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# Hamiltonian decomposition of recursive circulant graphs

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**Abstract**

The graph $G(N,d)$ has vertex set $V=\{0,1,\ldots,N-1\}$, with $\{v,w\}$ an edge if $v-w\equiv \pm d^i (\mathrm{mod}\, N)$ for some $0\leqslant i\leqslant \lceil \log_d N\rceil -1$. We show that the circulant graph $G(cd^m,d)$ is Hamilton decomposable for all positive integers $c,d$, and $m$ with $c<d$. This extends work of Micheneau and answers a special case of a question of Alspach. © 2000 Elsevier Science B.V. All rights reserved.

## 1. Introduction

Throughout this paper, the notation follows that of Micheneau [4]. Our results will concern the circulant graph $G(cd^m,d)$.

**Definition.** For two positive integers $N$ and $d$ with $N\geqslant d$, $G(N,d)$ is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_N;S)$ where $S=\{\pm d^i \mid 0\leqslant i\leqslant \lceil \log_d N\rceil -1\}$. In other words, the vertex set of $G(N,d)$ is $V=\{0,1,\ldots,N-1\}$, and two vertices $v$ and $w$ are connected if and only if $v-w=\pm d^i$ for some $0\leqslant i\leqslant \lceil \log_d N\rceil -1$.

We will study these graphs in the case $N=cd^m$ for some $0<c<d$. Micheneau treats the cases $d=4$ and $c=1,2$, that is, the case $G(2^k,4)$. This family of graphs has been studied as a possible topology for multicomputer networks [5].

**Definition.** A regular graph of degree $\delta$ is said to be *Hamilton decomposable* if it can be partitioned into $\delta/2$ edge-disjoint Hamiltonian cycles if $\delta$ is even, or $(\delta-1)/2$ edge-disjoint Hamiltonian cycles and a perfect matching if $\delta$ is odd.

In 1984, Alspach [1] asked whether every connected Cayley graph $\mathrm{Cay}(G;S)$ with $G$ abelian is Hamilton decomposable. Quite a bit of progress has been made on this

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problem [3]. Micheneau constructed Hamiltonian decompositions of $G(2^m, 4)$. In this paper, we construct Hamiltonian decompositions of $G(cd^m, d)$ for all positive integers $d$ and $m$, and all $0 < c < d$. The basic idea of the construction is inductive. In Section 2, we partition the vertices of $G(cd^m, d)$ into $d$ sets each of which induces $G(cd^{m-1}, d)$ as a subgraph; furthermore, it is shown that the edges of $G(cd^m, d)$ not contained in any of these subgraphs form a single Hamiltonian cycle. The basic idea of our proof will be to take a family of $k$ edge-disjoint Hamiltonian cycles on $G(cd^{m-1}, d)$ and construct from it $k$ edge-disjoint Hamiltonian cycles on $G(cd^m, d)$; each cycle will be constructed by 'linking' together the $d$ copies of one of the cycles on $G(cd^{m-1}, d)$ corresponding to the $d$ parts of the partition of the vertices of $G(cd^m, d)$. Since the degree of each vertex of $G(cd^m, d)$ is two greater than the degree of each vertex of $G(cd^{m-1}, d)$, we only need to find one more Hamiltonian cycle on $G(cd^m, d)$ to obtain a decomposition. This cycle is obtained by modifying the cycle constructed in Section 2.

The arguments used work most cleanly in the case $d \geqslant 4$. The proof of this case is the one which most closely mirrors Micheneau's proof. Sections 3 and 4 do the stickier cases $d = 2$ and 3, and the proof of the case $d \geqslant 4$ is presented in Section 5.

## 2. An inductive construction

Our construction of the Hamiltonian decomposition of $G(cd^m, d)$ uses induction on $m$. In order to make the induction work, we will need to find induced subgraphs of $G(cd^m, d)$ isomorphic to $G(cd^{m-1}, d)$. To do so, we write the vertices of $G(cd^m, d)$ in base $d$, so a vertex is denoted by an $(m+1)$-tuple $(x_0, x_1, \ldots, x_m)$ where $0 \leqslant x_0 < c$, and for all $i > 0$, $0 \leqslant x_i < d$. For example, Fig. 1 shows the graph $G(9, 3)$ with its vertices labeled in base 3 (since $x_0 = 0$ for all vertices, we label the vertices simply $x_1 x_2$).

For any integer $j$ with $0 \leqslant j < d$, we denote by $G_j(cd^m, d)$ (or, when it is clear from context, simply $G_j$) the induced subgraph of $G(cd^m, d)$ containing exactly those vertices $v$ with $v \equiv j \pmod d$; that is, those vertices $v$ whose last digit is $j$ in the base $d$ representation.

