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Discrete Mathematics 309 (2009) 64–76
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Cycles through 4 vertices in 3-connected graphs

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Received 3 November 2004; received in revised form 14 December 2007; accepted 17 December 2007
Available online 5 February 2008

Abstract

S.C. Locke proposed a question: If G is a 3-connected graph with minimum degree d and X is a set of 4 vertices on a cycle in G, must G have a cycle through X with length at least $\min\{2d, |V(G)|\}$? In this paper, we answer this question.

Keywords: Cycles; 3-Connected graphs; Prescribed vertex set

1. Introduction

All graphs considered here are finite, undirected, and without loops or multiple edges. Dirac has given two well-known results about cycles. One [3] says that a k-connected graph has a cycle through any given k vertices in the graph. The other [4] is that if G is a 2-connected graph with minimum degree d, then G contains a cycle with length at least $\min\{2d, |V(G)|\}$. Starting with the two results, many researchers have considered long cycles through a prescribed vertex set or a prescribed edge set. Egawa et al. [5] proved that if G is a k-connected graph with minimum degree d and X is a set of k vertices in G, then G has a cycle through X with length at least $\min\{2d, |V(G)|\}$. Locke and Zhang [6] proved that if G is a 2-connected graph with minimum degree d and X is a set of 3 vertices on a cycle in G, then G has a cycle through X with length at least $\min\{2d, |V(G)|\}$.

We prove Theorem 1 which gives the answer to the following question proposed by S.C. Locke in [7].

Question. *If G is a 3-connected graph with minimum degree d and X is a set of 4 vertices on a cycle in G, must G have a cycle through X with length at least $\min\{2d, |V(G)|\}$?*

Theorem 1. *Let G be a 3-connected graph with minimum degree d and X be a set of 4 vertices on a cycle in G, then G contains a cycle through X with length at least $\min\{2d, |V(G)|\}$.*

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2007.12.076

2. Some lemmas and results

Let G be a 3-connected graph with minimum degree d and X be a set of 4 vertices on a cycle in G. For any two vertices $u, v \in V(G)$ and an integer k, a $(u, v; k)$-path denotes a path connecting u and v with length at least k. For a path P in G, we denote by $|P|$ the number of vertices that P contains. Suppose C is a longest cycle through X and $R = G - C$. When we consider a cycle, we always consider its orientation. Let C^+ be an orientation of C and C^- be its reverse orientation. Let $C^+ = c_1c_2\cdots c_mc_1$. $C^+[c_i, c_j]$ and $C^-[c_i, c_j]$ denote the segments of C with $C^+[c_i, c_j] = c_ic_{i+1}\cdots c_{j-1}c_j$ and $C^-[c_i, c_j] = c_ic_{i-1}\cdots c_{j+1}c_j$, respectively. Denote by $|C^+[c_i, c_j]|$ the number of vertices that $C^+[c_i, c_j]$ contains, and $|C^-[c_i, c_j]|$ is similarly defined. Also, let $C^+[c_i, c_j)$ be the segment $C^+[c_i, c_j] - c_j$. Analogously, $C^+(c_i, c_j]$, $C^+(c_i, c_j)$, $C^-[c_i, c_j)$, $C^-(c_i, c_j]$, $C^-(c_i, c_j)$ are also defined. We also denote $c_i^+ = c_{i+1}, c_i^- = c_{i-1}, c_i^{++} = c_{i+2}, c_i^{--} = c_{i-2}$.

For a component H of R, let $W(H) = N_C(H)$, and label the vertices of $W(H)$ along C^+ as $u_1, u_2, \ldots, u_r$. Let

$$W_2(H) = \{u_i \in W(H) : |N_H(\{u_i, u_{i+1}\})| \geq 2\} \quad \text{and} \quad W_1(H) = W(H) - W_2(H).$$

Also let

$$W_{2,0}(H) = \{u_i \in W_2(H) : C(u_i, u_{i+1}) \cap X = \phi\} \quad \text{and} \quad W_{2,1}(H) = W_2(H) - W_{2,0}(H).$$

Denote $w(H) = |W(H)|$ and for an index I, $w_I(H) = |W_I(H)|$.

We use [1] for terminology and notation not defined here. Before proving the main result, we first give some lemmas.

Lemma 1 *([2]). Let B be a 2-connected graph on at least 4 vertices, x, y, z be 3 distinct vertices of B and $k > 0$ an integer. Suppose that every vertex of B, except possibly x, y, z, has degree at least k, then there exist an $(x, y; k)$-path, an $(x, z; k)$-path and a $(y, z; k)$-path in B.*

Alternatively, if B is nonseparable on $|V(B)| = 3$ vertices, then $B = K_3$ and there are an $(x, y; 2)$-path, an $(x, z; 2)$-path and a $(y, z; 2)$-path in B.

Since C is a longest cycle through X, we can easily get the following lemma.

Lemma 2. *Let $u, v \in W(H)$, then*

(i) $W(H) \cap W(H)^+ = \phi$;
(ii) *There exists no path connecting u^+ and v^+ with all internal vertices in $R - H$;*
(iii) *There exists no path connecting u^- and v^- with all internal vertices in $R - H$;*
(iv) *Suppose that $|N_H(\{u, v\})| \geq 2$ and $v^+ \notin X$, then there exists no path connecting u^+ and v^{++} with all internal vertices in $R - H$;*
(v) *Suppose that $|N_H(\{u, v\})| \geq 2$ and $v^- \notin X$, then there exists no path connecting u^- and v^{--} with all internal vertices in $R - H$.*

Theorem 2. *Let G be a 3-connected graph with minimum degree d and X be a set of 4 vertices on a cycle in G. Suppose C is a longest cycle through X, if there exists a component H of $R = G - C$ such that $1 \leq |V(H)| \leq 3$, then $|V(C)| \geq 2d$.*

Proof. Suppose $C^+ = c_1c_2\cdots c_mc_1$, we may assume $m < 2d$. Then by Lemma 2 (i), $w(H) < d$. Hence $|V(H')| \geq 2$ for any component H' of R. So $2 \leq |V(H)| \leq 3$. Since G is 3-connected, $w(H) \geq 3$ and so $d \geq 4$. Suppose $W(H) = \{u_1, u_2, \ldots, u_r\}$ that are arranged along C^+, and let $u_{r+1} = u_1$. For $i \neq j$, denote by $P_H(u_i, u_j)$ a longest path joining u_i, u_j with all internal vertices in H. First we prove the following claim.

Claim 1. *Suppose that $|C^+(u_i, u_{i+1})| = |C^+(u_j, u_{j+1})| = 1$, $X \cap C^+(u_k, u_{k+1}) = \phi (i < j < k \leq r)$, H_1 and H_2 are components of R such that $N_{H_1}(u_i^+) \neq \phi$ and $N_{H_2}(u_j^+) \neq \phi$. Then*

(i) *$u_{j+1} \notin W(H_1)$ or $u_i \notin W(H_2)$;*

(ii) *If $|C^+(u_k, u_{k+1})| < |P_H(u_{j+1}, u_i)|$, then $u_k \notin W(H_1)$ or $u_{k+1} \notin W(H_2)$;*

(iii) *If $|C^+(u_k, u_{k+1})| < |P_H(u_{k+1}, u_{i+1})| - 1$, then $u_k \notin W(H_1)$; If $|C^+(u_k, u_{k+1})| < |P_H(u_k, u_i)| - 1$, then $u_{k+1} \notin W(H_1)$.*

Proof. (i) Suppose $u_{j+1} \in W(H_1)$ and $u_i \in W(H_2)$, then there is a path $P_{H_1}(u_i^+, u_{j+1})$ joining u_i^+, u_{j+1} with all internal vertices in H_1 and a path $P_{H_2}(u_i, u_j^+)$ with all internal vertices in H_2. Hence

$$u_i^+ P_{H_1}(u_i^+, u_{j+1}) C^+(u_{j+1}, u_i) P_{H_2}(u_i, u_j^+) C^-(u_j^+, u_i^+)$$

is a cycle through X longer than C, a contradiction.

(ii) Suppose $u_k \in W(H_1)$ and $u_{k+1} \in W(H_2)$, then there is a path $P_{H_1}(u_i^+, u_k)$ joining u_i^+, u_k with all internal vertices in H_1 and a path $P_{H_2}(u_j^+, u_{k+1})$ with all internal vertices in H_2. Hence

$$u_i^+ P_{H_1}(u_i^+, u_k) C^-(u_k, u_{j+1}) P_H(u_{j+1}, u_i) C^-(u_i, u_{k+1}) P_{H_2}(u_{k+1}, u_j^+) C^-(u_j^+, u_i^+)$$

is a cycle through X longer than C since $|C^+(u_k, u_{k+1})| < |P_H(u_{j+1}, u_i)|$, a contradiction.

(iii) If $u_k \in W(H_1)$, there is a path $P_{H_1}(u_k, u_i^+)$ joining u_k, u_i^+ with all internal vertices in H_1. Hence

$$u_k P_{H_1}(u_k, u_i^+) C^-(u_i^+, u_{k+1}) P_H(u_{k+1}, u_{i+1}) C^+(u_{i+1}, u_k)$$

is a cycle through X longer than C since $|C^+(u_k, u_{k+1})| < |P_H(u_{k+1}, u_{i+1})| - 1$, a contradiction. Similarly if $|C^+[u_k, u_{k+1}]| < |P_H(u_k, u_i)| - 1$, then $u_{k+1} \notin W(H_1)$. ■

We divide the proof into two cases.

Case 1. $H = K_2$ or $H = K_3^-$.

Then there exist $u, v \in V(H)$ with $|N_H(u)| = |N_H(v)| = 1$. Since $\delta(G) \geq d$, $|N_C(u)| \geq d - 1$ and $|N_C(v)| \geq d - 1$. Obviously we must have $|N_C(u)| = |N_C(v)| = r = d - 1$ and $N_C(u) = N_C(v) = W(H)$, and then $W_2(H) = W(H)$. Since $w_2(H) = d - 1$, and $|X| = 4$, then $w_{2,0}(H) \geq d - 5$. And if $u_i \in W_{2,0}(H)$, $|C^+(u_i, u_{i+1})| \geq 2$ since C is a longest cycle through X. We first prove

Claim 2. $1 \leq |C^+(u_i, u_{i+1})| \leq 2$ *for* $1 \leq i \leq d - 1$; *and if* $|C^+(u_i, u_{i+1})| = 2$, *then* $|C^+(u_j, u_{j+1})| = 1$ *for* $j \neq i (1 \leq i, j \leq d - 1)$.

Proof. If $|C^+(u_i, u_{i+1})| \geq 3$ for some i, then $m \geq 4 + 2(d - 2) = 2d$, a contradiction. And if there exist $1 \leq i, j \leq d - 1$ and $i \neq j$ such that $|C^+(u_i, u_{i+1})| = 2$ and $|C^+(u_j, u_{j+1})| = 2$, then $m \geq 3 \times 2 + 2(d - 3) = 2d$, also a contradiction. ■

Then by Claim 2, $w_{2,0}(H) \leq 1$ and hence $4 \leq d \leq 6$. For any $1 \leq i \leq r$, if $|C^+(u_i, u_{i+1})| = 1$, we know that $C^+(u_i, u_{i+1}) \cap X \neq \phi$, say $x_i \in C^+(u_i, u_{i+1})$. By Claim 2 and Lemma 2(ii) and (iii), $N_C(x_i) \subseteq W(H)$. Then since $N_H(x_i) = \phi$ and $w(H) = d - 1$, there should exist a component H_i of R such that $N_{H_i}(x_i) \neq \phi$. And obviously $W(H_i) \subseteq \{x_i\} \cup W(H) - \{u_i, u_{i+1}\}$.

