SOE MINIMAL PAIRS OF 
$\alpha$-RECURSIVELY ENUMERABLE DEGREES

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§0. Introduction

Exploitation of the model theoretic properties of Gödel’s constructible sets led in [6] to a generalization of the Friedberg–Muchnik finite injury (or priority) method from $\omega$ to every $\Sigma_1$ admissible $\alpha$. In order to generalize, it was necessary to sacrifice the standard indexing of $\alpha$-recursively enumerable sets, and hence of the requirements associated with finite injury arguments. For some $\alpha$’s the indexing was demonstrably not $\alpha$-recursive. [3] gave an alternative view of [6] that centered on the nature of the indexing. This paper continues the study of indexing of requirements, and applies it to construct minimal pairs of $\alpha$-recursively enumerable sets for some, but not all, $\alpha$. The Friedberg–Muchnik solution of Post’s problem generalizes in a trivial fashion to every $\Sigma_2$ admissible ordinal. All the complications of [3] and [6] resulted from forcing a $\Sigma_1$ admissible ordinal $\alpha$ to do the work of a $\Sigma_2$ admissible ordinal. In this paper $\alpha$ is forced to do a much larger share of that work, and even

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some of the work of a $\Sigma_3$ admissible, since the Lachlan-Yates minimal pair construction lifts easily to every $\Sigma_3$ admissible.

From now on $\alpha$ is invariably a $\Sigma_1$ admissible ordinal. $A$ and $B$ form a minimal pair of subsets of $\alpha$ if neither is $\alpha$-recursive, and if every $C$ (a subset of $\alpha$) $\alpha$-recursive in each is $\alpha$-recursive. $a$ and $b$ form a minimal pair of $\alpha$-degrees if neither is $\mathbf{0}$, and if $c \leq a$ and $c \leq b$ imply $c = \mathbf{0}$.

Lachlan [2] and Yates [8] constructed a minimal pair of recursively enumerable sets, and Sukonick [7] lifted their construction to meta-recursion theory ($\alpha = \text{least nonrecursive ordinal } \omega^\alpha_{\text{CK}}$). Sukonick used an effective $\omega$-ordering of requirements, and consequently had no need of the $\alpha$-finite injury method. He did however introduce one new twist. His $A$ and $B$ were hyperregular by design. Thus for each $e$ and each metafinite $K$, if $\{e\}^A$ was total on $K$, then $\{e\}^A$ restricted to $K$ was a metafinite partial function. He needed the hyperregularity to lift some of the convergence lemmas from $\omega$ to $\omega^\alpha_{\text{CK}}$. Something like Sukonick's twist will be needed in our argument as well. It was not essential to the solution of Post's problem [6].

Section 1 contains a review of elementary definitions, that of $\alpha$-cardinal being typical. Section 2 is devoted to projecta, the means of indexing requirements in priority arguments, and in particular to the notion of tame $\Sigma_2$ projectum invented by Lerman [3]. Section 3 introduces refractory $\Sigma_1$ admissible $\alpha$'s, and constructs minimal pairs of $\alpha$-recursively enumerable sets for all nonrefractory $\alpha$'s. Section 4 discusses further results and open questions.
§ 1. Preliminaries

The following concepts are defined in [6]: \( \Sigma_1 \) admissible ordinal, partial \( \alpha \)-recursive function, \( \alpha \)-recursively enumerable set, \( \alpha \)-recursive set, bounded (below \( \alpha \)) set, \( \alpha \)-finite set, regular set, hyperregular set, \( \leq_{wa} \) (weakly \( \alpha \)-recursive in), \( \leq_\alpha \) (\( \alpha \)-recursive in), \( \equiv_\alpha \) (\( \leq_\alpha \) and \( \geq_\alpha \)), \( \alpha \)-degree, \( \alpha \)-recursively enumerable degree.

If \( A \) is a bounded subset of \( \alpha \), then \( \text{lub} \, A \) is the least \( \gamma < \alpha \) such that for all \( \delta \in A \), \( \delta < \alpha \). \( A \oplus B \) is \( \{2x \mid x \in A \} \cup \{2x + 1 \mid x \in B \} \). If \( a \) is the \( \alpha \)-degree of \( A \) and \( b \) is the \( \alpha \)-degree of \( B \), then \( a \oplus b \) is the \( \alpha \)-degree of \( A \oplus B \). \( A \otimes i \) is the \( \alpha \)-degree of \( A \otimes i \).

\( \Upsilon_i \mid i < \alpha \) is the sequence of all partial \( \alpha \)-recursive functionals. There exists an \( \alpha \)-recursively enumerable sequence of \( \alpha \)-finite partial functionals \( \Upsilon_i \mid i < \alpha \) such that for all \( i \) and all regular \( A \),

\[
\Psi_i(A) = \lim_{\gamma \to \alpha} \Psi^\gamma_i(A). \tag{1}
\]

Similarly, \( \langle \Phi_i, \theta \mid i < \alpha \rangle \) is the sequence of all pairs of partial \( \alpha \)-recursive functionals and is the limit of the \( \alpha \)-recursively enumerable sequence \( \langle \Phi_i, \theta \mid i < \alpha \rangle \).

\( L \) is Gödel's class of constructible sets, and \( L_\gamma \) is the set of all sets constructible via ordinals less than \( \gamma \).

\( \gamma \) is an \( \alpha \)-cardinal if \( \gamma < \alpha \) and there is no one-one \( \alpha \)-finite map of \( \gamma \) onto some lesser ordinal. \( \gamma \) is a regular \( \alpha \)-cardinal if \( \gamma \) is not of the form \( \bigcup K_\beta \mid \beta \in I \) where \( K_\beta \) is the \( \alpha \)-finite set of canonical index \( \beta \) and \( I \) is an \( \alpha \)-finite set of \( \alpha \)-cardinality less than \( \gamma \). \( \gamma \) is a singular \( \alpha \)-cardinal if it is not regular. If \( K \) is \( \alpha \)-finite, then \( \alpha \)-card \( K \) is the least \( \alpha \)-cardinal \( \gamma \) such that there is a one-one \( \alpha \)-finite correspondence between \( K \) and \( \gamma \). \( \text{gca} \) is the greatest \( \alpha \)-cardinal if there is one, and \( \alpha \) otherwise.

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1 The regularity of \( \alpha \) insures the consistency (or single-valuedness) of \( \Psi_i \) applied to \( A \). If \( A \) is regular, then computations based on \( \alpha \)-finite sets of membership facts about \( A \) can be replaced by computations based on \( \alpha \)-finite initial segments of the characteristic function of \( A \). And the consistency can be achieved, as in ordinary recursion theory, by preferring shorter to longer initial segments.
A relation \( R(x_1, \ldots, x_n) \) is \( \Sigma_2 \) over \( L_\alpha \) if there is an \( \alpha \)-recursive relation \( S(y, z, x_1, \ldots, x_n) \) such that \( R(x_1, \ldots, x_n) \leftrightarrow (E_{\forall})(z)S(y, z, x_1, \ldots, x_n) \).

\( f \) is a \( \Sigma_1 \) projection if \( f \) is a one-one \( \alpha \)-recursive map of \( \alpha \) into \( \alpha \). \( \alpha^* \), the \( \Sigma_1 \) projectum of \( \alpha \), is the least \( \beta \) for which there is a \( \Sigma_1 \) projection of \( \alpha \) into \( \beta \). (\( f \) need not be onto.)

\( g \) is a \( \Sigma_2 \) projection of \( \alpha \) into \( \beta \) if \( g \) is one-one and the graph of \( g \) is \( \Sigma_2 \) over \( L_\alpha \). The \( \Sigma_2 \)-ness of \( g \) is equivalent to the existence of an \( \alpha \)-recursive \( g' \) with the following property:

\[
(1.1) \quad gx = y \leftrightarrow (E_{\forall})[\tau \geq \sigma \rightarrow g'(\tau, x) = gx] ;
\]

i.e. \( gx = \lim_{\tau \to \alpha} g'(\tau, x) \). \( g \) is said to be tame \( \Sigma_2 \) if there exists a one-one \( \alpha \)-recursive \( g' \) such that:

\[
(1.2) \quad (z)_{z < \beta} (E_{\forall})(y)(\tau)[\tau \geq \sigma \& gx \leq z \rightarrow g'(\tau, x) = gx] ;
\]

\[
(1.3) \quad (z)_{z < \beta} (E_{\forall})(y)(\tau)[\tau \geq \sigma \& gx > z \rightarrow g'(\tau, x) > z] .
\]

\( g \) is said to be strong \( \Sigma_2 \) if \( g \) is \( \Sigma_2 \) and there exists a one-one \( \alpha \)-recursive \( g' \) such that:

\[
(1.4) \quad (z)(E_{\forall})(w)_{w \leq \tau}(\tau)[\tau \geq \sigma \rightarrow g'(\tau, w) = gw] ;
\]

i.e. \( g' \) defines \( g \) correctly on any proper initial segment of the domain of \( g \) for all sufficiently large \( \tau \).

Define \( p2\alpha \) (\( tp2\alpha \), \( sp2\alpha \)), the \( \Sigma_2 \) projectum of \( \alpha \) (tame \( \Sigma_2 \) projectum of \( \alpha \), strong \( \Sigma_2 \) projectum of \( \alpha \)), to be the least \( \beta \) such that there is a \( \Sigma_2 \) projection (tame \( \Sigma_2 \) projection, strong \( \Sigma_2 \) projection) of \( \alpha \) into \( \beta \).

Suppose \( \gamma \leq \alpha \), \( g : \gamma \to \alpha \) is a \( \Sigma_2 \) cofinality function if \( g \) is \( \Sigma_2 \) and its range is unbounded. Define \( \text{cf2}\alpha \), the \( \Sigma_2 \) cofinality of \( \alpha \), to be the least \( \gamma \leq \alpha \) for which there is a \( \Sigma_2 \) cofinality function \( g : \gamma \to \alpha \). Clearly \( \text{cf2}\alpha = \alpha \) if and only if \( L_\alpha \) satisfies \( \Sigma_2 \) replacement.