**Lemma 1.** *For every $0 \leqslant j < d$, we have $G_j(cd^m, d) \cong G(cd^{m-1}, d)$. Furthermore, the edges of $G(cd^m, d)$ not contained in any $G_j$ form a Hamiltonian cycle.*

**Proof.** Two vertices $v, w$ of $G_j$ are adjacent if and only if they differ by some $d^i$ with $0 \leqslant i \leqslant m + a(c)$, where $a(c) = -1$ if $c = 1$ and $a(c) = 0$ otherwise. However, they certainly never differ by 1, so we conclude that $\{v, w\}$ is an edge if and only if $v - w \equiv \pm d^i \pmod{cd^m}$ for some $1 \leqslant i \leqslant m + a(c)$. Now, let $\varphi_j(v) = 1/d(v - j)$. Then $\varphi_j$ maps the vertices of $G_j$ to the vertices of $G(cd^{m-1}, d)$. Furthermore, $\varphi_j(v - w) = 1/d(v - w)$, so $v - w \equiv \pm d^i \pmod{cd^m}$ if and only if $\varphi_j(v) - \varphi_j(w) \equiv \pm d^{i-1} \pmod{cd^{m-1}}$, so $\varphi_j$ is an isomorphism of graphs.

We now must examine the edges not contained in any $G_j$. Suppose $\{v, w\}$ is such an edge. Then $v - w \equiv \pm d^i \pmod{cd^m}$ for some $i$. But if $i > 0$, then $\{v, w\}$ is an edge

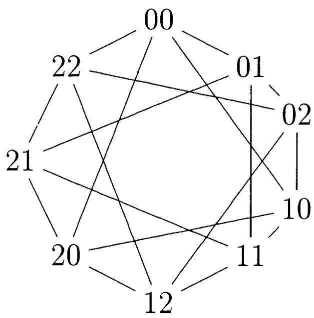

Fig. 1. The graph $G(9,3)$.

of $G_j$ for $v \equiv j \pmod d$. Thus, the edges we must consider are precisely the edges of the form $\{v, v+1\}$. But these obviously form a Hamiltonian cycle (in fact, they form the subgraph $\mathrm{Cay}(\mathbb{Z}_{cd^m}; \{\pm 1\})$), so the proof is complete. □

We call the Hamiltonian cycle constructed in Lemma 1 the *basic cycle*.

## 3. The case $d = 2$

Our general techniques will not apply to the case $d = 2$, so we must treat this case separately. The basic idea is to use Lemma 1 to inductively build our Hamiltonian cycles. However, we will need to consider only Hamiltonian cycles having an additional property.

**Definition.** A Hamiltonian cycle in $G(2^m, 2)$ is said to be a *$\lambda$-cycle* if it contains two consecutive vertices $X_1, X_2$, called a *$\lambda$-sequence*, such that $X_2 - X_1 = 1$ and $X_1 \equiv 0 \pmod 2$.

Notice that two $\lambda$-sequences must either be equal or disjoint. Therefore, two edge-disjoint $\lambda$-cycles contain vertex-disjoint $\lambda$-sequences.

Note that $G(2^m, 2)$ is regular of degree $2m-1$. We will show that it is decomposable into $m-1$ Hamiltonian $\lambda$-cycles and a perfect matching.

**Lemma 2.** *Given $k$ edge-disjoint Hamiltonian $\lambda$-cycles in $G(2^{m-1}, 2)$, we can construct $k+1$ edge-disjoint Hamiltonian $\lambda$-cycles in $G(2^m, 2)$.*

**Proof.** We first give an algorithm that, given a Hamiltonian $\lambda$-cycle $C$ in $G(2^{m-1}, 2)$, produces a Hamiltonian $\lambda$-cycle in $G(2^m, 2)$. Given such a cycle in $G(2^{m-1}, 2)$, let $X_1$, $X_2$ be a $\lambda$-sequence for $C$. Then $G(2^{m-1}, 2)$ has two copies, $G_0$ and $G_1$, in $G(2^m, 2)$, which are the induced subgraphs made up of the even and odd vertices, respectively.