Without loss of generality, we may assume $|C^+(u_1, u_2)| = 1$ and H_1 is component of R such that $N_{H_1}(x_1) \neq \phi$, $W(H_1) \subseteq \{x_1\} \cup W(H) - \{u_1, u_2\}$. Then if $d = 4$, we get $w(H_1) \leq 2$, a contradiction to that G is 3-connected. So we may assume $d = 5$ or 6. If $w_{2,0}(H) = 0$, then $d \leq 5$ and hence $d = 5$. By symmetry, we may assume $|C^+(u_2, u_3)| = 1$. Suppose $x_2 \in C^+(u_2, u_3) \cap X$ and H_2 is a component of R such that $N_{H_2}(x_2) \neq \phi$. Then $W(H_2) \subseteq \{x_2\} \cup W(H) - \{u_2, u_3\}$. By Claim 1 (i), $u_3 \notin W(H_1)$ or $u_1 \notin W(H_2)$, and hence $w(H_1) \leq 2$ or $w(H_2) \leq 2$, a contradiction to that G is 3-connected. If $w_{2,0}(H) = 1$, suppose $u_i \in W_{2,0}(H) (1 \leq i \leq r)$. Then by Claim 2, $|C^+(u_{i-2}, u_{i-1})| = 1$. Suppose $x' \in X \cap C^+(u_{i-2}, u_{i-1})$, H' is a component of R such that $N_{H'}(x') \neq \phi$ and then $W(H') \subseteq \{x'\} \cup W(H) - \{u_{i-2}, u_{i-1}\}$. By Claim 1(iii), $u_i, u_{i+1} \notin W(H')$. That means $w(H') \leq 2$, a contradiction to that G is 3-connected.

From the proof of Case 1, we may assume that each component of R has at least 3 vertices.

Case 2. $H = K_3$.

Suppose $V(H) = \{y_1, y_2, y_3\}$, then it is easy to know that $d - 2 \leq |N_C(y_i)| \leq d - 1$ for $i = 1, 2, 3$. Hence $|N_C(y_i) - N_C(\{y_j, y_k\})| \leq 1$ for any $1 \leq i, j, k \leq 3$. This implies $W(H) = W_2(H)$. Since $w_2(H) = r$, and $|X| = 4$, then $w_{2,0}(H) \geq r - 4$ and $w_{2,1}(H) \leq 4$. And if $u_i \in W_{2,0}(H)$, $|C^+(u_i, u_{i+1})| \geq 3$ since C is a longest cycle through X. Then if $w_{2,0}(H) \geq 2$, $m \geq 4 \times 2 + 2(r - 2) \geq 2d$. And if $r \geq 6$, we have $w_{2,0}(H) \geq 2$. We only need to prove the theorem when $w_{2,0}(H) \leq 1$ and $4 \leq d \leq 7$. We first prove

Claim 3. (i) $1 \leq |C^+(u_i, u_{i+1})| \leq 4$ for $1 \leq i \leq r$ *and if* $r = d - 1$, $1 \leq |C^+(u_i, u_{i+1})| \leq 2$ *for* $1 \leq i \leq r$;

(ii) *If* $|C^+(u_i, u_{i+1})| = 4$, *then* $|C^+(u_j, u_{j+1})| = 1$ *for* $j \neq i$;

(iii) *If* $|C^+(u_i, u_{i+1})| = 3$, *then* $|C^+(u_j, u_{j+1})| \le 2$ *for* $j \ne i$; *moreover, if* $|C^+(u_i, u_{i+1})| = 3$ *and* $|C^+(u_j, u_{j+1})| = 2$ *for* $j \ne i$, *then* $|C^+(u_k, u_{k+1})| = 1$ *for* $k \ne i, j$.

Proof. (i) If $|C^+(u_i, u_{i+1})| \ge 5$ for some $1 \le i \le r$, then $m \ge 6 + 2(r-1) \ge 2d$, a contradiction. If $r = d-1$ and $|C^+(u_i, u_{i+1})| \ge 3$ for some $1 \le i \le r$, then $m \ge 4 + 2(d-2) = 2d$, a contradiction.

(ii) If there exist $1 \le i, j \le r$ and $i \ne j$ such that $|C^+(u_i, u_{i+1})| = 4$ and $|C^+(u_j, u_{j+1})| \ge 2$, then $m \ge 5 + 3 + 2(r-2) \ge 2d$, a contradiction.

(iii) If there exist $1 \le i, j \le r$ and $i \ne j$ such that $|C^+(u_i, u_{i+1})| = 3$ and $|C^+(u_j, u_{j+1})| \ge 3$, then $m \ge 4 \times 2 + 2(r-2) \ge 2d$, a contradiction. If $|C^+(u_i, u_{i+1})| = 3$ and $|C^+(u_j, u_{j+1})| = 2$ for $1 \le i \ne j \le r$, then $m \ge 4 + 3 + 2(r-2) \ge 2d - 1$, so we must have $|C^+(u_k, u_{k+1})| = 1$ for $k \ne i, j$. ■

Then if $|C^+(u_i, u_{i+1})| = 1 (1 \le i \le r)$, say $x_i \in C^+(u_i, u_{i+1}) \cap X$, by Claim 3 and Lemma 2(ii)–(v), there exists a component H_i of R such that $N_{H_i}(x_i) \ne \phi$ and obviously $W(H_i) \subseteq \{x_i\} \cup W(H) - \{u_i, u_{i+1}\}$.

Subcase 2.1. $w_{2,0}(H) = 1$.

Then by Claim 3(i), $r = d - 2$ and hence $5 \le d \le 7$. By symmetry, we may assume $u_1 \in W_{2,0}(H)$ and $|C^+(u_2, u_3)| = 1$ by Claim 3(ii) or (iii). Suppose $x_1 \in C^+(u_2, u_3) \cap X$, H_1 is a component of R such that $N_{H_1}(x_1) \ne \phi$. Then $W(H_1) \subseteq \{x_1\} \cup W(H) - \{u_2, u_3\}$. Since G is 3-connected, $3 \le w(H_1) \le w(H) - 1$ and hence $r = w(H) \ge 4$. We may assume $d = 6$ and $r = 4$ or $d = 7$ and $r = 5$. Again by Claim 3(ii) or (iii), there exists another $u_j (j \ne 2)$ such that $|C^+(u_j, u_{j+1})| = 1$, say $x_2 \in C^+(u_j, u_{j+1})$. Suppose H_2 is a component of R such that $N_{H_2}(x_2) \ne \phi$ and then $W(H_2) \subseteq \{x_2\} \cup W(H) - \{u_j, u_{j+1}\}$.

If $|C^+(u_1, u_2)| = 3$, then by Claim 1(iii), $u_1 \notin W(H_1)$ and $u_1, u_2 \notin W(H_2)$. So if $d = 6$ and $r = 4$, we immediately get $w(H_1) \le 2$, a contradiction to that G is 3-connected. If $d = 7$ and $r = 5$, then by Claim 3(iii), at least one of $|C^+(u_3, u_4)| = 1$ and $|C^+(u_4, u_5)| = 1$ holds. Without loss of generality, let $j = 3$, then $W(H_2) \subseteq \{x_2\} \cup W(H) - \{u_1, u_2, u_3, u_4\}$. Thus we can get $w(H_2) \le 2$, also a contradiction to that G is 3-connected.

If $|C^+(u_1, u_2)| = 4$, then by Claim 3(ii), $|C^+(u_i, u_{i+1})| = 1$ for $\forall i \ne 1$. If $d = 6$ and $r = 4$, let $j = 3$, then by Claim 1(ii), $u_1 \notin W(H_1)$ or $u_2 \notin W(H_2)$. Thus we can get $w(H_1) \le 2$ or $w(H_2) \le 2$, a contradiction to that G is 3-connected. If $d = 7$ and $r = 5$, suppose $x_{i-1} \in C^+(u_i, u_{i+1}) \cap X$ for $i = 2, 3, 4, 5$ where $u_{5+1} = u_1$, and H_{i-1} is the component of R such that $N_{H_{i-1}}(x_{i-1}) \ne \phi$. By Lemma 2, we know that $W(H_{i-1}) \subseteq \{x_{i-1}\} \cup W(H) - \{u_i, u_{i+1}\}$. And by Claim 1(ii), if $u_1 \in W(H_1)$, then $u_2 \notin W(H_j)$ for $j = 2, 3, 4$. So if $u_1 \in W(H_1)$, we should have $W(H_2) = \{x_2, u_1, u_5\}$ and $W(H_3) = \{x_3, u_1, u_3\}$ since G is 3-connected. But by Claim 1(i), $u_5 \notin W(H_2)$ or $u_3 \notin W(H_3)$, a contradiction. So we may assume $u_1 \notin W(H_1)$, which means $W(H_1) = \{x_1, u_4, u_5\}$. And then by Claim 1(i), $u_2 \notin W(H_2)$ and $u_2 \notin W(H_3)$. Similarly we can get $W(H_2) = \{x_2, u_1, u_5\}$ and $W(H_3) = \{x_3, u_1, u_3\}$ and again by Claim 1 (i), $u_5 \notin W(H_2)$ or $u_3 \notin W(H_3)$, a contradiction.

Subcase 2.2. $w_{2,0}(H) = 0$.

Then $3 \le r = w(H) = w_{2,1}(H) \le 4$.

Subcase 2.2.1. $r = 3$. Then $d = 4$ or 5.

Claim 4. *If* $|C^+(u_i^+, u_{i+1})| \le 2$ *and* $C^+(u_i^+, u_{i+1}) \cap X = \phi$, *then* $u_j^- \notin N_C(u_i^+)$ *for any* $j \ne i$.

Proof. If $u_j^- \in N_C(u_i^+)$ for some $j \ne i$, then $u_i^+ u_j^- C^-(u_j^-, u_{i+1}) P_H(u_{i+1}, u_j) C^+(u_j, u_i^+)$ is a cycle through X and longer than C, a contradiction. ■

If $d = 4$, or $d = 5$ and $|C^+(u_i, u_{i+1})| \ge 3$ for some i, then by Claim 3(ii) or (iii), there exists u_j such that $|C^+(u_j, u_{j+1})| = 1$. Without loss of generality, assume $|C^+(u_1, u_2)| = 1$, say $x_1 \in X \cap C^+(u_1, u_2)$. Suppose H_1 is a component of R such that $N_{H_1}(x_1) \ne \phi$ and then $W(H_1) \subseteq \{x_1\} \cup W(H) - \{u_1, u_2\}$, which implies $w(H_1) \le 2$ since $w(H) = 3$, a contradiction. So we may assume $d = 5$ and $|C^+(u_i, u_{i+1})| \le 2$ for each $1 \le i \le 3$. Since $w(H) = 3$, there exists u_i such that $|C^+(u_i, u_{i+1}) \cap X| = 1$. Without loss of generality, suppose $|C^+(u_1, u_2) \cap X| = 1$ and $x_1 = u_1^+ \in X$. Then by Lemma 2 and Claim 4, $N_C(x_1) \subseteq W(H) \cup \{u_1^{++}\}$. Then there exists a component H_1 of R such that $N_{H_1}(x_1) \ne \phi$, say $y_1 \in N_{H_1}(x_1)$. By Lemma 2, we know that $W(H_1) \subseteq \{u_2, u_3, x_1\}$. Note that $|V(H_1)| \ge 3$, therefore $V(H_1) - \{y_1\} \ne \phi$. Since $x_1 = u_1^+ \in W(H_1)$, $|C^+(x_1, u_2)| \le 1$ and $C^+(x_1, u_2) \cap X = \phi$, it follows immediately that $u_2 \notin N_C(H_1 - \{y_1\})$. Then if $N_{H_1}(x_1) = y_1$, we have $|N_C(H_1 - \{y_1\})| \le 1$, a contradiction. Otherwise, if $|N_{H_1}(x_1)| \ge 2$, then $u_2 \notin W(H_1)$. Thus we have $w(H_1) \le 2$, a contradiction.

Subcase 2.2.2. $r = 4$. Then $d = 5$ or 6.

If $|C^+(u_i, u_{i+1})| \geq 2$ for $1 \leq i \leq 4$, then $m \geq 3r = 12 \geq 2d$, a contradiction. By symmetry, we may assume $|C^+(u_1, u_2)| = 1$ and $x_1 \in X \cap C^+(u_1, u_2)$. Suppose H_1 is a component of R such that $N_{H_1}(x_1) \neq \phi$ and hence $W(H_1) \subseteq \{x_1\} \cup W(H) - \{u_1, u_2\}$. Since G is 3-connected, $W(H_1) = \{x_1, u_3, u_4\}$.