**Proposition 1.1.** Every \( \Sigma_1 \) projection is tame \( \Sigma_2 \).

1.1 is a triviality but it does introduce an important point. If \( \alpha^* < \alpha \),
then the $\alpha$-finite injury method [6, section 4] is based on an $\alpha$-recursive indexing of requirements of length $\alpha^*$. If $\alpha^* = \alpha$ it is often necessary to use a tame $\Sigma_2$ indexing of requirements of length $tp2\alpha$. The virtue of tameness resides in: for each $\beta < tp2\alpha$ there is a $\gamma < \alpha$ and a $\Sigma_1$ projection of $\gamma$ into $\beta$. In short, tame $\Sigma_2$ projections can be approximated on proper initial segments of their ranges by $\Sigma_1$ projections, and consequently are suited to $\alpha$-finite injury arguments. 2

2 Similarly a tame $\Sigma_{n+1}$ projection can be approximated on any proper initial segment of its range by a $\Sigma_n$ projection.
§2. Tame $\Sigma_2$ projections

The lemmas of this section are needed to compute bounds on ordinals that crop up in the priority arguments of section 3, and (hopefully) in future priority developments. The first lemma says the tame $\Sigma_2$ projection could have been defined in terms of one-one onto maps.

**Lemma 2.1.** If $\beta = \text{tp}2\alpha$, then there exists a tame $\Sigma_2$ projection of $\alpha$ onto $\beta$.

**Proof.** Let $g : \alpha \to \beta$ be a tame $\Sigma_2$ projection, and let $\beta'$ be the ordertype of the range of $g$. For each $x$ in the range of $g$ define

$$h_x = \text{lub} \{ h_w \mid w < x \& w \in \text{range } g \} .$$

Thus $h$ is a one-one orderpreserving map of range $g$ onto $\beta'$. We will show $f = hg$ is a tame $\Sigma_2$ projection of $\alpha$ onto $\beta'$. Since $\beta' \leq \beta = \text{tp}2\alpha$, it will then follow $\beta' = \beta$.

Suppose $g'$ is an $\alpha$-recursive function such that $g$ and $g'$ satisfy (1.2) and (1.3). Let $r_\sigma = \{ g'(\sigma, x) \mid x < \sigma \}$. For each $y \in r_\sigma$ define

$$h'(\sigma, y) = \text{lub} \{ h'(\sigma, w) \mid w < y \& w \in r_\sigma \} .$$

If $y \notin r_\sigma$ let $h'(\sigma, y) = \sigma$. Define

$$f'(\sigma, x) = h'(\sigma, g'(\sigma, x)) .$$

Clearly $f'(\sigma, x)$ is $\alpha$-recursive. Fix $z < \beta'$. Choose $\sigma_0$ so that $\sigma_0 > \text{lub} \{ x \mid gx < h^{-1}z \}$ and

$$(x)(\sigma)[\sigma \geq \sigma_0 \& gx < h^{-1}z \to g'(\sigma, x) = gx] .$$

$$(x)(\sigma)[\sigma \geq \sigma_0 \& gx \geq h^{-1}z \to g'(\sigma, x) \geq h^{-1}z] .$$

Thus $r_\sigma \cap h^{-1}z = \text{range } g \cap h^{-1}z$ for all $\sigma \geq \sigma_0$. Consequently $h'(\sigma, x) = h_x$ for all $\sigma \geq \sigma_0$ and $x < h^{-1}z$. And so
(x)(σ)[σ ≥ σ₀ & fx < z → f' (σ, x) = fx] .

(x)(σ)[σ ≥ σ₀ & fx ≥ z → f' (σ, x) ≥ z] .

The tameness of f is now immediate if β' is a limit ordinal; if β' were a successor, there would be a tame Σ₂ projection of α into the greatest limit ordinal less than β, that projection differing only finitely from g.

The next lemma relates the Σ₂ cofinality of α to the tame Σ₂ projectum of α, and is the principal source of tame Σ₂ projections.

Lemma 2.2. Suppose \{ T_ξ | ξ < λ \} is a sequence of simultaneously α-recursively enumerable sets whose union is α. Let κ be an α-cardinal such that for each ξ < λ, T_ξ is α-finite and α-card T_ξ ≤ κ. Then there exists a tame Σ₂ projection of α into κ · λ₀ for some λ₀ ≤ λ.

Proof. Let λ₀ be the least z ≤ λ such that U \{ T_ξ | ξ < z \} is not α-finite. Then

\[ V_ξ = T_ξ \setminus U \{ T_ρ | ρ < ξ \} \]

is α-finite for all ξ < λ₀. Clearly \{ V_ξ | ξ < λ₀ \} is a partition of a non-α-finite, α-recursively enumerable set; assume it is a partition of α. Let

\[ K_ξ : V_ξ → κ \]

be an α-finite one-one map of least possible canonical index. Define

\[ g : α → κ · λ₀ \]

by

\[ gβ = \langle K_ξ (β), ξ \rangle \]

when β ∈ V_ξ. Let T_ξ^σ be the subset of T_ξ enumerated at stage σ of the simultaneous α-recursive enumeration of the T_ξ 's. Define

\[ V_ξ^σ = T_ξ^σ \setminus U \{ T_ρ^σ | ρ < ξ \} . \]

Let K_ξ^σ : V_ξ^σ → κ be an α-finite one-one map of least possible canonical index. Define
\[ g'(\sigma, \beta) = (K^\omega_{\xi}(\beta), \xi) \]

when \( \beta \in V_q^\omega \), and \( = \text{ lub } \{ g'(\sigma, \beta) | \beta \in U \{ V_q^\omega | \xi < \lambda_0 \} \} \) otherwise.

To check the tameness of \( g \), fix \( z < \lambda_0 \). Since \( U \{ T_\xi | \xi < z \} \) is \( \sigma \)-finite, there is a \( \sigma \) such that \( T_\tau^\sigma = T_\xi \) for all \( \tau \geq \sigma \) and \( \xi \leq z \). But then \( g'(\sigma, \beta) = g\beta \) for all \( \tau \geq \sigma \) and \( \beta \in U \{ V_\xi | \xi \leq z \} \). Consequently

\[ \tau \geq \sigma & \implies g_x \leq \aleph \cdot z \implies g'(\tau, x) = g_x, \]

\[ \tau \geq \sigma & \implies g_x > \aleph \cdot z \implies g'(\tau, x) > \aleph \cdot z. \]

Recall that \( gca \) denotes the greatest \( \alpha \)-cardinal if there is one, and \( \alpha \) otherwise. The next theorem is the most useful inequality relating the tame \( \Sigma_2 \) projectum of \( \alpha \) and the \( \Sigma_2 \) cofinality of \( \alpha \).

**Theorem 2.3.** \( \text{cf}2\alpha \leq \text{tp}2\alpha \leq gca \cdot \text{cf}2\alpha. \)

**Proof.** By 2.1 there exists a one-one \( \Sigma_2 \) map \( f \) of \( \alpha \) onto \( \text{tp}2\alpha \). But then \( f^{-1} \) is a \( \Sigma_2 \) map from \( \text{tp}2\alpha \) onto \( \alpha \) and so \( \text{cf}2\alpha \leq \text{tp}2\alpha \).

Let \( \lambda = \text{cf}2\alpha \) and assume \( gca < \alpha \). Thus there is a \( \Sigma_2 \) \( h : \lambda \rightarrow \alpha \) with unbounded range. Let \( R(u, v, \xi, y) \) be an \( \alpha \)-recursive relation such that

\[ h_\xi = y \iff (\mathcal{E}u)(v)R(u, v, \xi, y) \]

for all \( \xi < \lambda \) and all \( y \). Let \( i : \alpha \rightarrow \alpha \times \alpha \) be an \( \alpha \)-recursive onto map:

\[ i\beta = (u\beta, y\beta). \]

Define \( T_\xi \) to be the set of all \( \beta \) such that

\[ (\delta)_{\delta \leq \beta}(\mathcal{E}v) \sim R(u\delta, v, \xi, y\delta). \]

\( T_\xi \) is \( \alpha \)-finite since it is an initial segment of \( \alpha \) that omits \( \beta \) when \( h_\xi = y\beta \) and \( (v)R(u\beta, v, \xi, y\beta) \). \( U \{ T_\xi | \xi < \lambda \} = \alpha \) because the range of \( h \) is unbounded and \( i \) is \( \alpha \)-recursive. By 2.2 there is a tame \( \Sigma_2 \) projection of \( \alpha \) into \( gca \cdot \lambda \).

**Lemma 2.4.** \( \text{sp}2\alpha \geq gca. \)

**Proof.** Suppose \( \gamma = \text{sp}2\alpha < gca \). Then there must be an \( \alpha \)-cardinal \( \beta \)
such that $\gamma < \beta$. Let $g$ be a strong $\Sigma_2$ projection of $\alpha$ into $\gamma$, and let $g'$ be an $\alpha$-recursive function such that $g$ and $g'$ satisfy (1.4). Choose $\sigma$ so that

$$(w)_{w < \beta} \tau \geq \sigma \rightarrow g'(\tau, w) = gw.$$

Then $g$ restricted to $\beta$ is an $\alpha$-finite one-one map of $\beta$ into $\gamma < \beta$.

The next theorem, which was also proved independently by S. Simpson, will be used to describe those $\alpha$'s for which the existence of a minimal pair of $\alpha$-recursively enumerable $\alpha$-degrees is as yet unknown.

**Theorem 2.5.** Suppose $p2\alpha = gc\alpha < tp2\alpha \leq \alpha$. Then $tp2\alpha = gc\alpha \cdot cf2\alpha$.

**Proof.** Assume $tp2\alpha < \alpha$; then the tame $\Sigma_2$ projection is expressible as $\mathbb{N} \cdot \lambda + \gamma$, where $\mathbb{N}$ is the greatest $\alpha$-cardinal and $\gamma < \mathbb{N}$. Suppose $\gamma > 0$. By 2.1 there exists a one-one tame $\Sigma_2$ map $f$ of $\alpha$ onto $\mathbb{N} \cdot \lambda + \gamma$. Since $f$ is tame, $f^{-1} [\mathbb{N} \cdot \lambda]$ is $\alpha$-finite, and consequently $f$ maps the complement of an $\alpha$-finite set one-one into $(\mathbb{N} \cdot \lambda + \gamma) - (\mathbb{N} \cdot \lambda)$. It follows that $p2\alpha < gc\alpha$. Hence $\gamma = 0$.

By 2.3, $\lambda \leq cf2\alpha$. Let $g$ be a one-one tame $\Sigma_2$ map of $\alpha$ onto $\mathbb{N} \cdot \lambda$. For each $\delta < \lambda$, define

$$h\delta = \sup \{ g^{-1} \beta | \mathbb{N} \cdot \delta \leq \beta < \mathbb{N} \cdot (\delta + 1) \}.$$

The tameness of $g$ implies $h\delta < \alpha$ for all $\delta < \lambda$. Clearly $h : \lambda \rightarrow \alpha$ is unbounded. In addition $h$ is $\Sigma_2$, since $g$ is $\Delta_2$. Hence $cf2\alpha \leq \lambda$. 