Let $C^0$ and $C^1$ be the images of $C$ in $G_0$ and $G_1$ respectively, and let $X_1^0$, $X_1^1$ and $X_2^0$, $X_2^1$ denote the images of $X_1$ and $X_2$. Then let $C^+$ be the cycle in $G(2^m, 2)$ which is the union of $C^0$ and $C^1$, except that we replace the edges $\{X_1^0, X_2^0\}$ and $\{X_1^1, X_2^1\}$ by the edges $\{X_1^0, X_1^1\}$ and $\{X_2^0, X_2^1\}$. Then $C^+$ is certainly a Hamiltonian cycle, and $X_1^0$, $X_1^1$ forms a $\lambda$-sequence for $C^+$.

Now, suppose that we begin with two edge-disjoint Hamiltonian $\lambda$-cycles $C_1$ and $C_2$ in $G(2^{m-1}, 2)$; we would like to show that $C_1^+$ and $C_2^+$ are disjoint. Since $C_1^0 \cup C_1^1$ and $C_2^0 \cup C_2^1$ are edge-disjoint, we need only consider the two other edges we are adding into each cycle. But since the $\lambda$-sequences for $C_1$ and $C_2$ are necessarily disjoint, the new edges are also all distinct.

Hence, we have constructed $k$ edge-disjoint Hamiltonian $\lambda$-cycles in $G(2^m, 2)$; we still must construct one more. We begin with the basic cycle of Lemma 1. We needed to use some of the edges of the basic cycle in constructing the first $k$ cycles; however, every use of edges from the basic cycle consisted of taking the edges $\{c00, c01\}$ and $\{c10, c11\}$ from the basic cycle and using them to replace the edges $\{c00, c10\}$ and $\{c01, c11\}$ (here, $c$ is any binary string of length $m-2$). Thus, we can take the edges $\{c00, c10\}$ and $\{c01, c11\}$ and add them into the basic cycle. The result of this switch is that instead of reading $(\ldots, c00, c01, c10, c11, \ldots,)$, the modified basic cycle now reads $(\ldots, c00, c10, c01, c11, \ldots)$. Hence, we still have a Hamiltonian cycle. We must check that it is a $\lambda$-cycle. In the unmodified basic cycle, there are $2^{m-1}$ pairs of vertices that could serve as a $\lambda$-sequence. Each modification removes 2 of these pairs; however, the number of modifications is obviously bounded by the number of distinct edge-disjoint Hamiltonian cycles in $G(2^{m-1}, 2)$, namely $m-2$. Hence, there are at least $2^{m-1} - 2(m-2) > 0$ pairs of vertices that could serve as a $\lambda$-sequence remaining after all the modifications. Therefore, the modified basic cycle is a $\lambda$-cycle and the proof is complete. □

**Theorem 1.** *For all $m$, $G(2^m, 2)$ is decomposable into $m-1$ edge-disjoint Hamiltonian $\lambda$-cycles and one perfect matching.*

**Proof.** Certainly if we can find $m-1$ edge-disjoint Hamiltonian cycles, then the remaining edges will form a perfect matching. We construct the $m-1$ cycles by induction. The case $m=2$ is easy because $G(4, 2)$ is just a $K_4$, and Lemma 2 takes care of the induction step.

## 4. The case $d = 3$

Our general construction will also not work in the case $d=3$, so we must treat this case separately as well. However, as before, our induction will not work for arbitrary Hamiltonian cycles; we must introduce an additional bit of structure analogous to the $\lambda$-cycle property.

**Definition.** A Hamiltonian cycle in $G(3^m c, 3)$ is said to be a $\mu$*-cycle* if it contains three consecutive vertices $X_0, X_1, X_2$, called a $\mu$*-sequence*, such that $X_2 - X_1 = X_1 - X_0 = 1$.

Notice that two $\mu$-sequences are either equal or disjoint, or share exactly 1 vertex. Therefore, two edge-disjoint $\mu$-cycles must either contain disjoint $\mu$-sequences or $\mu$-sequences of the form $X_0$, $X_1$, $X_2$ and $X_2$, $X_3$, $X_4$ with $X_1 - X_0 = X_2 - X_1 = X_3 - X_2 = X_4 - X_3 = 1$.

We will show that $G(3^m c, 3)$ is decomposable into Hamiltonian $\mu$-cycles (and a perfect matching if $c = 2$).