If $d = 5$, or $d = 6$ and $|C^+(u_i, u_{i+1})| \geq 3$ for some $i (2 \leq i \leq 4)$, then by Claim 3(ii) or (iii), there exists $u_j (2 \leq j \leq 4)$ such that $|C^+(u_j, u_{j+1})| = 1$, say $x_2 \in X \cap C^+(u_j, u_{j+1})$. Suppose H_2 is a component of R such that $N_{H_2}(x_2) \neq \phi$ and then $W(H_2) \subseteq \{x_2\} \cup W(H) - \{u_j, u_{j+1}\}$. Since $u_3, u_4 \in W(H_1)$, then by Claim 1(i), $u_1 \notin W(H_2)$ if $j = 2, 3$ or $u_2 \notin W(H_2)$ if $j = 4$. In either case we have $w(H_2) \leq 2$, a contradiction. Hence we may assume $d = 6$ and $|C^+(u_i, u_{i+1})| \leq 2$ for each $i (2 \leq i \leq 4)$. Since $w(H) = 4$ and $w_{2,0}(H) = 0$, $|C^+(u_i, u_{i+1}) \cap X| = 1$ for $2 \leq i \leq 4$. Without loss of generality, we may assume $x_2 = u_3^+ \in X$. By Lemma 2 and Claim 4, $N_C(x_2) \subseteq W(H) \cup \{u_3^{++}\}$. Then there exists a component H_2 of R such that $N_{H_2}(x_2) \neq \phi$ and then $W(H_2) \subseteq \{x_2, u_1, u_2, u_4\}$. Since $W(H_1) = \{x_1, u_3, u_4\}$, $|C^+(x_2, u_4)| \leq 1$ and $C^+(x_2, u_4) \cap X = \phi$, then by Lemma 2, $u_1, u_2 \notin W(H_2)$. That means $w(H_2) \leq 2$, a contradiction. ■

From the proof of Theorem 2, we know that the condition of $H = K_2$ or K_3^- can be replaced by that there are two vertices $y_1, y_2 \in V(H)$ such that $|N_H(y_1)| = |N_H(y_2)| = 1$. And the condition of $H = K_3$ can be replaced by that there are three vertices $y_1, y_2, y_3 \in V(H)$ such that $|N_H(y_i)| \leq 2$ and there is a $(y_i, y_j; 2)$-path in H for $1 \leq i \neq j \leq 3$.

3. Proof of Theorem 1

Let $C = c_1c_2 \cdots c_mc_1$ be a longest cycle through $X = \{x_1, x_2, x_3, x_4\}$ in G and assume $m \leq 2d - 1$. If there is a component H' of R such that $|V(H')| \leq 3$ or H' is separable and there are two end blocks of H' with no more than 3 vertices, then Theorem 1 follows directly from Theorem 2. Then if a component H' of R is separable, we may assume at least one end block B of H' with not less than 4 vertices and b is the unique cut vertex in B. And then we can get a new graph G' by contracting $H - B$ to b and adding all the edges in $\{bu : u \in N_C(H - B)\}$. It is easy to see that G' is 3-connected and C is still a longest cycle through X in G': If there exists a component H' of R such that $w_{2,0}(H') \geq 2$. Choose $y \in V(H')$ such that $n(y) = |N_C(y)| = \max\{|N_C(x)| : x \in V(H')\}$. Then for any two vertices $y_1, y_2 \in V(H')$, there is a $(y_1, y_2; d-n(y))$-path by Lemma 1. Thus we have $m \geq 2(d-n(y)+2)+2(n(y)-2) = 2d$. From the above, we only need to prove Theorem 1 when every component H' of R has at least 4 vertices, 2-connected and $w_{2,0}(H') \leq 1$. Suppose H is a component of R such that $w_{2,0}(H) \geq w_{2,0}(H')$ for any component H' of R and then $X \cap W(H)$ is as maximal as possible. Since G is 3-connected and $|V(H)| \geq 4$, we can choose three disjoint edges y_1v_1, y_2v_2 and y_3v_3 in $E(H, C)$ where y_1, y_2, y_3 are three distinct vertices in H, v_1, v_2, v_3 are arranged along C^+. Suppose $y, y' \in V(H)$ such that $n(y) = |N_C(y)| = \max\{|N_C(x)| : x \in V(H) \setminus \{y_1, y_2, y_3\}\}$ and $n(y') = \max\{n(y), n(y_1)\}$. Then by Lemma 1, there exist a $(y_i, y_j; d - n(y))$-path $(1 \leq i \neq j \leq 3)$ and a $(y_i, y; d - n(y'))$-path $(i = 2, 3)$ in H, denoted by $P(y_i, y_j)$ and $P(y_i, y)$ respectively. Suppose $A = N_C(y) \cap \{v_1, v_2, v_3\}$ and $a = |A|$.

We divide the proof of Theorem 1 into two parts according to $w_{2,0}(H) = 0$ or 1.

Part I. $w_{2,0}(H) = 0$. Then $|\{v_1, v_2, v_3\} \cap X| \leq 1$.

Suppose $\{x_1, x_2, x_3, x_4\}$ are arranged along C^+. If $X \cap W(H) \neq \phi$, we may choose v_1, v_2, v_3 such that $\{v_1, v_2, v_3\} \cap X \neq \phi$, suppose $v_1 = x_1$ by symmetry. And then $v_2 \in C^+(x_2, x_3)$, $v_3 \in C^+(x_3, x_4)$. If $X \cap W(H) = \phi$, we may assume $v_1 \in C^+(x_1, x_2)$, $v_2 \in C^+(x_2, x_3)$ and $v_3 \in C^+(x_3, x_4)$ by symmetry. In either case, we have $\{x_2\} = C^+(v_1, v_2) \cap X$, $\{x_3\} = C^+(v_2, v_3) \cap X$ and $N_C(y) \cap C^+[x_1, x_4] \subseteq \{v_1, v_2, v_3\}$. Suppose $N_C(y) \cap C^+(x_4, x_1) = \{w_1, w_2, \ldots, w_q\}$. We first prove:

Claim 1. $|N_C(x_2)| + |N_C(x_3)| < 2d$.

Proof. Suppose $C_1 = C^+[v_1, x_2)$, $C_2 = C^+(x_2, v_2]$, $C_3 = C^+[v_2, x_3)$, $C_4 = C^+(x_3, v_3]$, $C_5 = C^+(v_3, x_4)$. If $v_1 \neq x_1$, suppose $C_6 = C^+[x_4, x_1]$, $C_7 = C^+(x_1, v_1)$. Otherwise suppose $C_6 = C^+[x_4, x_1)$. Denote $c_i = |C_i|$. Note that $m = \sum_{i=1}^{7} c_i + 1$ if $v_1 \neq x_1$ and $m = \sum_{i=1}^{6} c_i + 1$ if $v_1 = x_1$.

(i) $x_2x_3 \notin E(G)$.

If $x_2x_3 \in E(G)$, we know that $x_2x_3C^+(x_3, v_1)y_1P(y_1, y_2)v_2C^-(v_2, x_2)$ and $x_2x_3C^-(x_3, v_2)y_2P(y_2, y_3)v_3C^+(v_3, x_2)$ are two cycles through X. Thus we have $|C^+(v_1, x_2)| + |C^+(v_2, x_3)| \geq d - n(y) + 1$ and $|C^+(x_2, v_2)| + |C^+(x_3, v_3)| \geq d - n(y) + 1$. Then we have $m \geq 2d$, a contradiction.

(ii) *If* $x_2v_3^- \in E(G)$, then $N_{C_1}(x_3) = \phi$; if $x_3v_1^+ \in E(G)$, then $N_{C_4}(x_2) = \phi$.

If $x_2v_3^- \in E(G)$ and $s \in N_{C_1}(x_3)$, then $x_2v_3^-C^-(v_3^-, v_2)y_2P(y_2, y_3)v_3C^+(v_3, x_2)$ and $x_3sC^-(s, v_3)y_3P(y_3, y_2)v_2C^-(v_2, x_2)v_3^-C^-(v_3^-, x_3)$ are two cycles through X. Thus we have $|C^+(x_2, v_2)| \geq d - n(y) + 1$ and $|C^+(s, x_2)| + |C^+(v_2, x_3)| \geq d - n(y) + 1$, then we can get $m \geq 2d$, a contradiction to $m < 2d$. By symmetry, we can prove if $x_3v_1^+ \in E(G)$, then $N_{C_4}(x_2) = \phi$.

(iii) *If* $x_2v_2^+ \in E(G)$, *then* $N_{C_2}(x_3) = \phi$; *if* $x_3v_2^- \in E(G)$, *then* $N_{C_3}(x_2) = \phi$.

If $x_2v_2^+ \in E(G)$ and $s \in N_{C_2}(x_3)$, then $x_2v_2^+C^+(v_2^+, v_1)y_1P(y_1, y_2)v_2C^-(v_2, x_2)$ and $x_3sC^+(s, v_2)v_2y_2P(y_2, y_3)v_3C^+(v_3, x_2)v_2^+C^+(v_2^+, x_3)$ are two cycles through X. Thus we have $|C^+(v_1, x_2)| \geq d - n(y) + 1$ and $|C^+(x_2, s)| + |C^+(x_3, v_3)| \geq d - n(y) + 1$. So we can get $m \geq 2d$, a contradiction. By symmetry, we can prove if $x_3v_2^- \in E(G)$, then $N_{C_3}(x_2) = \phi$.

(iv) *If* $x_2v_3^+ \in E(G)$, $N_{C_5}(x_3) = \phi$; *if* $v_1 \neq x_1$ *and* $x_3v_1^- \in E(G)$, $N_{C_7}(x_2) = \phi$.

If $x_2v_3^+ \in E(G)$ and $s \in N_{C_5}(x_3)$, then $x_2v_3^+C^+(v_3^+, v_1)y_1P(y_1, y_3)v_3C^-(v_3, x_2)$ and $x_3sC^+(s, v_2)y_2P(y_2, y_3)v_3C^-(v_3, x_3)$ are two cycles through X. Thus we have $|C^+(v_1, x_2)| \geq d - n(y) + 1$ and $|C^+(v_2, x_3)| + |C^+(v_3, s)| \geq d - n(y) + 1$. So we can get $m \geq 2d$, a contradiction. If $v_1 \neq x_1$, we can prove if $x_3v_1^- \in E(G)$, then $N_{C_7}(x_2) = \phi$ by symmetry.

(v) $|N_{C_1 \cup C_4}(x_2)| + |N_{C_1 \cup C_4}(x_3)| \leq c_1 + c_4 + 2$.

If there exist $s_1 \in N_{C_1 - \{v_1\}}(x_2)$ and $s_2 \in C^+(s_1, x_2) \cap N_C(x_3)$, then $x_2s_1C^+(s_1, s_2)x_3C^+(x_3, v_1)y_1P(y_1, y_2)v_2C^-(v_2, x_2)$ and $x_2s_1C^-(s_1, v_3)y_3P(y_3, y_2)v_2C^+(v_2, x_3)s_2C^+(s_2, x_2)$ are two cycles through X. Thus we have $|C^+(v_1, s_1)| + |C^+(s_2, x_2)| + |C^+(v_2, x_3)| \geq d - n(y) + 1$ and $|C^+(s_1, s_2)| + |C^+(x_2, v_2)| + |C^+(x_3, v_3)| \geq d - n(y) + 1$. Then $m \geq 2d$, a contradiction. That means if $s \in N_{C_1 - \{v_1\}}(x_2)$, then $N_C(x_3) \cap C^+(s, x_2) = \phi$. Suppose $N_{C_1}(x_2) = \{s_1, s_2, \ldots, s_p\}$ and they are arranged along C^+, then if $s_1 \neq v_1$, $N_{C_1}(x_3) \subseteq C^+[v_1, s_1]$ and if $s_1 = v_1$, $N_{C_1}(x_3) \subseteq C^+[v_1, s_2]$. Then we can get $|N_{C_1}(x_2)| + |N_{C_1}(x_3)| \leq p + c_1 - (p - 2) = c_1 + 2$ and the equality holds only if $x_3v_1^+ \in E(G)$ and $N_{C_1}(x_2) \neq \phi$. By symmetry, we can prove $|N_{C_4}(x_2)| + N_{C_4}(x_3)| \leq c_4 + 2$ and the equality holds only if $x_2v_3^- \in E(G)$ and $N_{C_4}(x_3) \neq \phi$. Note the results of (ii), we know that $|N_{C_1 \cup C_4}(x_2)| + |N_{C_1 \cup C_4}(x_3)| \leq c_1 + c_4 + 2$.