\section{Tame $\Sigma_2$ projections}
§3. Existence of minimal pairs

\( \alpha \) is said to be refractory if \( p^2\alpha = gca < tp^2\alpha \leq \alpha \). If \( \alpha \) is refractory, many theorems about recursively enumerable sets fail to lift readily to \( \alpha \), and in particular those theorems whose proofs permit requirements to be injured infinitely often. Theorem 2.5 pins down \( tp^2\alpha \) when \( \alpha \) is refractory, but gives no hint of how to perform nontrivial priority arguments.

**Theorem 3.1.** If \( \alpha \) is not refractory, then there exists a minimal pair of \( \alpha \)-recursively enumerable \( \alpha \)-degrees.

**Proof.** \( \alpha \)-recursively enumerable sets \( A \) and \( B \) are to be constructed so that neither is \( \alpha \)-recursive, and so that \( C \) is \( \alpha \)-recursive whenever \( C \) is \( \alpha \)-recursive in \( A \) and in \( B \). \( A^\alpha(B^\alpha) \) will be the \( \alpha \)-finite set of ordinals put in \( A \) (\( B \)) prior to stage \( \sigma \) of the construction. Let \( p_0 \) be a one-one tame \( \Sigma_2 \) map of \( \alpha \) onto \( tp^2\alpha \), and let \( p'_0 : \alpha \times \alpha \rightarrow tp^2\alpha \) be a one-one \( \alpha \)-recursive function such that for each \( z < tp^2\alpha \):

\[
(E\sigma)(x)(\tau)[\tau \geq \sigma \& p^*_0 \leq z \rightarrow p^*_0(\tau, x) = p^*_0 x]
\]

\[
(E\sigma)(x)(\tau)[\tau \geq \sigma \& p^*_0 > z \rightarrow p^*_0(\tau, x) > z].
\]

\( p_0 \) will define priorities for the positive requirements, which insure that neither \( A \) nor \( B \) are \( \alpha \)-recursive. Let \( p_1 \) be a one-one \( \Sigma_2 \) map of \( \alpha \) into \( p^2\alpha \), and let \( p'_1 : \alpha \times \alpha \rightarrow p^2\alpha \) be a one-one \( \alpha \)-recursive function such that for all \( x \) and \( y \):

\[
p_1 x = y \iff (E\sigma)(\tau)[\tau \geq \sigma \rightarrow p'_1(\tau, x) = p_1 x].
\]

\( p_1 \) will define priorities for the negative requirements, which insure that the only sets \( \alpha \)-recursive in both \( A \) and \( B \) are the \( \alpha \)-recursive sets.

The positive requirements are \( \{ \Psi_i \neq A \mid i < \alpha \} \cup \{ \Psi_i \neq B \mid i < \alpha \} \) and (after being interlaced) are denoted by \( \{ R_i \mid i < \alpha \} \). \( R_i \) has higher priority than \( R_j \) if \( p^*_0 i < p^*_0 j \). \( R_i \) has higher priority than \( R_j \) at stage \( \sigma \) if

\[
p^*_0(\sigma, i) < p^*_0(\sigma, j).
\]

Followers are appointed for the sake of \( R_e \) at certain stages; they are
subject to cancellation at later stages. At every stage a follower is either realized or unrealized and each $R_e$ has at most one unrealized follower. $p$ follows $R_e$ if $p$ is appointed to follow $R_e$ and is never cancelled. $p$ follows $R_e$ at stage $\sigma$ if $p$ was appointed prior to stage $\sigma$ and was not cancelled prior to stage $\sigma$. $p$ has higher rank than $q$ (at stage $\sigma$) if $p$ follows $R_i$ (at stage $\sigma$), $q$ follows $R_j$ (at stage $\sigma$), and $R_i$ has higher priority than $R_j$ (at stage $\sigma$). $p$ has higher order than $q$ (at stage $\sigma$) if $p$ and $q$ both follow $R_i$ (at stage $\sigma$) and $p$ was appointed before $q$ was.

$R_i$ is persistent at stage $\sigma$ if there is a $\lambda < \sigma$ such that $(\tau)[\lambda \leq \tau \leq \sigma \rightarrow p'_0(\tau, i) = p'_0(\sigma, i)]$.

Suppose $R_e$ is $\Psi_i \neq A$. $p$ satisfies $R_e$ at stage $\sigma$ if $p$ follows $R_e$ at stage $\sigma$, $\Psi_i^\sigma(p)$ is defined, $\Psi_i^\sigma(p) \neq A^\sigma(p)$, and either $A^\sigma(p) = 0$ and $p$ was realized at stage $\sigma$, or $A^\sigma(p) = 1$ and $p \notin \bigcup \{A_q \mid \gamma < \sigma\}$. $R_e$ is satisfied at stage $\sigma$ if there is a $p$ such that $p$ satisfies $R_e$ at stage $\sigma$. $R_e$ is satisfied (before stage $\sigma$) if there is a $\tau (\tau < \sigma)$ such that $R_e$ is satisfied at stage $\tau$. Similar definitions are made if $R_e$ is $\Psi_i \neq B$.

Two auxiliary functions are needed, $L$ and $M$. $L(\sigma, e)$ is the least $x < \sigma$ such that either $\Phi_e^\sigma(A^\sigma, x)$ is undefined, or $\theta_e^\sigma(B^\sigma, x)$ is undefined, or $\Phi_e^\sigma(A^\sigma, x)$ and $\theta_e^\sigma(B^\sigma, x)$ are defined and unequal; if no such $x$ exists, $L(\sigma, e) = \sigma$. $M(\sigma, e)$ is the least $x$ such that $L(\tau, e) \leq x$ for all $\tau \leq \sigma$.

The negative requirements are $\{\Psi_i(A) = \emptyset \mid i < \sigma\}$. They are denoted by $\{Q_i \mid i < \sigma\}$. $Q_i$ has higher priority than $R_i$ if $p_i < p_i$. $Q_i$ has higher priority than $R_j$ at stage $\sigma$ if $p_i(\sigma, i) < p_i(\sigma, j)$.

$Q_i$ is persistent at stage $\sigma$ if there is a $\lambda < \sigma$ such that $(\tau)[\lambda \leq \tau \leq \sigma \rightarrow p'_i(\tau, i) = p'_i(\sigma, i)]$.

A follower $p$ is associated with $Q_i$ (at stage $\sigma$) if there is a stage $\tau (\tau < \sigma)$ such that $p$ is associated with $Q_i$ at stage $\tau$ of the construction and the association is not cancelled at any stage subsequent to $\tau$ (and prior to $\sigma$).

Suppose $Q_i$ is $\Phi_i(A) = \emptyset$. $\sigma$ satisfies $Q_i$ if $L(\sigma, i) = M(\sigma, i)$ or $Q_i$ is not persistent at stage $\sigma$.

Let $R_e$ be $\Psi_i \neq A$ or $\Psi_i \neq B$. $R_e$ requires attention through $p$ at stage $\sigma$ if $p$ follows $R_e$ at stage $\sigma$, $R_e$ is not satisfied prior to stage $\sigma$, $e \leq \sigma$ and at least one of the next three clauses holds.

(3.1) $p$ is a realized follower of $R_e$ at stage $\sigma$, and $p$ is not associated with any $Q_j$ at stage $\sigma$;
(3.2) $p$ is a realized follower of $R_e$ at stage $\sigma$, and $p$ is associated with some $Q_j$ at stage $\sigma$ and $\sigma$ satisfies $Q_j$;

(3.3) $p$ is an unrealized follower of $R_e$ at stage $\sigma$ and $\Psi_\sigma^p(p)$ is defined.

$R_e$ requires attention at stage $\sigma$ if for some $p$, $R_e$ requires attention via $p$ at stage $\sigma$; or if $e \leq \sigma$, $R_e$ is not satisfied prior to stage $\sigma$, and

(3.4) $R_e$ has no unrealized follower at stage $\sigma$.

A review of the minimal pair construction for ordinary recursively enumerable sets will speed comprehension of the proof of Theorem 3.1. Thus the requirements are $\{R_i\}_{i < \omega}$ and $\{Q_i\}_{i < \omega}$. Suppose $R_j$ is $\Psi_i \neq A$. In order to satisfy $R_j$ a follower $p$ of $R_j$ is sought such that $\Psi_j(p)$ is defined; suppose such a $p$ is put in $A$ if and only if $\Psi_j(p) = 0$. Then $A$ is not recursive via Gödel number $i$.

The $Q_i$'s oppose the deposit of followers in $A$ and $B$. If $\Phi_i^r(A^r, x) = \theta_i^r(B^r, x) = q$, then for the sake of $Q_i$, it is preferable to add members to $A$ or $B$ at stage $r \geq \sigma$ only if $\Phi_i^r(A^r, x) = \theta_i^r(B^r, x) = q$. Honoring the preference results in $\Phi_i(A, x) = \theta_i(B, x) = q$, and ultimately in $\Phi_i(A)$ and $\theta_i(B)$ being recursive. The preference must occasionally be ignored in order to satisfy $R_j$ but not too often if $Q_i$ is to be satisfied. Thus followers of $R_j$ are associated only with $Q_i$'s of higher priority than $R_j$, and at most one follower of $R_j$ is associated with any particular $Q_i$ at any stage. Followers of $R_k$ are cancelled at stage $s$ only if $R_k$ or some $R_j$ of higher priority than $R_k$ receives attention at stage $s$. Cancellation of associations of followers with negative requirements is also allowed. After such a cancellation the follower can be associated only with negative requirements of higher priority than those it was formerly associated with.

It then can be shown that each $R_e$ receives attention only finitely often. Fix $e$ and suppose $R_i$ fails to receive attention after stage $s$ for any $i < e$. The follower of $R_e$ of highest order after stage $s$ remains forever unrealized or is put in $A$ or $B$ (in either event $R_e$ never again requires attention), or is associated with $Q_i$ for some $i < e$ for all but finitely many stages. Once the follower in question is associated with some $Q_i$, a new unrealized follower is appointed and never cancelled. After finitely many such appointments, a follower $p$ of $R_e$ is developed such that $p$ is never realized or $p$ is placed in $A$ or $B$. In either case $R_e$ is met. (The
nonrealization of \( p \) means that \( \Psi_e \) is not total.) Hence \( R_e \) receives attention only finitely often.

