Diffraction Problems and Inversion of Infinite Structured Matrices

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Many model diffraction problems, generated by the Helmholtz equation, can be reduced to solving the infinite systems of linear algebraic equations $SX = F$. It proves that often the operators $S$ in these problems satisfy some operator identities of the form $AS - SB = \Pi_1 \Pi_2^*$, where $A$ and $B$ are diagonal matrices and matrices $\Pi_1$ and $\Pi_2$ have a finite number of columns. By the rigorous regularization and simultaneous application of the operator identity method one can justify and significantly improve the procedure for solving such systems. The case of the diffraction in the planar waveguide with a cross-sectional jump, absolutely soft upper boundary, and absolutely hard lower boundary is thoroughly considered in the paper as an example. In this case $S$ is an infinite matrix, acting as an unbounded operator. The reflection and transmission coefficients are expressed explicitly via the action of $S^{-1}$ on two fixed columns, and the ways of obtaining $S^{-1}$ on these columns are investigated.

0. INTRODUCTION

Many model diffraction problems in acoustics and electrodynamics generated by the Helmholtz equation, can be reduced to solving the infinite systems of linear algebraic equations

$$SX_m = F_m$$

(0.1)

(see [1, 7, 13] and references therein). It proves that operators $S$ in [1, 7, 13] satisfy the operator identity

$$AS - SB = \Pi_1 \Pi_2^*,$$

(0.2)
where \( A \) and \( B \) are diagonal matrices and matrices \( \Pi_1 \) and \( \Pi_2 \) have a finite number of columns. Identities of the form (0.2) were introduced in [10] as an important tool for the inversion of structured operators (see also [12] and references therein). In case of the finite structured matrices this identity was successfully used in many works (see, for instance, [5, 8]). Here we shall consider an example, when \( S \) is an infinite matrix acting as an unbounded operator. By the rigorous regularization and simultaneous application of the operator identity method one can justify and significantly improve the procedure for solving such systems. Some of the results were announced earlier in [9].

1. SOLVABILITY AND REGULARIZATION

Consider a planar waveguide with cross-sectional jump (Fig. 1). The lower boundary \( \Gamma_1 \) of the waveguide, given as a set

\[
\Gamma_1 = \{(x, -a): -\infty < x < 0\} \cup \{(0, y): -a \leq y \leq -b\} \\
\cup \{(x, -b): 0 < x < \infty\} \quad (a > b > 0),
\]

is absolutely hard. The upper boundary \( \Gamma_2 = \{(x, 0): -\infty < x < \infty\} \) is absolutely soft. Suppose that the wide part of the waveguide contains the incident \( m \)th proper wave \( P = \sin[\pi(m + \frac{1}{2})y/a]e^{2\pi i\omega_{ma}x/a}, \) where \( \omega_{ma} = \sqrt{\kappa^2 - (m + \frac{1}{2})^2}/4, \) the branch of the square root is chosen so that

![FIG. 1. Planar waveguide with a cross-sectional jump.](image_url)
\[ \omega_{na} + \bar{\omega}_{na} \geq 0, \quad i(\omega_{na} - \omega_{na}) \geq 0, \quad \text{and} \quad \kappa \text{ is determined by the wavelength and the properties of media.} \]

Let \( R_{lm} \) and \( T_{lm} \) denote the complex amplitudes of the reflected and transmitted eigenwaves, respectively. In our problem the reflection coefficients \( R_{lm} \) are obtained from the solution \( X_m = \text{col}[R_{0m} \ R_{1m} \ \ldots] \) of system (0.1) (col means column). The matrix \( S \) and the vector \( F_m \) in (0.1) have a block form

\[
S = \begin{bmatrix} C_1D_1 + D_2C_1 \ D_3 \\ C_2D_1 \end{bmatrix}, \quad F_m = \begin{bmatrix} f_{1m} \\ f_{2m} \end{bmatrix},
\]

where \( D_p \) (\( 1 \leq p \leq 3 \)) are diagonal matrices

\[
D_1 = \text{diag}\{\omega_{0a}, \omega_{1a}, \ldots\}, \quad D_2 = \text{diag}\{\omega_{0b}, \omega_{1b}, \ldots\}, \quad D_3 = \text{diag}\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\},
\]

with \( \omega_{mb} \) given by

\[
\omega_{mb} = \sqrt{\kappa^2 - (m + \frac{1}{2})^2/(2\theta)^2}, \quad i(\omega_{mb} - \omega_{mb}) \geq 0, \quad \theta = b/a.
\]