(vi) $|N_{C_2 \cup C_3}(x_2)| + |N_{C_2 \cup C_3}(x_3)| \leq c_2 + c_3$.

If there exist $s_1 \in N_{C_2}(x_3)$ and $s_2 \in C^+(s_1, v_2) \cap N_C(x_2)$, then $x_3s_1C^+(s_1, s_2)x_2C^-(x_2, v_3)y_3P(y_3, y_2)v_2C^+(v_2, x_3)$ and $x_3s_1C^-(s_1, x_2)s_2C^+(s_2, v_2)y_2P(y_2, y_1)v_1C^-(v_1, x_3)$ are two cycles through X. Thus we have $|C^+(x_2, s_1)| + |C^+(s_2, v_2)| + |C^+(x_3, v_3)| \geq d - n(y) + 1$ and $|C^+(v_1, x_2)| + |C^+(s_1, s_2)| + |C^+(v_2, x_3)| \geq d - n(y) + 1$. Then $m \geq 2d$, a contradiction. That means if $s \in N_{C_2}(x_3)$, then $N_C(x_2) \cap C^+(s, v_2) = \phi$. Then we can get $|N_{C_2 - \{v_2\}}(x_2)| + |N_{C_2 - \{v_2\}}(x_3)| \leq c_2$ and the equality holds only if $x_3v_2^- \in E(G)$. By symmetry, we can prove $|N_{C_3 - \{v_2\}}(x_2)| + N_{C_3 - \{v_2\}}(x_3)| \leq c_3$ and the equality holds only if $x_2v_2^+ \in E(G)$. Note the results of (iii), we can get $|N_{C_2 \cup C_3}(x_2)| + |N_{C_2 \cup C_3}(x_3)| \leq c_2 + c_3$.

(vii) $|N_{C_5}(x_2)| + |N_{C_5}(x_3)| \leq c_5$; $|N_{C_7}(x_2)| + |N_{C_7}(x_3)| \leq c_7$ if $v_1 \neq x_1$.

If there exist $s_1 \in N_{C_5}(x_3)$ and $s_2 \in C^+(s_1, x_4] \cap N_C(x_2)$, then $x_3s_1C^+(s_1, v_2)y_2P(y_2, y_3)v_3C^-(v_3, x_3)$ and $x_3s_1C^-(s_1, v_3)y_3P(y_3, y_1)v_1C^-(v_1, s_2)x_2C^+(x_2, x_3)$ are two cycles through X. Thus we have $|C^+(v_2, x_3)| + |C^+(v_3, s_1)| \geq d - n(y) + 1$ and $|C^+(v_1, x_2)| + |C^+(x_3, v_3)| + |C^+(s_1, s_2)| \geq d - n(y) + 1$. Then $m \geq 2d$, a contradiction. That means if $s \in N_{C_5}(x_3)$, then $N_C(x_2) \cap C^+(s, x_4] = \phi$. Then we can get $|N_{C_5}(x_2)| + |N_{C_5}(x_3)| \leq p + c_5 - (p - 1) = c_5 + 1$ and the equality holds only if $x_2v_3^+ \in E(G)$ and $N_{C_5}(x_3) \neq \phi$. Note the results of (iv), we have $|N_{C_5}(x_2)| + |N_{C_5}(x_3)| \leq c_5$. If $v_1 \neq x_1$, we can prove $|N_{C_7}(x_2)| + N_{C_7}(x_3)| \leq c_7$ by symmetry.

(viii) *If* $v_1 = x_1$, *then* $|N_{C_6}(x_2)| + |N_{C_6}(X_3)| \leq c_6 - 1$.

If $v_1 = x_1$, then $q = 0$, which means $n(y) = a$. If $x_2x_4 \in E(G)$, then $x_2x_4C^-(x_4, v_2)y_2P(y_2, y_1)v_1C^+(v_1, x_2)$ and $x_2x_4C^+(x_4, v_1)y_1P(y_1, y_3)v_3C^-(v_3, x_2)$ are two cycles through X, thus $|C^+(x_2, v_2)| + |C^+(x_4, x_1)| \geq d - a + 1$ and $|C^+(x_1, x_2)| + |C^+(v_3, x_4)| \geq d - a + 1$. Then we can get $m \geq 2d$, a contradiction. Similarly we can prove $x_3x_4 \notin E(G)$.

If $s \in N_{C_6}(x_3)$, then $x_3sC^-(s, v_3)y_3P(y_3, y_1)v_1C^+(v_1, x_3)$ is a cycle through X, thus $|C^+(x_3, v_3)| + |C^+(s, x_1)| \geq d - a + 1$. Similarly if $t \in N_{C_6}(x_2)$, then $|C^+(x_2, v_2)| + |C^+(t, x_1)| \geq d - a + 1$. If there exist vertices $s, s^+ \in C^+(x_4, x_1)$ such that $x_3s, x_3s^+ \in E(G)$, then $x_3sC^-(s, v_3)y_3P(y_3, y_2)v_2C^-(v_2, s^+)x_3$ is a cycle through X, thus $|C^+(v_2, x_3)| + |C^+(x_3, v_3)| \geq d - a + 1$. Similarly if there exist vertices $t, t^+ \in C^+(x_4, x_1)$ such that $x_2t, x_2t^+ \in E(G)$, then $|C^+(v_1, x_2)| + |C^+(x_2, v_2)| \geq d - a + 1$. Hence if there exist vertices $s, s^+ \in C^+(x_4, x_1)$ such that $x_3s, x_3s^+ \in E(G)$, then $N_{C_6}(x_2) = \phi$, thus $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq c_6 - 1$. Similarly if there exist vertices $t, t^+ \in C^+(x_4, x_1)$ such that $x_2t, x_2t^+ \in E(G)$, then $N_{C_6}(x_3) = \phi$, which also means $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq c_6 - 1$.

So we may assume there exist no such vertices. We know if $x_2x_1^- \in E(G)$, $|C^+(x_2, v_2)| \geq d - a + 1$. And then $N_{C_6}(x_3) = \phi$, thus $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq c_6 - 2$. Similarly if $x_3x_1^- \in E(G)$, then $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq c_6 - 2$. If $x_2x_1^-, x_3x_1^- \notin E(G)$, then $|N_{C_6}(x_2)| \leq \frac{|C^+(x_4,x_1)|}{2}$ and $|N_{C_6}(x_3)| \leq \frac{|C^+(x_4,x_1)|}{2}$, and hence $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq |C^+(x_4, x_1)| = c_6 - 1$.

(ix) *If* $v_1 \neq x_1$, *then* $|N_{C_6 \cup C_7}(x_2)| + |N_{C_6 \cup C_7}(x_3)| \leq c_6 + c_7 - 1$.

Under this case, if $x_2x_1 \in E(G)$, then $x_2x_1C^-(x_1, v_2)y_2P(y_2, y_1)v_1C^+(v_1, x_2)$ is a cycle through X. Thus $|C^+(x_1, v_1)| + |C^+(x_2, v_2)| \geq d - n(y) + 1$. If $x_2x_4 \in E(G)$, then $x_2x_4C^+(x_4, v_1)y_1P(y_1, y_3)v_3C^-(v_3, x_2)$ is a cycle through X. Thus $|C^+(v_1, x_2)| + |C^+(v_3, x_4)| \geq d - n(y) + 1$. So $x_2x_1 \notin E(G)$ or $x_2x_4 \notin E(G)$. By symmetry, we can prove $x_3x_1 \notin E(G)$ or $x_3x_4 \notin E(G)$.

If $x_2x_1 \in E(G)$, then $N_C(x_3) \cap C^+[x_4, x_1) = \phi$. Otherwise, suppose $x_2x_1 \in E(G)$ and $s \in N_C(x_3) \cap C^+[x_4, x_1)$. Then if $q = 0$ or $C^+(s, x_1) \cap N_C(y) = \phi$, then $x_3sC^-(s, v_3)y_3P(y_3, y_1)v_1C^-(v_1, x_1)x_2C^+(x_2, x_3)$ is a cycle through X, thus $|C^+(v_1, x_2)| + |C^+(x_3, v_3)| + |C^+(s, x_1)| \geq d - n(y) + 1$; if $q \geq 1$ and $C^+(s, x_1) \cap N_C(y) \neq \phi$, choose a vertex $w_j \in C^+(s, x_1) \cap N_C(y)$ such that $C^+(s, w_j) \cap N_C(y) = \phi$. Then $x_3sC^-(s, v_3)y_3P(y_3, y)w_jC^+(w_j, x_3)$ is a cycle through X, thus $|C^+(x_3, v_3)| + |C^+(s, w_j)| \geq d - n(y') + 1$. In either case, together with $|C^+(x_1, v_1)| + |C^+(x_2, v_2)| \geq d - n(y) + 1$, we can get a contradiction that $m \geq 2d$. Similarly we can prove if $x_3x_4 \in E(G)$, then $N_C(x_2) \cap C^+(x_4, x_1] = \phi$; if $x_2x_4 \in E(G)$, then $N_C(x_3) \cap C^+(x_4, x_1] = \phi$; if $x_3x_1 \in E(G)$, then $N_C(x_2) \cap C^+[x_4, x_1) = \phi$.

Similarly as in the proof of (viii), if there exist two vertices $t, t^+ \in C^+(x_4, x_1) \cap N_C(x_2)$, then $|C^+(v_1, x_2)| + |C^+(x_2, v_2)| \geq d - n(y) + 1$. If $|N_C(x_3) \cap C^+(x_4, x_1)| \geq 2$, suppose $s, s' \in N_C(x_3) \cap C^+(x_4, x_1)$ and $s' \in C^+(s, x_1)$. Then if $q = 0$ or $C^+(s, s') \cap N_C(y) = \phi$, $x_3s'C^+(s', v_2)y_2P(y_2, y_3)v_3C^+(v_3, s)x_3$ is a cycle through X, thus $|C^+(v_2, x_3)| + |C^+(x_3, v_3)| + |C^+(s, s')| \geq d - n(y) + 1$. If $C^+(s, s') \cap N_C(y) \neq \phi$, choose a vertex $w_j \in C^+(s, s') \cap N_C(y)$ such that $C^+(s, w_j) \cap N_C(y) = \phi$. Then $x_3sC^-(s, v_3)y_3P(y_3, y)w_jC^+(w_j, x_3)$ is a cycle through X, thus $|C^+(x_3, v_3)| + |C^+(s, w_j)| \geq d - n(y') + 1$. So if there exist vertices $t, t^+ \in C^+(x_4, x_1)$ such that $x_2t, x_2t^+ \in E(G)$, then $|N_C(x_3) \cap C^+(x_4, x_1)| \leq 1$. Otherwise we can get a contradiction that $m \geq 2d$.

Then if $N_C(\{x_2, x_3\}) \cap \{x_1, x_4\} = \phi$, and there exist two vertices $t, t^+ \in C^+(x_4, x_1) \cap N_C(x_2)$, then $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq c_6 - 1$. By symmetry, if there exist vertices $s, s^+ \in C^+(x_4, x_1)$ such that $x_3s, x_3s^+ \in E(G)$, then $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq c_6 - 1$. And if there exists no such vertices, $|N_{C^+(x_4,x_1)}(x_2)| \leq \frac{|C^+(x_4,x_1)|+1}{2}$ and $|N_{C^+(x_4,x_1)}(x_3)| \leq \frac{|C^+(x_4,x_1)|+1}{2}$, and hence $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq c_6 - 1$. Together with the results of (vii), we know that $|N_{C_6 \cup C_7}(x_2)| + |N_{C_6 \cup C_7}(x_3)| \leq c_6 + c_7 - 1$.

