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INTERVAL GRAPHS AND RELATED TOPICS

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Interval graphs have engaged the interest of researchers for over twenty-five years. The scope of current research in this area now extends to the mathematical and algorithmic properties of interval graphs, their generalizations, and graph parameters motivated by them. One main reason for this increasing interest is that many real-world applications involve solving problems on graphs which are either interval graphs themselves or are related to interval graphs in a natural way.

The problem of characterizing interval graphs was first posed independently by Hajos [45] in combinatorics and by Benzer [17] in genetics. By definition, an *interval graph* is the intersection graph of a family of intervals on the real line, i.e., to every vertex of the graph there corresponds an interval and two vertices are connected by an edge of the graph if and only if their corresponding intervals have nonempty intersection. However, this suggests a more general paradigm for studying various classes of graphs which can be described as follows.

Let $\mathcal{F} = [S_1, \ldots, S_n]$ be a family of nonempty subsets of a set S. The subsets are not necessarily distinct. We will call S the *host* and the subsets S_i the *objects*. In addition, there may or may not be certain *constraints* placed on the objects such as not allowing one subset to properly contain another or requiring that each subset satisfies a special property. The intersection graph of $\mathcal{F}$ has vertices $v_1, \ldots, v_n$ with v_i and v_j joined by an edge if and only if $S_i \cap S_j \neq \emptyset$. We call the pair $\mathcal{H} = (S, \mathcal{F})$ an *intersection representation hypergraph* for G or more simply a *representation*. When $\mathcal{F}$ is a family of intervals on a line, G is an interval graph and $\mathcal{H}$ is an interval hypergraph. If we add the constraint that no interval may properly contain another, then we obtain the class of proper interval graphs. When $\mathcal{F}$ is allowed to be an arbitrary family of sets, the class obtained as intersection graphs is all undirected graphs.

On one hand, research has been directed towards intersection graphs of families having some specific topological or other structure. These include circular-arcs, paths in trees, chords of circles, cliques of graphs, and others [35]. On the other hand, certain well-known classes of graphs have subsequently been characterized in terms of intersection graphs. For example, triangulated graphs (every cycle of length ≥ 4 has a chord) are the intersection graphs of subtrees of a tree [21, 33,

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Class–name	Interaction	Objects	Host	Constraints
Interval graphs	Intersection	Intervals	Line	—
Proper interval graphs	Intersection	Intervals	Line	No proper containment allowed
Circle containment graphs	Containment	Circles	Plane	—
Path graphs	Intersection	Paths	Tree	—
Edge path graphs	Edge intersection	Paths	Tree	—
Local edge path graphs	Edge Intersection	Paths	Tree	All paths must share a vertex
Circular-arc graphs	Intersection	Arcs	Circle	—
Proper circular-arc graphs	Intersection	Arcs	Circle	No proper containment allowed
Interval containment graphs	Containment	Intervals	Line	—
Rectangle intersection graphs	Intersection	Rectinlinear rectangles or boxes	Plane	—
Circle graphs	Intersection	Chords	Circle	—
Clique graphs	Intersection	Maximum cliques	Undirected graph	—
Triangulated graphs	Intersection	Subtrees	Tree	—
Comparability graphs	Intersection	Curves	Plane	Each curve is the graph of a function
Rectangle containment graphs	Containment	Rectilinear rectangles or boxes	Plane	—

Fig. 1. The table describing the classes of interval graphs, proper interval graphs, circle containment graphs, and others.

72] and complements of comparability (cocomparability) graphs are the intersection graphs of curves of continuous functions [42].

However, intersection is only one type of *interaction* between the objects which gives rise to a graph obtained from a representation $\mathcal{H}$. Other types of interaction, such as *overlap* (S_i and S_j intersect but neither contains the other), *containment* (either $S_i \subset S_j$ or $S_i \supset S_j$), or a *measured intersection* (see [40, 41]), give yet other kinds of graphs from $\mathcal{H}$. Any of our classes of graphs can be described by filling in a table as in Fig. 1.

This special issue of *Discrete Mathematics* brings together significant new results on a variety of problems involving interval graphs and other related classes of graphs as well as their representations by special families of sets. We will give an overview of these results followed by a discussion of a number of other directions of research.

An overview of this issue

Special classes of interval graphs

Benzaken, Hammer and de Werra [1] give a short proof that the class of interval graphs whose complements are interval graphs is equivalent to the split graphs having Dilworth number at most 2. A forbidden subgraph characterization is given for this class, and a linear-time recognition algorithm is also presented for them.