Now suppose that \( \Phi_i(A) \neq \theta_i(B) \) and that \( Q_i \) is \( \Phi_i(A) = \theta_i(B) \). Go to a stage after which no requirement of higher priority than \( Q_i \) receives attention. Then all followers appointed from now on are subject to association with \( Q_i \). There are of course only finitely many negative requirements of higher priority than \( Q_i \); some will always have their associations with followers cancelled, and some will not. For the latter there is a stage after which no association with a follower is cancelled. \( \Phi_i(A, x) \) can be computed effectively as follows. Go to a stage \( s \) such that

\[
\Phi_i^s(A^s, y) = \Psi_i^s(B^s, y)
\]

for all \( y \leq x \), and such that no followers associated with negative requirements of higher priority than \( Q_i \) can interfere with the computation of the above equation for any \( y \leq x \). The computation is protected at all subsequent stages in the sense that for all \( t \geq s \), either \( \Phi_i^t(A^t, y) = \Phi_i^s(A^s, y) \) or \( \Psi_i^t(B^t, y) = \Psi_i^s(B^s, y) \). Hence \( \Phi_i(A, x) \) must equal \( \Phi_i^s(A^s, x) \). The protection leads to cancellation of certain followers, and the cancellation of the association of \( p \) with \( Q_i \) at stage \( t \) only if \( L(i, t) = M(i, t) \), i.e. both sides of the requirement are equal on at least as long an initial segment as at the previous stage.

The problems encountered in lifting the minimal pair construction from \( \omega \) to \( \alpha \) have two sources: certain details peculiar to the construction; and the somewhat more general priority method used, to be termed the finite injury, infinite preservation method. The details of the construction rely on the following equality,

\[
\Phi_i(A) = \lim_{\sigma \to \alpha} \Phi_i^\sigma(A^\sigma),
\]

which can fail if \( A \) is not regular. \(^1\) So some further details, routine in

\(^1\) The regularity of \( A \) insures the consistency (or single-valuedness) of \( \Psi_I \) applied to \( A \). If \( A \) is regular, then computations based on \( \alpha \)-finite sets of membership facts about \( A \) can be replaced by computations based on \( \alpha \)-finite initial segments of the characteristic function of \( A \). And the consistency can be achieved, as in ordinary recursion theory, by preferring shorter to longer initial segments.
nature, will be added to insure the regularity of $A$ and $B$. (Sukonick [7], faced with the same difficulty for $\alpha = \omega^\omega$, made $A$ and $B$ hyperregular.) Another peculiar detail is made more complicated by the presence of limit ordinals less than $\alpha$. Suppose $\lambda < \alpha$ is a limit ordinal and $\Phi^\beta_\lambda(A^\beta, x) = q$ or $\theta^\beta_\lambda(A^\beta, x) = q$ for all $\beta < \lambda$. Then under certain conditions it will be necessary to have $\Phi^\lambda_\lambda(A^\lambda, x) = q$ or $\theta^\lambda_\lambda(A^\lambda, x) = q$, and this will be accomplished by permitting only finitely many changes of heart in deciding which of the two computations to protect.

The problems arising from the priority method itself are more severe than the two above. The most immediate problem is a consequence of the fact that followers of $R_e$ are subject to association with negative requirements in order of increasing priority; i.e. followers associated with $Q_i$ precede followers associated with $Q_j$ if $Q_j$ has higher priority than $Q_i$. Thus if the priority of $R_e$ is infinite, then the ordering of followers of $R_e$ is not a wellordering. If the ordering were reversed, it would become a wellordering, but the information needed to compute $\Phi_i(A) = \theta_i(B)$ recursively would be lost. When $\alpha = \omega$ the needed information is finite; when $\alpha > \omega$ it is bounded but not always $\alpha$-finite if the ordering is not reversed. The compromise adopted below consists of reversing the ordering and repeating the process of associating followers with negative requirements $\omega$ times. The compromise works for two reasons: only finitely many changes of heart are permitted in deciding which side of a computation to preserve; each follower can be associated with a fixed negative requirement at all stages in a sequence cofinal with the stage at which the follower is put in $A$ or $B$.

The most severe problem of all arises from the assignment of priorities. Recall the role of the priorities. First it was argued that if $s$ is a stage after which no requirement of priority higher than that of $R_e$ receives attention, then $R_e$ receives attention at only finitely many stages after stage $s$. Then it was argued that if $s$ is a stage after which no requirement of priority greater than that of $Q_e$ receives attention, then $\Phi_e(A) = \theta_e(B)$ can be computed from the finite state of affairs at stage $s$. The first argument can be lifted to $\alpha$ by weakening the process of cancelling followers, thereby obviating the need for all requirements of higher priority than $R_e$ to cease receiving attention at stage $s$. The second argument is less amenable; lifting it seems to require that each proper initial segment of the priority ordering of $\{R_i | i < \alpha\}$ be correct from some stage onward.
Consequently the priorities for the $R_i$'s are generated by a tame $\Sigma_2$ projection. Curiously a $\Sigma_2$ projection suffices for the priorities of the $Q_i$'s, because it is enough for each $Q_i$ to attain its correct priority from some stage onward.

The assignment of priorities guarantees that each $R_e$ receives attention only $\alpha$-finitely often if $\alpha$ is not refractory. A preliminary indication of the reasoning behind the last assertion will prove helpful. If $\alpha > \omega$, then it is possible for $R_e$ to receive attention infinitely often after all positive requirements of higher priority than $R_e$ have ceased to receive attention. That infinite set must be $\alpha$-finite if the construction is to succeed. The $\alpha$-cardinality approach of [3] (or the $\Sigma_1$ substructure approach of [6]) seems to work only if there is an $\alpha$-cardinal $\gamma$ such that $R_e$ receives attention less than $\gamma$ times after some stage. Such a $\gamma$ can be found when there is no greatest $\alpha$-cardinal, or when the tame $\Sigma_2$ projectum of $\alpha$ does not exceed the greatest $\alpha$-cardinal (if it exists), or when the $\Sigma_2$ projectum of $\alpha$ is less than some $\alpha$-cardinal. Such a $\gamma$ is not needed when $\alpha$ equals the $\Sigma_2$ projectum of $\alpha$. Suppose the worst: there is a set $S$ of stages cofinal with $\alpha$ and a proper initial segment $I$ of requirements such that some member of $I$ requires attention at every stage of $S$, but such that each member of $I$ receives attention only boundedly often. Then $\alpha > \text{tp}2\alpha$. In addition the association of followers of a given $R_e$ with negative requirements yields $\alpha > \text{sp}2\alpha$. Thus all is well when $\alpha = p2\alpha$. If $\alpha$ is refractory, then the desired $\gamma$ does not exist and $\alpha > \text{sp}2\alpha$.

The construction of $A$ and $B$ is by stages.

Stage 0: $A^0 = B^0 = 0$.

Stage $\sigma > 0$: Let $R_e$ be the positive requirement of highest priority at stage $\sigma$ which requires attention at stage $\sigma$. If no such $R_e$ exists, cancel all followers of all requirements that are not satisfied before stage $\sigma$ and not persistent at stage $\sigma$, and all associations of such followers to negative requirements. Let $A^\sigma = U \{ A^\delta | \delta < \sigma \}$ and $B^\sigma = U \{ B^\delta | \delta < \sigma \}$ and go to the next stage.

Suppose such an $R_e$ exists. Let $S$ be $\{ x | R_x$ has lower priority than $R_e$ at stage $\sigma \}$. Cancel all followers of $R_x$ for all $x \in S$, and all associations of such followers with negative requirements. $R_e$ is said to receive attention at stage $\sigma$.

Let $p$ be the follower of $R_e$ of highest order at stage $\sigma$ such that $R_e$ requires attention through $p$ as defined earlier in terms of clauses (3.1)—
(3.3). (If no such \( p \) exists, adopt case 4 below.) Assume such a \( p \) exists. Cancel all followers of \( R_e \) of lower order than \( p \) at stage \( \sigma \), and all associations of such followers with negative requirements. \( R_e \) is said to receive attention through \( p \) at stage \( \sigma \). Adopt case 1, case 2 or case 3 respectively if \( R_e \) requires attention through \( p \) at stage \( \sigma \) and clause (3.1), clause (3.2) or clause (3.3) respectively holds.

**Case 1.** Let \( T_e^\sigma = \{ (y, n) \mid y < p^0_0(\sigma, e) \& n < \omega \} \). Wellorder \( T_e^\sigma \) by: \( (y, n) \leq (u, m) \) if and only if \( n < m \) or \( n = m \) and \( y \leq u \). Let \( V_e^\sigma(p) \) be the set of all \( (y, n) \) in \( T_e^\sigma \) such that for some \( z, r, u \) and \( m: r < \sigma \) and \( p \) is associated with \( Q_z \) at stage \( \tau \) through \( (u, m) \) and \( (y, n) \leq (u, m) \). (The association of a follower will always take place through some \( (u, m) \) as specified below.) Let \( \langle y_0, n_0 \rangle \) be the least member of \( T_e^\sigma \setminus V_e^\sigma(p) \) such that \( (\exists z)(z < \sigma \& p^1_0(\sigma, z) = y_0) \). If \( \langle y_0, n_0 \rangle \) is welldefined, then associate \( p \) with \( Q_{z_0} \) through \( \langle y_0, n_0 \rangle \); \( z_0 \) is the unique \( z \) such that \( z < \sigma \) and \( p^1_0(\sigma, z_0) = y_0 \) (recall that \( p^1_0 \) is one-one). Let \( A^\sigma = \bigcup \{ A^\delta \mid \delta < \sigma \} \), \( B^\sigma = \bigcup \{ B^\delta \mid \delta < \sigma \} \), and go to the next stage.

Suppose \( \langle y_0, n_0 \rangle \) is not welldefined. If \( R_e \) is \( \Psi_i \neq A \), let \( A^\sigma = \bigcup \{ A^\delta \mid \delta < \sigma \} \cup \{ p \} \) and \( B^\sigma = \bigcup \{ B^\delta \mid \delta < \sigma \} \cup \{ p \} \). If \( R_e \) is \( \Psi_i \neq B \), let \( A^\sigma = \bigcup \{ A^\delta \mid \delta < \sigma \} \) and \( B^\sigma = \bigcup \{ B^\delta \mid \delta < \sigma \} \). Cancel all followers of \( R_e \) at stage \( \sigma \) save for \( p \), and all associations of such followers with negative requirements, and go to the next stage.

**Case 2.** Suppose \( p \) is associated with \( Q_j \) at stage \( \sigma \). Cancel the association of \( p \) with \( Q_j \). Proceed as in Case 1.

**Case 3.** \( p \) is now realized. If \( \Psi_i(p) \neq 0 \), add nothing to \( A \) or \( B \), and cancel all followers of \( R_e \) at stage \( \sigma \) save for \( p \), and all associations of such followers with negative requirements, and go to the next stage. If \( \Psi_i(p) = 0 \), proceed as in Case 1.

