A Retarded Gronwall-Like Inequality and Its Applications

Olivia Lipovan

Str. Carpați 29, Timisoara 1900, Romania

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We prove a Gronwall-like inequality and present some of its applications to the qualitative study of retarded differential equations. The problems of global continuation of the solutions and the existence of nonoscillatory solutions are considered. By means of examples we show the usefulness of our results.

Key Words: integral inequality; retarded differential equation; global existence; nonoscillatory solution.

1. INTRODUCTION

Integral inequalities play an important role in the qualitative analysis of the solutions to differential and integral equations; cf. [11].

In this paper we obtain a slight generalization of the celebrated Gronwall inequality and we present some of its applications.

Our result can be used to prove the global existence and boundedness of the solutions to certain integral equations. With respect to the problem of global existence of the solutions to functional differential equations we show that our inequality leads to an improvement of some recent investigations [13]. The obtained criterion for global existence can be applied to the generalized pantograph equation and to the generalized Liénard equation with time delay.

In the last section we show that the Gronwall-like inequality can be applied to the analysis of the behavior of the solutions of some retarded nonlinear differential equations. We give sufficient conditions ensuring the nonoscillatory character of some retarded differential equations, extending the results in [7]. The existence of nonoscillatory solutions to certain impulsive differential equations is also considered.
2. AN INTEGRAL INEQUALITY

In this section we prove a generalization of the Gronwall inequality.

**Theorem.** Let \( u, f \in C([t_0, T], \mathbb{R}_+) \). Moreover, let \( w \in C(\mathbb{R}_+ \times \mathbb{R}_+) \) be nondecreasing with \( w(u) > 0 \) on \((0, \infty)\) and \( \alpha \in C^1([t_0, T], [0, T]) \) be non-decreasing with \( \alpha(t) \leq t \) on \([t_0, T]\).

If

\[
  u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} f(s) w(u(s)) \, ds, \quad t_0 \leq t < T,
\]

where \( k \) is a nonnegative constant, then for \( t_0 \leq t < t_1 \),

\[
  u(t) \leq G^{-1}\left( G(k) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) \, ds \right),
\]

where \( G(r) = \int_{r}^{\infty} \frac{dr}{w(s)} \), \( r > 0 \), and \( t_1 \in (t_0, T) \) is chosen so that

\[
  G(k) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) \, ds \in \text{Dom}(G^{-1}),
\]

for all \( t \) lying in the interval \([t_0, t_1]\).

**Proof.** Assume first that \( k > 0 \) and let us denote by \( U(t) \) the right-hand side of (2.1). Then \( U(t_0) = k \) and

\[
  U'(t) = f(\alpha(t))w(u(\alpha(t)))\alpha'(t)
\geq f(\alpha(t))w(U(\alpha(t)))\alpha'(t).
\]

As \( \alpha(t) \leq t \) on \([t_0, T]\), we deduce that

\[
  U'(t) \leq f(\alpha(t))w(U(t))\alpha'(t).
\]

From the definition of \( G \) and the above relation we infer

\[
  \frac{d}{dt} G(U(t)) \leq f(\alpha(t))\alpha'(t).
\]

An integration on \([t_0, t]\) shows now that

\[
  G(U(t)) \leq G(k) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) \, ds.
\]

Since \( G^{-1} \) is increasing on \( \text{Dom}(G^{-1}) \), the previous inequality yields

\[
  U(t) \leq G^{-1}\left( G(k) + \int_{\alpha(t_0)}^{\alpha(t)} f(s) \, ds \right), \quad t_0 \leq t < t_1.
\]
The required inequality (2.2) is obtained in view of the relation \( u(t) \leq U(t) \) on \([t_0, T]\). If \( k = 0 \), we carry out the above procedure with \( \varepsilon > 0 \) instead of \( k \) and subsequently let \( \varepsilon \to 0 \).

**Remark.** (i) For \( \alpha(t) \equiv t \) in the theorem we obtain Bihari’s inequality [2].

(ii) Note that if \( \int_1^\infty \frac{ds}{w(t)} = \infty \), then \( G(\infty) = \infty \) and (2.2) is valid on \([t_0, T]\). Examples of such functions are \( w(u) \equiv u \) and \( w(u) \equiv u \ln(1 + u) \).