**Lemma 3.** *Suppose $m \geqslant 3$ or $m = c = 2$. Then given $k$ edge-disjoint Hamiltonian $\mu$-cycles in $G(3^{m-1} c, 3)$, we can construct $k + 1$ edge-disjoint Hamiltonian $\mu$-cycles in $G(3^m c, 3)$.*

**Proof.** Suppose we are given a Hamiltonian $\mu$-cycle $C$ in $G(3^{m-1} c, 3)$ with $\mu$-sequence $X_0$, $X_1$, $X_2$; we will construct a cycle $C^+$ in $G(3^m c, 3)$. Recall that $G(3^{m-1} c, 3)$ has three disjoint copies, $G_0$, $G_1$, and $G_2$ in $G(3^m c, 3)$, which are the induced subgraphs made up of vertices which are 0, 1, and 2 modulo 3, respectively. Denote by $X_i^j$ and $C^i$ the copies of $X_i$ and $C$ in $G_j$. Then let $C^+$ be the cycle $C^0 \cup C^1 \cup C^2$, with the edges $\{X_0^0, X_1^0\}, \{X_0^1, X_1^1\}, \{X_1^1, X_2^1\}$, and $\{X_1^2, X_2^2\}$ replaced by the edges $\{X_0^0, X_0^1\}, \{X_1^0, X_1^1\}$, $\{X_1^1, X_1^2\}$, and $\{X_2^1, X_2^2\}$. Then $C^+$ is a Hamiltonian cycle, and $X_1^0, X_1^1, X_1^2$ serves as a $\mu$-sequence for $C^+$.

Now, suppose we began with two disjoint Hamiltonian $\mu$-cycles $C_1$ and $C_2$ in $G(3^{m-1} c, 3)$. Then we must show that $C_1^+$ and $C_2^+$ are also disjoint. Certainly the edges of $C_1^0 \cup C_1^1 \cup C_1^2$ are disjoint from the edges of $C_2^0 \cup C_2^1 \cup C_2^2$, so we need only consider the four edges we added in. But since $C_1$ and $C_2$ are disjoint, their $\mu$-sequences must either be disjoint or of the form $X_0, X_1, X_2$ and $X_2, X_3, X_4$ with $X_1 - X_0 = X_2 - X_1 = X_3 - X_2 = X_4 - X_3 = 1$. If the $\mu$-sequences are disjoint, then the added edges are clearly disjoint; in the second case, the edges added to the first cycle are $\{X_0^0, X_0^1\}$, $\{X_1^0, X_1^1\}$, $\{X_1^1, X_1^2\}$, and $\{X_2^1, X_2^2\}$, and the edges added to the second cycle are $\{X_2^0, X_2^1\}$, $\{X_3^0, X_3^1\}$, $\{X_3^1, X_3^2\}$, and $\{x_4^1, X_4^2\}$ so we see that the added edges are disjoint in this case as well.

So, we have constructed $k$ of the desired $k+1$ edge disjoint Hamiltonian $\mu$-cycles in $G(3^m c, 3)$. To construct the last one, consider the basic cycle of Lemma 1. The construction of the previous paragraph uses some edges from the basic cycle, but for each $C^+$ that we construct, we use 4 edges from the basic cycle to replace 4 edges from $C^0 \cup C^1 \cup C^2$. We then add these 4 replaced edges to the basic cycle to obtain a modified basic cycle. The modified basic cycle is certainly still a cycle; the construction of the cycle $C^+$ replaces the sequence $(\ldots, X_0^0, X_0^1, X_0^2, X_1^0, X_1^1, X_1^2, X_2^0, X_2^1, X_2^2, \ldots)$ of consecutive vertices of the basic cycle by the sequence $(\ldots, X_0^0, X_1^0, X_0^2, X_0^1, X_1^1, X_2^1, X_2^0, X_1^2, X_2^2, \ldots)$, and the rest of the cycle is uneffected. We must show that this modified basic cycle is a $\mu$-cycle. Now, if $X$ is any vertex of $G(3^{m-1} c, 3)$ not contained in any of the $\mu$-sequences of the $k$ Hamiltonian cycles in $G(3^{m-1} c, 3)$, then $X^0, X^1, X^2$ is a $\mu$-sequence of the modified basic cycle. Thus, we need only show that there exists such a vertex. But there are at most $m-1$ disjoint Hamiltonian cycles in $G(3^{m-1}c,3)$, so there are at most $3m-3$ vertices contained in their $\mu$-sequences. However, there are $3^{m-1}c$ vertices in $G(3^{m-1}c,3)$, so it suffices to show $3^{m-1}c > 3m-3$, or $3^{m-2}c > m-1$, which is true for all $m \geqslant 3$ and for $m=c=2$. □

**Theorem 2.** *Flator all m, $G(3^m c,3)$ is decomposable into m edge-disjoint Hamiltonian $\mu$-cycles (and one perfect matching if $c=2$).*