Suppose that \( \theta \) is an irrational number. Then matrices \( C_p \) and vectors \( f_{pm} \) (\( p = 1, 2 \)) are defined by the following expressions for their elements:

\[
\begin{align*}
\cos(\theta(k + \frac{1}{2})) & \quad \frac{(1 + \frac{1}{2})^2 \theta^2 - (k + \frac{1}{2})^2}{(l + \frac{1}{2})^2 (1 - \theta)^2 - (k + \frac{1}{2})^2},
\end{align*}
\]

\[
\begin{align*}
\cos(\theta(k + \frac{1}{2})) & \quad \frac{(1 + \frac{1}{2})^2 \theta^2 - (k + \frac{1}{2})^2}{(l + \frac{1}{2})^2 (1 - \theta)^2 - (k + \frac{1}{2})^2},
\end{align*}
\]

\[
\begin{align*}
f_{1km} = (\omega_{ma} - \omega_{kb})(m + \frac{1}{2})c_{1km}, & \quad f_{2km} = \omega_{mb}c_{2km},
\end{align*}
\]

Notice that relations (0.1) and (1.1)–(1.4) are drawn by the standard matching method which was also used in [7, 13]. Let \( \ell_2 \) and \( \tilde{\ell}_2 \) denote the sequence spaces of vectors \( h = (h_k)_{k=0}^\infty \) with the norms given by

\[
\|h\|^2 = \sum_{k=0}^\infty |h_k|^2, \quad \|h\|_\alpha^2 = \sum_{k=0}^\infty (k + \frac{1}{2})|h_k|^2,
\]

respectively. Taking into account energetical considerations, we see that vectors \( X_m \) have to belong to \( \tilde{\ell}_2 \). As we remarked above, \( S \) is an unbounded operator defined on linear manifold \( M \subset \tilde{\ell}_2 \):

\[
M = \left\{ h: h \in \tilde{\ell}_2, \lim_{n \to \infty} \sum_{k=0}^n h_k \cos(\pi \theta(k + \frac{1}{2})) = c < \infty \right\}.
\]
PROPOSITION 1.1. There exists the unique solution $X \in M$ of the system

$$SX = F,$$  \hspace{1cm} (1.6)

with $F \in l_2 \oplus l_2$.

Proof. The sets of functions

$$\left\{ \sqrt{2/a} \sin\left( \pi a^{-1} \left( l + \frac{1}{2} \right) y \right) \right\}_{l=0}^{\infty}, \quad \left\{ \sqrt{2/b} \sin\left( \pi b^{-1} \left( l + \frac{1}{2} \right) y \right) \right\}_{l=0}^{\infty}, \quad \text{and} \quad \left\{ \sqrt{2/(a-b)} \sin\left( \pi (a-b)^{-1} \left( l + \frac{1}{2} \right) (a-y) \right) \right\}_{l=0}^{\infty}$$

form orthonormal bases in $L_2(0,a)$, $L_2(0,b)$, and $L_2(b,a)$, respectively. Let $(\cdot, \cdot)_{L_2}$ denote the scalar product in the corresponding space. Let us introduce operators

$$U_1 = \left\{ \left( -1 \right)^{l+1} \sqrt{2/a} \sin\left( \pi a^{-1} (k + \frac{1}{2}) y \right), \right. \sqrt{2/b} \sin\left( \pi b^{-1} \left( l + \frac{1}{2} \right) y \right) \right\}_{k,l=0}^{\infty},$$

$$U_2 = \left\{ \left( -1 \right)^{l+1} \sqrt{2/a} \sin\left( \pi a^{-1} (k + \frac{1}{2}) y \right), \right. \sqrt{2/(a-b)} \sin\left( \pi (a-b)^{-1} \left( l + \frac{1}{2} \right) (a-y) \right) \right\}_{k,l=0}^{\infty}.$$