Setting \( w(u) \equiv u \) in the theorem we obtain the following

**Corollary.** Let \( u, f \in C([t_0, T], \mathbb{R}_+) \). Further, let \( \alpha \in C^1([t_0, T], [t_0, T]) \) be nondecreasing with \( \alpha(t) \leq t \) on \([t_0, T] \), and let \( k \) be a nonnegative constant. Then the inequality

\[
    u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} f(s) u(s) \, ds, \quad t_0 \leq t < T
\]

implies that

\[
    u(t) \leq k \exp\left( \int_{\alpha(t_0)}^{\alpha(t)} f(s) \, ds \right), \quad t_0 \leq t < T.
\]

**Remark.** (i) With \( \alpha(t) \equiv t \) in the Corollary we obtain the celebrated Gronwall–Bellman inequality [1, 6].

(ii) Let us assume that \( t_0 = 0 \) and \( T = \infty \). In this case note that \( \alpha(0) = 0 \) and the hypothesis (2.1) of the theorem implies that

\[
    u(t) \leq k + \int_0^t f(s) w(u(s)) \, ds, \quad 0 \leq t.
\]

Hence Bihari’s result [2] could also be applied in order to obtain an upper estimate for \( u(t) \). However, the estimate provided by our theorem is sharper. To see this, take \( w(u) \equiv u, \, \alpha(t) \equiv \ln(t + 1), \) and \( f(t) \equiv \frac{1}{t+1} \). Bihari’s inequality yields

\[
    u(t) \leq k(t + 1), \quad 0 \leq t,
\]

while our theorem gives the estimate

\[
    u(t) \leq k(\ln(t + 1) + 1), \quad 0 \leq t.
\]

Along the same lines of the proof of the theorem one obtains

**Proposition 1.** Let \( u, f, g \in C([t_0, T], \mathbb{R}_+), \) and \( w \in C(\mathbb{R}_+, \mathbb{R}_+) \) be nondecreasing with \( w(u) > 0 \) for \( u > 0 \), and \( \alpha \in C^1([t_0, T], [t_0, T]) \) be nondecreasing with \( \alpha(t) \leq t \) on \([t_0, T]\).
If
\[ u(t) \leq k + \int_{t_0}^{t} f(s)w(u(s)) \, ds + \int_{\alpha(t_0)}^{\alpha(t)} g(s)w(u(s)) \, ds, \quad t_0 \leq t < T, \]
where \( k \) is a nonnegative constant, then for \( t_0 \leq t < t_1 \),
\[ u(t) \leq G^{-1}\left(G(k) + \int_{t_0}^{t} f(s) \, ds + \int_{\alpha(t_0)}^{\alpha(t)} g(s) \, ds\right), \]
with \( G \) as in the theorem and \( t_1 \) chosen so that the right-hand side is well-defined.

3. APPLICATIONS TO THE GLOBAL EXISTENCE OF SOLUTIONS

In this section we will show that our main result is useful in investigating the global existence of solutions to certain integral equations and functional differential equations.

We first consider the integral equation
\[ u(t) = k(t) + \int_{0}^{t} f(s)w(u(s)) \, ds, \quad t \geq 0, \quad (3.1) \]
where \( k, f, w \in C(\mathbb{R}_+, \mathbb{R}_+) \) with \( w(0) = 0 \) and \( \alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) is nondecreasing with \( \alpha(t) \leq t \) on \( \mathbb{R}_+ \). Under these assumptions, Eq. (3.1) has a solution \( u \in C([0, T], \mathbb{R}_+) \) on some maximal interval of existence \([0, T]\); cf. [4]. Moreover, if \( T < \infty \), then
\[ \limsup_{t \to T} u(t) = \infty. \quad (3.2) \]

Let us prove

**Proposition 2.** Assume that
\[ |w(x) - w(y)| \leq z(|x - y|), \quad x, y \in \mathbb{R}_+, \]
with \( z \in C(\mathbb{R}_+, \mathbb{R}_+) \) nondecreasing, \( z(x) > 0 \) for \( x > 0 \).