**Proof.** Certainly if we can find $m$ disjoint Hamiltonian cycles then we are done, since if $c=2$ the remaining edges will necessarily form a perfect matching. We construct the cycles by induction: for $c=2$, $G(6,3)$ is a $K_{3,3}$, which divides into one Hamiltonian cycle and one perfect matching. Lemma 3 then provides the inductive step. In the case $c=1$, $G(3,3)$ is the 3-cycle $C_3$, which is clearly a single Hamiltonian cycle. However, we cannot apply Lemma 3 to this case; the base case is $G(9,3)$. Recall that we denote the vertices by their base 3 representation. Then we can partition $G(9,3)$ into two $\mu$-cycles as follows: $C_1=(00,01,02,22,12,11,21,20,10,00)$ and $C_2=(02,10,11,01,21,22,00,20,12,02)$ (compare Fig. 1). Now, Lemma 3 provides the inductive step. □

## 5. The case $d \geqslant 4$

As in the previous sections, we would like to use Lemma 1 to inductively build Hamiltonian cycles. Once again, we will need to restrict the cycles that we consider.

**Definition.** A Hamiltonian cycle in $G(cd^m,d)$ is said to be a $\gamma$*-cycle* if it contains three consecutive vertices $X_0,X_1,X_2$, called a $\gamma$*-sequence*, such that $X_2-X_1=X_1-X_0=1$, and $X_0 \equiv 0 \pmod d$.

Notice that two $\gamma$-sequences must either be equal or disjoint. Therefore, two edge-disjoint $\gamma$-cycles contain vertex-disjoint $\gamma$-sequences.

We will show that with the exception of the case $c=3$, $m=1$, $G(cd^m,d)$ is decomposable into Hamiltonian $\gamma$-cycles (and a perfect matching in the odd degree case).

**Lemma 4.** *Given k edge-disjoint Hamiltonian $\gamma$-cycles in $G(cd^{m-1},d)$, we can construct $k+1$ edge-disjoint Hamiltonian $\gamma$-cycles in $G(cd^m,d)$.*

**Proof.** First of all, we give an algorithm for producing a Hamiltonian $\gamma$-cycle in $G(cd^m,d)$ from one in $G(cd^{m-1},d)$. Suppose $C$ is a $\gamma$-cycle in $G(cd^{m-1},d)$ with $\gamma$-sequence $X_0,X_1,X_2$. Then we have $d$ disjoint cycles of length $cd^{m-1}$ in $G(cd^m,d)$,

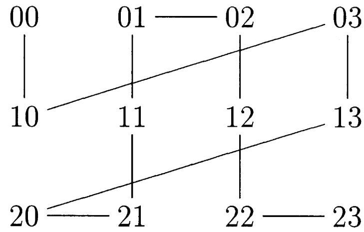


Fig. 2. The modified basic cycle in the case $d=4$.

corresponding to the $d$ disjoint copies of $G(cd^{m-1},d)$ sitting inside $G(cd^m,d)$; we denote the copy of $C$ contained in $G_j$ by $C^j$, and we denote the images of the vertices $X_0,X_1,X_2$ in $C^j$ by $X_0^j,X_1^j,X_2^j$. We now 'link' these $d$ disjoint cycles to produce a single Hamiltonian $\gamma$-cycle in the following manner. For every even integer $l$ with $0\leqslant l\leqslant d-2$, we throw out the edges $\{X_0^l,X_1^l\}$ and $\{X_0^{l+1},X_1^{l+1}\}$ from the cycles $C^l$ and $C^{l+1}$ and link the cycles by adding the edges $\{X_0^l,X_0^{l+1}\}$ and $\{X_1^l,X_1^{l+1}\}$. We now have $\lceil d/2\rceil$ cycles. Now, for every odd integer $l$ with $1\leqslant l\leqslant d-2$, we throw out the edges $\{X_1^l,X_2^l\}$ and $\{X_1^{l+1},X_2^{l+1}\}$ and replace them by $\{X_1^l,X_1^{l+1}\}$ and $\{X_2^l,X_2^{l+1}\}$. This produces a single Hamiltonian cycle $C^+$, and $X_1^1,X_1^2,X_1^3$ form a $\gamma$-sequence for the cycle.

Now, suppose we began with two disjoint Hamiltonian $\gamma$-cycles $C_1$ and $C_2$ in $G(cd^{m-1},d)$. We must show that the two Hamiltonian $\gamma$-cycles $C_1^+$ and $C_2^+$ in $G(cd^m,d)$ are also disjoint. Certainly the edges of $C_1^{j_1}$ are disjoint from the edges of $C_2^{j_2}$ for all $j_1,j_2$, so we need only consider the edges that we added in. But since $C_1$ and $C_2$ are disjoint, their $\gamma$-sequences must also be disjoint, so these added edges are also disjoint.

