Global Attractivity and Oscillations in a Periodic Delay-Logistic Equation

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Sufficient conditions are obtained for the global attractivity of a positive periodic solution of the delay-logistic equation

\[
\frac{dx(t)}{dt} = r(t)x(t) \left[ 1 - \frac{x(t-n)}{K(t)} \right],
\]

where \( r \) and \( K \) are positive periodic functions of period \( T \) and \( n \) is a positive integer; sufficient conditions are also obtained for all solutions to be oscillatory about \( K \).

1. INTRODUCTION

One of the elementary and mathematically simple models used to describe the temporal evolution of a single species population in a constant environment is described by the autonomous scalar ordinary differential equation

\[
\frac{du(t)}{dt} = ru(t) \left[ 1 - \frac{u(t)}{K} \right]
\]  

(1.1)

in which \( r \) and \( K \) are positive numbers; \( r \) is related to the reproduction of the species while \( K \) is related to the capacity of the environment to sustain the population. It is assumed that there is no immigration or emigration and other characteristics such as age dependence and interactions with other species are assumed to be not significant. Elementary analysis of (1.1) indicates that the solutions of (1.1) are monotonic functions of \( t \) with \( \lim_{t \to \infty} u(t) = K \) if \( u(0) > 0 \). Since solutions of (1.1) do not show some

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of the experimentally observed oscillations of laboratory populations, a modification of (1.1) has been suggested in the form

$$\frac{du(t)}{dt} = ru(t) \left[ 1 - \frac{u(t-\tau)}{K} \right]$$

(1.2)

by Hutchinson [6]. Equation (1.2) is known as the delay-logistic equation. The time delay \( \tau \) (assumed to be a positive valued parameter) in (1.2) can cause oscillations in solutions of (1.2) and there have been numerous investigations of the qualitative behaviour of (1.2) and some of its generalisations.

If the environment is not temporally constant, the parameters \( r \) and \( K \) in (1.1) and hence in (1.2) become \( t \)-dependent. One of the ways of studying the analogues of (1.1) and (1.2) in a changing environment is to consider \( r \) and \( K \) to be periodic in \( t \) of certain positive period. We refer to Pianka [9] for a discussion of the relevance of periodic environments to evolutionary theory. The purpose of this article is to show that if \( r \) and \( K \) are positive periodic functions of period \( \tau \) then there exists a unique solution of (1.2) which is globally attractive with respect to all other positive solutions when suitable sufficient conditions hold. Our sufficient conditions will collapse to those already known when \( r \) and \( K \) are positive constants.

In Section 3 below, we have obtained sufficient conditions for all positive solutions of the ‘periodic delay-logistic’ equation (2.1) to be “oscillatory about \( K(t) \).”

We note that a periodic logistic equation of the form

$$\frac{dv(t)}{dt} = r(t)v(t) \left[ 1 - \frac{v(t)}{K(t)} \right]$$

(1.3)

with periodic \( r \) and \( K \) has been considered by Coleman [3, 4] and Boyce and Daley [1]. We refer to Hallam and Clark [5] for further discussion of (1.3) when \( r \) and \( K \) are not necessarily periodic in \( t \).

2. A Periodic Delay-Logistic Equation

We now consider the periodic delay-logistic equation

$$\frac{dx(t)}{dt} = r(t)x(t) \left[ 1 - \frac{x(t-n\tau)}{K(t)} \right]$$

(2.1)

with the assumption that \( n \) is a positive integer, \( \tau \) is a positive constant, and \( r, K \) are positive continuous periodic functions of period \( \tau \). One of the questions for (2.1) is the following. Does there exist a strictly positive
periodic solution of (2.1)? If (2.1) has a periodic solution of period \( \tau \) then such a solution is also a periodic solution of the periodic logistic equation

\[
\frac{dy(t)}{dt} = r(t) y(t) \left[ 1 - \frac{y(t)}{K(t)} \right].
\]  

(2.2)

Conversely if (2.2) has a periodic solution of period \( \tau \), then such a solution is also a periodic solution of (2.1). The periodic logistic equation (2.2) has been discussed by Coleman [2, 3] and Coleman et al. [4] and it has been shown that (2.2) has a unique periodic solution \( y^*(t) \), where

\[
y^*(t) = \left[ \int_0^\infty \frac{r(t-s)}{K(t-s)} \exp \left\{ -\int_0^s r(t-\sigma) \, d\sigma \right\} \, ds \right]^{-1}
\]

(2.3)

\[
y^*(t) = \frac{1}{\int_0^\tau \frac{r(t-s)}{K(t-s)} \exp \left[ -\int_0^s r(t-\sigma) \, d\sigma \right] ds}
\]

(2.4)

and

\[
\langle r \rangle = \int_0^\tau r(s) \, ds.
\]

