CHARACTERIZING HILBERT CUBE MANIFOLDS BY THEIR HOMOLOGICAL STRUCTURE

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A hierarchy of disjoint Čech carriers properties is introduced; and each is shown to be characteristic of ANR's whose products with 2-cells are Hilbert cube manifolds.

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1. Introduction

The analysis in [5] isolated a homological condition that is characteristic of ANR's (absolute neighborhood retracts) whose products with a 2-cell are Hilbert cube manifolds. The role of the 2-cell is to establish the Disjoint Disks Property, the need for which can be understood by studying [1, 7]. The homological condition specified is a disjoint Čech carriers property, labeled Property (DCC) below. The property captures the global and local infinite dimensionality of the ANR; and [5] records that it is detectable in many interesting situations. The discussion in [5; Section 10] centers on an, apparently, weaker condition, namely

(*) \( H_\ast(X, X - \{x\}; \mathbb{Z}) = 0 \) for each \( x \in X \).

The condition is a natural extension to infinite dimensions of that used to specify finite dimensional ANR homology manifolds. Introduced below is a hierarchy of disjoint Čech carriers properties, namely Property (DCC)_n for \( n = 2, 3, \ldots \). The thrust of the paper is towards establishing that each of these properties is characteristic of ANR's whose products with a 2-cell are Hilbert cube manifolds. While apparently none of these properties quite captures condition (*), the latter lies somewhere between them and Property (DCC)_\omega; see the discussion in Section 5.

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Main Theorem. Let $X$ be a locally compact ANR satisfying Property (DCC)$_n$ for some $n$; that is, for open pairs $(U_i, V_i)$ in $X$ and homology elements $z_i \in H_q(i)(U_i, V_i)$, $1 \leq i \leq n$, there are compact pairs $(C_i, \partial C_i) \subset (U_i, V_i)$ with $z_i \in \text{Im}(i_*: H_q(i)(C_i, \partial C_i) \to H_q(i)(U_i, V_i))$ such that $C_1 \cap \cdots \cap C_n = \emptyset$. Then

(a) $X \times I^2$ is a Hilbert cube manifold; and

(b) $X$ is a Hilbert cube manifold provided $X$ satisfies the Disjoint Disks Property.

There are a variety of homological conclusions that follow once this geometric result has been established; we point out two that appear difficult to verify directly.

Corollary. Let $X$ be a locally compact ANR.

1. If $X$ satisfies Property (DCC)$_n$ for some $n \geq 2$, then $X$ satisfies Property (DCC)$_2$.

2. If $X$ satisfies Property (DCC)$_n$, then $X \times I$ satisfies Property (DCC)$_n$.

The specific catalyst for considering these properties (specifically, Property (DCC)$_3$) was the work of the first author in [9] that produced in the setting of $Q$-manifolds examples analogous to those constructed in [2] and [6]. The process of determining that $X \times I$ was a $Q$-manifold for a carefully constructed ANR $X$ included establishing that $X$ satisfied Property (DCC)$_2$; the argument turned out to be surprisingly complicated in spite of the fact that $X$ evidently satisfied Property (DCC)$_3$.

Spaces are assumed to be separable and metric, and ANR's are assumed to be locally compact. The interval is denoted by $I$ and the Hilbert cube is denoted by $Q$. The coefficient group for homology is always the integers and is suppressed. Singular homology is denoted by $H_*$ and Čech homology is denoted by $\check{H}_*$. A map $f: X \to Y$ is a near homeomorphism provided, for each open cover $\mathcal{U}$ of $Y$, there is a homeomorphism $h: X \to Y$ such that $h$ and $f$ are $\mathcal{U}$-close.