If $N_C(\{x_2, x_3\}) \cap \{x_1, x_4\} \neq \phi$, without loss of generality, suppose $x_2x_1 \in E(G)$, then $x_2x_4 \notin E(G)$ and $N_C(x_3) \cap C^+[x_4, x_1) = \phi$. If $x_3x_1 \notin E(G)$, then $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq c_6 - 1$. If $x_3x_1 \in E(G)$, we know that $N_C(x_2) \cap C^+[x_4, x_1) = \phi$. Thus we have $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| = 2$. So if $c_6 \geq 3$, we have $|N_{C_6}(x_2)| + |N_{C_6}(x_3)| \leq c_6 - 1$. Then we can get $|N_{C_6 \cup C_7}(x_2)| + |N_{C_6 \cup C_7}(x_3)| \leq c_6 + c_7 - 1$. So we may assume $c_6 = 2$ and $x_2x_1, x_3x_1 \in E(G)$. Then if $c_7 = 0$, we know that $|C^+(x_2, v_2)| \geq d - n(y) + 1$ and $|C^+(x_3, v_3)| \geq d - n(y) + 1$, thus we can get a contradiction that $m \geq 2d$. Thus we know that $c_7 > 0$. If $s \in N_{C_7}(x_2)$, then $x_2sC^-(s, v_2)y_2P(y_2, y_1)v_1C^+(v_1, x_2)$ and $x_2sC^+(s, v_1)y_1P(y_1, y_3)v_3C^+(v_3, x_1)x_3C^-(x_3, x_2)$ are two cycles through X, then $|C^+(s, v_1)| + |C^+(x_2, v_2)| \geq d - n(y) + 1$ and $|C^+(x_1, s)| + C^+(v_1, x_2)| + C^+(x_3, v_3)| \geq d - n(y) + 1$, thus we can get $m \geq 2d$, a contradiction. Similarly we can prove if $x_2x_1 \in E(G)$, then $N_{C_7}(x_3) = \phi$. So if $x_2x_1, x_3x_1 \in E(G)$, then $|N_{C_7}(x_2)| + |N_{C_7}(x_3)| = 0 \leq c_7 - 1$ and then $|N_{C_6 \cup C_7}(x_2)| + |N_{C_6 \cup C_7}(x_3)| \leq c_6 + c_7 - 1$.

From the above, we know that $|N_C(x_2)| + |N_C(x_3)| \leq m < 2d$. ■

Since $w_{2,0}(H) = 0$, we know that $N_H(x_2) = N_H(x_3) = \phi$. By Claim 1, $|N_C(x_2)| + |N_C(x_3)| < 2d$, so there exists a component H_1 of R such that $N_{H_1}(x_i) \neq \phi$ for $i = 2$ or 3. Without loss of generality, suppose $N_{H_1}(x_2) \neq \phi$ which means $X \cap W(H_1) \neq \phi$. Then if $X \cap W(H) = \phi$, $|X \cap W(H_1)| > |X \cap W(H)|$. It contradicts to the choice of H. So we may assume $X \cap W(H) \neq \phi$. Then $v_1 = x_1$, $N_C(y) \subseteq \{v_1, v_2, v_3\}$ and thus $n(y) = a$. For H_1, we also can choose three vertex disjoint edges z_1x_2, z_2s and z_3s' in $E(H_1, C)$ where $s \in C^+(x_3, x_4)$ and $s' \in C^+(x_4, x_1)$. Suppose $z \in V(H_1) \setminus \{z_1, z_2, z_3\}$ such that $n(z) = |N_C(z)| = \max\{|N_C(x)| : x \in V(H_1) \setminus \{z_1, z_2, z_3\}\}$. Then by Lemma 1, there exists a $(z_i, z_j; d - n(z))$-path in H_1. For simplification, we denote such a path by $Q(z_i, z_j)(1 \leq i \neq j \leq 3)$. Now $x_2z_1Q(z_1, z_3)s'C^-(s', v_2)v_2y_2P(y_2, y_1)v_1C^+(v_1, x_2)$ is a cycle through X. Thus we have $|C^+(x_2, v_2)| + |C^+(s', v_1)| \geq d - a + 1 + d - n(z) + 1$. Then $m \geq (d - a + 1) + (d - n(z) + 1) + 2(\max\{a, n(z)\} - 2) + 2 \geq 2d$, a contradiction.

Part II. $w_{2,0}(H) = 1$.

Then we may choose v_1, v_2, v_3 such that $C^+(v_1, v_2) \cap X = \phi$, $|\{v_1, v_2\} \cap X|$ as large as possible and then $N_H(C^+(v_2, v_3)) \subseteq \{y_1, y_2\}$.

Case 1. $|C^+(v_2, v_3) \cap X| = 1$, say $x_1 \in C^+(v_2, v_3) \cap X$.

Since $w_{2,0}(H) = 1$, $N_C(y) \cap C^+[v_1, v_3] \subseteq \{v_1, v_2, v_3\}$ and $N_H(x_1) = \phi$. Suppose $N_C(y) \cap C^+(v_3, v_1) = \{w_1, w_2, \ldots, w_q\}$ where $q = n(y) - a$, and they are arranged along C^+. Let x'_2, x'_3 denote vertices lying in $C^+(v_3, v_1) \cap X$ such that $C^+(v_3, x'_2) \cap X = \phi$ and $C^+(x'_3, v_1) \cap X = \phi$ (It is possible that $x'_2 = x'_3$). Then if $q \neq 0$, $x'_2 \in C^+(v_3, w_1)$ and $x'_3 \in C^+(w_q, v_1)$ since $w_{2,0}(H) = 1$. We first prove

Claim 2. (1) *There exists no path connecting x_1 and a vertex in $C^+(v_1, v_2) \cup C^+(v_3, x'_2] \cup C^+[x'_3, v_1)$ with all internal vertices in $R - H$;*

(2) *If $C^+(w_j, w_{j+1}) \cap X = \phi$ for $1 \leq j \leq q - 1$, then there exists no path connecting x_1 and a vertex in $C^+[w_j, w_{j+1})$ with all internal vertices in $R - H$;*

(3) *If $|C^+(w_j, w_{j+1}) \cap X| = 1 (1 \leq j \leq q - 1)$, then there exists no path connecting x_1 and a vertex in $C^+(w_j, w_{j+1})$ with all internal vertices in $R - H$;*

(4) *If $C^+(x'_2, w_1) \cap X = \phi$, there exists no path connecting x_1 and a vertex in $C^+[x'_2, w_1)$ with all internal vertices in $R - H$;*

(5) *If $C^+(w_q, x'_3) \cap X = \phi$, there exists no path connecting x_1 and a vertex in $C^+(w_q, x'_3]$ with all internal vertices in $R - H$;*

(6) *If $C^+(s_1, s_2) \cap (X \cup N_C(y)) = \phi$ where $s_1, s_2 \in C^+[x'_2, x'_3]$, then there do not exist two disjoint paths connecting x_1 and two vertices in $C^+[s_1, s_2]$ with all internal vertices in $R - H$.*

Proof. (1) Otherwise suppose Q is a path connecting x_1 and a vertex $z \in C^+(v_1, v_2) \cup C^+(v_3, x'_2] \cup C^+[x'_3, v_1)$ with all internal vertices in $R - H$. Then if $z \in C^+(v_1, v_2)$, $x_1 QzC^-(z, v_3)y_3 P(y_3, y_2)v_2 C^+(v_2, x_1)$ and $x_1 QzC^+(z, v_2)y_2 P(y_2, y_1)v_1 C^-(v_1, x_1)$ are two cycles through X. Then we know that $|C^+(z, v_2)| + |C^+(x_1, v_3)| \geq d - n(y) + 1$ and $|C^+(v_1, z)| + |C^+(v_2, x_1)| \geq d - n(y) + 1$. Thus $m \geq (d - n(y) + 1) \times 2 + 4 + 2(n(y) - 3) = 2d$, a contradiction. If $z \in C^+(v_3, x'_2]$, then $x_1 QzC^+(z, v_2)y_2 P(y_2, y_3)v_3 C^-(v_3, x_1)$ is a cycle through X. So $|C^+(v_2, x_1)| + |C^+(v_3, z)| \geq d - n(y) + 1$. Together with $|C^+(v_1, v_2)| \geq d - n(y) + 1$, we have $m \geq 2d$, a contradiction. If $z \in C^+[x'_3, v_1)$, then $x_1 QzC^-(z, v_3)y_3 P_H(y_3, y_1)v_1 C^+(v_1, x_1)$ is a cycle through X. Similarly we have $|C^+(z, v_1)| + |C^+(x_1, v_3)| \geq d - n(y) + 1$, and then $m \geq 2d$, a contradiction.

(2) If $C^+(w_j, w_{j+1}) \cap X = \phi (1 \leq j \leq q - 1)$, suppose Q is a path connecting x_1 and a vertex $z \in C^+[w_j, w_{j+1})$ with all internal vertices in $R - H$, then $x_1 QzC^-(z, v_3)y_3 P(y_3, y)w_{j+1} C^+(w_{j+1}, x_1)$ is a cycle through X and we get $|C^+(x_1, v_3)| + |C^+(z, w_{j+1})| \geq d - n(y') + 1$. Together with $|C^+(v_1, v_2)| \geq d - n(y) + 1$, we have $m \geq (d - n(y') + 1) \times 2 + 5 + 2(n(y') - 3) > 2d$ since $N_C(y_1) \cap C^+(w_j, w_{j+1}) = \phi$, a contradiction.

(3) Say $x'_4 \in C^+(w_j, w_{j+1})$ for $1 \leq j \leq q - 1$, if there exists a path Q connecting x_1 and a vertex $z \in C^+(w_j, x'_4]$ with all internal vertices in $R - H$, then $x_1 QzC^+(z, v_2)y_2 P(y_2, y)w_j C^-(w_j, x_1)$ is a cycle through X. Thus we have $|C^+(v_2, x_1)| + |C^+(w_j, z)| \geq d - n(y') + 1$. Together with $|C^+(v_1, v_2)| \geq d - n(y) + 1$, we can get $m \geq (d - n(y') + 1) \times 2 + 4 + 2(n(y') - 3) = 2d$ since $N_C(y_1) \cap C^+(w_j, x'_4) = \phi$, a contradiction. Similarly we can prove there exists no path connecting x_1 and a vertex $z \in C^+(x'_4, w_{j+1})$ with all internal vertices in $R - H$.

(4) If $C^+(x'_2, w_1) \cap X = \phi$ and there exists a path Q connecting x_1 and a vertex $z \in C^+[x'_2, w_1)$, then $x_1 QzC^-(z, v_3)y_3 P(y_3, y)w_1 C^+(w_1, x_1)$ is a cycle through X. Thus we have $|C^+(x_1, v_3)| + |C^+(z, w_1)| \geq d - n(y') + 1$. Then together with $|C^+(v_1, v_2)| \geq d - n(y) + 1$, we have $m \geq 2d$, a contradiction.

(5) If $C^+(w_q, x'_3) \cap X = \phi$ and there exists a path Q connecting x_1 and a vertex $z \in C^+(w_q, x'_3]$, then $x_1 QzC^+(z, v_2)y_2 P(y_2, y)w_1 C^-(w_1, x_1)$ is a cycle through X. Thus we have $|C^+(v_2, x_1)| + |C^+(w_q, z)| \geq d - n(y') + 1$. Then together with $|C^+(v_1, v_2)| \geq d - n(y) + 1$, we have $m \geq 2d$, a contradiction.

(6) If $C^+(s_1, s_2) \cap (X \cup N_C(y)) = \phi$ and there are two disjoint paths Q_1, Q_2 connecting x_1 and two vertices $z, z' \in C^+[s_1, s_2]$ with all internal vertices in $R - H$. Without loss of generality, assume $z' \in C^+(z, s_2)$, then $x_2 Q_1 zC^-(z, v_3)y_3 P(y_3, y_2)v_2 C^-(v_2, z')Q_2 x_1$ is a cycle through X. Thus we have $|C^+(v_2, x_1)| + |C^+(x_1, v_3)| + |C^+(z, z')| \geq d - n(y) + 1$. Then together with $|C^+(v_1, v_2)| \geq d - n(y) + 1$, we have $m \geq 2d$ since $C^+(s_1, s_2) \cap N_C(y) = \phi$, a contradiction. ■

Claim 3. *If $w_{2,0}(H) = 1$ and $C^+(v_2, v_3) \cap X = \{x_1\}$, then $|N_C(x_1)| < d$.*

Proof. Suppose $z \in N_C(x_1)$, then by Claim 2(1), $z \notin C^+(v_1, v_2) \cup C^+(v_3, x_2'] \cup C^+[x_3', v_1)$. If $x_2' = x_3'$, $N_C(x_1) \subseteq C^+[v_2, v_3] \cup \{v_1\}$ which means $|N_C(x_1)| \leq |C^+[v_2, v_3]|$ since $x_1 \in C^+[v_2, v_3]$. If $x_2' \neq x_3'$, then $|C^+(x_2', x_3') \cap X| \leq 1$. If $|C^+(x_2', x_3') \cap X| = 1$, say $x_4' \in C^+(x_2', x_3')$, then $|N_C(x_1) \cap C^+(x_2', x_4']| \leq 1$ and $|N_C(x_1) \cap C^+[x_4', x_3')| \leq 1$ by Claim 2(2)–(6). If $C^+(x_2', x_3') \cap X = \phi$, $|N_C(x_1) \cap C^+(x_2', x_3')| \leq 1$ by Claim 2(6). In either case, $|N_C(x_1)| \leq |C^+[v_2, v_3]| + 2$.

