r-1+\beta,i}(Y_i) := \int_{\mathbb{R}} x^{|r-1+\beta|} d |(F_{X_i} - F_{Y_i})(x)| < \infty \quad (i \in \mathbb{N})
\]

and that there exist constants \(C_j\) such that

\[
\sum_{i=1}^{n} x^{j} d(F_{X_i} - F_{Y_i})(x) \leq C_j \sum_{i=1}^{n} \nu_{r-1+\beta,i}(Y_i)
\]

\((0 \leq j \leq r - 1).\)

For each \(f \in \text{Lip}(r - 1 + \beta; r; C_B)\) there holds

\[
\left| \int_{\mathbb{R}} f(x + y) d[F_{S_n/n^{1/y}}(x) - F_{Y_n}(x)] \right| \leq (C_f + L_f((r - 1)!)) n^{-(r-1+\beta)/y} \sum_{i=1}^{n} \nu_{r-1+\beta,i}(Y_i)
\]

uniformly in \(y \in \mathbb{R}\).

(b) If there holds assumption (6.2) for \(\beta = 1\) and instead of (6.3) the stronger condition

\[
\int_{\mathbb{R}} x^j d(F_{X_i} - F_{Y_i})(x) = 0 \quad (0 \leq j \leq r - 1; i \in \mathbb{N}),
\]

then

\[
\left| \int_{\mathbb{R}} f(x + y) d[F_{S_n/n^{1/y}}(x) - F_{Y_n}(x)] \right| \leq 2c_{2,r} \omega_r \left( \left[ \sum_{i=1}^{n} \nu_{r,i}(Y_i) \right]^{1/r} ; f; C_B \right)
\]

uniformly in \(y \in \mathbb{R}\). In particular, \(f \in \text{Lip}(\alpha; r; C_n)\) implies that

\[
\| V_{S_n/n^{1/y}} f - V_{Y_n} f \| \leq 2c_{2,r} L_n n^{-\alpha/y} \left[ \sum_{i=1}^{n} \nu_{r,i}(Y_i) \right]^{\alpha/r}.
\]

(c) If the r.v. \(X_i\) are i.d., \(\nu_r(Y_i) := \int_{\mathbb{R}} x^r d |(F_{X_i} - F_{Y_i})(x)| < \infty \) and (6.5) holds, then

\[
\left| \int_{\mathbb{R}} f(x + y) d[F_{S_n/n^{1/y}}(x) - F_{Y_n}(x)] \right| \leq 2c_{2,r} \omega_r \left( \left[ n^{-(r-\gamma)/\nu_r(Y_i)} \right]^{1/r} ; f; C_B \right)
\]

\((6.6)\)
uniformly in \( y \in \mathbb{R} \). In particular, \( f \in \text{Lip}(\alpha; r; C_B) \) yields

\[
\left\| V_{S_n/n^{1/r}}f - V_{Y_r}f \right\| \leq 2c_{z,n}L_r n^{-a(r-\alpha)/r}\left[v_r(Y_r)\right]^{a/r}.
\]

The proof of this theorem is a simple application of the results of Section 3. First, one must determine the \( \varphi \)-decomposition \( Z_{i,n} \) of a stable r.v. For this purpose we set

\[
F_{Z_{i,n}} := F_{Y_r}, \quad q(n) = n^{-1/r}, \quad 1 \leq i \leq n, \quad n \in \mathbb{N}.
\]

Then

\[
F_{\varphi(n)\Sigma_{i=1}^{n} Z_{i,n}} = F_{Y_r}, \tag{6.7}
\]

which follows most easily by applying ch.f. to both sides. In the terminology of Section 3 this means that \( G_{i,n} = F_{X_i} - F_{Y_r} \). Then the proof of part (a) follows from Theorem 1, part (b), Theorem 2, and part (c) from Corollary 2.

Returning to the remark to Theorem 2, there is pure convergence in (6.4) provided \( \sum_{i=1}^{n} \nu_{r-1+\delta,i}(Y_i) = O(n^{(r-1+\delta)/\alpha}) \). In particular this is so in (6.6) if \( r \geq \gamma \).

Since the \( r \)'th moment of a stable r.v. only exists for \( r' < \gamma < 2 \), conditions (6.2), (6.3), and (6.5) have to be interpreted correctly. Recall the remarks at the end of Section 2. For the same reasons Theorem 5 (i) but not part (ii) may be applied, which leads to a \( o \)-theorem for stable distributions.

**Theorem 10.** Let \( (X_i)_{i \in \mathbb{N}} \) be a sequence of real independent r.v. and \( r \in \mathbb{N} \).

(a) Assume that \( \nu_{r,i}(Y_r) < \infty, \quad i \in \mathbb{N}, \) and

\[
n^{(r-1)/\gamma} \sum_{i=1}^{n} \int_{\mathbb{R}} x^j \text{d}(F_{X_i} - F_{Y_r})(x) = o_n \left( \sum_{i=1}^{n} \nu_{r,i}(Y_r) \right), \tag{6.8}
\]

for \( 1 \leq j \leq r, \) as well as

\[
\sum_{i=1}^{n} \int_{x \geq \delta n^{1/\gamma}} x^j \text{d} |(F_{X_i} - F_{Y_r})(x)| = o_n \left( \sum_{i=1}^{n} \nu_{r,i}(Y_r) \right) \tag{6.9}
\]

for \( n \to \infty, \) each \( \delta > 0. \)

Then \( f \in C_B^r \) implies that

\[
\int_{\mathbb{R}} f(x + y) \text{d}[F_{S_n/n^{1/r}}(x) - F_{Y_r}(x)] = o_n \left( n^{-r/\gamma} \sum_{i=1}^{n} \nu_{r,i}(Y_r) \right) \quad (n \to \infty)
\]

uniformly in \( y \in \mathbb{R} \).

(b) If the r.v. \( X_i \) are i.d., \( \nu_r(Y_r) < \infty, \) as well as

\[
\int_{\mathbb{R}} x^j \text{d}(F_{X_i} - F_{Y_r})(x) = 0 \quad (1 \leq j \leq r),
\]
then for \( f \in C^r \) one has
\[
\sup_{x \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + t) d[F_{S_{n}}/n^{1/y}(x) - F_{X}(x)] \right| = o_{p}(n^{-(r - y)/y}(Y)) \quad (n \to \infty).
\]

The proof follows directly by Theorem 5 (i), again using the fact that the \( Z_{i,n} \) are decomposed via (6.7).

Returning to pure convergence again, Corollary 3 (case \( r = 1 \)) delivers this fact for i.d. r.v. even for \( r = y, \) i.e., for the stable distribution with index \( y = 1, \) the Cauchy distribution, provided
\[
\int_{\mathbb{R}} x \ d(F_x - F_{Y})(x) = 0
\]
since (4.10) holds. Indeed,
\[
\sum_{i=1}^{n} \nu_{1; t, n}(Y) = n\nu_{1}(Y) = O(n).
\]

For classical results in this respect, compare with, e.g., [14, p. 76ff] or [11, p. 175ff].