2. Čech carriers and infinite codimension

Following [5; Section 3], a Čech carrier for an element $z \in H_q(U, V)$, where $V \subset U$ are open subsets of an ANR $X$, is a compact pair $(C, \partial C) \subset (U, V)$ such that $z \in \text{Im}(i_*: \check{H}_q(C, \partial C) \to H_q(U, V))$, $i_*$ being the inclusion induced homomorphism. A hierarchy of disjoint Čech carriers properties for ANR's extending that introduced in [5] is determined by defining, for $n \geq 2$;

Property (DCC)$_n$. For open pairs $(U_i, V_i)$ and homology elements $z_i \in H_q(i)(U_i, V_i)$, $1 \leq i \leq n$, there are Čech carriers $(C_i, \partial C_i)$ for $z_i$ such that $C_1 \cap \cdots \cap C_n = \emptyset$.

In the course of establishing the Main Theorem, a 'local' version of these properties is needed; namely, for a closed subset $A$, an ANR $X$ satisfies:
Property (DCC), at $A$. For open pairs $(U_i, V_i)$ and homology elements $z_i \in H_q(U_i, V_i)$, $1 \leq i \leq n$, there are Čech carriers $(C_i, \partial C_i)$ for $z_i$ such that $(C_1 \cap \cdots \cap C_n) \cap A = \emptyset$.

While the absolute Property (DCC), makes little sense, the local version Property (DCC), at $A$ is equivalent to $A$ having infinite codimension in $X$; that is, $H_q(U, U \setminus A) = 0$ for all integers $q \geq 0$ and all open sets $U \subset X$ (see [5; Lemma 3.1]). It is convenient to extend the latter definition by saying that an arbitrary subset $F \subset X$ has infinite codimension provided each closed (in $X$) set $A \subset F$ has infinite codimension. We shall need the following results, a proof of the first can be found in [5; Section 2].

**Lemma 2.1.** If a subset $A$ of an ANR $X$ has infinite codimension in $X$, then $A \times I$ has infinite codimension in $X \times I$.

**Lemma 2.2.** Let $f : X \to Y$ be a cell-like map between ANR's and suppose that a subset $A \subset Y$ has infinite codimension in $Y$ and that $f$ is one to one over $A$. Then the subset $f^{-1}(A)$ has infinite codimension in $X$.

**Proof.** Since each closed (in $X$) subset of $f^{-1}(A)$ is contained in $f^{-1}(C)$ for some closed (in $Y$) subset $C \subset A$, it suffices to consider the case that $A$ is closed.

Since $f$ is one to one over $A$, given an open subset $U \subset X$, there is an $f$-saturated open subset $V \subset U$ (i.e., $f^{-1}(V) = V$) such that $V \cap f^{-1}(A) = U \cap f^{-1}(A)$. The inclusion $(V, V \setminus f^{-1}(A)) \to (U, U \setminus f^{-1}(A))$ yields, by excision, an isomorphism

$$H_*(V, V \setminus f^{-1}(A)) \to H_*(U, U \setminus f^{-1}(A))$$

and, since the restrictions $f|V$ and $f|(V \setminus f^{-1}(A))$ are cell-like, there is an isomorphism

$$H_*(V, V \setminus f^{-1}(A)) \to H_*(f(V), f(V) \setminus A).$$

The last group is trivial by hypothesis and the two isomorphisms reveal that $H_*(U, U \setminus f^{-1}(A)) = 0$, completing the proof.

A resolution of an ANR $X$ is a cell-like map $\pi : M^O \to X$ from a Hilbert cube manifold onto $X$; ANR's have resolutions [11], for example, the projection $X \times Q \to X$ (see [4]).

**Remark 2.3.** Given a resolution $\pi : M^O \to X$ and a closed subset $A \subset X$, denote by $G(A)$ the decomposition of $M$ that is the trivial extension of $\{ \pi^{-1}(a) : a \in A \}$ and by $\pi_A : M \to M/G(A)$ the quotient map. Since the map $\pi$ is a hereditary shape equivalence and $A$ is closed, the map $\pi_A$ is a hereditary shape equivalence and, consequently, $M/G(A)$ is an ANR (see [8]). Since $\pi \circ \pi_A^{-1} : M/G(A) \to X$ is one to one over $A$, Lemma 2.2 records that $\pi_A \pi^{-1}(A)$ has infinite codimension in $M/G(A)$ whenever $A$ has infinite codimension in $X$. 
3. Disjoint disks property and a characterization of $Q$-manifolds