(2.5)

It has also been shown by Coleman [2, 3] (see also Coleman et al. [4]) that all other positive solutions of (2.2) have the asymptotic behaviour

\[
\lim_{t \to \infty} \left[ y(t) - y^*(t) \right] = 0.
\]

(2.6)

Thus \( y^* \) is globally attractive of all other positive solutions of (2.2). The principal result of this article is the following:

**Theorem 2.1.** Assume that \( r \) and \( K \) are continuous strictly positive periodic functions with period \( \tau > 0 \) such that

\[
n \langle r \rangle = \int_0^{\tau\pi} r(s) \, ds \leq \frac{3}{2}.
\]

(2.7)

Then the periodic delay-logistic equation (2.1) has a unique positive periodic solution \( y^*(t) \) given by (2.3) (or (2.4)) and all other solutions of (2.1) corresponding to initial conditions of the form

\[
x(s) = \varphi(s) \geq 0; \quad \varphi(0) > 0; \quad \varphi \in C[-\pi\tau, 0]
\]

are such that

\[
\lim_{t \to \infty} |x(t) - y^*(t)| = 0.
\]

(2.8)
Proof. The existence of \( y^* \) has been observed in our discussion preceding the statement of the theorem and the uniqueness of \( y^* \) follows from its attractivity as a solution of (2.2); for if (2.1) has another periodic solution then such a solution is also a periodic solution of (2.2) and hence \( y^* \) is unique. Let \( x(t) \) denote any positive solution of (2.1). We define \( v \) so that

\[
\ln[1 + v(t)] = \ln[x(t)] - \ln[y^*(t)]
\]

and note that \( v \) is governed by

\[
\frac{dv}{dt} = - \left( \frac{r(t) y^*(t)}{K(t)} \right) [1 + v(t)] v(t - n\tau).
\]

It is sufficient to prove that solutions of (2.10) corresponding to initial conditions of the form

\[
1 + v(s) \geq 0, \quad 1 + v(0) > 0; \quad v \in C[-n\tau, 0]
\]

have the asymptotic behaviour

\[
\lim_{t \to \infty} v(t) = 0.
\]

For convenience we let

\[
\frac{r(t) y^*(t)}{K(t)} \equiv a(t)
\]

and consider

\[
\frac{dv(t)}{dt} = -a(t)[1 + v(t)] v(t - n\tau).
\]

We define two new variables \( \omega \) and \( \sigma \) as follows:

\[
\omega = \sigma(t) = \int_{t_0}^{\sigma'(t)} a(s) \, ds,
\]

where \( t_0 \) is any nonnegative number and note that

\[
\omega \to \infty \text{ as } t \to \infty \quad \text{and} \quad \sigma^{-1}(t) \text{ exists.}
\]

Also

\[
\sigma(t - n\tau) = \int_{t_0}^{t - n\tau} a(s) \, ds = \omega - \int_{\sigma^{-1}(\omega)}^{\sigma^{-1}(\sigma(t - n\tau))} a(s) \, ds
\]
and hence
\[ t - n\tau = \sigma^{-1} \left( \omega - \int_{\frac{t}{\sigma} - n\tau}^{\sigma^{-1} \omega} a(s) \, ds \right). \tag{2.15} \]

If we now define
\[ v(t) = v(\sigma^{-1}(\omega)) = z(\omega) \tag{2.16} \]
than we have from (2.13), (2.15), (2.16) that
\[ \frac{dz(\omega)}{d\omega} = - \left[ 1 + z(\omega) \right] z(\omega - \eta(\omega)), \tag{2.17} \]

where
\[ \eta(\omega) = \int_{\frac{t}{\sigma} - n\tau}^{\sigma^{-1} \omega} a(s) \, ds = \int_{t - n\tau}^{t} a(s) \, ds = n 1 \langle a \rangle, \tag{2.18} \]

where \( n 1 \langle a \rangle = \int_{0}^{n \tau} a(s) \, ds \).

From the fact that \( y^* \) is a positive periodic solution of (2.2) of period \( \tau \)
we derive that
\[ 0 = \int_{0}^{n \tau} \frac{\dot{y}(s)}{y(s)} \, ds = \int_{0}^{n} \tau_0 r(s) \, ds - \int_{0}^{n} \tau_0 a(s) \, ds \]
\[ = \left[ n 1 \langle r \rangle - n 1 \langle a \rangle \right] \tau_0 \]
and hence \( n 1 \langle a \rangle = n 1 \langle r \rangle \). Thus (2.17) simplifies to
\[ \frac{dz(\omega)}{d\omega} = - \left[ 1 + z(\omega) \right] z(\omega - n 1 \langle r \rangle) \tag{2.19} \]

and (2.19) is a familiar autonomous (with respect to the new variables)
delay-logistic equation whose positive solutions are known to be bounded.
It is known from the work of Wright [10] (see Theorem 3 in [10]) that
if the delay \( n 1 \langle r \rangle \) in (2.19) satisfies (2.7) then all positive solutions of (2.19)
satisfy
\[ \lim_{\omega \to \infty} z(\omega) = 0. \tag{2.20} \]