If
\[ \int_{0}^{1} \frac{ds}{z(s)} = \int_{1}^{\infty} \frac{ds}{z(s)} = \infty \]
then Eq. (3.1) has a unique solution defined on \( \mathbb{R}_+ \). Moreover, if \( k \) is bounded on \( \mathbb{R}_+ \) and if either \( \alpha \) is bounded on \( \mathbb{R}_+ \) or \( \int_{0}^{\infty} f(s) \, ds < \infty \), then this solution is bounded on \( \mathbb{R}_+ \).
Proof. As the existence of a solution on some maximal interval $[0, T)$ is guaranteed [4] let us first prove the uniqueness statement.

Suppose that on some interval $[0, t_0]$ with $t_0 > 0$, Eq. (3.1) has two solutions $u_1, u_2 \in C([0, t_0], \mathbb{R}_+)$.

From the corresponding two equations we obtain

$$u_1(t) - u_2(t) = \int_0^t a(t)f(s)[w(u_1(s)) - w(u_2(s))] \, ds, \quad 0 \leq t \leq t_0.$$ 

Denote $v(t) = |u_1(t) - u_2(t)|$ for $t \in [0, t_0]$. Using the hypotheses we deduce that

$$v(t) \leq \int_0^t a(t)f(s)z(v(s)) \, ds, \quad 0 \leq t \leq t_0. \quad (3.3)$$

Let

$$G(r) = \int_1^r \frac{ds}{z(s)}, \quad r > 0,$$

and note that $G(0) = -\infty$ and $G(\infty) = \infty$. From (3.3) it is clear that for $\varepsilon > 0$,

$$v(t) \leq \varepsilon + \int_0^t a(t)f(s)z(v(s)) \, ds, \quad 0 \leq t \leq t_0.\quad (3.4)$$

By our theorem the above relation implies

$$v(t) \leq G^{-1}\left(G(\varepsilon) + \int_0^t a(t)f(s) \, ds\right), \quad 0 \leq t \leq t_0.$$ 

For every fixed $t \in [0, t_0]$, let $\varepsilon \to 0$ in the above inequality to infer $v(t) = 0$.

Therefore the uniqueness of the solution is proved.

Let us now show that the solution is global, i.e., $T = \infty$, where $T$ is the maximal time of existence. If $T < \infty$, relation (3.2) holds. With $k_0 = \max_{0 \leq t \leq T} \{k(t)\}$ one obtains from (3.1) that

$$u(t) \leq k_0 + \int_0^t a(t)f(s)z(u(s)) \, ds, \quad 0 \leq t < T, \quad (3.4)$$

as $w(u(s)) = w(u(s)) - w(0) \leq z(u(s))$ for $0 \leq s < T$, $u$ being nonnegative.

Applying our theorem to (3.4) we deduce that

$$u(t) \leq G^{-1}\left(G(k_0) + \int_0^t a(t)f(s) \, ds\right), \quad 0 \leq t < T. \quad (3.5)$$
The above relation contradicts (3.2) and therefore the global existence is proved.

If \( k \) is bounded, with \( k_0 = \sup_{t \in \mathbb{R}_+} \{ k(t) \} \) one obtains that (3.4) holds on \( \mathbb{R}_+ \). From (3.4), by means of our theorem, relation (3.5) is obtained for all \( t \in \mathbb{R}_+ \). The boundedness assertion is now plain.

Remark. The uniqueness and global existence statements in Proposition 2 follow also from Bihari’s inequality [2]. However, if \( \alpha \) and \( k \) are bounded on \( \mathbb{R}_+ \) and \( \int_0^t f(s) \, ds = \infty \) our result yields the boundedness of the solution on \( \mathbb{R}_+ \). This conclusion cannot be reached using Bihari’s result.

Consider now the functional differential equation
\[
\begin{cases}
    x'(t) = F(t, x(t), x(\alpha(t)), x'(q(t))), \\
    x(0) = x_0,
\end{cases}
\] (3.6)
with \( F \in C(\mathbb{R}_+ \times \mathbb{C}^{3n}, \mathbb{C}^n) \), and \( \alpha, q \in C^1(\mathbb{R}_+, \mathbb{R}_+) \), and \( \alpha(t) \leq t, q(t) < t \) for \( t > 0 \). By a result in [5] we have that there exists a maximal interval of existence \([0, T)\) of a solution to (3.6). Moreover, if \( T < \infty \), then
\[
\limsup_{t \to T} |x(t)| = \infty.
\] (3.7)
Using the integral inequality proved in Proposition 1 we will give sufficient conditions for the global existence of the solutions of (3.6).