A space $X$ satisfies the Disjoint Disks Property provided each pair of maps $f, g : B^2 \to X$ can be approximated arbitrarily closely by maps $f^*, g^* : B^2 \to X$ so that $f^*(B^2) \cap g^*(B^2) = \emptyset$. The analysis in [5] led to characterizations of $Q$-manifolds that combined this property, that had played a central role in the detection of finite dimensional manifolds [1], [7], with properties involving Čech carriers. The next result supplies the principal machinery used in the proof of the Main Theorem; it combines the characterization [5; Theorem 7.1] detecting that $X$ is a $Q$-manifold and the approximation result from [3] stating that cell-like maps between $Q$-manifolds are near homeomorphisms.

**Theorem 3.1** ([5] and [3]). Let $\pi : M^Q \to X$ be a cell-like map from a $Q$-manifold onto an ANR and suppose that the set $\{x \in X : \pi^{-1}(x) \neq \text{point}\}$ has infinite codimension in $X$ and that $X$ satisfies the Disjoint Disks Property. Then $\pi$ is near homeomorphism.

**Remark 3.2.** The Disjoint Disks Property is preserved under circumstances analogous to those described in Lemma 2.2. For our purposes, we only need the special case: if $\pi : M \to X$ is a resolution of an ANR $X$ satisfying the Disjoint Disks Property, $A \subset X$ is a closed subset, and $G(A)$ is the decomposition of $M$ that is the trivial extension of $\{\pi^{-1}(a) : a \in A\}$, then $M/G(A)$ satisfies the Disjoint Disks Property (see [7]).

4. Proof of Main Theorem

The strategy is similar to that employed by Edwards to show that a cell-like map from an $n$-manifold ($n \geq 5$) to an ANR satisfying the Disjoint Disks Property is a near homeomorphism. A resolution $\pi : M^Q \to X \times I^2$ will be improved successively by induction thereby producing a resolution $\hat{\pi} : M^Q \to X \times I^2$ such that the subset $\{x \in X \times I^2 : \pi^{-1}(x) \neq \text{point}\}$ has infinite codimension in $X \times I^2$ and, thus, permitting the use of a result from [5] included in the statement of Theorem 3.1 to conclude that $X \times I^2$ is a $Q$-manifold.

**Lemma 4.1.** An ANR $X$ satisfies property (DCC)$_n$ at a closed subset $A$ if and only if, for open pairs $(O_i, W_i)$ and homology elements $\tau_i \in H_{s(i)}(O_i, W_i)$, $1 \leq i \leq n - 1$, there are Čech carriers $(C_i, \partial C_i)$ for $\tau_i$ such that $(C_1 \cap \cdots \cap C_{n-1}) \cap A$ has infinite codimension.

**Proof.** One direction is evident since, having chosen $(C_1, \partial C_1), \ldots, (C_{n-1}, \partial C_{n-1})$ so that $(C_1 \cap \cdots \cap C_{n-1}) \cap A$ has infinite codimension, one can choose $(C_n, \partial C_n)$ so that $C_n \subset X \setminus (C_1 \cap \cdots \cap C_{n-1}) \cap A$. 
Conversely, the first step is to identify a countable subset that captures the singular homology structure of $X$. Specify a countable basis $\mathcal{U}$ of $X$, set $\mathcal{W} = \{\text{finite unions of elements from } \mathcal{U}\}$, and set

$$\Sigma = \{(U, V, z, q) : U, V \in \mathcal{W}; \ V \subset U; \ z \in H_q(U, V) \text{ for an integer } q \geq 0\}.$$ 

Let $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ be an ordering of the elements of $\Sigma$ where $\sigma_i = (U_i, V_i, z_i, q(i))$. The property of $\Sigma$ exploited in [5] and needed below is that a subset $A \subset X$ has infinite codimension provided each $z_i \in H_q(U_i, V_i)$ has a Čech carrier $(D_i, \partial D_i)$ with $D_i \subset X \setminus A$. (The critical observation is that, for an arbitrary element $z \in H_q(U, V)$ where $(U, V)$ is an open pair, there is a $\sigma_i$ with $q(i) = q$ such that every Čech carrier for $z_i$ is also a Čech carrier for $z$; see [5; Lemma 3.2].)