**Case 4.** Define \( p \) to be \( \sigma \). \( R_e \) receives attention through \( p \) at stage \( \sigma \). Make \( p \) an unrealized follower of \( R_e \). Add nothing to \( A \) or \( B \), and go to the next stage.

End of construction.

\( R_e \) is discharged (at stage \( \sigma \)) if \( R_e \) does not receive attention at stage \( \tau \) for any \( \tau (\geq \sigma) \). \( R_e \) is discharged by \( p \) (at stage \( \sigma \)) if \( R_e \) does not receive attention through \( p \) at stage \( \tau \) for any \( \tau (\geq \sigma) \). The next four lemmas establish that every positive requirement is satisfied.

**Lemma 3.2.** Suppose \( e \) and \( \sigma \) are such that \( p^0_0(\tau, e) = p_0^e \) for all \( \tau \geq \sigma \).
§ 3. Existence of minimal pairs

Let $\gamma > \omega$ be a regular $\alpha$-cardinal such that $\gamma > \min(p2\alpha, p_0e)$. Define $S_p = \{\tau | \tau \geq \sigma & R_e \text{ receives attention through } p \text{ at stage } \tau\}$. Then the ordertype of $S_p$ is less than $\gamma$.

Proof. If $p$ fails to follow $R_e$ at any stage $\tau \geq \sigma$, then $S_p = 0$. So suppose $\sigma_0$ is the least $\tau \geq \sigma$ such that $p$ follows $f_e$ at stage $\sigma_0$. Thus $p$ has been appointed an unrealized follower of $R_e$ prior to stage $\sigma_0 + 1$. (If $p$ is cancelled, $p$ can never be reappointed.) If $p$ is never realized, then $R_e$ never receives attention through $p$ at any stage after $\sigma_0$. So suppose $\sigma_1$ is the least $\tau \geq \sigma_0$ such that $p$ is realized at stage $\tau$. For each $\tau > \sigma_1$, if $R_e$ receives attention through $p$ at stage $\tau$, then either $R_e$ is satisfied at stage $\tau$ (and consequently never receives attention after stage $\tau$) or $p$ is associated with some $Q_i$ through some $\langle y, n \rangle$ (as in Case 2) at stage $\tau$.

Define a partial $\alpha$-recursive $f$ by: $f0$ is the least $\tau > \sigma_1$ such that $R_e$ receives attention through $p$ at stage $\tau$; $f\nu (\nu > 0)$ is the least $\tau > \text{lub}\{|f\delta| \delta < \nu\}$ such that $R_e$ receives attention through $p$ at stage $\tau$.

Define $g_\tau = \langle y, n \rangle$ if $R_e$ receives attention at stage $\tau$ through $p$, $R_e$ is not satisfied at stage $\tau$, and $p$ is associated with some $Q_i$ through $\langle y, n \rangle$ at stage $\tau$. Clearly $g$ is partial $\alpha$-recursive.

$g$ is one-one on its domain, because $\langle y, n \rangle < \langle z, m \rangle$ if $p$ is associated with $Q_i$ through $\langle y, n \rangle$ at stage $\tau_1$, and with $Q_j$ through $\langle z, m \rangle$ at stage $\tau_2 > \tau_1$. The domain of $g$ is an initial segment of $\alpha$, and its range is a subset of $e_1 \times \omega$ where $e_1 = \min (p2\alpha, p_0e)$. $g_\gamma$ is undefined, since otherwise $g$ would map $\gamma$ one-one onto $e_1 \times \omega$, a set whose $\alpha$-cardinality is less than $\gamma$. Thus the ordertype of the domain of $g$ is less than $\gamma$. The ordertype of $S_p$ is at worst 2 plus the ordertype of the domain of $f$, which is at worst 2 plus the ordertype of the domain of $g + 1$.

Lemma 3.3. Suppose $e$ and $\sigma$ are such that $p_0'(\tau, e) = p_0e$ for all $\tau \geq \sigma$. Assume $p_0e < \omega$. Define $S_p = \{\tau | \tau \geq \sigma & R_e \text{ receives attention through } p \text{ at stage } \tau\}$. Then the ordertype of $S_p$ is $\alpha$-finite.

Proof. Similar to that of Lemma 3.2.

Lemma 3.4. Suppose $\sigma, \sigma'$ and $e$ are such that $\sigma < \sigma' \leq \alpha$ and

\[(3.5) \quad (z)(\tau)(y)\{\sigma \leq \tau < \sigma' \& z \leq e \& p_0y = z \Rightarrow p_0'(\tau, y) = z\},\]
(3.6) \((z)(r)(y)[\sigma \leq \tau < \sigma' \& z < e \& p_0 y = z \rightarrow R_y \text{ does not receive attention at stage } r]\).

Let \(\gamma > \omega\) be a regular \(\alpha\)-cardinal such that \(\gamma > \min \{e, \rho \alpha\}\). Assume \(\beta < \gamma\) and define \(T = \{\tau | \sigma \leq \tau < \sigma' \& (E p)(E y)(p'_0(\tau, y) = e \& R_y \text{ receives attention through } p \text{ at stage } \tau \text{ and } p \text{ is the follower of order } \beta \text{ of } R_y \text{ at stage } \tau\}\}. Then the ordertype of \(T\) is less than \(\gamma\).

**Proof.** Fix \(y\) so that \(p_0(\sigma, y) = e\). A follower \(p\) of \(R_y\) can be cancelled at stage \(\tau\) in only one of the following ways: a requirement of higher priority than \(R_y\) requires attention at stage \(\tau\); \(R_y\) is not persistent at stage \(\tau\); \(R_y\) receives attention through \(q\) at stage \(\tau\) and \(q\) has higher order than \(p\) at stage \(\tau\); \(R_y\) is satisfied at stage \(\tau\). If the first way occurs, then the requirement of highest priority at stage \(\tau\), which requires attention at stage \(\tau\), receives attention at stage \(\tau\). Hence by (3.6) \(p\) cannot be cancelled in the first way if \(\sigma < \tau < \sigma'\). By (3.5) \(R_y\) is persistent at stage \(\tau\), so \(p\) cannot be cancelled in the second way if \(\sigma < \tau < \sigma'\). Consequently if \(\sigma < \tau < \sigma'\) and \(p\) follows \(R_y\) at stage \(\tau\), then \(p\) can be cancelled at stage \(\tau\) only if \(R_y\) is satisfied at stage \(\tau\) or \(R_y\) receives attention via \(q\) at stage \(\tau\) and \(q\) has higher order than \(p\) at stage \(\tau\).

Define \(R(x, \tau)\) by \(\sigma \leq \tau < \sigma'\) and \((E p)(R_y \text{ receives attention through } p \text{ at stage } \tau \text{ and } p \text{ has order } x \text{ at stage } \tau)\). \(R(x, \tau)\) is an \(\alpha\)-recursive relation. Let \(T_x = \{u | R(z, u)\}\).

Let \(\beta\) be the least ordinal such that the ordertype of \(T_\beta\) is at least \(\gamma\). Then a contradiction will follow from Lemma 2.3 cf [6]. Suppose \(\sigma_1\) and \(\sigma_2\) are such that \(\sigma_1 < \sigma_2 \leq \alpha\) and

\[
U \{ T_z | z < \beta \} \cap \{ w | \sigma_1 < w < \sigma_2 \} = 0.
\]

If \(\sigma_1 < \tau < \sigma_2\) and \(R_y\) receives attention through \(q\) at stage \(\tau\) and \(\tau \in T_\beta\), then \(q\) has order at least \(\beta\) at stage \(\tau\). Consequently if \(\tau_1, \tau_2 \in T_\beta\), \(\sigma_1 < \tau_1 < \sigma_2\) and \(\sigma_1 < \tau_2 < \sigma_2\), \(R_y\) has a follower \(p_1\) of order \(\beta\) at stage \(\tau_1\), and \(R_y\) has a follower \(p_2\) of order \(\beta\) at stage \(\tau_2\), then \(p_1 = p_2\). By (3.5) and Lemma 3.2, the ordertype of \(T_\beta \cap \{ \tau | \sigma_1 < \tau < \sigma_2 \}\) is less than \(\gamma\). Finally by Lemma 2.3 of [6], the ordertype of \(T_\beta\) is less than \(\gamma\).
**Lemma 3.5.** Suppose \( \alpha, \alpha' \) and \( \epsilon \) satisfy hypotheses (3.5) and (3.6) of Lemma 3.4. Assume \( \alpha \) is not refractory, \( \alpha = \alpha' \) and \( p_0 \gamma = e \). Then \( R_y \) is discharged. In addition if \( \gamma > \omega \) is a regular \( \alpha \)-cardinal greater than \( \min \{ e, p2 \alpha \} \), then the ordertype of \( T = \{ \tau \mid \alpha \leq \tau < \alpha' \land R_y \) receives attention at stage \( \tau \} \) is less than \( \gamma \).

**Proof.** Define a partial \( \alpha \)-recursive \( g' \) by: 
\[
g'(\tau, z) = v \text{ if } R_y \text{ has a follower } p \text{ of order } z \text{ at the end of stage } \alpha + \tau, 
\text{and } p \text{ is associated with } Q_u \text{ at the end of stage } \alpha + \tau, 
\text{and } p'_1(\alpha + \tau, u) = v.
\]
Define \( g \) by \( g(z) = v \) if
\[
(\exists \tau)(\rho)(\tau \leq \rho < \alpha' \rightarrow g'(\rho, z) = v).
\]
The conclusion of Lemma 3.5 will follow easily from four facts about \( g \).

**Fact 1.** The domain of \( g \) is an initial segment of \( \alpha \).

**Fact 2.** \( g \) is one-one on its domain.

**Fact 3.** If \( g \) is total and \( \alpha' = \alpha \), then \( g \) is a strong \( \Sigma_2 \) projection of \( \alpha \) into \( \epsilon \).

**Fact 4.** Assume \( \gamma = 2 \alpha < p2 \alpha = \alpha, \alpha' = \alpha \), and the domain of \( g \) is \( \beta < \alpha \). If \( R_y \) is not discharged, \( R_y \) is either \( \Psi_i \neq A \) or \( \Psi_i \neq B \), \( i \) is total, then \( tp2 \alpha < \alpha \).

The proofs of Fact 1--4 are momentarily deferred.