It can be easily seen that the operator with matrix representation

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \frac{2}{\pi} \left[ \theta^{3/2} D_3 C_1 \quad (1 - \theta)^{1/2} C_2 \right]$$

unitarily acts from $l_2 \oplus l_2$ onto $l_2$. Hence the operator

$$V = \frac{4}{\pi^2} \left[ \theta^{3/2} C_1 \quad (1 - \theta) C_2 \right]$$  \hspace{1cm} (1.7)

is bounded together with its inverse. Using the regularizer $V$ we rewrite system (1.6) in the form

$$S_0 Z = D_3^{-1/2} V F = \Phi,$$  \hspace{1cm} (1.8)

where

$$S_0 = D_3^{-1} D_1 + \left( 4\theta^3 / \pi^2 \right) D_3^{1/2} C_1 C_2 D_3^{1/2}, \quad Z = D_3^{1/2} X.$$  \hspace{1cm} (1.9)
In view of the boundedness of $V$, we see that the operator $D_3C_1^*$ is bounded. Therefore the operator

$$D_3^{1/2}C_1^*D_3^{1/2}$$

$$= \theta^{1/2}C_1D_3$$

$$- \left\{ \frac{\sqrt{l + \frac{1}{2}} \cos \pi \theta(l + \frac{1}{2})}{\left( \sqrt{\theta(l + \frac{1}{2})} + \sqrt{(k + \frac{1}{2})}(\theta(l + \frac{1}{2}) + (k + \frac{1}{2})) \right)} \right\}_{k,l=0}^{\infty}$$

is also bounded, which yields finally the boundedness of $S_0$.

Notice that $D_k + D_k^* \geq 0$, $i(D_k^* - D_k) \geq 0$ ($k = 1, 2$). Then by definitions (1.2) and (1.9) we have

$$\ker S_0 = 0. \quad (1.10)$$

The operator $\frac{i}{2}I + (2\theta^2/\pi^2)D_3^{1/2}C_1^*D_3C_1D_3^{1/2}$, where $I$ denotes the identity, is strictly positive and hence may be presented as $\nu(I + K_1)$, where $\|K_1\| < 1$, $\nu > 0$. Then the representation

$$S_0 = i\nu(I + K_1 + \tilde{K}) = i\nu(I + K_1)(I + K_2) \quad (1.11)$$

with compact operators $\tilde{K}$ and $K_2$ follows. Relations (1.10) and (1.11) mean that $S_0$ is boundedly invertible in $l_2$. Moreover, from $\Phi \in \tilde{l}_2$ it follows that

$$X = D_3^{-(1/2)}S_0^{-1}\Phi \in M. \quad (1.12)$$

Indeed, $X \in \tilde{l}_2$ and it remains to prove that

$$\lim_{n \to \infty} Y(n)D_3^{1/2}X < \infty \left( Y(n) = \begin{bmatrix} y_0 & y_1 & \cdots & y_n & 0 & \cdots \end{bmatrix}, \right.$$

$$y_i = (l + \frac{1}{2})^{-(1/2)}\cos(\pi\theta(l + \frac{1}{2})). \quad (1.13)$$

With rather long but trivial calculations one can show that

$$\text{w-lim} \lim_{n \to \infty} D_3^{1/2}C_1^*D_2C_1D_3^{1/2}Y(n)^\# = 0, \quad (1.14)$$

where w-lim is the weak limit in $l_2$. According to (1.14) we get

$$\lim_{n \to \infty} Y(n)D_3^{1/2}C_1^*D_2C_1D_3^{1/2}D_3^{1/2}X = 0. \quad (1.15)$$
Taking into account that \( \lim_{n \to \infty} Y(n)S_0D_3^{1/2}X = \lim_{n \to \infty} Y(n)\Phi < \infty \), from (1.9) and (1.15) we obtain
\[
\lim_{n \to \infty} Y(n)D_3^{-1}D_1D_3^{1/2}X < \infty; \tag{1.16}
\]
i.e., (1.13) is also true.

Notice also that by (1.11) the solution of system (1.8) may be approximated by the finite selection method.