The Čech carriers for the $r_i$'s arise as infinite intersections, say $(C_i, \partial C_i) = (\bigcap N_{i,h} \bigcap \partial N_{i,j})$, of compact pairs chosen so that $(N_{i,h} \partial N_{i,j})$ is a Čech carrier for $r_i$ in $N_{i,j-1} \subset \operatorname{Int} N_{i,j}$ and $\partial N_{i,j} \circ N_{i,j-1} \subset \partial N_{i,j}$ and $[(N_{1,j} \cap \cdots \cap N_{n,j-1}) \cap D_j] \cap A = \emptyset$ for some Čech carrier $(D_j, \partial D_j)$ of the element $z_j$ determined by $\sigma_j \in \Sigma$. The pairs $(N_{i,h} \partial N_{i,j})$ are specified successively as neighborhoods of Čech carriers $(K_{i,h} \partial K_{i,j})$ for $r_i$ chosen, using the hypothesis, so that $[(K_{i,j} \cap \cdots \cap K_{n,j-1}) \cap D_j] \cap A = \emptyset$ for some Čech carrier $(D_j, \partial D_j)$ of $z_j$. The latter carriers detect that $(C_1 \cap \cdots \cap C_{n-1}) \cap A$ has infinite codimension.

**Proof of Main Theorem.** The proof of part (a) consists of establishing inductively a strong 'local' version of the theorem; the proof of part (b) is identical except that the $I^2$ factor is not needed, its sole role being to pick up the Disjoint Disks Property.

We begin by specifying a set $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ with $\sigma_i = (U_i, V_i, z_i, q(i))$ that captures the homological structure of $X$ exactly as in the proof of the preceding lemma.

**Inductive Hypothesis ($k$).** If $X$ satisfies property (DCC)$_k$ at a closed subset $A$, $\pi : M^Q \to X \times I^2$ is a resolution that is one to one over a closed subset $K \subset X \times I^2$, and $\epsilon > 0$, then there is a resolution $\pi^* : M \to X \times I^2$ that is $\epsilon$-close to $\pi$ and that is one to one over $K \cup (A \times I^2)$.

For $k = 1$, the hypothesis states that $A$ has infinite codimension in $X$ and, by virtue of Lemma 2.1, $A \times I^2$ has infinite codimension in $X \times I^2$. Denote by $G(A)$ the decomposition of $M$ that is the trivial extension of $\{\pi^{-1}(a) : a \in A \times I^2\}$ and by $\pi_A$ the quotient map $M \to M/G(A)$. The decomposition space $M/G(A)$ is an ANR (Remark 2.3) and satisfies the Disjoint Disks Property (Remark 3.2); furthermore, $\pi_A$ is one to one over the complement of a set, namely $\pi_A^{-1}(A \times I^2)$, that has infinite codimension (Lemma 2.2 and Remark 2.3). The combined result of [5] and [3], Theorem 3.1 in Section 3, states that $\pi_A$ is a near homeomorphism. The resolution $\pi^*$ is obtained by choosing a homeomorphism $h : M \to M/G(A)$ approximating $\pi_A$ and setting $\pi^* = \pi \pi_A^{-1} h$.