Hence, given $k$ edge-disjoint Hamiltonian $\gamma$-cycles in $G(cd^{m-1},d)$, we have constructed $k$ edge-disjoint Hamiltonian $\gamma$-cycles in $G(cd^m,d)$. We need only find one more. Consider the basic cycle constructed in Lemma 1, whose edges are precisely those contained in no $G_j$. We have used some of these edges in our construction of the first $k$ Hamiltonian $\gamma$-cycles, but every operation consisted of taking two edges of the form $\{X_i^l,X_i^{l+1}\}$ and $\{X_{i+1}^l,X_{i+1}^{l+1}\}$ from the basic cycle and using them to replace the edges $\{X_i^l,X_{i+1}^l\}$ and $\{X_i^{l+1},X_{i+1}^{l+1}\}$ (here, if $l$ is even then $i=0$; if $l$ is odd then $i=1$). We are now free to take these two discarded edges and include them in the basic cycle. The only vertices of the basic cycle that this construction effects are those whose penultimate digit is 0, 1, or 2. Thus, we need only consider the last two digits of each vertex, since the construction only influences one such block at once. The exchanges we make remove all edges of the form $\{1l,1(l\pm1)\}$; $\{0l,0(l+1)\}$ where $l$ is even; and $\{2l,2(l+1)\}$ where $l$ is odd. Also, the edges $\{1l,(1\pm1)l\}$ are added, with the exception of $\{10,20\}$ and $\{1(d-1),(1+(-1)^d)(d-1)\}$. As is easily seen, the basic cycle remains a Hamiltonian cycle after these operations are carried out (the cases $d=4$ and are shown in Figs. 2 and 3), and $(3,3,\dots,3,0),(3,\dots,3,1),(3,\dots,3,2)$ make up a $\gamma$-sequence for the cycle. □

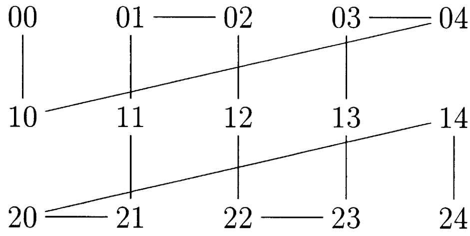


Fig. 3. The modified basic cycle in the case $d=5$.

The construction of the Hamiltonian decomposition of $G(cd^m,d)$ will use an inductive argument akin to those of the previous two sections. However, one of the base cases is significantly more difficult than the rest.

**Lemma 5.** *For all* $d\geqslant 4$, $G(3d^2,d)$ *is decomposable into three Hamiltonian* $\gamma$*-cycles.*

**Proof.** One can easily see that $G(3d,d)$ need not have a decomposition into Hamiltonian $\gamma$-cycles. However, since the degree of this graph is 4, we do know that it has a Hamiltonian decomposition [2]. Hence, we know that we have two edge-disjoint Hamiltonian cycles on $G(3d,d)$. We will construct three edge-disjoint Hamiltonian $\gamma$-cycles on $G(3d^2,d)$ inductively, using techniques similar to our other inductive arguments. So, let $C$ be one of the Hamiltonian cycles on $G(3d,d)$, and fix any sequence of 3 consecutive vertices in $C$, say $X_0,X_1,X_2$. Then the Hamiltonian cycle on $G(3d^2,d)$ that we construct from $C$ will contain the $d$ images of $C$, one in each $G_j$, with all edges $\{X_i^j,X_{i+1}^j\}$ except for $\{X_0^0,X_1^0\}$ and $\{X_1^{d-1},X_i^{d-1}\}$ where $i=1+(-1)^d$ removed; also, we add in all edges $\{X_1^j,X_1^{j+1}\}$, $\{X_0^{2j+1},X_0^{2j+2}\}$, and $\{X_2^{2j},X_2^{2j+1}\}$. Then these edges make up a cycle, and, furthermore, $X_1^0,X_1^1,X_1^2$ make up a $\gamma$-sequence for the cycle.