Concerning the literature, Paulauskas [15] (for earlier papers see Banys [1] and Satybaldina [21]) established an \( O \)-approximation theorem for i.d. r.v. \( X_{i} \) for the stable limit law; however, for the function class \( \mathcal{H} = \{\chi_{(-\infty, u]}(x); x, u \in \mathbb{R}\}; \) in other words, he studied the convergence of the d.f. \( F_{S_{n}/n^{1/y}} \) towards \( Y_{\gamma}, \) giving pointwise error estimates. Under somewhat stronger assumptions he obtained the same rates as ours for \( 0 < r - y < 1, \) but just the rate \( n^{-1/y} \) for \( r - y \geq 1. \) This phenomenon already occurs with the CLT (see [8]), i.e., in the particular case of stable distributions with index \( y = 2. \) Indeed, when considering the class \( \mathcal{H} \) one cannot achieve convergence rates better than \( n^{-1/2} \) without supplementary assumptions upon the r.v. \( X_{i} \) such as Cramér's condition. This stands in contrast to rates with respect to the function class \( C^r \). The reason is that in this case the smoothness conditions upon the d.f. \( F_{X_{i}} \) of the given \( X_{i} \) have been transferred to the class \( C^r \). In a newer paper [16] Paulauskas also examines function classes other than \( \mathcal{H} \) (in the multivariate case) and deduces convergences rates that depend directly upon the function \( h \) he considers, namely, in the form \( 1/h(n^{1/y}). \)

7. THE CENTRAL LIMIT THEOREM WITH RATES

Theorems 9 and 10 dealing with stable distributions also enable one to handle the CLT, the case \( y = 2. \) Since all of the moments exist in case the limiting distribution is the normal distribution, contrary to the case of stable
distributions with $\gamma < 2$, it is possible to disentangle conditions (6.2), (6.3), etc., according to the individual d.f. For this reason one needs to pose the existence of the $r$th moment only upon the given sequence of r.v. and not upon the difference; moreover, one can detach the matter from the particular choice $\varphi(n) = n^{-1/\gamma}$ with $\gamma = 2$, choosing now $\varphi(n) = 1/s_n$, so that $T_n = S_n/s_n$.

So let us set

$$
\eta_{i,t} := \mathbb{E}(X_{i,t}) := \int_{\mathbb{R}} x^2 \, dF_{X_{i,t}}(x), \quad \xi_{i,t} := \int_{\mathbb{R}} |x|^2 \, dF_{X_{i,t}}(x),
$$

$$
n_{i,t}^2 := \sum_{t=1}^{n} \eta_{i,t}^2 =: \sum_{t=1}^{n} \sigma_i^2.
$$

In this instance the $\varphi$-decomposition components are given by $F_{z_{i,n}} = F_{\varphi_{i,n}X_{i,t}}$, and so

$$
F_{(1/s_n)\Sigma_{i=1}^{n}X_{i,n}} = F_{X_{i,t}}.
$$

**Theorem 11.** Let $(X_{i,n})_{i\in\mathbb{N}}$ be a sequence of real independent r.v. and $r \in \mathbb{N}$, $r \geq 3$, $0 < \beta \leq 1$, $0 < \alpha \leq r$.

(a) Assume that

$$
\xi_{r-1+\beta,i} < \infty \quad (i \in \mathbb{N})
$$

and that there exist constants $C_j$ such that ($0 \leq j \leq r - 1$)

$$
s_{n}^{r+\beta-j-1} \sum_{t=1}^{n} |\eta_{i,t} - \sigma_i^jE(X_{i,t})| \leq C_j \sum_{t=1}^{n} |\int_{\mathbb{R}} |x|^{r-1+\beta} \, dF_{X_{i,t}}(x) - F_{\varphi_{i,n}X_{i,t}}(x)|. \quad (7.4)
$$

Then one has for $f \in \text{Lip}(r - 1 + \beta; r; C_B)$

$$
\sup_{x \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) \, d[F_{S_n/s_n}(x) - F_{X_{i,t}}(x)] \right| 
\leq (C_f + L_f)(r - 1)!s_n^{r-1+\beta} \sum_{t=1}^{n} \left[ \xi_{r-1+\beta,i} + \sigma_i^{r-1+\beta}E(|X_{i,t}^{*}||r-1+\beta)| \right] \quad (7.4)
$$

(b) If condition (7.3) holds, and instead of (7.4) the stronger condition

$$
\eta_{i,j} = \sigma_i^jE(X_{i,t}^{*}) \quad (0 \leq j \leq r - 1; i \in \mathbb{N}), \quad (7.5)
$$

then $f \in \text{Lip}(r - 1 + \beta; r; C_B + C_B)$ implies that

$$
\sup_{x \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) \, d[F_{S_n/s_n}(x) - F_{X_{i,t}}(x)] \right| 
\leq (L_f(r - 1)!s_n^{r-1+\beta} \sum_{t=1}^{n} \left[ \xi_{r-1+\beta,i} + \sigma_i^{r-1+\beta}E(|X_{i,t}^{*}||r-1+\beta)| \right].
$$
If there holds (7.3) for $\beta = 1$ as well as (7.5), then
\[
\sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) \, d[F_{S_{n}/n}(x) - F_{X*}(x)] \right| 
\leq 2c_{n}^{*} \omega_{r} \left\{ \int_{\mathbb{R}} \left[ \sum_{i=1}^{n} \left\{ a_{i}^{*} E[|X^{*}|^p] \right\} \right]^{1/r} \right\}^{1/r} ; f; C_{b} + C_{b}^{*}.
\]

If $f \in \text{Lip}(\alpha; r; C_{b} + C_{b}^{*})$, then
\[
\sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) \, d[F_{S_{n}/n}(x) - F_{X*}(x)] \right| 
\leq 2c_{n}^{*} \omega_{r} \left\{ \int_{\mathbb{R}} \left[ \sum_{i=1}^{n} \left\{ a_{i}^{*} E[|X^{*}|^p] \right\} \right]^{1/r} \right\}^{1/r}.
\]

If the r.v. $X_{i}$ are i.d., $\zeta_{r-1+\beta} < \infty$, and
\[
E(X_{i}^{j}) = E(X^{*j}) \quad (0 \leq j \leq r - 1),
\]
then $f \in \text{Lip}(r - 1 + \beta; r; C_{b} + C_{b}^{*})$ implies
\[
\sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) \, d[F_{S_{n}/n}(x) - F_{X*}(x)] \right| 
\leq (L_{f}(r - 1)! n^{-(r-2+\beta)/2} [\zeta_{r-1+\beta} + E(|X^{*}|^r)]^{1/r}) ; f; C_{b} + C_{b}^{*}. \tag{7.7}
\]