Skrien and Gimble [10] call an interval graph G *homogeneously representable* if for every vertex v there exists an interval representation of G in which the interval representing v is the left-most (or right-most) interval. They prove that an interval graph is homogeneously representable if and only if it has no induced subgraph isomorphic to the path on 5 vertices or the 'A'-graph on 5 vertices.

Interval graphs and interval orders

Fishburn [3] presents new and recent developments on the relationship between interval orders and interval graphs. An *interval order* is a partial order which is isomorphic to a family of intervals under the natural partial ordering where $I < J$ if I lies totally to the left of J. Fishburn considers several problems: the number of different interval lengths required to represent an interval order or an interval graph (the *interval count*), bounds on the lengths of the representing intervals, the number of distinct left (or right) endpoints that have to be used in a representation, and the number of distinct interval orders that have the same comparability graph. The interested reader will certainly wish to continue study on this topic from Fishburn's current book [30].

Interval thickness

Kirousis and Papadimitriou [7] define the *interval thickness* of a graph G to be the smallest clique number $\theta(H)$ over all supergraphs H of G, i.e., H is an interval graph on the same vertices as G and every edge of G is an edge of H. A node searching game played on an undirected graph G is introduced by the authors, and the *node search number* of G is the least number of searchers (or pebbles) needed to successfully clear all edges of G according to the rules of the game. It is shown that the interval thickness of G is equal to the node search number. As a corollary of previous results, they prove that computing the interval thickness of a graph is an NP-complete problem, but that checking whether the interval thickness is less than a constant can be done in polynomial time. These complexity results have also been discovered independently in [14]. The interval thickness would be a very useful parameter in the application to electrical circuits described in [62].

Multiple interval graphs

A *t-interval graph* is the intersection graph of sets each of which is the union of t intervals on the real line. We often call these graphs *multiple interval graphs.* The *interval number* of a graph G is the least number t such that G is a t-interval graph. The interval number of a graph on n vertices is bounded above by $\lceil (n+1)/4 \rceil$ and this is best possible. In their paper, Erdös and West [2] investigate lower bounds on the interval number including the result that for almost all graphs the interval number is bounded below by $n/4 \lg n$. Lower bounds on the interval number are also discussed in [58]. Scheinerman [9] considers the generalized notion of multiple intersection for higher dimensional analogues of intervals and for arbitrary families of sets. In most cases the generalized parameter goes to infinity as n goes to infinity.

Intersection classes of graphs

We have seen many different classes of intersection graphs being studied. Scheinerman [8] takes this one step further by investigating under what conditions an arbitrary class of graphs can be characterized as the class of intersection graphs of some specified family of sets.

Edge intersection graphs

As discussed earlier, one can vary both the type of sets that make up a representation $\mathscr{H}$ and the type of interaction between the sets that gives rise to the edges of the graph obtained from $\mathscr{H}$. Examples of this are the *overlap graphs* of intervals on a line [35], the *containment graphs* of boxes in d-space [25, 37], and the *edge intersection graphs* of paths in a tree (*EPT graphs*) [39]. Two paths in a tree are said to *edge intersect* if they share an edge. In Golumbic and Jamison [4] it is shown that the recognition problem for EPT graphs is NP-complete. A characterization of the graphs which are both EPT and path graphs is also given. (Path graphs are the vertex intersection graphs of paths in a tree.) Syslo [11] then proves that every triangulated EPT graph is a path graph. This relationship is illustrated in Fig. 2.

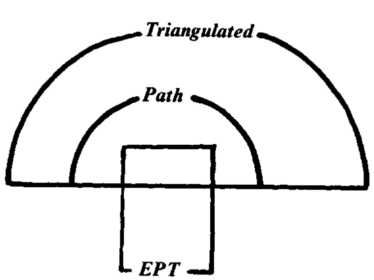

Fig. 2. The relationship between triangulated graphs, path graphs and EPT graphs.

Applications of clique separators

The paper by Tarjan [12] contains an $O(nm)$-time algorithm to decompose an n vertex m edge graph by means of clique separators, i.e., finding complete subgraphs whose removal disconnects the graph. Such a decomposition can be used with divide and conquer techniques to obtain algorithms for exact or approximate solutions to the maximum stable set, minimum fill-in, chromatic number and maximum clique problem on certain classes of graphs. The results are then applied to triangulated graphs, clique separable graphs, and EPT graphs. In particular, the methods provide polynomial time solutions to the maximum weighted stable set problem on EPT graphs and to an approximation algorithm to color an EPT graph to within 1.5 times the minimum number of colors.