Assuming Inductive Hypothesis ($k - 1$), we proceed and establish Inductive hypothesis ($k$). Lemma 4.1 states that, for each $(k - 1)$-tuple $r = (\sigma_1, \ldots, \sigma_{k-1}) \in \Sigma^{k-1}$, there are Čech carriers $(C_i, \partial C_i)$ for $\sigma_i$ so that the subset $C_r = \bigcap_{\sigma_i} C_i$...
(C_1 \cap \cdots \cap C_k) \cup A$ has infinite codimension in $X$. An application of [5; Lemma 3.3] produces Čech carriers $(D_i, \partial D_i)$ for each $\sigma_i \in \Sigma$ such that $D_i \cap A \subset A \cup C_i$. Evidently, $X$ satisfies Property $(\text{DCC})_{k-1}$ at each of the subsets $D_i \cap A$. Inductive Hypothesis $(k-1)$ enables us to specify a sequence of cell-like maps $\pi_i : M \to X \times I^2$ such that $\pi_i$ is one to one over each $(D_i \cap A) \times I^2$. Denote by $\hat{G}(A)$ the decomposition of $M$ that is the trivial extension of $\{\hat{\pi}^{-1}(a) : a \in A \times I^2\}$ and by $\hat{\pi}_A$ the quotient map $M \to M/G(A)$. The final step is to establish that $\hat{\pi}_A$ is a near homeomorphism, for then the composition $\pi^* = \hat{\pi} \hat{\pi}_A^{-1} h$ is one to one over $K \cup (A \times I^2)$ where $h$ is a homeomorphism approximating $\hat{\pi}_A$. Since $M/\hat{G}(A)$ is an ANR (Remark 2.3) and satisfies the Disjoint Disks Property (Remark 3.3), the combined result of [5] and [3], Theorem 3.1 in Section 3, detects this feature of $G(A)$ provided the nondegeneracy set $\pi = \{y \in M/\hat{G}(A) : \pi_A^{-1}(y) \neq \text{point}\}$ has infinite codimension. Lemma 2.2 reduces this to verifying that the set

$$N = \hat{\pi} \hat{\pi}_A^{-1}(\hat{N}) = \{a \in A \times I^2 : \hat{\pi}^{-1}(a) \neq \text{point}\}$$

has infinite codimension in $X \times I^2$. Let $p : X \times I^2 \to X$ be the coordinate projection. Then $p(N)$ has infinite codimension in $X$ since $p(N) \subset A \setminus D_i$; consequently, Lemma 2.1 reveals that $p(N) \times I^2$ has infinite codimension in $X \times I^2$; and in turn, [5; Lemma 2.1] reveals that $N \subset p(N) \times I^2$ has infinite codimension in $X \times I^2$.

The Theorem follows from Inductive Hypothesis $(n)$ by starting with any resolution $\pi : M^O \to X \times I^2$ [11] (or [4]) and setting $A = X$.

5. Discussion

A condition was isolated in the Introduction that is discussed in [5; Section 10] and that appears to be the most natural candidate for a homological condition that might characterize $I^2$-factors of Hilbert cube manifolds; namely

$$(*) \quad H_i(X, X - \{x\}; Z) = 0 \quad \text{for each } x \in X.$$

The next lemma translates $(*)$ into a disjoint Čech carriers property.

**Lemma 5.1.** An ANR $X$ satisfies $(*)$ if and only if, for each open pair $(U, V)$ in $X$ and each homology element $z \in H_q(U, V)$, there is an integer $n$ and Čech carriers $(C_1, \partial C_1), \ldots, (C_n, \partial C_n)$ for $z$ with $C_1 \cap \cdots \cap C_n = \emptyset$.

**Proof.** Condition $(*)$ and excision assure that, for each $x \in X$, $z$ has a Čech carrier $(C_x, \partial C_x)$ with $C_x \subset X - \{x\}$. Consequently, $\bigcap_{x \in X} C_x = \emptyset$ and, since the $C_x$'s are compact, some finite intersection is empty. The converse follows since, if $C_1 \cap \cdots \cap C_n = \emptyset$, then some $C_i \subset X - \{x\}$ for each $x \in X$. 

The last result exposes some potential failings of condition (*). For this reason and since it is a natural extension of the properties analyzed in the body of the paper, we end by introducing:

Property (DCC). For each countable number of open pairs \((U_i, V_i)\) in \(X\) and homology elements \(z_i \in H_q(i)(U_i, V_i)\), there are Čech carriers \((C_i, \partial C_i)\) for \(z_i\) such that \(C_1 \cap C_2 \cap \cdots = \emptyset\).

Lemma 5.1 records that condition (*) is the requirement that the property hold for the special case that \(z_1 = z_2 = \cdots\).

References