Let \( e_1 = \min \{ e, p2 \alpha \} \), and let \( \gamma > \omega \) be a regular \( \alpha \)-cardinal greater than \( e_1 \). By (3.5) and (3.6): if \( \alpha \leq \tau < \alpha' \), then no requirement of higher priority than \( R_y \) at stage \( \tau \) receives attention at stage \( \tau \); and if \( R_y \) receives attention at stage \( \tau \), then there is a follower \( p \) of \( R_y \) at stage \( \tau \) such that \( R_y \) receives attention through \( p \) at stage \( \tau \). Assume that the ordertype of \( T \) is at least \( \gamma \). Then there must be a \( \sigma_1 \) such that \( \sigma < \sigma_1 < \alpha' \) and \( T \cap \{ \tau \mid \tau < \sigma_1 \} \) has ordertype \( \gamma \). At stage \( \sigma_1 \), \( R_y \) has followers of all orders \( \beta < \gamma \) for some \( \beta \leq \gamma \).

Suppose \( \beta = \gamma \). Let \( p \) be a follower of order \( x < \gamma \). \( p \) cannot be unrealized at stage \( \sigma_1 \), since otherwise \( R_y \) would have no follower of order \( x + 1 \) at stage \( \sigma_1 \). \( R_y \) cannot be satisfied before stage \( \sigma_1 \), since otherwise \( R_y \) would have only one follower at stage \( \sigma_1 \). The cofinality of \( \sigma_1 \) (in \( L_\gamma \)) must be \( \gamma \), because \( \gamma \) is regular. By Lemma 3.2 the set of stages prior to \( \sigma_1 \) at which \( R_y \) receives attention through \( p \) has ordertype less than \( \gamma \), and so cannot be cofinal with \( \sigma_1 \). Hence there is a \( \sigma_2 < \sigma_1 \) such that \( R_y \) does not receive attention through \( p \) at any stage after \( \sigma_2 \) and prior to \( \sigma_1 \). Hence there is a \( Q_u \) such that \( p \) is associated with \( Q_u \) at stages \( \sigma_2 \) and \( \sigma_1 \). \( Q_u \) must be persistent at every stage \( \tau \) such that...
\( \sigma_2 < \tau < \sigma_1 \), since otherwise \( R_y \) would require attention through \( p \) at stage \( \tau \), and either \( R_y \) would receive attention through \( p \) at stage \( \tau \) or \( p \) would be cancelled at stage \( \tau \). Define \( h \) by: \( hz = v \) if \( R_y \) has a follower of order \( z \) at stage \( \sigma_1 \), \( p \) is associated with \( Q_v \) at stage \( \sigma_1 \), and \( v \) satisfies \( (E \rho)(\tau)[\rho < \sigma_1 \land (\rho \leq \tau < \sigma_1 \rightarrow p'_1(\tau, u) = v)] \). Clearly \( h z = \lim_{\tau \to \sigma_1} g'(\tau, z) \). \( h \) is a one-one, \( \alpha \)-finite map of \( \gamma \) into \( e_1 < \gamma \).

Thus \( \beta < \gamma \). By Lemma 3.4 the ordertype of \( (\tau \mid \sigma < \tau < \sigma_1 \land R_y \text{ receives attention through a follower of order } \beta) \) is less than \( \gamma \). Hence there is a \( \sigma_3 < \sigma_1 \) such that \( R_y \) does not receive attention through a follower of order \( \beta \) at any stage between \( \sigma_3 \) and \( \sigma_1 \). In addition there is a \( \sigma_4 \) such that \( \sigma_3 < \sigma_4 < \sigma_1 \) and \( R_y \) has no follower of order \( \beta \) at any stage between \( \sigma_4 \) and \( \sigma_1 \). The defining properties of \( \sigma_1 \) imply there is a \( v < \gamma \) such that \( R_y \) never receives attention through a follower of order \( \beta \) at any stage between \( \sigma \) and \( \sigma_1 \). Define \( R(x, z) \) by \( \sigma_1 < z < \sigma_1 \) and \( R_y \) receives attention through a follower of order \( x \) at stage \( z \). Let \( T_x \) be \( \{ z \mid R(x, z) \} \). By Lemma 3.4 (and Lemma 2.3 of [6]), the ordertype \( U \{ T_z \mid z < v \} \) is less than \( \gamma \). But \( T \cap (\tau \mid \tau < \sigma_1) \) has ordertype \( \gamma \) and \( T = U \{ T_z \mid z < v \} \). Hence the ordertype of \( T \) is less than \( \gamma \).

The proof that \( R_y \) is discharged breaks into four cases. Remember that \( R_y \) receives attention at stage \( \tau \geq \sigma \) if \( R_y \) requires attention at stage \( \tau \).

Case 1. \( \alpha = \omega \). Hence there is a regular \( \alpha \)-cardinal \( \gamma \) such that \( \gamma > \omega \) and \( \gamma > \omega \). Consequently the set of stages at which \( R_y \) receives attention has ordertype \( \omega \) less than \( \gamma \) and is \( \alpha \)-finite, since (as was just shown) \( T \) has ordertype less than \( \gamma \).

Case 2. \( p2\alpha < gc\alpha < \alpha \). If \( gc\alpha \) is regular, let \( \gamma = gc\alpha \); if \( gc\alpha \) is singular, then there is a regular \( \alpha \)-cardinal \( \gamma \) such that \( p2\alpha < \gamma < gc\alpha \). Proceed as in Case 1.

Case 3. \( p2\alpha = tp2\alpha = gc\alpha < \alpha \). If \( gc\alpha > \omega \), then the argument of Case 1 succeeds. Suppose \( gc\alpha = \omega \). Then \( e \) is finite. Suppose \( R_y \) is not satisfied at any stage. Each realized follower of \( R_y \) at stage \( \tau \geq \sigma \) is associated with a different \( Q_n \) at the end of stage \( \tau \). Consequently \( R_y \) has at most \( e \) realized followers at the end of stage \( \tau \geq \sigma \), and at most one unrealized follower at stage \( \tau \). Let \( q_0 \) be the first follower of \( R_y \) of order \( 0 \) at any stage after \( \sigma \). Then \( q_0 \) is never cancelled, and \( R_y \) is never satisfied. If \( q_0 \) is always unrealized, then \( R_y \) is discharged. Otherwise \( q_0 \) is associated with some \( Q_w \) for all sufficiently large stages. Let \( \sigma_0 \) be the stage at which \( q_0 \) is last associated with \( Q_w \). At stage \( \sigma_0 + 1 \) a follower
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$q_1$ of $R_y$ of order 1 is appointed, never to be cancelled. And so on until termination with at worst $q_{e+1}$. Either $R_y$ is satisfied, or some $q_i$ is never realized. In either event $R_y$ is discharged.

Case 4. $g_c a < p_2 a = a$. Hence $e < tp_2 a = a$. If $g$ were total, then $p_2 a$ would be at most $e$ by Fact 3. So $g$ must be partial. If $R_y$ were not discharged, then $tp_2 a$ would be less than $a$ by Fact 4.

If $a$ is not refractory, then one of the above four cases must apply.

Only the proofs of Facts 1–4 remain.

Proof of Fact 1: If $R_y$ is satisfied at some stage $\tau^*$, then $R_y$ has just one follower $p$ at the end of stage $\tau$ for each $\tau \geq \tau^*$, and $p$ is not associated with any negative requirement. So the domain of $g$ is 0.

Assume $R_y$ is not satisfied. Suppose $z_1 < z_2$ and $g(z_2)$ is defined with the intent of seeing that $g(z_1)$ is defined. Suppose $\tau_0 < \sigma'$ and $g'(\tau, z_2) = g'(\tau_0, z_2)$ whenever $\tau_0 \leq \tau < \sigma'$. Fix $\tau$ so that $\tau_0 \leq \tau < \sigma'$.

It is impossible for some requirement of higher priority than $R_y$ at stage $\tau$ to receive attention at stage $\tau$, or for $R_y$ to receive attention at stage $\tau$ through a follower of higher order than $z_2$. Otherwise the follower of $R_y$ of order $z_2$ at stage $\tau$ is cancelled at the end of stage $\tau$ and no new follower of $R_y$ of order $z_2$ at stage $\tau + 1$ is appointed at stage $\tau$. Consequently $R_y$ has a follower $p$ of order $z_1$ at the end of stage $\tau$. $p$ cannot be unrealized at stage $\tau$; otherwise $R_y$ would have no follower of higher order than $z_1$ at the end of stage $\tau$. Since $R_y$ is not satisfied, $p$ must be associated with some $Q_u$ at the end of stage $\tau$ and some $Q_v$ at the end of stage $\tau_0$. If $u \neq v$, then a requirement of higher priority than $R_y$ at stage $p (\tau_0 < p \leq \tau)$ would have received attention at stage $p$, or $R_y$ would have received attention at stage $p$ through some follower of order at most $z_1$ at stage $p$. Each of the last two conclusions is impossible (recall hypothesis (3.655)), hence $p$ is associated with $Q_u$ at the end of stage $\tau$ and $u = v$. $Q_u$ is persistent at stage $\tau$, and $p'_1 (\tau, u) = p_1 u$. Thus $g'(\tau, z_1) = g'(\tau_0, z_1)$, and so $g(z_1)$ is defined.

Proof of Fact 2: Fix $\tau$. The function $p'_1 (\tau, x)$ is one-one, and no two followers of $R_y$ are associated with the same $Q_u$ at the end of stage $\tau$. Hence $g'(\tau, x)$ is one-one on its domain.

Proof of Fact 3: The proof of Fact 1 established: if $z_1 < z_2$ and $g'(\rho, z_2) = g'(\tau, z_2)$ for all $\rho \geq \tau$, then $g'(\rho, z_1) = g'(\tau, z_1)$ for all $\rho \geq \tau$.

Proof of Fact 4: Define $f' (\tau, u) = v$ if $v$ is lub $\{ p \mid p \leq \tau \text{ and } R_y \text{ receives attention at stage } p \text{ through a follower of order } u \text{ at stage } p \}$. Define
fu = v if there is a \( \tau \) such that \( f'(\rho, u) = v \) for all \( \rho \geq \tau \). Clearly the domain of \( f' \) is \( \beta \). Since \( R_y \) is not discharged, \( R_y \) is never satisfied; and since \( \Psi_i \) is total, every unrealized follower of \( R_y \) is either cancelled or realized. Hence \( R_y \) must receive attention unboundedly often through a follower of order less than \( \beta \); otherwise \( g\beta \) would be defined. Thus \( f \) is a \( \Sigma_2 \) cofinality function and \( \text{cf} 2\alpha \leq \beta \). By 2.5 \( \text{tp}2\alpha < \alpha \).