**Proposition 3.** Assume that \( \alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is an increasing diffeomorphism of \( \mathbb{R}_+ \) and
\[
|F(t, x, y, z)| \leq a(t)w(|x|) + b(t)w(|y|) + c(t) d(|z|),
\]
t \geq 0, (x, y, z) \in \mathbb{C}^{3n},
where \( a, b, c, d, w \in C(\mathbb{R}_+, \mathbb{R}_+) \) and \( w(u) > 0 \) for \( u > 0 \), \( w \) is nondecreasing with \( \int_0^\infty \frac{ds}{w(s)} = \infty \). Then all solutions of (3.6) are global in time.

**Proof.** If the assertion is not true, there is some \( x_0 \in \mathbb{C}^n \) such that the problem (3.6) has a solution \( x(t) \) which blows-up in finite time \( T \). If \( M = \sup_{u \in [0, T]} \{ q(u) \} \), let
\[
k_0 = |x_0| + \sup_{t \in [0, M]} \left\{ d\left(|x'(t)|\right) \right\} \left( \int_0^T c(s) \, ds \right).
\]
With \( u(t) = |x(t)|, t \in [0, T) \), we deduce from (3.6) and our hypothesis on \( F \) that
\[
u(t) \leq k_0 + \int_0^t a(s) w(u(s)) \, ds + \int_0^t b(s) w(u(\alpha(s))) \, ds,
\]
\( 0 \leq t < T \),
after integration. Changing variables in the second integral of the right-hand side in the above inequality, we obtain

\[ u(t) \leq k_0 + \int_0^t a(s) w(u(s)) \, ds + \int_0^t b(\alpha^{-1}(s)) w(u(s)) (\alpha^{-1})'(s) \, ds, \]

\[ t_0 \leq t < T. \]

From Proposition 1 one can now infer that \( u(t) \) is bounded on \([0, T]\), which contradicts (3.7) and completes the proof. \( \square \)

**Remark.** (i) In [13] a continuation theorem for (3.6) is obtained under the hypothesis that \( F \) has at most linear growth in all the variables other than \( t \). We allow a much larger class of nonlinearities.

(ii) In the particular case when \( F \) in (3.6) does not depend upon the last variable, Proposition 3 yields global existence of solutions for retarded differential equations and we recover some results from [3, 9].

**Example.** Consider the generalized pantograph equation [10]

\[
\begin{align*}
    x'(t) &= Ax(t) + Bx(pt) + Cx'(pt), \\
    x(0) &= x_0,
\end{align*}
\]

where \( A, B, C \) are complex matrices and \( p \in (0, 1) \). Proposition 3 shows that all solutions are global in time.

**Example.** Consider the generalized Liénard system with time delay

\[
\begin{align*}
    x' &= y - F(x) \\
    y' &= g(t, x(t - \tau(t))),
\end{align*}
\]

(3.8)

where \( F \in C(\mathbb{R}, \mathbb{R}), g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R}), \) and \( \tau(t) \leq t \) on \( \mathbb{R}_+ \). If \( a(t) = t - \tau(t) \) is an increasing diffeomorphism of \( \mathbb{R}_+ \) and

\[ |F(x)| \leq w(|x|), \quad |g(t, x)| \leq a(t) w(|x|), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \]

with \( a, w \in C(\mathbb{R}_+, \mathbb{R}_+), w(u) > 0 \) for \( u > 0, w \) nondecreasing and such that \( \int_1^\infty \frac{dt}{a(t)} = \infty \), then all solutions of (3.8) are global.