Let the r.v. $X_{i}$ be i.d., $\zeta_{r} < \infty$, and let (7.6) hold. Then
\[
\sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) \, d[F_{S_{n}/n}(x) - F_{X*}(x)] \right| 
\leq 2c_{n}^{*} \omega_{r} (n^{-(r-2)/2} [\zeta_{r} + E(|X^{*}|^r)]^{1/r} ; f; C_{b} + C_{b}^{*}).
\]

In particular, $f \in \text{Lip}(\alpha; r; C_{b} + C_{b}^{*})$ yields
\[
\| V_{S_{n}/n} f - V_{X*} f \| \leq 2c_{n}^{*} L_{f} n^{-(r-2)/2} [\zeta_{r} + E(|X^{*}|^r)]^{1/r}.
\]

**Proof.** (a) This part follows from Theorem 1 since ($i \in \mathbb{N}$)
\[
\nu_{r-1+\beta,i} = \int_{\mathbb{R}} |x|^{r-1+\beta} \, d[F_{X*}(x) - F_{S_{n}/n}(x)] 
\leq \zeta_{r-1+\beta,i} + \sigma_{i}^{r-1+\beta} E(|X^{*}|^r) < \infty. \tag{7.8}
\]

(b) First, because of (2.5) and since $f \in \text{Lip}(r - 1 + \beta; r; C_{b} + C_{b}^{*})$, it follows that $f^{(r-1)} \in \text{Lip}(\beta; 1; C_{b} + C_{b}^{*})$, which in turn implies that
\[
f^{(r-1)}(x) = O(|x|^\beta). \tag{7.9}
\]
Now using the fact that for $x > 0$ (case $x < 0$ is analogous), $0 < u_i \leq x$, $0 \leq i \leq r - 2$, one has

$$f(x) = \int_0^x \left\{ \int_0^{u_{r-3}} \cdots \left( \int_0^{u_{r-2}} f^{(r-1)}(u_{r-1}) \, du_{r-1} + a_{r-2} \right) \, du_{r-2} + a_{r-3} \right\} \, du_{r-3}$$

$$\cdots + a_1 \right\} \, du_1 + a_0,$$

with $a_i := f^{(i)}(0)$, and therefore $f(x) = O(|x|^{r-1+\beta})$. Now $\zeta_{r-1+\beta,i} < \infty$, $i \in \mathbb{N}$ by (7.3), so $E(|S_n| + X^*|^{r-1+\beta}) < \infty$; hence $f \in L(F_{S_n/X^*})$. The result now follows by Theorem 3, noting (7.8).

Parts (c), (e), and (d) follow readily by the corresponding parts of Theorem 4. In case the r.v. are i.d., one takes $\sigma^2 = 1$ without loss of generality. Then $s_n = n^{1/2}$.

Observe that, in analogy with condition (7.8), Theorem 11 (a) remains true if condition (7.4) is weakened to

$$s_n^{r-\beta-1} \sum_{i=1}^n |\eta_{j,i} - \sigma_i E(X^{\alpha_i})| \leq C_j \sum_{i=1}^n [\zeta_{r-1+\beta,i} + \sigma_i^{r-1+\beta} E(|X^*|^{r-1+\beta})]. \quad (7.4')$$

Theorem 5 enables one to deduce two $o$-error estimates in this instance.

**Theorem 12.** Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of real independent r.v. and $r \in \mathbb{N}$, $r \geq 2$. Assume that $\zeta_{r,i} < \infty$, $i \in \mathbb{N}$, and

$$s_n^{r-j} \sum_{i=1}^n |\eta_{j,i} - \sigma_i E(X^{\alpha_i})| = o_j \left( \sum_{i=1}^n \int_{\mathbb{R}} |x|^{r} \, d|F_{X_i}(x) - F_{\alpha_i X^*}(x)| \right) \quad (7.10)$$

for $1 \leq j \leq r$, $n \to \infty$, together with

$$\sum_{i=1}^n \int_{|x| > \delta s_n} |x|^{r} \, d|F_{X_i}(x) - F_{\alpha_i X^*}(x)| = o_\delta \left( \sum_{i=1}^n \int_{\mathbb{R}} |x|^{r} \, d|F_{X_i}(x) - F_{\alpha_i X^*}(x)| \right) \quad (7.11)$$

for $n \to \infty$, each $\delta > 0$. Then $f \in C^{r}_{\beta}$ yields for $n \to \infty$

$$\sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) \, d[F_{S_n/X^*}(x) - F_{X^*}(x)] \right|$$

$$= o_f \left( s_n^{r-\beta} \sum_{i=1}^n \int_{\mathbb{R}} |x|^{r} \, d|F_{X_i}(x) - F_{\alpha_i X^*}(x)| \right). \quad (7.12)$$

The proof follows at once from Theorem 5 (a)(i), noting (7.8) with $\beta = 1$.

If one applies Theorem 5(ii) instead of Theorem 5(i) it is even possible to disentangle the hypotheses (7.10) and (7.11) in such a fashion that conditions are only posed on the d.f. $F_{X_i}$ and no more on the difference $F_{X_i} - F_{\alpha_i X^*}$. 


For this purpose, however, one needs an additional condition, namely, the Feller condition

\[ \lim_{n \to \infty} \max_{1 \leq i \leq n} (\sigma_i/s_n) = 0. \]  

(7.13)

**Theorem 13.** Let \((X_i)_{i \in \mathbb{N}}\) be a sequence of real independent r.v. and \(r \in \mathbb{N}, \) \(r \geq 2.\)

(a) Assume that there holds \(\zeta_{r,i} < \infty, \) \(i \in \mathbb{N},\)

\[ s_{n}^{-r} \sum_{i=1}^{n} \eta_{j,i} - \sigma_{r}E(X^{*r}) = o_{j} \left( \sum_{i=1}^{n} \left[ \zeta_{r,i} + \sigma_{r}E(|X^{*}|^r) \right] \right) \]  

for \(1 \leq j \leq r, \) \(n \to \infty,\) together with

\[ L_{n} = \left( \frac{1}{\sum_{i=1}^{n} \zeta_{r,i}} \right) \sum_{i=1}^{n} \int_{|x| \geq \delta_{n}} |x|^r dF_{X_{r}}(x) = o_{\delta}(1) \]  

(7.15)

for \(n \to \infty,\) each \(\delta > 0,\) as well as the Feller condition (7.13). Then \(f \in C_{0}^{r}\) yields for \(n \to \infty\)

\[ \sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) d[F_{S_{n}/s_{n}}(x) - F_{X_{r}}(x)] \right| = o_{f} \left( s_{n}^{-r} \sum_{i=1}^{n} \left[ \zeta_{r,i} + \sigma_{r}E(|X^{*}|^r) \right] \right). \]  

(7.16)

(b) If the r.v. are i.d., \(\zeta_{r} < \infty,\) as well as

\[ \int_{\mathbb{R}} x^{j} d[F_{X_{1}}(x) - F_{X_{r}}(x)] = 0 \quad (1 \leq j \leq r), \]  