But \( \omega \to \infty \) implies that \( t \to \infty \) from (2.14). Thus it follows from (2.16) that
\[ \lim_{t \to \infty} v(t) = 0. \tag{2.21} \]
PERIODIC DELAY-LOGISTIC EQUATION

It is now easy to see from (2.9) that

$$\lim_{t \to \infty} [x(t) - y^*(t)] = 0$$

(2.22)

and this completes the proof.

The following question is of some interest: Is the condition (2.7) the best possible? One can see from the work of Wright [10] that the right side of (2.7) can be replaced by $37/24$ and even by 1.567. It appears to these authors that the best condition could be

$$[\langle n \rangle] < \frac{\pi}{2};$$

(2.23)

but a condition such as (2.23) has remained a conjecture for the autonomous delay-logistic equation since the publication of the article of Wright [10]. Another related question is whether our technique of proving the existence and attractivity is applicable to more general classes of equations considered by Coleman [3, 4]; the authors intend to pursue this in the future.

3. OSCILLATORY DELAY-LOGISTIC EQUATIONS

Oscillations in laboratory and field population sizes have been commonly observed phenomena. In this section we obtain sufficient conditions for all solutions of the delay-logistic equation

$$\frac{dx(t)}{dt} = r(t)x(t) \left[ 1 - \frac{x(t-nt)}{K(t)} \right]$$

(3.1)

to be "oscillatory," where $r$ is not necessarily periodic and $K$ is periodic in $t$ of period $\tau$. A strictly positive solution of (3.1) is said to be "oscillatory about $K$" if there exists a sequence $\{t_n\} \to \infty$ as $n \to \infty$ such that $x(t_n) - K(t_n) = 0$. The system (3.1) is said to be oscillatory about $K$ if every positive solution of (3.1) is oscillatory about $K$. We remark that our analysis of oscillation is valid even if $r$ is periodic with a period different from that of $K$.

**Theorem 3.1.** Assume the following:

(i) $K$ is a nonconstant positive differentiable periodic function with period $\tau$. 

(ii) \( r \) is a positive and continuous for \( t \geq 0 \) such that
\[
\liminf_{t \to \infty} r(t) > 0 \quad \text{and} \quad \liminf_{t \to \infty} \int_{t-n\tau}^{t} r(s) \, ds > 1/e. \tag{3.2}
\]

Then every positive solution of (3.1) is oscillatory about \( K \).

**Proof.** If we define \( y(t) = \ln \left[ x(t)/K(t) \right] \) then \( y \) is governed by
\[
\dot{y}(t) = r(t) \left[ 1 - \exp \{ y(t-n\tau) \} \right] - \left[ \dot{K}(t)/K(t) \right] \tag{3.3}
\]
and the oscillation of \( x \) about \( K \) becomes equivalent to that of \( y \) about zero and thus it is sufficient to consider the usual oscillation of \( y \). We simplify (3.3) further by letting
\[
\ln \left[ K(t_0)/K(t) \right] = Q(t) \tag{3.4}
\]
and note that (3.3) becomes
\[
\dot{y}(t) + r(t) \left[ \exp \{ y(t-n\tau) \} - 1 \right] = \dot{Q}(t). \tag{3.5}
\]

Suppose now the conclusion of the theorem is false. Then there exists an eventually positive or eventually negative solution for (3.5). For instance let us first assume that (3.5) has an eventually positive solution \( y \). Since \( Q \) is a nonconstant periodic function, there exist two sequences \( \{ t'_n \} \) and \( \{ t''_n \} \) such that
\[
\lim_{n \to \infty} t'_n = \infty; \quad \lim_{n \to \infty} t''_n = \infty
\]
and
\[
-\infty < q_1 \leq Q(t) \leq q_2 < \infty
\]
\[
q_1 = Q(t'_n); \quad q_2 = Q(t''_n) \tag{3.6}
\]
\[
n = 1, 2, 3, \ldots
\]

Let us define \( u \) so that
\[
u(t) = y(t) - Q(t) \quad \text{for} \quad t \geq T \tag{3.7}
\]
(where \( y(t-\tau) > 0 \) for \( t > T \)). It is found that (3.5) becomes
\[
\dot{u}(t) = r(t) \left[ 1 - \exp \{ y(t-n\tau) \} \right] < 0. \tag{3.8}
\]