Indeed, we have that

\[ |(y - F(x), 0)| \leq |(x, y)| + w(|(x, y)|), \quad (x, y) \in \mathbb{R}^2, \]

\[ |(0, g(t, x))| \leq a(t) w(|(x, y)|), \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2, \]

and our hypotheses on \( w \) guarantee that \( \int_1^\infty \frac{dt}{a(t)} = \infty \) (see [3]). The statement follows from Proposition 3.
Remark. The problem of the global existence of the solutions to (3.8) was the object of the paper [14]. Therein, global existence criteria were given provided

\[ \{ t > 0 : \tau(t) = 0 \} = \emptyset. \]  

(3.9)

For \( g(x) = F(x) = x \ln(1 + |x|), x \in \mathbb{R}, \) and \( \tau[t] = \frac{1}{2} \ln(t^2 - 2t + 2), t \geq 1, \) we proved that all solutions of (3.8) are global. Note that the results from [14] are not applicable in this case, as condition (3.9) is not satisfied.

4. APPLICATIONS TO RETARDED AND IMPULSIVE DIFFERENTIAL EQUATIONS: NONOSCILLATORY SOLUTIONS

In this section we apply our main results to the qualitative analysis of solutions to retarded differential equations. Delay differential equations arise in the theory of control, mathematical biology, mathematical economics, and the theory of systems which communicate through lossless channels (see [8]). The oscillatory behaviour of the solutions of delay differential equations has been the subject of many investigations; cf. [7, 12] and the citations therein.

Our first aim is to give sufficient conditions under which the retarded differential equation

\[ x'(t) = F(t, x(\alpha(t))), \quad t \geq 0, \]  

(4.1)

has a positive solution on \( \mathbb{R}_+. \) Here \( \alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) is a diffeomorphism of \( \mathbb{R}_+ \) with \( \alpha(t) \leq t \) on \( \mathbb{R}_+. \)

**Proposition 4.** Consider (4.1) and assume that \( F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) is a continuous function for which there exists a constant \( \delta > 0 \) and a continuous function \( a(t) > 0 \) for \( t \in \mathbb{R}_+ \) such that

\[ |F(t, x)| \leq a(t)x \quad \text{for } t \in \mathbb{R}_+ \text{ and } 0 < x < \delta. \]

If

\[ \int_{\alpha(t)}^{t} a(s) \, ds < \frac{1}{e} \quad \text{for all } t \geq 0, \text{ and } \int_{0}^{\infty} a(s) \, ds < \infty, \]

then for every initial data \( x_0 \) such that \( 0 < x_0 < \delta \exp(-e\int_{0}^{\infty} a(s) \, ds) \), Eq. (4.1) has a positive solution \( x(t) \) on \( \mathbb{R}_+ \) which satisfies

\[ x_0 \exp\left( -e\int_{0}^{\infty} a(s) \, ds \right) < x(t) \leq x_0 \exp\left[ \int_{0}^{t} a(s) \exp\left( e\int_{\alpha(s)}^{s} a(\sigma) \, d\sigma \right) \, ds \right], \]

\( t \geq 0. \)
Proof. Under our conditions it is known (see [8, Chaps. 3, 4]) that for every \( x_0 \in \mathbb{R} \) there exists a solution \( x(t) \) of (4.1) with initial data \( x(0) = x_0 \), defined on some maximal interval \( [0, T) \) (note that \( \alpha(0) = 0 \)). Moreover, if \( T < \infty \) then

\[
\limsup_{t \to T} |x(t)| = \infty. \tag{4.2}
\]

Following an idea introduced in [7], let us perform the transformation

\[
y(t) = x(t) \exp \left( e \int_0^t a(s) \, ds \right), \quad 0 \leq t < T.
\]

Observe that

\[
y(0) = x(0) = x_0\]

and

\[
y'(t) = x'(t) \exp \left( e \int_0^t a(s) \, ds \right) + e x(t) a(t) \exp \left( e \int_0^t a(s) \, ds \right), \tag{4.3}
\]

for all \( t \in [0, T) \).

In particular,

\[
y'(0) = F(0, x_0) + e x_0 a(0) \geq a(0) x_0 (e - 1) > 0.
\]

Therefore, for any \( t > 0 \) near zero we have

\[
x_0 < y(t) < \delta \exp \left( e \int_0^t a(s) \, ds \right). \tag{4.4}
\]

We will show that (4.4) holds on \( (0, T) \). If not, there are two possible cases:

(A) There exists \( t_1 \in (0, T) \) such that (4.4) holds for all times \( t \in (0, t_1) \) and

\[
y(t_1) = \delta \exp \left( e \int_0^{t_1} a(s) \, ds \right).
\]

(B) There exists \( t_2 \in (0, T) \) such that (4.4) holds for \( t \in (0, t_2) \), and

\[
y(t_2) = x_0.
\]

Let us first show that case (A) does not occur.