(7.17)

then one has for \(f \in C_{0}^{r}\) and \(n \to \infty\)

\[ \sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) d[F_{S_{n}/s_{n}}(x) - F_{X_{r}}(x)] \right| = o_{f}(n^{-(r-2)/2} \zeta_{r} + E(|X^{*}|^r)). \]

Proof. (a) To make use of Theorem 5 (a)(ii), one only has to show that condition (7.15) implies (4.6ii), since (7.14) corresponds directly to (4.5ii). Indeed,

\[ \sum_{i=1}^{n} \lambda_{r,i,n} \leq \sum_{i=1}^{n} \int_{|x| \geq \delta_{n}} |x|^r d[F_{X_{r}}(x) + F_{\sigma_{r}x}(x)] \]

\[ = o_{\delta} \left( \sum_{i=1}^{n} \zeta_{r,i} \right) \sum_{i=1}^{n} \int_{|x| \geq \delta_{n}} |x|^r dF_{\sigma_{r}x}(x). \quad (n \to \infty). \]
Now because of Feller's condition (7.13), it was shown in [7] that the r.v. \( \sigma_i X^* \) also satisfy the Lindeberg condition of order \( r \), i.e.,

\[
\sum_{i=1}^{n} \int_{|x| \geq \delta s_i} x^r dF_{\sigma_i X^*}(x) = o_\delta \left( \sum_{i=1}^{n} \sigma_i^r E(|X^*|^r) \right) \quad (n \to \infty),
\]

which concludes the proof, noting that \( \sum_{i=1}^{n} \text{Var}(\sigma_i X^*) = \sum_{i=1}^{n} \sigma_i^2 = s_n^2 \).

(b) This follows by Theorem 5 (c) since \( C_{Z_{1,n}}^r \subset L(F_{(Z_{1,n}/\text{var} + X^*)}) \) because \( E((S_n/n^{1/2}) + X^* | r) < \infty \), and since the r.v. \( Z_{i,n} \) with \( F_{Z_{i,n}} = F_{X^*} \) are i.d. with respect to \( i, n \in \mathbb{N} \).

8. THE CLASSICAL CLT; GENERALIZED LINDEBERG CONDITION; DISCUSSION

The well-known Lindeberg CLT for not necessarily i.d. r.v. states that

\[
\lim_{n \to \infty} F_{S_n/s_n}(x) = F_{X^*}(x) \quad (8.1)
\]

for any \( x \in \mathbb{R} \) provided the classical Lindeberg condition, i.e., (7.15) holds for \( r = 2 \). This result also follows from Corollary 3. Indeed, \( E(X_i) = 0 \), \( i \in \mathbb{N} \), without loss of generality, so that (7.14) is trivially satisfied for \( j = 1, 2 \). On the other hand, (7.15) for \( r = 2 \) is known to imply the Feller condition. Now, as in the proof of Theorem 13 (a), one can show that the assumptions of Theorem 5 (a)(ii) are satisfied, and therefore those of Corollary 3 apart from (4.11). But

\[
\sum_{i=1}^{n} [\xi_{2,i} + \sigma_i^2] = 2s_n^2 = O(s_n^2),
\]

which yields (4.12) (with \( T_n = S_n/s_n \) and \( X = X^* \)) which in turn is known (see, e.g., [5, pp. 42 and 18]) to be equivalent to (8.1).

Let us consider condition (7.15), introduced in [7] in greater detail. It may be considered to be a generalized Lindeberg condition of order \( r \) since it reduces to the classical one in the case \( r = 2 \). At first a sufficient condition is given such that a sequence of r.v. satisfies such a generalized Lindeberg condition. Although a Lindeberg condition of higher order does not necessarily imply one of lower order, this will be shown to be so provided a supplementary condition is satisfied.

**Proposition 2.** (a) If there exists an \( \epsilon > 0 \) such that

\[
t_{r+\epsilon,n}(t_{r,n}s_n^r) = o(1) \quad (n \to \infty),
\]

(8.2)
then \( L_n^r(\delta) \to 0, n \to \infty \), for each \( \delta > 0 \), where

\[
t_{s,n} := \sum_{i=1}^{n} \xi_{s,i} \quad (s \in \mathbb{R}^+, n \in \mathbb{N}).
\]

(b) If the Lindeberg condition of order \( r + \epsilon, r \geq 2, 0 < \epsilon \leq 1 \), is satisfied, then that of order \( r \) holds provided

\[
t_{r+\epsilon,n}/(t_{r,n}^{s_n^r}) = O(1) \quad (n \to \infty).
\]

Proof. (a) Noting that \( |x| \geq \delta s_n \) implies \( 1 \leq |x|/\delta s_n \), it follows together with (8.2) that for each \( \delta > 0 \)

\[
L_n^r(\delta) \leq \frac{1}{t_{r,n}^{s_n^r}} \sum_{i=1}^{n} \int_{|x| \geq \delta s_n} |x|^{r+\epsilon} \, dF(x) \leq \frac{t_{r+\epsilon,n}}{t_{r,n}^{s_n^r}} \, \delta^{-\epsilon} \to 0 \quad (n \to \infty).
\]

(b) Because \( |x| \geq \delta s_n \) also implies \( |x|^{r+\epsilon} \geq |x|^{r \delta s_n^r} \), and (8.3) means that \( s_n^{1/r+\epsilon,n} \geq 1/(M r_{r,n}) \), \( n \to \infty \), for some \( M > 0 \), one has

\[
L_n^{r+\epsilon}(\delta) \geq \frac{s_n^{r-\epsilon}}{t_{r+\epsilon,n}} \sum_{i=1}^{n} \int_{|x| \geq \delta s_n} |x|^r \, dF(x) \geq \frac{s_n}{M r_{r,n}} \sum_{i=1}^{n} \int_{|x| \geq \delta s_n} |x|^r \, dF(x) = \frac{s_n^r}{M} L_n^r(\delta)
\]

for \( n \to \infty \) and each \( \delta > 0 \). Since \( L_n^{r+\epsilon}(\delta) \to 0 \) for each \( \delta > 0 \), the proof follows.

Note that Proposition 2 (a) in the particular case \( r = 2 \) is the well-known fact that condition (8.2), then called the (classical) Ljapounov condition, implies the classical Lindeberg condition since \( t_{2,n} = s_n^2 \). This suggests calling (8.2) the Ljapounov condition of order \( r \).

In this terminology Proposition 2 (a) states that the Ljapounov condition of order \( r \) implies the Lindeberg condition of the same order.

Note that Brown [6] gave another generalization of the Lindeberg and Ljapounov conditions of order \( r \) in such fashion that both are equivalent if \( r > 2 \). See also Basu [2], who follows Brown.