Then if $|N_C(x_1)| \geq d$, we have $|C^+[v_2, v_3]| + 2 \geq d$. Together with $C^+(v_1, v_2)| \geq d - n(y) - 1 = d - (a+q) + 1$, we have $m \geq (d - a - q + 2) + (d - 2) + 3 + 2q = 2d$, a contradiction. ■

Since $N_H(x_1) = \phi$ and $|N_C(x_1)| < d$, there should exist a component H_1 of R such that $N_{H_1}(x_1) \neq \phi$ and then $W(H_1) \subseteq \{x_1\} \cup N_C(x_1)$ by Claim 2. For H_1, we can choose three disjoint edges z_1x_1, z_2s, z_3s' in $E(H_1, C)$ where z_1, z_2, z_3 are three disjoint vertices in $V(H_1)$. Suppose $z \in V(H_1) \setminus \{z_1, z_2, z_3\}$ such that $n(z) = |N_C(z)| = \max\{|N_C(x)| : x \in V(H_1) \setminus \{z_1, z_2\}\}$. Then by Lemma 2, we know that there exists a $(z_i, z_j; d - n(z))$-path for $1 \leq i \neq j \leq 3$ in H_1, denote it by $Q(z_i, z_j)$. If $s \in C^+[v_2, v_3]$ or $s' \in C^+[v_2, v_3]$, without loss of generality, suppose $s \in C^+(x_1, v_3]$, then $|C^+(x_1, s)| \geq d - n(z) + 1$. Together with $|C^+(v_1, v_2)| \geq d - n(y) + 1$, we can get $m \geq (d - n(y) + 1) + (d - n(z) + 1) + 2(\max\{n(y), n(z)\} - 2) \geq 2d$, a contradiction. So we assume $s, s' \notin C^+[v_2, v_3]$. If $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1\}$, then $s \in C^+[v_2, v_3]$ or $s' \in C^+(v_2, v_3]$. So we may assume $W(H_1) \nsubseteq C^+[v_2, v_3] \cup \{v_1\}$.

Note $X = \{x_1, x_2, x_3, x_4\}$, we suppose x_1, x_2, x_3, x_4 to be arranged along C^+ in the following proof. According to the different positions of $\{v_1, v_2, v_3\}$ on C, we prove the theorem in seven subcases by symmetry.

Subcase 1.1. $v_1 = x_3$, $v_2 = x_4$ and $v_3 \in C^+(x_1, x_2)$.

Subcase 1.2. $v_1 = x_4$, $v_2 \in C^+(x_4, x_1)$ and $v_3 = x_2$.

Subcase 1.3. $v_1 \in C^+(x_3, x_4)$, $v_2 = x_4$, and $v_3 = x_2$.

Under the above three subcases, we know that $N_C(x_1) \subseteq C^+[v_2, v_3] \cup \{v_1\}$ and then $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1\}$ by Claim 2, a contradiction.

Subcase 1.4. $v_1 = x_4$, $v_2 \in C^+(v_1, x_1)$ and $v_3 \in C^+(x_1, x_2)$.

Then $N_C(y) \subseteq C^+(x_2, x_3) \cup \{v_1, v_2, v_3\}$ since $w_{2,0}(H) = 1$. If $q \geq 2$, by Claim 2, $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1\}$, a contradiction. If $q \leq 1$, by Claim 2, $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1, w\}$ where $w \in C^+(x_2, x_3)$. Without loss of generality, we may assume $s = w$ and $s' = v_1 = x_4$. Then $w_{2,0}(H_1) = 1$ and $|X \cap W(H_1)| = 2 > |X \cap W(H)|$, which contradict the choice of H.

Subcase 1.5. $v_1 \in C^+(x_4, x_1)$, $v_2 \in C^+(v_1, x_1)$ and $v_3 = x_2$.

Then $N_C(y) \subseteq C^+(x_3, x_4) \cup \{v_1, v_2, v_3\}$ since $w_{2,0}(H) = 1$. If $q \geq 2$, then $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1\}$ by Claim 2, a contradiction. If $q \leq 1$, $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1, w\}$ where $w \in C^+(x_3, x_4)$ by Claim 2. Without loss of generality, we may assume $s = w$ and $s' = v_1$. Then we can choose H_1 instead of H and reverse the orientation of C, thus we can prove the theorem similarly as in Subcase 1.4.

Subcase 1.6. $v_1 \in C^+(x_3, x_4)$, $v_2 = x_4$, and $v_3 \in C^+(x_1, x_2)$.

Then $N_C(y) \subseteq C^+(x_2, x_3) \cup \{v_1, v_2, v_3\}$ since $w_{2,0}(H) = 1$. If $q \geq 2$, then $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1\}$ by Claim 2, a contradiction. If $q \leq 1$, $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1, w\}$ where $w \in C^+(x_2, x_3)$ by Claim 2. Without loss of generality, we may assume $s = w$ and $s' = v_1$. Note that $z \in V(H_1) - \{z_1, z_2, z_3\}$ with $n(z) = |N_C(z)| = \max\{|N_C(v)| : v \in V(H_1) - \{z_1, z_2, z_3\}$. It is easy to see that $N_C(z) \subseteq \{v_1, w\}$. Then $x_1z_1Q(z_1, z_3)v_1C^-(v_1, v_3)y_3P(y_3, y_2)v_2C^+(v_2, x_1)$ is a cycle through X. Thus we know that $|C^+(v_1, x_4)| + |C^+(x_1, v_3)| \geq d - n(y) + 1 + d - n(z) + 1$. Thus we can get $m \geq (d - n(y) + 2) + (d - n(z) + 2) + 2(\max\{n(y), n(z)\} - 2) \geq 2d$, a contradiction.

Subcase 1.7. $v_1 \in C^+(x_4, x_1)$, $v_2 \in C^+(v_1, x_1)$, and $v_3 \in C^+(x_1, x_2)$.

Then $N_C(y) \subseteq C^+(x_2, x_4) \cup \{v_1, v_2, v_3\}$ and $N_H(x_2) \subseteq \{y_3\}$ since $w_{2,0}(H) = 1$. Suppose $N_C(y) \cap C^+(x_2, x_3] = \{w_1, w_2, \ldots, w_{q_1}\}$, $N_C(y) \cap C^+(x_3, x_4) = \{w_{q_1+1}, \ldots, w_q\}$. If $q_1 \geq 2$ and $q - q_1 \geq 2$, then by Claim 2, $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1\}$, a contradiction. If $q_1 \geq 2$ and $q - q_1 \leq 1$, then $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1, w\}$ where $w \in C^+(x_3, x_4)$. Without loss of generality, suppose $s = w$ and $s' = v_1$. Then choose H_1 instead of H and reverse the orientation of C, we can prove the theorem similarly as in Subcase 1.4. If $q_1 \leq 1$ and $q - q_1 \geq 2$, then $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1, w\}$ where $w \in C^+(x_2, x_3)$. Without loss of generality, suppose $s = w$ and $s' = v_1$. Then choose H_1 instead of H, we can prove the theorem similarly as in Subcase 1.6. If $q_1 \leq 1$ and $q - q_1 \leq 1$, then $W(H_1) \subseteq C^+[v_2, v_3] \cup \{v_1, w, w'\}$ where $w \in C^+(x_2, x_3)$, $w' \in C^+(x_3, x_4)$. If $s = v_1$ or $s' = v_1$, we can prove the theorem similarly as in Subcase 1.4 or Subcase 1.6. So we may assume $s = w$ and $s' = w'$. Note that

$z \in V(H_1) - \{z_1, z_2, z_3\}$ with $n(z) = |N_C(z)| = \max\{|N_C(v)| : v \in V(H_1) - \{z_1, z_2, z_3\}$. It is easy to see that $N_C(z) \subseteq \{x_1, w, w'\}$. We can prove the following claim.

Claim 4. *There exists no path connecting x_2 and a vertex in $C^+(s, x_1) - \{s'\}$ with all internal vertices in $R - \{H, H_1\}$.*

Proof. Otherwise suppose K is a path connecting x_2 and a vertex $t \in C^+(s, x_1) - \{s'\}$ with all internal vertices in $R - \{H, H_1\}$. Then if $t \in C^+(s, x_3]$, then $x_2KtC^+(t, x_1)z_1Q(z_1, z_2)sC^-(s, x_2)$ is a cycle through X. So we have $|C^+(x_1, x_2)| + |C^+(s, t)| \geq d - n(z) + 1$. If $t \in C^+(x_3, s')$, then $x_2KtC^-(t, s)z_2Q(z_2, z_3)s'C^+(s', x_2)$ is a cycle through X. So we have $|C^+(x_2, s)| + |C^+(t, s')| \geq d - n(z) + 1$. If $t \in C^+(s', x_4)$, $x_2KtC^+(t, x_1)z_1Q(z_1, z_3)s'C^-(s', x_2)$ is a cycle through X. Then we know that $|C^+(x_1, x_2)| + |C^+(s', t)| \geq d - n(z) + 1$. If $t \in C^+[v_2, x_1)$, $x_2KtC^-(t, s)z_2Q(z_2, z_1)x_1C^+(x_1, x_2)$ is a cycle through X. Then we know that $|C^+(x_2, s)| + |C^+(t, x_1)| \geq d - n(z) + 1$. In any of the above four cases, together with $|C^+(v_1, v_2)| \geq d - n(y) + 1$, we can get $m \geq 2d$, a contradiction. If $t \in C^+[x_4, v_2)$, $x_2KtC^-(t, s)z_2Q(z_2, z_1)x_1C^-(x_1, v_2)y_2P(y_2, y_3)v_3C^+(v_3, x_2)$ is a cycle through X. Then we know that $|C^+(t, v_2)| + |C^+(x_1, v_3)| + |C^+(x_2, s)| \geq d - n(y) + 1 + d - n(z) + 1$. Thus we have $m \geq 2d$, a contradiction.

Then $N_C(x_2) \subseteq C^+[x_1, s] \cup \{s'\}$. So if $|N_C(x_2)| \geq d - 1$, then $m \geq 2d$, a contradiction. Since $|N_H(x_2)| \leq 1$ and $N_{H_1}(x_2) = \phi$ and $|N_C(x_2)| < d - 1$, there should exist a component H_2 of R such that $N_{H_2}(x_2) \neq \phi$ and then $W(H_2) \subseteq C^+[x_1, s] \cup \{s'\}$ by Claim 4. Thus we know that $w_{2,0}(H_2) \geq 1$, and if suppose $t \in W_{2,0}(H_2)$ and t' is the next vertex after t along C^+ in $W(H_2)$, then $C^+(t, t') \cap C^+(v_1, v_2) = \phi$. Then it is easy to get $m \geq 2d$, a contradiction.

Case 2. $|C^+(v_3, v_1) \cap X| = 1$, say $x_1 \in C^+(v_3, v_1) \cap X$.

Then we have $|N_C(y)| = a$. And in fact for any vertex $y' \in V(H) - \{y_1, y_2, y_3\}$, we have $N_C(y') \subseteq \{v_1, v_2, v_3\}$ by the choice of v_1, v_2, v_3. Thus we can reverse the orientation of C and then we can prove the theorem similarly as in Case 1.

Case 3. $|C^+(v_2, v_3) \cap X| \geq 2$ and $|C^+(v_3, v_1) \cap X| \geq 2$.