Now, consider the two edge-disjoint Hamiltonian cycles $C_1$ and $C_2$ on $G(3d,d)$, and let $X_0,X_1,X_2$ and $Y_0,Y_1,Y_2$ be consecutive vertices on $C_1$ and $C_2$, respectively. The Hamiltonian cycles that we construct from $C_1$ and $C_2$ on $G(3d^2,d)$ depend on the sequences $X_0,X_1,X_2$ and $Y_0,Y_1,Y_2$; in fact, one easily sees that the sequences are disjoint if and only if $\{X_0,X_1,X_2\}\cap\{Y_0,Y_1,Y_2\}=\emptyset$. So we must show that we can pick these sets to be disjoint. Suppose we have already chosen $X_0,X_1,X_2$. Then $X_0$ and $X_2$ are each adjacent to 3 vertices other than the $X_i$, and $X_1$ is adjacent to 2. Therefore, there are at most 11 vertices whose distance from the set $\{X_0,X_1,X_2\}$ is at most 1. However, since $c=3$, we have $d\geqslant 4$, so $cd\geqslant 12$, so there is a vertex $Y_1$ with distance at least two from the set $\{X_0,X_1,X_2\}$. Then if $Y_0$ and $Y_2$ are any two vertices adjacent to $Y_1$, we have $\{X_0,X_1,X_2\}\cap\{Y_0,Y_1,Y_2\}=\emptyset$, so the cycles in $G(3d^2,d)$ built from these two sequences are disjoint.

Now we must build the third Hamiltonian $\gamma$-cycle in $G(3d^2,d)$. As usual, it will be the basic cycle, modified as necessitated by the construction of the first two cycles. Certainly if this is a cycle, then it will be a $\gamma$-cycle, because for any $Z\notin\{X_0,X_1,X_2,Y_0,Y_1,Y_2\}$, $Z^0$, $Z^1$, $Z^2$ is a $\gamma$-sequence. It remains to show that we actually

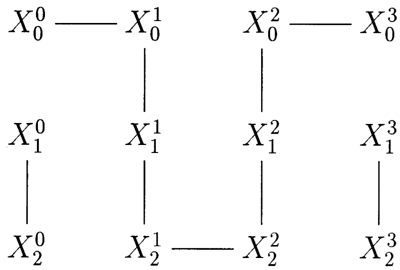


Fig. 4. The modified basic cycle without boundary edges in the case $d = 4$.

obtain a cycle. As usual, adding the cycle $C_1$ only effects the part of the basic cycle whose vertices are $X_i^j$, and adding the cycle $C_2$ only effects the vertices $Y_i^j$. The effect of the modification depends on the parity of $d$ modulo 2, as is seen from the figures below. Notice that the figures do not contain any edges of the form $\{V_1^0, V_2^{d-1}\}$. This is because we have no information about $X_i - X_j$ for any $0 \leqslant i, j \leqslant 2$. We call edges of basic cycle having that form *boundary edges*.

The unmodified basic cycle with boundary edges removed simply contains edges of the form $\{V^j, V^{j+1}\}$; in the figures, this corresponds to having all possible horizontal edges and no other edges. So, we have a collection of paths, the union of whose endpoints makes up the set $\{V^0, V^{d-1} \mid V \in G(3d, d)\}$; adding in the perfect matching made up of the boundary edges makes this into a single Hamiltonian cycle. The modification of the basic cycle simply replaces these paths with another collection of paths the union of whose endpoints makes up the same set. But in proving that the modified basic cycle is Hamiltonian, the only relevant data are the endpoints of the paths, since we are only concerned with the way that these paths join together to form cycles.

Therefore, in the case that $d$ is even (see Fig. 4), the fact that the path from $X_0^0$ to $X_0^{d-1}$ is different than the path $(X_0^0, X_0^1, \ldots, X_0^{d-1})$ of the basic cycle is of no import. The only vertices that we need concern ourselves with are $X_1^0, X_2^0, X_1^{d-1}$, and $X_2^{d-1}$. So let us follow the cycle from $X_1^{d-1}$. First, a boundary edge takes us to $(X_1 + 1)^0$, then a path takes us to $(X_1 + 1)^{d-1}$, then a boundary edge takes us to $(X_1 + 2)^0$, and the pattern goes on until we reach $X_2^0 = (X_1 + r)^0$ for some $r$ (all addition and subtraction is taken modulo $3d$). Then we go to $X_1^0$, then $(X_1 - 1)^{d-1}$, $(X_1 - 1)^0$, $(X_1 - 2)^{d-1}$, and so on, until we reach $X_2^{d-1} = (X_1 - s)^{d-1}$, which finally takes us back to $X_1^{d-1}$. But since $X_1 + r = X_2 = X_1 - s$, we have $r + s = 3d$, so all the vertices are traversed, and the cycle is Hamiltonian.