Concerning the discussion of the material of Section 7, let us mention that Theorems 11 and 13, especially Theorem 11, are improvements of those of [7]. Indeed, the hypothesis in Theorem 1 of [7], namely, that \( \zeta_{s,i} < \infty \), is replaced by the weaker condition \( \zeta_{r-1+i,\beta} < \infty \) for \( \beta \in (0, 1] \) in parts (a), (b), and (d) of Theorem 8, and that additionally in part (a) the assumption on the equality of the moments [namely, (7.5)] is replaced by the further weaker condition...
(7.4) or (7.4') on the rate of the difference of the moments \( \eta_{i,i} - \sigma_i^j E(X^*) \)
(see example below). Furthermore, the five assertions (a)-(e) of Theorem 11
are sharpenings of those of [7] since either the function class \( C_{\beta'} \)
is replaced by the larger class \( C_b + C_{\gamma'} \), or the rate of convergence as well as the constants
involved are improved. This is particularly evident in part (e). Indeed, instead of
\[
O_f(n^{-\frac{(r-3+\beta)}{2}}[\zeta_r + E(|X^*|^\gamma) + 1]) \quad (0 < \beta \leq 1)
\]
one now has
\[
O_f(n^{-\frac{(r-2)}{2}}[\zeta_r + E(|X^*|^\gamma)]^{a/r}) \quad (0 < \alpha \leq r).
\]
Comparing both estimates for \( \alpha = r - 1 + \beta \), then \( n^{-a(r-2)/2r} \leq n^{-\frac{(r-3+\beta)}{2}} \)
iff \( \beta \leq 1 \); moreover, \( [\zeta_r + E(|X^*|^\gamma)]^{a/r} \leq \zeta_r + E(|X^*|^\gamma) + 1 \).

The fact that the hypotheses of Theorem 8 (a) are actually weaker than those of Theorem 1 [7] is revealed by the following example. It also justifies
that posing a condition of type (7.4') is meaningful in the sense that there
exist r.v. satisfying (7.4') but not (7.5).

Choose \( r = 6, \beta = 1 \), and symmetric r.v. \( X_i \), the moments of which differ
"slightly" from those of the standard normal r.v. \( X^* \) as follows: define the
r.v. \( X_i \) via their d.f. \( F_{X_i} \) by
\[
F_{X_i}(x) = a_i F_{-c_i}(x) + (1 - 2a_i) F_{0}(x) + a_i F_{c_i}(x),
\]
where
\[
F_{0}(x) = \begin{cases} 0, & x < a \\ 1, & x \geq a \end{cases} \quad (a \in \mathbb{R}),
\]
and
\[
a_i = \frac{1 - 3^{-i}}{2(3 + 2^{-i})}, \quad c_i = \frac{(3 + 2^{-i})^{1/2}}{(1 - 3^{-i})^{1/2}} \quad (i \in \mathbb{N}).
\]
Now \( F_{X_i} \) is a d.f. since the sum of its coefficients, which are all non-negative,
is equal to one. One has \( E(X_i^j) = 0 \) for \( j = 1, 3 \) and \( 5, i \in \mathbb{N} \). Moreover,
\[
E(X_i^0) = \eta_{2,i} - (1 - 3^{-i})^{1/2},
\]
\[
E(X_i^j) = \eta_{4,i} = 3 + 2^{-i}, \quad (i \in \mathbb{N})
\]
\[
E(X_i^0) = \eta_{6,i} = (3 + 2^{-i})^{1/2}(1 - 3^{-i})^{1/2}.
\]
This gives the estimates \( \frac{8}{3} \leq \eta_{2,i} \leq 1, \frac{8}{3} \leq \eta_{2,i} \leq 1, 9 \leq \eta_{6,i} \leq 31, \) and
\( (2/3)n \leq s_i^2 \leq n \). Then it can be shown that condition (7.4') is satisfied with
C_1 = C_2 = C_3 = C_5 = 0, C_4 = 1. Therefore Theorem 11 (a) yields for $f \in C_b^g$ that

$$
\| V_{S_n/S_n} f - V_{X^*} f \| \leq \left( \frac{1}{4} \left\| f^{(4)} \right\| + \frac{1}{120} \left\| f^{(6)} \right\| \right) s_n^2 \sum_{i=1}^n \left[ \epsilon_{i,i} + \sigma_i^6 \mathbb{E}(|X^*|^8) \right]
$$

$$
= O_r(n^{-a}),
$$

where

$$
s_n = \left\{ \sum_{i=1}^n \left( 1 - 3^{-i} \right)^{1/2} \right\}^{1/2}.
$$

Similar but somewhat more difficult examples can be constructed that give convergence rates of order higher than $O(n^{-2})$.

Let us finally recall that when examining convergence rates even in the case of i.d. r.v. for the d.f. themselves, thus when considering the indicator function $\chi_{(-\infty, u]}$, $u \in \mathbb{R}$, instead of the function $f \in C_b + C_b^*$, one cannot do without a supplementary condition in the assumptions of Theorem 11d, namely, Cramér's (C) condition.

On the other hand, let us mention here that the assertions of Theorem 11 (d), for example, are none the less best possible for the function classes under consideration in the sense that condition (7.6) is not only sufficient but even necessary such that (7.7) holds with respect to the class Lip($r - 1 + \beta; r; C_b + C_b^*$). For partial results see [8], for sharper ones [12].

### 9. Weak Law of Large Numbers with Rates

As a final application of the general theorems of Sections 3 and 4 we shall deduce the WLLN together with error estimates. These will not be considered for the law in the form that $\varphi(n)S_n$ converges in probability to zero, i.e., for each $\varepsilon > 0$

$$
P(\| \varphi(n)S_n \| \geq \varepsilon) \to 0 \quad (n \to \infty),
$$

(9.1)

but in one that is known to be equivalent to it (see [5, p. 25], [3, p. 220]), namely,

$$
\left\| \int \mathbb{R} f(x + \cdot) \, dF_n(x) - f \right\| \to 0 \quad (n \to \infty)
$$

(9.2)

for each $f \in C_b^r$, any $r \in P$. In case $(X_i)_{i \in \mathbb{N}}$ is a sequence of real independent, not necessarily i.d. r.v. that is square integrable, the well-known sufficient condition for either to hold is that $\varphi(n) = 1/n$ and

$$
\sum_{i=1}^n \text{Var}(X_i) = o(n^2) \quad (n \to \infty).
$$

(9.3)

For our goal, the limiting r.v. will be taken as $X_0$ with d.f. $F_0$ (recall (8.4)).
Theorem 14. Let \((X_i)_{i \in \mathbb{N}}\) be a sequence of real independent r.v. and \(r \in \mathbb{N}\), \(0 < \beta \leq 1\), \(0 < \alpha \leq 2\).