We now show that \( u(t) + q_1 > 0 \); for instance suppose for some \( t > T \), \( u(t) + q_1 \leq 0 \). Since \( y(t) > 0 \) we have \( u(t) + Q(t) = y(t) > 0 \) and hence \( u(t'_n) + q_1 = y(t'_n) > 0 \) showing that \( u(t) + q_1 \leq 0 \) is not possible. Therefore,
\[
\dot{u}(t) + q_1 > 0 \quad \text{for large} \quad t > T. \tag{3.9}
\]
Let us now set
\[ z(t) = u(t) + q, \quad (3.10) \]
and derive that
\[
\dot{z}(t) = \dot{u}(t) = \dot{y}(t) - \dot{\bar{Q}}(t) \\
= r(t)[1 - \exp\{y(t - n\tau)\}] \\
= r(t)[1 - \exp\{u(t - n\tau) + \bar{Q}(t - n\tau)\}] \\
\leq -r(t)[u(t - n\tau) + \bar{Q}(t - n\tau)] \\
\leq -r(t)z(t - n\tau). \quad (3.11)
\]

It is found from (3.9)–(3.11) that (3.11) has an eventually positive solution and this is impossible due to the hypothesis (3.2) (for details of this we refer to Ladas and Stavroulakis [7]).

Let us now consider the other alternative by supposing that (3.3) has an eventually negative solution \( y(t) < 0 \); this will imply
\[
\frac{x(t)}{K(t)} < 1 \quad \text{for large } t. \quad (3.12)
\]
The boundedness of \( K \) (due to its periodicity) and (3.12) together imply that \( x \) is bounded. It follows from (3.1) that \( \dot{x}(t) > 0 \) eventually implies the existence of the limit
\[
\lim_{t \to \infty} x(t) = l > 0. \quad (3.13)
\]
Integrating (2.1) we have
\[
\log \frac{1}{x(t_0)} = \int_{t_0}^{\infty} r(s) \left( 1 - \frac{x(s - n\tau)}{K(s)} \right) ds < \infty. \quad (3.14)
\]
Hence
\[
\liminf_{t \to \infty} r(t) \left( 1 - \frac{x(t - n\tau)}{K(t)} \right) = 0.
\]
But \( \liminf_{t \to \infty} r(t) > 0 \), so
\[
\limsup_{t \to \infty} \frac{x(t - n\tau)}{K(t)} = 1;
\]
i.e., there exists a sequence \( \{t_k\} \) such that
\[
\lim_{k \to \infty} \frac{x(t_k - n\tau)}{K(t_k)} = 1. \quad (3.15)
\]
Since $x(t) < K(t)$, (3.13) and (3.15) lead to

$$\lim_{t \to \infty} x(t) = l - \min_{t \in [0, \tau]} K(t).$$

But then

$$\int_{t_0}^{\infty} r(t) \left[ 1 - \frac{x(t - n\tau)}{K} \right] dt \geq \frac{\inf r(t)}{\max_{t \in [0, \tau]} K(t)} \int_{t_0}^{\infty} [K(t) - x(t - n\tau)] dt$$

$$\geq \frac{\inf r(t)}{\max_{t \in [0, \tau]} K(t)} \int_{t_0}^{\infty} [K(t) - \min_{t \in [0, \tau]} K(t)] dt = \infty$$

which contradicts (3.14). This completes the proof.

4. SOME REMARKS

We have considered a special case of the periodic delay-logistic equation in which the period of the environmental parameter is related to the time delay in the self regulating negative feedback mechanism. For such a special case, our analysis has been analytical and in some sense complete in the sense that we have been able to demonstrate the global attractivity of the periodic solution in one case (small delay) and the oscillation of all solutions in the other case (large delay). It remains open to investigate whether or not (2.1) has a nonoscillatory solution $x(t)$ satisfying the requirement that $x(t) - K(t)$ is either eventually positive or eventually negative when $K$ is periodic. The existence of a nonoscillatory solution of the nonautonomous delay-logistic equation

$$\frac{dx(t)}{dt} = r(t)x(t) \left[ 1 - \frac{x(t - \tau(t))}{K} \right],$$

where $K \in (0, \infty)$ has been recently considered by Zhang and Gopalsamy [11]. Analysis of a more general periodic delay-logistic equation in which the time delay is not an integer multiple of the environmental periodicity also remains an open problem; a complete mathematical analysis of such an equation appears quite difficult and only an approximate (linear) and numerical analysis seems to be possible. We refer to Boyce and Daley [1] and Nisbet and Gurney [8] for such approximate and numerical analyses of periodic logistic equations. It has not been our purpose to discuss the ecological and evolutionary consequences of either periodicity of the
environmental parameters or those of the time delay in the self-regulating (or resource-limiting) negative feedback mechanism; these aspects have been discussed by Pianka [9].

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