The case of $d$ odd is somewhat more complicated (see Fig. 5). Indeed, let us trace the cycle from $X_1^{d-1}$. A boundary edge takes us to $(X_1+1)^0$, then we go to $(X_1+1)^{d-1}$, and so on, until we reach $Y^0 = (X_1 + r)^0$, where $Y = X_0$ or $X_2$. Now, if $Y = X_0$, then all is well. This takes us to $X_2^{d-1}$, then $(X_2 + 1)^0$, and so on. But since by increasing $X_1$ by increments of 1, we obtained $X_0$ before $X_2$, we know that increasing $X_2$ by increments of 1 will get us to $X_1$ before $X_0$. Thus, the next relevent vertex in our cycle

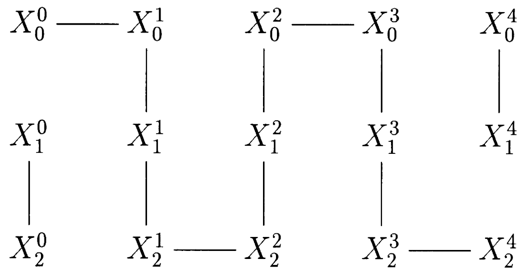


Fig. 5. The modified basic cycle without boundary edges in the case $d=5$.

will be $X_1^0=(X_2+s)^0$, followed by $X_2^0$, $(X_2-1)^{d-1}$, $(X_2-1)^0$, and so on. But since increasing $X_2$ got us to $X_1$ before $X_0$, decreasing $X_2$ will get us to $X_0$, so we obtain $X_0^{d-1}=(X_2-t)^0$, which finally takes us back to $X_1^{d-1}$. Now, we have the equations $X_0=X_1+r$, $X_1=X_2+s$, and $X_0=X_2-t$. Hence, $X_2+s=X_1=X_2-r-t$, so $r+s+t=3d$, and so all the vertices have been traversed, and the cycle is Hamiltonian.

However, if $Y=X_2$, then we will get into trouble. In this case, the sequence becomes $X_2^0$, $X_1^0$, $(X_1-1)^{d-1}$, and so on. But since incrementing $X_1$ brought us to $X_2$, decrementing $X_1$ will bring us to $X_0$, so the sequence then goes to $X_0^{d-1}$ and then back to $X_1^{d-1}$ without ever having passed $X_0^0$. If we encounter this problem then we have to alter the way we chose the cycle. Recall that $X_0, X_1, X_2$ were chosen to be any three consecutive vertices of the cycle in $G(3d,d)$ that we began with. In particular, if we reverse the order of the three, that is, if we let $Z_0=X_2, Z_1=X_1$, and $Z_2=X_0$, then we obtain different Hamiltonian cycle on $G(3d^2,d)$ corresponding to the sequence $Z_0, Z_1, Z_2$. Obviously, for any $\{Y_0,Y_1,Y_2\}$, we have $\{X_0,X_1,X_2\}\cap\{Y_0,Y_1,Y_2\}=\emptyset \Rightarrow \{Z_0,Z_1,Z_2\}\cap\{Y_0,Y_1,Y_2\}=\emptyset$, so all of our constructions still go through. But now, since incrementing $Z_1=X_1$ brought us to $X_2=Z_0$, the previous paragraph shows that the basic cycle remains Hamiltonian. This completes the proof. □

We are now ready to prove our main theorem.

**Theorem 3.** *For all $d\geqslant 4$ and $0<c<d$, $G(cd^m,d)$ is decomposable into Hamiltonian $\gamma$-cycles (and a perfect matching if $G(cd^m,d)$ has odd degree), except for $G(3d,d)$ which is also Hamilton decomposable.*

**Proof.** We proceed by induction on $m$. Lemma 4 handles the induction step, so we need only treat the base case. For $c\geqslant 4$, the base case is $m=0$ (because to invoke Lemma 4, all we need is for the numbers 0, 1, 2, and 3 to all appear in the last digit), which is trivial, since $G(c,d)=C_c$. For $c=1$, the base case is $m=1$ which is trivial since $G(d,d)=C_d$. For $c=2$, the base case is $m=1$, which is also easy, since $G(2d,d)$ is just a $2d$-cycle with diameters: the cycle itself forms the Hamiltonian cycle, and the diameters make up a perfect matching. For $c=3$, we know that $G(3,d)=C_3$, so it has a Hamiltonian decomposition, and $G(3d,d)$ has a Hamiltonian decomposition because its degree is 4. Also, Lemma 5 constructs a decomposition into Hamiltonian $\gamma$-cycles of $G(3d^2, d)$, which provides the base case for our induction. □


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