**Lemma 3.6.** For each \( y \), \( R_y \) is discharged.

**Proof.** By induction on \( e < \text{tp}2\alpha \). Step \( e \) of the induction shows \( R_y \) is discharged for \( y = \rho \circ \nu e \). Recall \( \rho_0 \) maps \( \alpha \) onto \( \text{tp}2\alpha \).

**Case 1.** \( \text{gca} = \alpha \). Let \( \sigma' = \alpha \), and assume \( \sigma' \), \( \sigma \) and \( e \) satisfy (3.5). Further assume \( \sigma \) and \( e \) satisfy (3.7).

(3.7) For each \( x < \text{tp}2\alpha \) and \( \tau \), define \( T^\tau_x \) to be \( \{ \rho \mid \rho \geq \tau \) and \( R_{\rho \circ \nu e x} \) receives attention at stage \( \rho \} \). If \( \gamma > \omega \) is an infinite \( \alpha \)-cardinal and \( e < \gamma \), then the ordertype of \( T^\sigma_x \) is less than \( \gamma \) for all \( x < e \).

Fix \( \gamma \) as the least \( \alpha \)-cardinal greater than \( \max(e, \omega) \). (\( \gamma \) exists because \( \text{gca} = \alpha \).) Clearly \( \gamma \) is regular. According to (3.7) \( T^\gamma_x \) has ordertype less than \( \gamma \) for every \( x < e \). It follows from Lemma 2.3 of [6] that \( T^\gamma = \bigcup \{ T^\gamma_z \mid z < e \} \) has ordertype less than \( \gamma \) and so is \( \alpha \)-finite. Thus for some \( \sigma_3 \geq \sigma \), it is the case that \( \sigma' (= \alpha) \), \( \sigma_3 \) and \( e \) satisfy (3.5) and (3.6). Suppose \( \sigma_1 < \sigma_2 \leq \alpha \) and \( \{ T^\gamma_z \mid z < e \} \cap \{ \delta \mid \sigma_1 \leq \delta < \sigma_2 \} = \emptyset \). By Lemma 3.5, \( T^\gamma_e \cap \{ \delta \mid \sigma_1 \leq \delta < \sigma_2 \} \) has ordertype less than \( \gamma \). But then by Lemma 3.5 and Lemma 2.3 of [6], \( R_{\rho \circ \nu e} \) is discharged.

**Case 2.** \( \text{p}2\alpha < \text{gca} < \alpha \). If \( \text{tp}2\alpha \leq \text{gca} \), then proceed as in Case 1. If \( \text{tp}2\alpha > \text{gca} \), then \( \text{tp}2\alpha \leq \text{gca} \cdot \text{cf}2\alpha \) by Theorem 2.3. In the hope of a contradiction, let \( e \) be the least \( x < \text{tp}2\alpha \) such that \( R_{\rho \circ \nu e x} \) is not discharged. Then \( e = \text{gca} \cdot \nu + \eta \) for some \( \nu < \text{cf}2\alpha \) and \( \eta < \text{gca} \). Let \( \sigma' = \alpha \), and let \( \sigma_0 \) be the least \( \sigma \) such that \( \sigma' \), \( \sigma \) and \( e \) satisfy (3.5). There is no \( \tau \) such that \( \sigma_0 \leq \tau < \alpha \) and \( \sigma' \), \( \tau \) and \( e \) satisfy (3.6) (with \( \sigma = \tau \)); otherwise Lemma 3.5 implies \( R_{\rho \circ \nu e} \) is discharged. Thus the set of stages after stage \( \sigma_0 \) at which \( R_{\rho \circ \nu e x} \) receives attention for some \( x < e \) must be cofinal with \( \alpha \).
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If \( \eta = 0 \), define \( f: \nu \mapsto \alpha \) by: \( f^\nu \) is the least \( \tau \) such that \( R^{p_0^{-1} \omega} \) does not receive attention at any stage after stage \( \tau \) for any \( z < \text{gca} \cdot x \). Then \( f \) is a \( \Sigma_2 \) cofinality function and \( \text{cf} 2 \alpha \leq \nu < \text{cf} 2 \alpha \).

Suppose \( \eta > 0 \). There is a \( \sigma_1 \) such that \( \sigma' (= \alpha) \), \( \sigma_1 \) and \( \text{gca} \cdot \nu \) satisfy (3.5) and (3.6) (with \( \sigma_1 \) as \( \sigma \) and \( \text{gca} \cdot \nu \) as \( \epsilon \)). But no requirement of priority greater than \( R^{p_0^{-1} (\text{gca} \cdot \nu)} \) receives attention after stage \( \sigma \). For each \( s \) such that \( \text{gca} \cdot \nu \leq x \leq e \) and each \( \tau \), define \( T^\tau_x = \{ \rho \mid \rho \geq \tau \) and \( R^{p_0^{-1} (\text{gca} \cdot \nu + x)} \) receives attention at stage \( \rho \} \). There is a regular \( \alpha \)-cardinal \( \gamma \) such that \( \text{p} 2 \alpha < \gamma \leq \text{gca} \). Proceed as in Case 1 to show \( R^{p_0^{-1} e} \) is discharged.

Case 3. \( \text{tp} 2 \alpha = \text{gca} < \alpha \). Hence for each \( e < \text{tp} 2 \alpha \), there is a regular \( \alpha \)-cardinal \( \gamma \) such that \( e < \gamma \leq \text{gca} < \alpha \). Proceed as in Case 1.

Case 4. \( \text{gca} < \text{p} 2 \alpha = \text{tp} 2 \alpha = \alpha \). Let \( e \) be the least \( x < \text{tp} 2 \alpha \) such that \( R^{p_0^{-1} x} \) is not discharged. As in Case 2, the set of stages at which \( R^{p_0^{-1} x} \) receives attention for some \( x < e \) must be cofinal with \( \alpha \) and so \( \text{cf} 2 \alpha \leq e \). But then \( \text{tp} 2 \alpha < \alpha \) by Theorem 2.3.

If none of the last four cases apply, then \( \alpha \) is refractory.

Lemma 3.7. Neither \( A \) nor \( B \) is \( \alpha \)-recursive.

Proof. Suppose \( A \) is \( \alpha \)-recursive. Then \( A = \Psi_i \) for some \( i \). If \( p \) is an unrealized follower of requirement \( \Psi_i \neq A \) at stage \( \sigma \), then \( p \) is either cancelled or eventually realized. By 3.6 requirement \( \Psi_i \neq A \) is discharged. Hence there is a stage \( \sigma \) such that \( \Psi_i \neq A \) does not require attention at stage \( \tau \) for any \( \tau \geq \sigma \). \( \Psi_i \neq A \) does not have an unrealized follower at stage \( \sigma \); otherwise it would not be satisfied at stage \( \sigma \), hence never satisfied, and either \( p \) would be cancelled or \( \Psi_i(p) \) would be defined, and so \( \Psi_i \neq A \) would require attention after stage \( \sigma \). Since \( \Psi_i \neq A \) has no unrealized follower at stage \( \sigma \) and does not require attention at stage \( \sigma \), \( \Psi_i \neq A \) must be satisfied prior to stage \( \sigma \). Hence there is a \( p \) that follows \( \Psi_i \neq A \) such that \( p \in A \iff \Psi_i(p) = 0 \). Thus \( A(p) \neq \Psi_i(p) \).

Lemma 3.8. Suppose \( p \) follows \( R_i \) at stage \( \sigma \), \( q \) follows \( R_j \) at stage \( \sigma \), \( p \in (A \cup B) \setminus (A^\sigma \cup B^\sigma) \) and \( q \in (A \cup B) \setminus (A^\sigma \cup B^\sigma) \). Let \( \sigma_1 \) (\( \sigma_2 \) respectively) be the first stage such that \( p \) (\( q \) respectively) is put in \( A \cup B \). Assume \( \sigma_1 < \sigma_2 \). Then \( p_0(\sigma, i) < p_0'(\sigma, j) \).
Proof. Since \( q \) is not cancelled at stage \( \sigma_1 \), it must be that 
\[
p_0'(\sigma_1, i) < p_0'(\sigma_1, j).
\]
If \( p_0'(\sigma_1, i) \neq p_0'(\sigma, i) \), then there is a \( \tau \) such that 
\( \sigma < \tau \leq \sigma_1 \), \( R_i \) is not persistent at stage \( \tau \), and \( p \) is cancelled at stage \( \tau \).
A cancelled follower can never be reappointed, hence \( p \) is never cancelled.
Thus \( p_0'(\sigma_1, i) = p_0'(\sigma, i) \).
Similarly \( p_0'(\sigma_2, j) = p_0'(\sigma_1, j) = p_0'(\sigma, j) \).

Lemma 3.9. A and B are regular.

Proof. Fix \( x \) to see \( A \cap x \) is \( \alpha \)-finite. If \( z \in A \cap x \), then \( z \in A^x \) or \( z \) is a follower at stage \( x \). Let \( \sigma_0 \geq x \) be the least stage such that some \( z < x \) is placed in \( A \) at stage \( \sigma_0 \). For each \( i < \omega \), let \( \sigma_{i+1} \geq \sigma_i \) be the least stage such that some \( z < x \) is placed in \( A \) at stage \( \sigma_{i+1} \). Suppose \( \sigma_i \) is welldefined for all \( i < \omega \). Let \( R_{k_i} \) be the requirement satisfied at stage \( \sigma_i \). By Lemma 3.8, \( p_0'(x, k_0) > p_0'(x, k_1) > \ldots \), an impossibility. If \( \sigma_0 \) is not defined, then \( z \in A \cap x \iff z \in A^x \cap x \). If \( \sigma_n \) is the last welldefined \( \sigma_i \), then \( z \in A \cap x \iff z \in A^{\sigma_n} \cap x \). The proof for \( B \) is similar.

Lemma 3.10. If \( C \) is \( \alpha \)-recursive in \( A \), and in \( B \), then \( C \) is \( \alpha \)-recursive.

Proof. There is an \( i \) such that \( C = \Phi_i A = \Theta_i B \). Let \( \sigma_1 \) be the least \( \sigma \) such that \( p_1'(\tau, i) = p_1 i \) for all \( \tau \geq \sigma \). Let \( \sigma_2 \) be the least \( \sigma \geq \sigma_1 \) such that \( R_y \) has been discharged prior to stage \( \sigma \) for every \( y \) with the property that \( p_0 y \leq p_1 i \). The existence of \( \sigma_2 \) follows from the proof of Lemma 3.6. The latter established the existence of a \( \sigma \) such that all requirements of higher priority than \( R_\varepsilon \) are discharged prior to stage \( \sigma \), whenever \( e < \text{tp}2\alpha \). \( \sigma_2 \) exists because \( p2\alpha \leq \text{tp}2\alpha \). Any requirement that receives attention at stage \( \tau \geq \sigma_2 \) has its followers subject to association with \( Q_i \) at stage \( \tau \).