(a) Assume that (7.3) holds and there exist constants \(C_i\) such that

\[
\varphi(n)^{j-1 - \beta} \sum_{i=1}^{n} |\eta_{j,i}| \leq C_i \sum_{i=1}^{n} \zeta_{r-1+i, i} \quad (0 \leq j \leq r - 1).
\]  

(9.4)

Then one has for \(f \in \text{Lip}(r - 1 + \beta; \alpha; C_\beta)\)

\[
\sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) \, d[F_{\varphi(n)}(x) - F_0(x)] \right| 
\leq (C_F + L_f/(r - 1)! \varphi(n))^{r-1+\beta} \sum_{i=1}^{n} \zeta_{r-1+i, i}.
\]  

(9.5)

(b) If there holds (7.3) and instead of (9.4)

\[
|\eta_{j,i}| = 0 \quad (j = 0, 1; i \in \mathbb{N}),
\]  

(9.6)

then \(f \in \text{Lip}(\beta + 1; \alpha; C_\beta)\) implies (9.5) with \(r = 2\) and \(C_\gamma = 0\).

(c) If the r.v. \(X_i\) are i.d., \(\zeta_2 < \infty\) and \(\eta_1 = 0\), then

\[
\sup_{y \in \mathbb{R}} \left| \int_{\mathbb{R}} f(x + y) \, d[F_{\varphi(n)}(x) - F_0(x)] \right| 
\leq 2c_{2,2} \omega_2(n^{1/2} \varphi(n))^{1/2} \beta; C_0 + C_\beta^2).
\]  

(9.7)

In particular, if \(f \in \text{Lip}(\alpha; 2; C_0 + C_\beta^2)\), then

\[
\| V_{\varphi(n)} f - V_{x_0} f \| \leq 2c_{2,2} L_f \beta^{1/2}(n^{1/2} \varphi(n))^{1/2}.
\]  

The proofs follow from the corresponding theorems in Sect. 3. Indeed, part (a) follows from Theorem 1, (b) from Theorem 3, and (c) from Theorem 4 (part 2). One just sets \(F_{Z_{i,n}} = F_0\) and has

\[
F_{\varphi(n)} \Sigma_{i=1}^{n} \zeta_{i,n} = F_0.
\]

With \(G_i = F_{X_i} - F_0\) one has further \(\mu_{j,i} = \eta_{j,i}\) and \(\nu_{j,i} = \zeta_{j,i}\).

Concerning the associated \(\alpha\)-error estimates for the WLLN, it does not matter whether one uses the (i) or the (ii) version of Theorem 5, the hypotheses coinciding. So one has

Theorem 15. Let \((X_i)_{i \in \mathbb{N}}\) be a sequence of real independent r.v. and \(r \in \mathbb{N}\).

(a) Assume that \(\zeta_{r,i} < \infty, i \in \mathbb{N},\) that (7.15), as well as

\[
\varphi(n)^{j-1} \sum_{i=1}^{n} |\eta_{j,i}| = o_j \left( \sum_{i=1}^{n} \zeta_{r,i} \right) \quad (1 \leq j \leq r; n \to \infty)
\]  

(11.9)
holds. If \( f \in C_B^2 \), then for \( n \to \infty \)

\[
\left\| \int \mathbb{R} (x + y) \, dF_{x\,n}(y) - f(y) \right\| = o_n \left( \varphi(n)^r \sum_{i=1}^{n} \eta_{r,i} \right).
\]  

(9.8)

(b) If the r.v. \( X_i \) are i.d., \( \epsilon_1 < \infty \), and \( \eta_1 = 0 \), then \((X_i)_{i \in \mathbb{N}}\) satisfies the WLLN, i.e., (9.1) holds with \( \varphi(n) = n^{-1} \).

Part (a) follows from Theorem 5 (a), noting that concerning (4.6) one has for an arbitrary measurable set \( A \)

\[
\mathbb{P} \left( \left\| \int_A |x|^r \, dF_{x\,n}(x) - F_0(x) \right\| = \int_A |x|^r \, dF_{x\,n}(x) \right) = 0,
\]

Part (b) follows from Corollary 3 (case \( r = 1 \)) with \( \varphi(n) = n^{-1} \) since (7.15) is always satisfied for r.v. that are i.d. and (4.10) holds in this instance. Therefore (9.2) holds, and so (9.1).

First note that concerning the question of convergence in (9.7) of Theorem 14 (c), \( \varphi(n)S_n \) converges in probability to zero if \( \varphi(n) = o(n^{-1/2}) \), \( n \to \infty \).

Second, in the case of not necessarily i.d. r.v., Theorem 14 (a) and (b) subsumes the WLLN in the form (9.2). Indeed, without loss of generality let the r.v. be such that \( E(X_i) = 0 \), \( i \in \mathbb{N} \), i.e., (9.6) holds; (7.3) is satisfied for \( \beta = 1 \), \( r = 2 \) since \( \text{Var}(X_i) < \infty \). Herewith Theorem 14 (b) implies with \( \varphi(n) = n^{-1} \) that for \( f \in C_B^2 \)

\[
\int \mathbb{R} f(x + y) \, d[F_{x\,n}(x) - F_0(x)] \leq (L/(r - 1)!)n^{-2} \sum_{i=1}^{n} \text{Var}(X_i),
\]

which gives (9.2) in view of (9.3).

Concerning previous results on rates in the matter, apart from the fact that one could interprete the d.f. of the r.v. \( X_0 \) in the WLLN as a degenerate stable d.f. (case \( c = 0 \)), it seems that the question of rates has only been studied in connection with the law in the form (9.1). Thus, for example, Baum and Katz [4] (see also Révész [20], Petrov [17]) gave equivalent conditions such that

\[
P(\left\{ |S_n| \geq \varepsilon \right\} = o_\varepsilon(n^{-t}) \quad (n \to \infty) \]

(9.9)

for \( t \geq 0 \) in the case of i.d. r.v. (For the case \( \varphi(n) = n^{-1/2} \), \( \gamma \in (0, 2) \), see Heyde and Rohatgi [13]). A conclusion of this result (see [17, p. 286]) is that \( E(X_1) = 0 \) and \( \eta_r < \infty \) for \( r \in \mathbb{N} \) implies assertion (9.9) for \( t = r - 1 \). This means that the rate in (9.9) is arbitrarily good provided the moments of sufficiently high order are finite. This is not so for our theorems; in \[12\] it will be shown that \( O(n^{-1}) \) is the best possible order for our estimates in the above particular case (i.e., for i.d. r.v. with \( \varphi(n) = n^{-1} \), unless \( F_{x\,i} = F_0 \), \( i \in \mathbb{N} \).

However it should be noted that the small- \( o \) term in (9.9) depends decisively upon \( \varepsilon \).
Let us finally remark that in the case of not necessarily i.d. r.v., however, our estimates yield rates that are better than $O(n^{-1})$, as the following example shows.