Assume that (A) holds. From (4.4) it follows that

\[
0 < x(t) < \delta, \quad 0 \leq t < t_1.
\]

Using the hypothesis on \( F \) we therefore obtain from (4.3) that for \( 0 \leq t < t_1 \),

\[
y'(t) \leq a(t) x(\alpha(t)) \exp \left( e \int_0^t a(s) \, ds \right) + e x(t) a(t) \exp \left( e \int_0^t a(s) \, ds \right).
\]
Taking into account the way we defined \( y \), we infer
\[
y'(t) \leq a(t)y(\alpha(t)) \exp\left( e \int_{\alpha(t)}^{t} a(s) \, ds \right) + ce(t)y(t), \quad 0 \leq t < t_1.
\]
Integrating on \([0, t]\) we deduce that for \(0 \leq t < t_1\),
\[
y(t) \leq x_0 + e \int_0^t a(s)y(s) \, ds + \int_0^t a(s)y(\alpha(s)) \exp\left( e \int_{\alpha(s)}^{s} a(\sigma) \, d\sigma \right) ds.
\]
A change of variables transforms the above inequality into
\[
y(t) \leq x_0 + e \int_0^t a(s)y(s) \, ds + \int_0^t a(s)y(\alpha(s)) \exp\left( e \int_{\alpha(s)}^{s} a(\sigma) \, d\sigma \right) ds.
\]
Applying Proposition 1 we obtain that for \(0 \leq t < t_1\),
\[
y(t) \leq x_0 \exp\left( e \int_0^t a(s) \, ds + \int_0^t a(s) \exp\left( e \int_{\alpha(s)}^{s} a(\sigma) \, d\sigma \right) ds \right), \quad (4.5)
\]
after performing a change of variables. Now letting \( t \to t_1 \) in the above relation we would have
\[
y(t_1) \leq x_0 \exp\left( e \int_0^{t_1} a(s) \, ds + \int_0^{t_1} a(s) \exp\left( e \int_{\alpha(s)}^{s} a(\sigma) \, d\sigma \right) ds \right)
\]
\[
\leq x_0 \exp\left( e \int_0^{t_1} a(s) \, ds + \int_0^{t_1} a(s) \exp\left( e \int_{\alpha(s)}^{s} a(\sigma) \, d\sigma \right) ds \right)
\]
\[
\leq \delta \exp\left( e \int_0^{t_1} a(s) \exp\left( e \int_{\alpha(s)}^{s} a(\sigma) \, d\sigma \right) ds \right)
\]
\[
< \delta \exp\left( e \int_0^{t_1} a(s) \, ds \right).
\]
The obtained contradiction proves case (A) never holds. Therefore we have
\[
x(t) < \delta, \quad 0 \leq t < T, \quad (4.6)
\]
and then (4.5) also holds on \([0, T]\). It follows that
\[
x(t) \leq x_0 \exp\left( e \int_0^t a(s) \exp\left( e \int_{\alpha(s)}^{s} a(\sigma) \, d\sigma \right) ds \right), \quad 0 \leq t < T.
\]
Consider now case (B). We will prove that
\[
y'(t) > 0 \quad \text{for } 0 \leq t < t_2. \tag{4.7}
\]
As \(y'(0) > 0\) it is clear that (4.7) holds for any \(t \geq 0\) near zero. If (4.7) does not hold there exists \(t_3 \in (0, t_2)\) such that \(y'(t_3) = 0\) and \(y'(t) > 0\) for \(t \in (0, t_3)\). Taking into account (4.3) and (4.6), we would obtain for \(0 < t < t_3\) that
\[
y'(t) = F(t, x(\alpha(t))) \exp \left( e \int_0^t a(s) \, ds \right) + e \alpha(t) \exp \left( e \int_0^t a(s) \, ds \right)
\geq -a(t) x(\alpha(t)) \exp \left( e \int_0^t a(s) \, ds \right) + e \alpha(t) y(t)
= -a(t) y(\alpha(t)) \exp \left( e \int_{\alpha(t)}^t a(s) \, ds \right) + e \alpha(t) y(t).
\]
Letting \(t \to t_3\) in the above inequality and taking into account the monotonicity of \(y\) on \([0, t_3]\) we would deduce that
\[
y'(t_3) \geq a(t_3) y(t_3) \left[ e - \exp \left( e \int_{\alpha(t_3)}^t a(s) \, ds \right) \right] > 0.
\]
This contradicts our assumption \(y'(t_3) = 0\) and completes the proof that case (B) does not hold.