Chromatic number and other parameters

The chromatic number problem is also analyzed for relatives of interval graphs in the three remaining papers of this special issue. Teng and Tucker [13] give an $O(qn)$-time algorithm to determine if a proper family of arcs on a circle is q-colorable. This algorithm gives rise to an $O(n^{\frac{3}{2}} \log n)$ algorithm for minimum coloring a proper family of arcs. Gyárfás and Lehel [6] and Gyárfás [5] provide a variety of results for bounding the clique cover number in terms of the stability number and bounding the chromatic number in terms of the clique number for several classes of graphs, including circular-arc, multiple interval, circle, d-dimensional box, and multiple box graphs.

Directions of current research

The topics addressed in the papers comprising this issue represent a number of current research directions. We briefly survey several other recent results and potential applications which have appeared elsewhere.

Applications of interval graphs

Numerous applications of interval graphs have appeared in the literature including applications to genetic structure, sequential storage, scheduling, seriation in archaeology and others summarized in [35]. More recent real world applications include pavement deterioration analysis [32], macro substitution [36, 38], file organization [34], protein sequencing [49], computer storage overlay [26], and circuit routing [50].

Triangulated graphs and acyclic hypergraphs

It is by now well known that every interval graph is a triangulated graph, and that the triangulated graphs are exactly the intersection graphs of subtrees of a tree. Recently, triangulated graphs have been found to have a connection with the theory of *relational databases* [15, 16, 18, 19, 24, 27, 28, 43, 59, 68].

With issues such as inconsistency, computational efficiency and database distributivity in mind, a number of researchers have defined and studied several basic, desirable properties of relational database schemes. What is most remarkable is that all of these properties have been shown to be equivalent and, moreover, *such a database scheme is equivalent to a family of subtrees of a tree.* Furthermore, these *acyclic* or *tree schemes* characterize exactly when good behavior is achieved since database schemes which fail to satisfy any of these properties exhibit certain unpleasant, pathological behavior. This connection is described in more detail in [37].

Containment graphs

The notion of the containment graph of a family $\mathcal{F}$ of subsets of a set was introduced earlier. Until recently only a few classes of containment graphs have been studied, for example, interval containment graphs. It is known that the containment graphs of rectilinear boxes (with sides parallel to the axes) in d-dimensional Euclidean space are precisely the graphs having partial order dimension at most $2d$. As a corollary one obtains the result that recognizing whether a graph G is the containment graph of rectangles in the plane is an NP-complete problem [37].

Boxicity and sphericity

Roberts [64] has shown that every undirected graph on n vertices is the intersection graph of boxes in $\lceil n/2 \rceil$-dimensional Euclidean space. Furthermore, the minimum dimensional space for which such an intersection representation is possible for a given graph G is called the *boxicity* of G. It has been shown that determining the boxicity of G is an NP-complete problem [23] and, moreover, that recognizing boxicity ≤ 3 is NP-complete [73]. Boxicity $= 1$ is simply the class of interval graphs. The complexity of boxicity ≤ 2 is still an open problem. The problem of *sphericity*, representing a graph as the intersection of spheres or unit spheres in higher dimensional Euclidean space, is studied in [29, 60, 61].

Computational geometry of rectangles

There has been great interest in several other computational problems concerning the geometry of rectangles. This interest is generated both for its own sake and by a number of practical applications including computer-aided design of VLSI circuits, numerical analysis and image processing [31, 44, 48, 51–57, 67, 69–71]. One such problem is the following: Given a set of n rectilinear boxes in the plane, report (in some order) all containments, (i.e., all m edges of the containment graphs.) This problem can be solved in $O(n \log^2 n + m)$ time [53, 69].

Acknowledgment

It has been my pleasure to serve as editor of this special issue. First and foremost, I would like to express my deep appreciation to the authors whose papers comprise this collection. It is upon the merit of these papers that this special issue rests. I was assisted in my editorship by a number of colleagues who served as referees, and I thank them very much. My communication with several authors and referees was greatly enhanced by the use of electronic mail via networks such as ARPANET, BITNET and CSNET. The speed with which I was able to receive final reports, especially toward the end of 1984, enabled this issue to go to press months earlier than would have otherwise been possible.

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