Then we may suppose $\{x_1, x_2\} = C^+(v_2, v_3) \cap X$ and $\{x_3, x_4\} = C^+(v_3, v_1) \cap X$ since $|X| = 4$. It is easy to see that $m \geq 9$ and hence $d \geq 5$. Since $w_{2,0}(H) = 1$ and by the choice of v_1, v_2, v_3, we know that $N_C(y) \subseteq C^+(x_3, x_4) \cup \{v_1, v_2, v_3\}$, $N_H(x_i) = \phi$ for $i = 1, 2, 4$ and $N_H(x_3) \subseteq \{y_3\}$. But if $N_C(y) \cap C^+(x_3, x_4) \neq \phi$, we can reverse the orientation of C and prove the theorem just as in Subcase 1.7. And if $y_3x_3 \in E(H, C)$, we can also reverse the orientation of C and prove the theorem similarly as in Subcase 1.5 or 1.6. So we may assume $N_C(y) \cap C^+(x_3, x_4) = \phi$, which means $n(y) = a$, and $N_H(x_i) = \phi$ for $1 \leq i \leq 4$.

Claim 5. $\sum_{i=1}^{4} |N_C(x_i)| < 4d$.

Proof. Denote $C_1 = C^+(v_1, v_2)$, $C_2 = C^+[v_2, x_1)$, $C_3 = C^+[x_1, x_2)$, $C_4 = C^+[x_2, x_3]$, $C_5 = C^+(x_3, x_4]$ and $C_6 = C^+(x_4, v_1]$. For $1 \leq i \leq 6$, let $c_i = |C_i|$.

(i) *$x_1v_1^+ \notin E(G)$, $x_4v_2^- \notin E(G)$, $x_1v_1^+ \notin E(G)$ and $x_2v_2^- \notin E(G)$; $x_3v_2 \notin E(G)$ or $x_1x_4 \notin E(G)$; $x_2v_1 \notin E(G)$ or $x_1x_4 \notin E(G)$; if $x_1x_4 \in E(G)$, then $N_{C_3-\{x_1\}}(x_3) = \phi$ and $N_{C_5-\{x_4\}}(x_2) = \phi$; $N_C(x_2) \cap C^+[x_4, v_1) = \phi$ and $N_C(x_3) \cap C^+(v_2, x_1] = \phi$.*

If $x_1v_1^+ \in E(G)$, then $x_1v_1^+C^+(v_1^+, v_2)y_2P(y_2, y_1)v_1C^-(v_1, x_1)$ is a cycle through X. Then we have $|C^+(v_2, x_1)| \geq d-a+1$, together with $|C^+(v_1, v_2)| \geq d-a+1$, we can get $m \geq 2d$, a contradiction. So $x_1v_1^+ \notin E(G)$. By symmetry, we can prove $x_4v_2^- \notin E(G)$. If $x_3v_1^+ \in E(G)$, then $x_3v_1^+C^+(v_1^+, v_3)y_3P(y_3, y_1)v_1C^-(v_1, x_3)$ is a cycle through X. Then we have $|C^+(v_3, x_3)| \geq d - a + 1$, together with $|C^+(v_1, v_2)| \geq d - a + 1$, we can get $m \geq 2d$, a contradiction. So $x_3v_1^+ \notin E(G)$. By symmetry, we can prove $x_2v_2^- \notin E(G)$.

If $x_3v_2, x_1x_4 \in E(G)$, then $x_4x_1C^+(x_1, v_3)y_3P(y_3, y_1)v_1C^+(v_1, v_2)x_3C^+(x_3, x_4)$ is a cycle through X. Thus we have $|C^+(v_2, x_1)| + |C^+(v_3, x_3)| + |C^+(x_4, v_1)| \geq d - a + 1$, together with $|C^+(v_1, v_2)| \geq d - a + 1$, we can get $m \geq 2d$, a contradiction. By symmetry, we can prove $x_2v_1 \notin E(G)$ or $x_1x_4 \notin E(G)$.

If $x_1x_4 \in E(G)$ and $s \in N_{C_3-\{x_1\}}(x_3)$, then $x_4x_1C^-(x_1, v_1)y_1P(y_1, y_3)v_3C^-(v_3, s)x_3C^+(x_3, x_4)$ is a cycle through X. Thus we have $|C^+(x_1, s)| + |C^+(v_3, x_3)| + |C^+(x_4, v_1)| \geq d - a + 1$, then $m \geq 2d$, a contradiction. By symmetry, we can prove if $x_1x_4 \in E(G)$, then $N_{C_5-\{x_4\}}(x_2) = \phi$.

If there is a vertex $s \in N(x_2) \cap C^+[x_4, v_1)$, then $x_2sC^-(s, v_3)y_3P(y_3, y_1)v_1C^+(v_1, x_2)$ is a cycle through X. Thus we have $|C^+(x_2, v_3)| + |C^+(s, v_1)| \geq d - a + 1$, then $m \geq 2d$, a contradiction. By symmetry, we can prove $N_C(x_3) \cap C^+(v_2, x_1] = \phi$.

(ii) $\sum_{i=1}^{4}|N_{C_1}(x_i)| \le 2c_1 - 2$;

Note that $c_1 \ge d-a+1 \ge 3$. If $s_1 \in N_{C_1}(x_1)$ and $s_2 \in N_C(x_2) \cap C^+(s_1, v_2)$, then $x_1s_1C^-(s_1, v_3)y_3P(y_3, y_2)v_2C^-(v_2, s_2)x_2C^-(x_2, x_1)$ and $x_1s_1C^+(s_1, s_2)x_2C^+(x_2, v_1)y_1P(y_1, y_2)v_2C^+(v_2, x_1)$ are two cycles through X. Thus we have $|C^+(x_2, v_3)| + |C^+(s_1, s_2)| + |C^+(v_2, x_1)| \ge d-a+1$ and $|C^+(x_1, x_2)| + |C^+(v_1, s_1)| + |C^+(s_2, v_2)| \ge d-a+1$, then $m \ge (d-a+1)\times 2+4 \ge 2d$, a contradiction. If $t_1 \in N_{C_1}(x_2)$ and $t_2 \in N_C(x_1) \cap C^+(t_1, v_2)$, then $x_2t_1C^-(t_1, v_3)y_3P(y_3, y_2)v_2C^-(v_2, t_2)x_1C^+(x_1, x_2)$ and $x_2t_1C^+(t_1, t_2)x_1C^-(x_1, v_2)y_2P(y_2, y_1)v_1C^-(v_1, x_2)$ are two cycles through X. Thus we have $|C^+(x_2, v_3)| + |C^+(t_1, t_2)| + |C^+(v_2, x_1)| \ge d-a+1$ and $|C^+(x_1, x_2)| + |C^+(v_1, t_1)| + |C^+(t_2, v_2)| \ge d-a+1$, then $m \ge (d-a+1)\times 2+4 \ge 2d$, a contradiction. Suppose $N_{C_1}(x_1) = \{s_1, s_2, \ldots, s_p\}$. Then if $p \ge 2$, $N_{C_1}(x_2) = \phi$ and thus $|N_{C_1}(x_1)| + |N_{C_1}(x_2)| \le p \le c_1 - 1$ since $x_1v_1^+ \notin E(G)$. If $p = 1$, then $N_{C_1}(x_2) \subseteq \{s_1\}$, thus $|N_{C_1}(x_1)| + |N_{C_1}(x_2)| \le 2 \le c_1 - 1$. If $p = 0$, then $|N_{C_1}(x_1)| + |N_{C_1}(x_2)| \le c_1 - 1$ since $x_2v_2^- \notin E(G)$. Thus we have $|N_{C_1}(x_1)| + |N_{C_1}(x_2)| \le c_1 - 1$. Similarly we can prove $|N_{C_1}(x_3)| + |N_{C_1}(x_4)| \le c_1 - 1$.

(iii) $\sum_{i=1}^{4}|N_{C_2}(x_i)| \le 2c_2 + 2$ *and the equality holds only if* $x_1x_4 \notin E(G)$; $\sum_{i=1}^{4}|N_{C_6}(x_i)| \le 2c_6 + 2$ *and the equality holds only if* $x_1x_4 \notin E(G)$;

By (i), $N_C(x_3) \cap C^+(v_2, x_1] = \phi$, so obviously $|N_{C_2}(x_1)| + |N_{C_2}(x_3)| \le c_2 + 1$, and the equality holds only if $x_3v_2 \in E(G)$. If $s_1 \in N_{C_2}(x_2)$, $s_2 \in N_C(x_4) \cap C^+(s_1, x_1)$, then $x_2s_1C^-(s_1, v_1)y_1P(y_1, y_3)v_3C^+(v_3, x_4)s_2C^+(s_2, x_2)$ is a cycle through X. Thus we have $|C^+(x_2, v_3)| + |C^+(x_4, v_1)| + |C^+(s_1, s_2)| \ge d-a+1$, together with $|C^+(v_1, v_2)| \ge d-a+1$, we can get $m \ge 2d$, a contradiction. So we know that $|N_{C_2}(x_2)| + |N_{C_2}(x_4)| \le c_2 + 1$. So $\sum_{i=1}^{4}|N_{C_2}(x_i)| \le 2c_2 + 2$ and the equality holds only if $x_3v_2 \in E(G)$, which means $x_1x_4 \notin E(G)$ by (i).

By symmetry, we can prove $\sum_{i=1}^{4}|N_{C_6}(x_i)| \le 2c_6 + 2$ and the equality holds only if $x_1x_4 \notin E(G)$.

(iv) $\sum_{i=1}^{4}|N_{C_3}(x_i)| \le 2c_3$ *and the equality holds only if* $x_1x_4 \in E(G)$ *or* $x_3x_2^- \in E(G)$; $\sum_{i=1}^{4}|N_{C_5}(x_i)| \le 2c_5$ *and the equality holds only if* $x_1x_4 \in E(G)$ *or* $x_2x_3^+ \in E(G)$;

If $s_1 \in N_{C^+(x_1,x_2)}(x_3)$ and $s_2 \in C^+(s_1, x_2) \cap N_C(x_1)$, then $x_3s_1C^-(s_1, x_1)s_2C^+(s_2, v_3)y_3P(y_3, y_2)v_2C^-(v_2, x_3)$ is a cycle through X. Thus we have $|C^+(s_1, s_2)| + |C^+(v_3, x_3)| + |C^+(v_2, x_1)| \ge d-a+1$, together with $|C^+(v_1, v_2)| \ge d-a+1$, we can get $m \ge 2d$, a contradiction. Since $x_1x_3 \notin E(G)$, so $|N_{C_3}(x_1)| + |N_{C_3}(x_3)| \le c_3$ and the equality holds only if $x_3x_2^- \in E(G)$. If $t_1 \in N_{C_2}(x_2)$, $t_2 \in C^+(t_1, x_2) \cap N_C(x_4)$, then $x_2t_1C^-(t_1, v_1)y_1P(y_1, y_3)v_3C^+(v_3, x_4)t_2C^+(t_2, x_2)$ is a cycle through X. Thus we have $|C^+(t_1, t_2)|+|C^+(x_2, v_3)|+|C^+(x_4, v_1)| \ge d-a+1$, together with $|C^+(v_1, v_2)| \ge d-a+1$, we can get $m \ge 2d$, a contradiction. So $|N_{C_3}(x_2)| + |N_{C_3}(x_4)| \le c_3 + 1$ and the equality holds only if $x_1x_4 \in E(G)$. But by (i), if $x_1x_4 \in E(G)$, then $x_3x_2^- \notin E(G)$. Thus $\sum_{i=1}^{4}|N_{C_3}(x_i)| \le 2c_3$ and the equality holds only if $x_1x_4 \in E(G)$ or $x_3x_2^- \in E(G)$.

By symmetry, we can prove $\sum_{i=1}^{4}|N_{C_5}(x_i)| \le 2c_5$ and the equality holds only if $x_1x_4 \in E(G)$ or $x_2x_3^+ \in E(G)$.

(v) $N_C(x_1) \cap C^+(v_3, x_3] = \phi$ *and* $N_C(x_4) \cap C^+[x_2, v_3) = \phi$; *if* $x_2v_3^+ \in E(G)$, *then* $N_C(x_1) \cap C^+(x_2, v_3] = \phi$ *and* $N_C(x_4) \cap C^+(x_2, v_3] = \phi$; *if* $x_3v_3^- \in E(G)$, *then* $N_C(x_1) \cap C^+[v_3, x_3) = \phi$ *and* $N_C(x_4) \cap C^+[v_3, x_3) = \phi$; $x_2x_3^+ \notin E(G)$ *or* $x_3x_4 \notin E(G)$; $x_3x_2^- \notin E(G)$ *or* $x_1x_2 \notin E(G)$.