To decide whether or not \( x \in C \), search for a stage \( \sigma_3 \geq \sigma_2 \) such that 
\[
L(\sigma_3, i) = M(\sigma_3, i) > x. \sigma_3 \text{ exists by Lemma 3.9. Clearly}
\]
\[
\Phi_i^{\sigma_3}(A^{\sigma_3}, x) = \Theta_i^{\sigma_3}(B^{\sigma_3}, x) = q
\]
for some \( q. A \) and \( B \) are regular, so \( \Phi_i(A, x) = \lim_{\sigma \to \alpha} \Phi_i^\sigma(A, x) \) and 
\[
\Theta_i(B, x) = \lim_{\sigma \to \alpha} \Theta_i^\sigma(B, x). \text{Thus to show } C(x) = q, \text{ it suffices to show}
\]
\[
\Phi_i^\tau(A^\tau, x) = q \text{ or } \Theta_i^\tau(B^\tau, x) = q \text{ for all } \tau \geq \sigma_3.
\]
Let $c_0$ be the computation of $q^{3}(A^{03}, x)$ and $d_0$ be the computation of $q^{3}(B^{03}, x)$. $c_0$ ($d_0$ respectively) will be invalid at stage $\tau > \sigma_3$ only if some $z < \sigma_3$ is put in $A$ ($B$ respectively) before stage $\tau$ and after stage $\sigma_3$. Let $\tau_1$ be the least $\tau \geq \sigma_3$ such that some $z < \sigma_3$ is put in $A$ or $B$ at stage $\tau$. Let $z_1$ be a $z$ put in $A$ at stage $\tau_1$, let $z_1$ follow $R_{y_1}$ and let $p'(\tau_1, y_1) = u_1$. Computation $d_0$ is still valid at stage $\tau_1 + 1$. $R_{y_1}$ must be persistent at every stage $\tau$ such that $\sigma_3 \leq \tau \leq \tau_1$, otherwise $z_1$ would be cancelled and would not be put in $A$ at stage $\tau_1$. Let $R_\nu$ be a requirement such that $R_\nu$ has a follower $p$ at the end of stage $\tau$ and is not satisfied before the end of stage $\tau$. Then $p'(\tau_1, v) < p'(\tau_1, y_1)$. To see that $p < \sigma_3$, assume $p \geq \sigma_3$. Then there is a $\delta$ such that $\sigma_3 \leq \delta < \tau$ and $p$ is appointed to follow $R_\nu$ at stage $\delta$. Since $z_1$ is not cancelled at stage $\delta$, $p'(\delta, y_1) < p'(\delta, v)$. Since $R_{y_1}$ is persistent for all $\tau$ such that $\sigma_3 \leq \tau \leq \tau_1$, $R_{y_1}$ cannot be persistent for all $\tau$ such that $\delta < \tau \leq \tau_1$. Hence $p$ is cancelled before the end of stage $\tau_1$. Thus all followers in existence at the end of stage $\tau_1$, and not satisfied before the end of stage $\tau_1$, are less than $\sigma_3$.

If no $z < \sigma_3$ is put in $B$ after stage $\tau_1$, then the computation $d_0$ is valid forever, and $q^3(B^\tau, x) = z$. So assume there is a $\tau > \tau_1$ such that some $z < \sigma_3$ is put in $B^\tau$. Let the least such $\tau$ be $\tau_2$, and let $z_2$ be such a $z$. Suppose $z_2$ follows $R_{y_2}$. Two cases occur.

**Case 1.** $z_2$ is not associated with $Q_i$ at any stage $\tau$ such that $\tau_1 < \tau \leq \tau_2$. Since $z_2 < \sigma_3$, $z_2$ existed as a follower at the end of stage $\tau_1$. If $z_2$ was unrealized at the end of stage $\tau_1$, then $z_2$ was eligible for association with $Q_i$ through $(i, 0)$; and if $z_2$ was associated with some $Q_j$ through $(i, n)$ at the end of stage $\tau_1$, then $z_2$ was eligible for association with $Q_i$ through $(i, n + 1)$. Since $z_2$ is not associated with $Q_i$, some follower $r$ of $R_{y_2}$ of higher order than that of $z_2$ was associated with $Q_i$ at stage $\lambda$, where $\lambda$ is the least stage such that $z_2$ was associated with some $Q_j$ through some $(u, k)$ (if there is no such stage, let $\lambda = \tau_2$). $(u, k) > (i, 0)$ if $z_2$ was unrealized at stage $\tau_1$, and $(u, k) > (i, n + 1)$ if $z_2$ was realized at stage $\tau_1$. But since $r$ has higher order than $z_2$ at stage $\lambda$, $r < z_2$, so $r$ existed at stage $\sigma_3$. If $r$ were not associated with $Q_i$ at stage $\sigma_3$, then $R_{y_2}$ must have received attention through $r$ at stage $\delta$ ($\sigma_2 \leq \delta < \lambda$), but then $z_2$ would have been cancelled at stage $\delta$ and could not have been a follower at stage $\tau_2$. Hence $r$ is associated with $Q_i$ at stage $\sigma_3$. But $L(\sigma_3, i) = M(\sigma_3, i)$, hence either $R_{y_2}$ is not persistent at
stage \( \sigma_3 \) (impossible because \( z_2 \) would be cancelled at stage \( \sigma_3 \)), or \( R_{y_2} \) requires attention through \( r \) at stage \( \sigma_3 \). There are now only three possibilities at stage \( \sigma_3 \): \( R_{y_2} \) receives attention through \( r \) and \( z_2 \) is cancelled; \( R_{y_2} \) receives attention through a follower of higher order than \( r \) and \( z_2 \) is cancelled; and some \( R_v \) of higher priority than \( R_{y_2} \) receives attention and \( z_2 \) is cancelled. None of the three can occur, hence case 1 cannot occur.

Case 2. \( z_2 \) is associated with \( Q_i \) at stage \( \tau \) for some \( \tau \) such that \( \tau_1 \leq \tau \leq \tau_2 \); let \( \tau'_1 \) be the least such \( \tau \). Since \( z_2 \in B \), there is a first stage \( \tau''_1 \) such that \( \tau'_1 < \tau''_1 \leq \tau_2 \) and the association of \( z_2 \) with \( Q_i \) is cancelled at stage \( \tau''_1 \). Since \( z_2 \) is still a follower at the end of stage \( \tau''_1 \), \( L(\tau''_1, i) = M(\tau''_1, i) \) and there is a computation \( c_1 \) of

\[
\Phi_{A(\tau''_1, x)} = q.
\]

If \( c_1 \) were invalid at the end of stage \( \tau_2 \), then some follower \( x \) would land in \( A \) at some stage \( \lambda(\tau''_1 < \lambda < \tau_2) \). Suppose \( x \) follows \( R_v \). \( x < \tau''_1 \) and \( x \) exists as a follower at stage \( \tau''_1 \). Also \( p'_0(\tau''_1, v) < p'_0(\tau''_1, y_2) \). For each \( \tau (\tau''_1 \leq \tau \leq \lambda) \), \( R_v \) must be persistent at stage \( \tau \) (otherwise \( x \) would be cancelled by the end of stage \( \lambda \) and so could not land in \( A \)), and \( R_{y_2} \) must be persistent at stage \( \lambda \) (otherwise \( z_2 \) would be cancelled by the end of stage \( \lambda < \tau_2 \)). Hence \( p'_0(\lambda, v) < p'_0(\lambda, y_2) \). Since \( R_v \) receives attention at stage \( \lambda \), \( z_2 \) is cancelled at stage \( \lambda \). Consequently \( c_1 \) is valid at the end of stage \( \tau_2 \). All followers in existence at the end of stage \( \tau_2 \) are less than \( \sigma_3 \). Let \( c_2 = c_1 \).

If no \( z < \sigma_3 \) is placed in \( A \) after stage \( \tau_2 \), then computation \( c_2 \) is valid forever. Assume some \( z < \sigma_3 \) is placed in \( A \) after stage \( \tau_2 \).

Continue to alternate as above between \( A \) and \( B \). If for some \( n < \omega \), \( z_n \) fails to be defined, then the lemma is proved. Suppose \( z_n \) is defined for all \( n < \omega \). \( z_n \) follows \( R_{y_n} \) at stage \( \sigma_3 \). By Lemma 3.8, \( p'_0(\sigma_3, y_1) > p'_0(\sigma_3, y_2) > \ldots \), an impossibility.

Theorem 3.1 is a consequence of Lemmas 3.7 and 3.10.
§4. Further results and open questions

The following Theorem will appear in Lerman [4]. Its proof is not an injury argument.

**Theorem 4.1.** Let $\alpha$ be any $\Sigma_1$ admissible ordinal. Suppose $A$ and $B$ are non-$\alpha$-recursive, $\alpha$-recursively enumerable sets whose $\alpha$-recursive disjoint union is complete. Then there exists a non-$\alpha$-recursive, $\alpha$-recursively enumerable $C$ such that $C$ is $\alpha$-recursive in $A$, and in $B$.

Let $T_\alpha$ be the elementary theory of the partial ordering of the $\alpha$-recursively enumerable degrees. Nothing is known about the dependence (if any) of $T_\alpha$ on $\alpha$. Lerman [3] proved that the $\Sigma_1$ sentences of $T_\alpha$ are independent of $\alpha$.

**Question 1.** Is there a minimal pair of $\alpha$-recursively enumerable degrees for every $\Sigma_1$ admissible $\alpha$?

Theorem 3.1 provides such a pair when $\alpha$ is not refractory. It might be wise to study those refractory $\alpha$ with the following properties: $\omega < p2\alpha = gca < \alpha; tp2\alpha = gca \cdot \omega; cf2\alpha = \omega$.

**Question 2.** Are the $\alpha$-recursively enumerable degrees dense for every $\Sigma_1$ admissible $\alpha$?

The answer is yes when $\alpha = \omega$ by Sacks [5], and when $\alpha^* = \omega$ by Driscoll [1]. An affirmative answer to Question 1 will probably include an account of the $\alpha$-infinite injury method, a method as yet unknown.

**Question 3.** For which $\Sigma_1$ admissible $\alpha$'s can every finite distributive lattice be embedded in the $\alpha$-recursively enumerable degrees?

Lerman and Thomason showed every one could be embedded when $\alpha = \omega$. Their arguments extended the minimal pair construction.
References