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of r.v. with d.f.

$$F_{x_i}(x) = \frac{1}{2^{a_{i+1}}} F_{-2^{i}}(x) + \left(1 - \frac{1}{2^{a_{i+1}}} \right) F_0(x) + \frac{1}{2^{a_{i+1}}} F_{2^{i}}(x) \quad (i \in \mathbb{N}).$$

This is actually a distribution since its coefficients are non-negative and the sum is one; for the moments one has $E(X_i) = \eta_{1,i} = 0$, $E(X_i^2) = 2^{-i} (\neq 0)$, and $E(|X_i|^3) = 1, i \in \mathbb{N}$. Setting $\phi(n) = n^{-1}, \beta = 1$, and $r = 3$, then hypothesis (9.4) of Theorem 14(a) is trivially satisfied for $j = 1$ since $\eta_{1,i} = 0, i \in \mathbb{N}$; for $j = 2$ it takes on the form

$$\sum_{i=1}^{n} 2^{-i} \leq n - C_2 \sum_{i=1}^{n} 2^{-i}$$

with $C_2 = 1$. Therefore for $f \in C_{ij}^3$ one has by (8.5)

$$\| V_{\phi(n)/n} f - f \| \leq (1/2)( \| f^{(3)} \| + \| f^{(3)} \|) n^{-2} = O(n^{-2}).$$

This example could be modified so that one has orders better than $O(n^{-2})$.

It is of course possible to apply our general limit theorems to other limiting r.v. that are \(\phi\)-decomposable in order to deduce further concrete limit theorems. An example is the "one-sided stable distribution with index $\alpha$" and corresponding r.v. $X'$ defined by its d.f. (compare [10, p. 51])

$$F_{X'}(\alpha, x) := 2 \left\{ F_{X'} \left( \frac{x}{\alpha} \right) \right\} \quad (x > 0), \quad (9.10)$$

where $\alpha \in \mathbb{R}$ is arbitrary.

It is not hard to verify that if $Z_1, \ldots, Z_n$ are independent r.v. with d.f. (9.10), then $n^{-2}(Z_1 + \cdots + Z_n)$ has the same distribution. Thus, our limit theorem now works with $\phi(n) = n^{-2}$ and $F_{Z_{n\alpha}}(x) = F_{X'}(\alpha, x)$, any fixed $\alpha$.

10. The Multivariate Case

Consider now $m$-dimensional r.v. $X_i = (X_{i1}, \ldots, X_{im})$: $\Omega \rightarrow \mathbb{R}^m$. The three limiting r.v. in question are $Y_\alpha, X_{\alpha}'$, and $X_0$ defined via their ch.f. $f_{Y_\alpha}$, $f_{X_{\alpha}'}$, and $f_{X_0}$, respectively, namely,

(i) $f_{Y_\alpha}(u) = \exp \left[ -c \sum_{j \in J} (u D_j u') \gamma / 2 \right] \quad (0 < \gamma < 2)$.
where \( c > 0 \), \( J \) is a finite or countable set of indices, \( D_j \) are symmetric positive definite \( m \times m \) matrices, and \( u^t \) denotes the transpose of a vector \( u = (u_1, \ldots, u^m) \) (compare [19]);

\[
(ii) \quad f_{X_1^*}(u) = \exp[-\frac{1}{2} u^t \Gamma u],
\]

where \( \Gamma \) is a symmetric positive definite \( m \times m \) matrix. It is well known that \( X_r^* \) has variance-covariance matrix \( \Gamma \). If \( \Gamma = I \), the identity matrix, then we write \( X^* \) instead of \( X_r^* \);

\[
(iii) \quad f_{X^*}(u) = 1.
\]

If \( (X_i)_{i \in \mathbb{N}} \) is a given sequence of r.v. with zero mean, provided it exists, and positive definite (variance-) covariance matrix \( A_i \), provided it exists, then the \( \varphi \)-decompositions of the corresponding limiting r.v. are the following:

\[
(i) \quad \varphi(n) = n^{-1/\gamma}; \quad F_{Z_{1,n}} = F_{X_{1^*}},
\]

(ii)(a) If the r.v. \( X_i \) are i.d. with covariance matrix \( A \), then take for the normally distributed r.v. \( X_{A^*}^* \)

\[
\varphi(n) = n^{-1/2}; \quad F_{Z_{1,n}} = F_{X_{A^*}}.
\]

(b) If the r.v. \( X_i \) are not necessarily i.d. with covariance matrix \( A_i \), take \( \sigma_i^2 := \text{Tr}(A_i) = \sum_{j=1}^m E[(X_i)_j^2] \) and \( s_n^2 = \sum_{i=1}^n \sigma_i^2 \). Then

\[
\varphi(n) = s_n^{-1}; \quad F_{Z_{1,n}} = F_{s_n X^*}.
\]

(iii) Here \( \varphi(n) \) can be chosen arbitrarily and

\[
F_{Z_{1,n}} = F_{X_0}.
\]

Since four (different) versions of general limit theorems (with orders) leading to ten concrete applications for the various limiting r.v. were presented in [9], here there is just room for a choice of the different versions. In particular, the counterparts of Theorems 9 (a), 11 (c), 11 (d), 14 (a), and 13 (a) read as follows:

**Theorem 16** (Stable limit law). Let \( (X_i)_{i \in \mathbb{N}} \) be a sequence of \( m \)-dimensional independent r.v. and \( r \in \mathbb{N}, 0 < \beta \leq 1 \). Assume that

\[
\nu_{r, 1, \beta, \alpha} (Y_r) := \int_{\mathbb{R}^m} |x|^{r-1+\beta} \, d |(F_{X_i} - F_{X_0})(x)| \quad (i \in \mathbb{N}),
\]
and that there exist constants $C_\kappa = C_{\kappa_1,\ldots,\kappa_m}$ such that
\[
218 - \nu_{r-1+\beta,i}(Y_j)
\]
for all $\kappa^* = (\kappa_1,\ldots,\kappa_m)^1$ and $\kappa_k \in \{0,\ldots,j\}$, $0 \leq j \leq r - 1$. Then for $f \in \text{Lip}(r - 1 + \beta, r) \cap C_B^{-1}(\mathbb{R}^m)$ there holds
\[
\sup_{y \in \mathbb{R}^m} \left| \int_{\mathbb{R}^m} f(x + y) \, d[F_{S_n/n^{1/r}}(x) - F_{Y_j}(x)] \right| \leq (C_\beta + C(r, m)L_f).
\]

**Theorem 17 (CLT).** Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of $m$-dimensional independent r.v. and $r \in \mathbb{N}$, $0 < \alpha \leq r$.