If $s \in N_C(x_1) \cap C^+(v_3, x_3]$, then $x_1sC^+(s, v_2)y_2P(y_2, y_3)v_3C^-(v_3, x_1)$ is a cycle through X. Thus we have $|C^+(v_3, s)| + |C^+(v_2, x_1)| \ge d-a+1$. Together with $|C^+(v_1, v_2)| \ge d-a+1$, we can get $m \ge 2d$, a contradiction. By symmetry, we can prove $N_C(x_4) \cap C^+[x_2, v_3) = \phi$.

If $x_2v_3^+ \in E(G)$ and $s \in N_C(x_1) \cap C^+(x_2, v_3]$, then $x_2v_3^+C^+(v_3^+, v_2)y_2P(y_2, y_3)v_3C^-(v_3, s)x_1C^+(x_1, x_2)$ is a cycle through X. Thus we have $|C^+(x_2, s)| + |C^+(v_2, x_1)| \ge d-a+1$, together with $|C^+(v_1, v_2)| \ge d-a+1$, we can get $m \ge 2d$, a contradiction. If $x_2v_3^+ \in E(G)$ and $t \in N_C(x_4) \cap C^+(x_2, v_3]$, then $x_2v_3^+C^+(v_3^+, x_4)tC^+(t, v_3)y_3P(y_3, y_1)v_1C^+(v_1, x_2)$ is a cycle through X. Thus we have $|C^+(x_2, t)|+|C^+(x_4, v_1)| \ge d-a+1$, and then $m \ge 2d$, a contradiction. By symmetry, we can prove if $x_3v_3^- \in E(G)$, then $N_C(x_1)\cap C^+[v_3, x_3) = \phi$ and $N_C(x_4) \cap C^+[v_3, x_3) = \phi$.

If $x_2x_3^+, x_3x_4 \in E(G)$, then $x_2x_3^+C^+(x_3^+, x_4)x_3C^-(x_3, v_3)y_3P(y_3, y_1)v_1C^+(v_1, x_2)$ is a cycle through X. Thus we have $|C^+(x_2, v_3)| + |C^+(x_4, v_1)| \ge d-a+1$, together with $|C^+(v_1, v_2)| \ge d-a+1$, we have $m \ge 2d$, a contradiction. By symmetry, we can prove $x_3x_2^- \notin E(G)$ or $x_1x_2 \notin E(G)$.

(vi) $\sum_{i=1}^{4}|N_{C_4}(x_i)| \le 2c_4$; if $x_2x_3^+ \in E(G)$ or $x_3x_2^- \in E(G)$, $\sum_{i=1}^{4}|N_{C_4}(x_i)| \le 2c_4 - 1$; if $x_2x_3^+, x_3x_2^- \in E(G)$, $\sum_{i=1}^{4}|N_{C_4}(x_i)| \le 2c_4 - 2$;

Suppose $C_{41} = C^+[x_2, v_3)$, $C_{42} = C^+(v_3, x_4]$ and $c_{41} = |C_{41}|$, $c_{42} = |C_{42}|$.

By (v), $N_{C_{41}}(x_4) = \phi$ and $N_{C_{42}}(x_1) = \phi$. So we have $|N_{C_{41}}(x_2)| + |N_{C_{41}}(x_4)| \le c_{41} - 1$ and $|N_{C_{42}}(x_1)| + |N_{C_{42}}(x_3)| \le c_{42} - 1$. If $s_1 \in N_{C_{41}}(x_3)$ and $s_2 \in C^+(s_1, v_3) \cap N_C(x_1)$, then

$x_3s_1C^-(s_1,x_1)s_2C^+(s_2,v_3)y_3P(y_3,y_2)v_2C^-(v_2,x_3)$ is a cycle through X, thus $|C^+(s_1,s_2)|+|C^+(v_3,x_3)|+|C^+(v_2,x_1)|\geq d-a+1$. Together with $|C^+(v_1,v_2)|\geq d-a+1$, we can get $m\geq 2d$, a contradiction. So $|N_{C_{41}}(x_1)|+|N_{C_{41}}(x_3)|\leq c_{41}+1$ and the equality holds only if $x_1x_2\in E(G)$ and $x_3v_3^-\in E(G)$. By symmetry, we can prove $|N_{C_{42}}(x_2)|+|N_{C_{42}}(x_4)|\leq c_{42}+1$ and the equality holds only if $x_3x_4\in E(G)$ and $x_2v_3^+\in E(G)$. So $\sum_{i=1}^4|N_{C_{41}}(x_i)|\leq 2c_{41}$ and the equality holds only if $x_1x_2, x_3v_3^-\in E(G)$. $\sum_{i=1}^4|N_{C_{42}}(x_i)|\leq 2c_{42}$ and the equality holds only if $x_3x_4, x_2v_3^+\in E(G)$.

By (v), if $x_2v_3^+\in E(G)$ or $x_3v_3^-\in E(G)$, then $x_1v_3, x_4v_3\notin E(G)$. Then if one and only one of $x_3v_3^-\in E(G)$ and $x_2v_3^+\in E(G)$ holds, we have $\sum_{i=1}^4|N_{C_4}(x_i)|\leq 2c_4-1$. If both of $x_3v_3^-\in E(G)$ and $x_2v_3^+\in E(G)$ hold or both of $x_3v_3^-\notin E(G)$ and $x_2v_3^+\notin E(G)$ hold, we have $\sum_{i=1}^4|N_{C_4}(x_i)|\leq 2c_4$ and the equality holds only if $x_1x_2\in E(G)$ and $x_3x_4\in E(G)$. By (v), $x_2x_3^+\notin E(G)$ or $x_3x_4\notin E(G)$, $x_3x_2^-\notin E(G)$ or $x_1x_2\notin E(G)$, so if $x_2x_3^+\in E(G)$ or $x_3x_2^-\in E(G)$, then $\sum_{i=1}^4|N_{C_4}(x_i)|\leq 2c_4-1$; if $x_2x_3^+, x_3x_2^-\in E(G)$, then $\sum_{i=1}^4|N_{C_4}(x_i)|\leq 2c_4-2$.

From the above, we can get $\sum_{i=1}^4|N_C(x_i)|\leq 2m<4d$ easily. ■

Then there should exist a component H_1 of R such that $N_{H_1}(x_i)\neq\phi$ for some $i\in\{1,2,3,4\}$ which means $X\cap W(H_1)\neq\phi$. Then if $w_{2,0}(H_1)=1$, we can choose H_1 instead of H and prove the theorem similarly as in Case 1. So we may assume $w_{2,0}(H_1)=0$. We can choose three disjoint edges z_1x_i, z_2s and z_3s' in $E(H_1,C)$ where z_1,z_2,z_3 are three different vertices in $V(H_1)$. Suppose $z\in V(H_1)\setminus\{z_1,z_2,z_3\}$ such that $n(z)=|N_C(z)|=\max\{|N_C(v)|: v\in V(H_1)\setminus\{z_1,z_2,z_3\}\}$. Then by Lemma 1, there exists a $(z_i,z_j;d-n(z))$-path in H_1, denoted by $Q(z_i,z_j)$ for $1\leq i\neq j\leq 3$. By symmetry, we only need to prove the theorem for $i=1$ or 2.

For $i=1$, we may assume $N_{H_1}(x_3)=\phi$, $s\in C^+(x_2,x_3)$ and $s'\in C^+(x_3,x_4)$ since $w_{2,0}(H_1)=0$. And it is easy to see $N_C(z)\subseteq\{x_1,s,s'\}$.

Claim 6. *There exists no path connecting x_3 and a vertex in $C^+(v_1,x_2)$ with all internal vertices in $R-\{H,H_1\}$.*

Proof. Otherwise suppose K is a path connecting x_3 and a vertex $t\in C^+(v_1,x_2)$ with all internal vertices in $R-\{H,H_1\}$. Then if $t\in C^+(v_1,v_2]$, $x_3KtC^+(t,v_3)y_3P(y_3,y_1)v_1C^-(v_1,x_3)$ and $x_3KtC^-(t,s')z_3Q(z_3,z_1)x_1C^+(x_1,x_3)$ are two cycles through X. Then we know that $|C^+(v_1,t)|+|C^+(v_3,x_3)|\geq d-a+1$ and $|C^+(t,x_1)|+|C^+(x_3,s')|\geq d-n(z)+1$. Thus we can get $m\geq 2d$, a contradiction. And by Claim 5, $N_C(x_3)\cap C^+(v_2,x_1]=\phi$ which means $t\notin C^+(v_2,x_1]$. If $t\in C^+(x_1,x_2)$, then $x_3KtC^+(t,s)z_2Q(z_2,z_1)x_1C^-(x_1,x_3)$ is a cycle through X. So we have $|C^+(x_1,t)|+|C^+(s,x_3)|\geq d-n(z)+1$. Together with $|C^+(v_1,v_2)|\geq d-a+1$, we have $m\geq 2d$, a contradiction. ■

Then $N_C(x_3)\subseteq C^+[x_2,v_1]$. So if $|N_C(x_3)|\geq d$, $|C^+[x_2,v_1]|\geq d+1$, together with $|C^+(v_1,v_2)|\geq d-a+1$, we can get $m\geq 2d$, a contradiction. So we may assume $|N_C(x_3)|<d$. Since $N_H(x_3)=N_{H_1}(x_3)=\phi$, there should exist a component H_2 of R such that $N_{H_2}(x_3)\neq\phi$. Then $W(H_2)\subseteq C^+[x_2,v_1]$, thus $w_{2,0}(H_2)=1$ and $W(H_2)\cap X\neq\phi$, we can choose H_2 instead of H and prove the theorem similarly as in Subcase 1.4, 1.5 or 1.6.

For $i=2$, we may assume $N_{H_1}(x_3)=\phi$, $s\in C^+(x_3,x_4)$ and $s'\in C^+(x_4,x_1)$.

Claim 7. *There exists no path connecting x_3 and a vertex in $C^+(x_4,x_2)-\{v_2\}$ with all internal vertices in $R-\{H,H_1\}$.*

Proof. Otherwise suppose K is a path connecting x_3 and a vertex $t\in C^+(x_4,x_2)-\{v_2\}$ with all internal vertices in $R-\{H,H_1\}$. If $t\in C^+(x_4,v_2)$, $x_3KtC^-(t,s)z_2Q(z_2,z_1)x_2C^-(x_2,v_2)y_2P(y_2,y_3)v_3C^+(v_3,x_3)$ is a cycle through X. Then we know that $|C^+(t,v_2)|+|C^+(x_2,v_3)|+|C^+(x_3,s)|\geq d-a+1+d-n(z)+1$. Thus we can get $m\geq 2d$, a contradiction. By Claim 5, $N_C(x_3)\cap C^+(v_2,x_1]=\phi$ which means $t\notin C^+(v_2,x_1]$. If $t\in C^+(x_1,x_2)$, then $x_3KtC^-(t,s)z_2Q(z_2,z_1)x_2C^+(x_2,x_3)$ is a cycle through X. So we have $|C^+(t,x_2)|+|C^+(x_3,s)|\geq d-n(z)+1$. Together with $|C^+(v_1,v_2)|\geq d-a+1$, we can get $m\geq 2d$, a contradiction. ■

Then $N_C(x_3)\subseteq C^+[x_2,x_4]\cup\{v_2\}$. So if $|N_C(x_3)|\geq d$, $|C^+[x_2,x_4]|\geq d$, together with $|C^+(v_1,v_2)|\geq d-a+1$, we can get $m\geq 2d$, a contradiction. So we may assume $|N_C(x_3)|<d$. Since $N_H(x_3)=N_{H_1}(x_3)=\phi$, there should exist a component H_2 of R such that $N_{H_2}(x_3)\neq\phi$. Then $W(H_2)\subseteq C^+[x_2,x_4]\cup\{v_2\}$, thus $w_{2,0}(H_2)=1$ and $W(H_2)\cap X\neq\phi$, we can choose H_2 instead of H and prove the theorem similarly as in Subcase 1.4 or 1.6.

Acknowledgements

We are very grateful to the referees for their helpful suggestions.

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