In conclusion (4.4) holds on \([0, T]\). This shows in particular that \(x(t)\) does not explode in finite time. By (4.2) we infer that \(T = \infty\) and the proof is complete.

Regarding the existence of positive solutions to Eq. (4.1), we also have

**Proposition 5** [7]. Assume that \(F: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}\) is a continuous function for which there exists a constant \(\delta > 0\) and a continuous function \(a(t) > 0\) for \(t \in \mathbb{R}_+\) such that
\[
0 > F(t, x) \geq -a(t) x \quad \text{for } t \in \mathbb{R}_+ \text{ and } 0 < x < \delta.
\]
If \(\alpha \in C(\mathbb{R}_+, \mathbb{R}_+)\) with \(\alpha(t) \leq t\) for \(t \geq 0\), and \(\lim_{t \to \infty} \alpha(t) = \infty\), while
\[
\int_{\alpha(t)}^t a(s) \, ds \leq \frac{1}{e}, \quad t \geq 0,
\]
then (4.1) has a positive solution \(x(t)\) on \(\mathbb{R}_+\) satisfying
\[
x_0 \exp \left( -e \int_0^t a(s) \, ds \right) < x(t) \quad \text{for all } t \geq 0.
\]
There are cases when our result yields the existence of a positive solution to (4.1) but Proposition 5 is not applicable.

**Example.** Take $\alpha(t) \equiv \ln(t + 1)$ and $F(t, x) \equiv (t + 1)^{-4} x^2$. By Proposition 4, Eq. (4.1) has in this case positive solution on $\mathbb{R}_+$. 

With respect to the conditions we imposed in Proposition 5 to guarantee the existence of a positive solution the following example is relevant.

**Example.** Consider the delay differential equation

$$x'(t) + p(t)x(\alpha(t)) = 0, \quad (4.8)$$

where $p \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing diffeomorphism of $\mathbb{R}_+$ such that $\alpha(t) \leq t$ on $\mathbb{R}_+$.

By a result in [12], if

$$\int_0^\infty p(s) \, ds < \infty \quad \text{and} \quad \liminf_{t \to \infty} \int_{\alpha(t)}^{t} p(s) \, ds > \frac{1}{e}$$

then all solutions of (4.8) are oscillatory.

Let us now show that Proposition 4 can be applied to prove the existence of nonoscillatory solutions to certain impulsive differential equations.

Let $\alpha : \mathbb{R} \to \mathbb{R}_+$ be an increasing diffeomorphism of $\mathbb{R}_+$ such that $\alpha(t) \leq t$ on $\mathbb{R}_+$ and such that $\alpha(t) = t - m \tau$ far out where $\tau > 0$ and $m \in \mathbb{N}$. Consider the delay impulsive differential equation

$$\begin{cases}
y'(t) + p(t)y(\alpha(t)) = 0, \\
y(t_k^+) - y(t_k) = b_k(t_k), \quad k \geq 1,
\end{cases} \quad (4.9)$$

where $b > -1$, and $t_1 > 0$, $t_{k+1} - t_k = \tau$, and $p \in C(\mathbb{R}_+ \times \mathbb{R}_+)$. 

By a result in [15], all solutions of (4.9) are oscillatory if and only if all solutions of the delay differential equation

$$x'(t) + p(t)(1 + b)^{-m} x(\alpha(t)) = 0$$

are oscillatory. In view of Proposition 4 we obtain

**Proposition 6.** If

$$\int_0^\infty |p(s)| \, ds < \frac{(1 + b)^m}{e},$$

then the impulsive differential equation (4.9) has a nonoscillatory solution.
REFERENCES