(a) **Assume that**
\[
E(|X_i|^r) < \infty \quad (i \in \mathbb{N})
\]
**as well as**
\[
\int_{\mathbb{R}^m} (x^1)^{\kappa_1} \cdots (x^m)^{\kappa_m} \, dF_{X_i}(x) = \int_{\mathbb{R}^m} (x^1)^{\kappa_1} \cdots (x^m)^{\kappa_m} \, dF_{\kappa_i x}.
\]
for all $\kappa^* = (\kappa_1,\ldots,\kappa_m)$ and $\kappa_k \in \{0,\ldots,j\}$, $0 \leq j \leq r - 1$. Then one has
\[
\sup_{y \in \mathbb{R}^m} \left| \int_{\mathbb{R}^m} f(x + y) \, d[F_{S_n/n^{1/r}}(x) - F_{X_i}(x)] \right| \leq C(r, m)\omega_r \left( \frac{c^{-r}}{n} \sum_{i=1}^{n} \left| x^i \right|^r \, d \left| (F_{X_i} - F_{\kappa_i x}(x)) \right| \right)^{1/r} f.
\]
In particular, if $f \in \text{Lip}(\alpha; r)$, then
\[
\sup_{y \in \mathbb{R}^m} \left| \int_{\mathbb{R}^m} f(x + y) \, d[F_{S_n/n^{1/r}}(x) - F_{X_i}(x)] \right| \leq C(r, m) L_f c^{-r} \left( \sum_{i=1}^{n} \left| x^i \right|^r \, d \left| (F_{X_i} - F_{\kappa_i x}(x)) \right| \right)^{\alpha/r}.
\]

(b) **Let the r.v.** $X_i$ **be i.d. with covariance matrix** $A$ **and let (10.1) hold as well**
\[
\int_{\mathbb{R}^m} (x^1)^{\kappa_1} \cdots (x^m)^{\kappa_m} \, dF_{X_A}(x) = \int_{\mathbb{R}^m} (x^1)^{\kappa_1} \cdots (x^m)^{\kappa_m} \, dF_{X_A}(x)
\]

$^1$ To save space here and below, $\kappa^*$ means that in addition to $\kappa_k \in \{0,\ldots,j\}$, also $\sum_{k=1}^{m} \kappa_k = j$. 
for all $\kappa^* = (\kappa_1, \ldots, \kappa_m)$ and $\kappa_k \in \{0, \ldots, j\}$, $0 \leq j \leq r - 1$. Then one has for $f \in \text{Lip}(\alpha; r)$

$$
\sup_{y \in \mathbb{R}^m} \left| \int_{\mathbb{R}^m} f(x + y) \, d[F_{S_n/n^1/s}(x) - F_{X^*_n}(x)] \right| \leq C(r, m) L_f n^{-a(r-2)/2} \left\{ \int_{\mathbb{R}^m} |x| r \, d[(F_{X^*_n} - F_{X^*_n})(x)] \right\}^{\alpha/r}.
$$

**Theorem 18 (WLLN).** Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of $m$-dimensional independent r.v. and $r \in \mathbb{N}$, $0 < \beta \leq 1$. Assume that

$$
E(|X_i|^{r-1+\beta}) < \infty \quad (i \in \mathbb{N}),
$$

and that there exist constants $C_{\kappa} = C_{\kappa_1, \ldots, \kappa_m}$ such that

$$
\varphi(n)^{r+1-\beta} \sum_{i=1}^{n} \left| \int_{\mathbb{R}^m} (x^1)^{\kappa_1} \cdots (x^m)^{\kappa_m} \, dF_{X^*_n}(x) \right| \leq C_{\kappa} \sum_{i=1}^{n} \int_{\mathbb{R}^m} |x|^{r-1+\beta} \, dF_{X^*_n}(x)
$$

for all $\kappa^* = (\kappa_1, \ldots, \kappa_m)$ and $\kappa_k \in \{0, \ldots, j\}$, $0 \leq j \leq r - 1$. Then one has for $f \in \text{Lip}(r - 1 + \beta; r)$

$$
\sup_{y \in \mathbb{R}^m} \left| \int_{\mathbb{R}^m} f(x + y) \, d[F_{\varphi(n) S_n}(x) - F_{X^*_n}(x)] \right| \leq (C_f + C(r, m)L_f) \varphi(n)^{r-1+\beta} \sum_{i=1}^{n} \int_{\mathbb{R}^m} |x|^{r-1+\beta} \, dF_{X^*_n}(x).
$$

Let us end with one small $o$-version, namely, for the CLT:

**Theorem 19.** Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of $m$-dimensional independent r.v. and $r \in \mathbb{N}$, $r \geq 2$. Assume that there holds (10.1) as well as for $n \to \infty$

$$
s_n^{-j} \sum_{i=1}^{n} \left| \int_{\mathbb{R}^m} (x^1)^{\kappa_1} \cdots (x^m)^{\kappa_m} \, d[F_{X^*_n}(x) - F_{X^*_n}(x)] \right| = o_j \left( \sum_{i=1}^{n} \left[ \int_{\mathbb{R}^m} |x| r \, dF_{X^*_n}(x) + \int_{\mathbb{R}^m} |x| r \, dF_{X^*_n}(x) \right] \right)
$$

for all $\kappa^* = (\kappa_1, \ldots, \kappa_m)$ and $\kappa_k \in \{0, \ldots, j\}$, $1 \leq j \leq r$. Further let

$$
\left( \sum_{i=1}^{n} \int_{\mathbb{R}^m} |x| r \, dF_{X^*_n}(x) \right)^{-1} \sum_{i=1}^{n} \int_{|x| \geq \delta S_n} |x| r \, dF_{X^*_n}(x) = o_j(1) \quad (10.2)
$$
for \( n \to \infty \), each \( \delta > 0 \), together with the Feller condition

\[
\lim_{n \to \infty} \max_{1 \leq i \leq n} (\sigma_i/s_n) = 0.
\]  

(10.3)

Then \( f \in C_{B'}(\mathbb{R}^m) \) yields for \( n \to \infty \)

\[
\sup_{y \in \mathbb{R}^m} \left| \int_{\mathbb{R}^m} f(x + y) \, d[F_{n/s_n}(x) - F_{X}(x)] \right| = o\left( \left( \sum_{i=1}^{n} \int_{\mathbb{R}^m} |x|^r \, dF_{X_i}(x) + \int_{\mathbb{R}^m} |x|^r \, dF_{\sigma_iX_i}(x) \right)^{r} \right).
\]

Analogously to the one-dimensional case, the proofs of the concrete multivariate \( O \)-limit theorems follow at once from the corresponding general theorems of Section 5 provided one has the associate \( \varphi \)-decompositions. Concerning the \( o \)-theorem, one has to prove that if \((X_i)_{i \in \mathbb{N}}\) satisfies the Lindeberg condition (10.2) of order \( r \) as well as the Feller condition (10.3), then \((\sigma_iX^*_i)_{i \in \mathbb{N}}\) satisfies the Lindeberg condition of order \( r \) too; this can be shown similarly as in the real case.

In Section 7 we have shown in great detail that for real r.v. our new results concerning the CLT are sharper than those given in [7].

For the same reasons, our present results are a great improvement of those of [18] who just extended our results of [7] to the multivariate case. In addition, our present ones are better than those of [18] for he had to assume that \( E(|X_i|^{r+m-1}) \) is finite instead of just (10.1).

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References