2. STRUCTURE OF S

Put now
\[
A = \begin{bmatrix}
\theta^2D_3^{-2} & 0 \\
0 & (1 - \theta)^2D_3^{-2}
\end{bmatrix}, \quad B = D_3^{-2}. \tag{2.1}
\]

Then operator \( S \) satisfies the operator identity
\[
AS - SB = \Pi_1\Pi_2^*, \tag{2.2}
\]
where \( \Pi_1 \) and \( \Pi_2 \) are the three-column matrices:
\[
\Pi_1 = \begin{bmatrix}
D_2g & g & 0 \\
0 & (1 - \theta)D_3g & -D_3g
\end{bmatrix}, \tag{2.3}
\]
\[
\Pi_2 = \begin{bmatrix}
D_2D_4 & D_5^*D_4 & -D_3D_4 \\
D_4 & D_5^*D_4 & D_3D_4
\end{bmatrix},
\]
\[
D_4 = \text{diag}\{\cos(\pi\theta/2), \ldots, \cos(\pi\theta(k + 1/2)), \ldots\},
\]
\[
D_5 = \text{diag}\{1, -1, 1, -1, \ldots\},
\]
\[
g = \text{col}\left[2^2, (2/3)^2, \ldots, (2/(2k + 1))^2, \ldots\right].
\]

From (2.2) it follows that
\[
S^{-1}A - BS^{-1} = S^{-1}\Pi_1\Pi_2^*S^{-1}. \tag{2.4}
\]

Hence, putting
\[
S^{-1}\Pi_1\Pi_2^*S^{-1} = \left[\begin{array}{c}
\{q_{1lm}\}_{l,m=0}^w \\
\{q_{2lm}\}_{l,m=0}^w
\end{array}\right], \tag{2.5}
\]
we have

\[ S^{-1} = \begin{bmatrix} \mathcal{F}_1 & \mathcal{F}_2 \end{bmatrix}, \tag{2.6} \]

\[
\mathcal{F}_1 = \begin{cases} 
q_{1\text{lin}}(l + 1/2)^2(m + 1/2)^2 \\ 
\theta^2(l + 1/2)^2 - (m + 1/2)^2 
\end{cases},
\]

\[
\mathcal{F}_2 = \begin{cases} 
q_{2\text{lin}}(l + 1/2)^2(m + 1/2)^2 \\ (1 - \theta)^2(l + 1/2)^2 - (m + 1/2)^2 
\end{cases},
\]

Therefore we can express \( S^{-1} \) explicitly via \( S^{-1} \Pi_1 \) and \( \Pi_2^* S^{-1} \).

However, one can find all the information that is necessary to construct \( X_m \) in the two-column matrix \( S^{-1} \Pi \), where \( \Pi \) consists of the first and third columns of \( \Pi_i \):

\[
\tilde{\Pi} = \Pi_i \alpha^* = \begin{bmatrix} D_2 g & 0 \\
0 & -D_3 g \end{bmatrix}, \quad \alpha = \begin{bmatrix} 1 & 0 & 0 \\
0 & 0 & 1 \end{bmatrix}.
\]

To construct \( X_m \) in this way we shall need some preparation. Let us introduce matrix functions \( W_A \) and \( W_B \),

\[
W_A(\lambda) = I_3 - \Pi_2^* S^{-1} (A - \lambda I)^{-1} \Pi_1, \tag{2.7}
\]

\[
W_B(\lambda) = I_3 + \Pi_2^* (B - \lambda I)^{-1} S^{-1} \Pi_1,
\]

where \( I \) is the identity matrix, and \( I_3 \) is the finite identity matrix of order 3. (Notice that the columns of \( \Pi_1 \) and \( (A - \lambda I)^{-1} \Pi_1 \) belong to \( l^2 \otimes l^2 \).)

Matrix functions of the form (2.7) are the so-called transfer functions from the system theory; in the case of the additional property (2.2) they were introduced by L. Sakhnovich [11] and are called the transfer functions of the \( S \)-node. By (2.4) we have

\[
(B - \lambda I) S^{-1} = S^{-1} (A - \lambda I) - S^{-1} \Pi_1 \Pi_2^* S^{-1}. \tag{2.8}
\]

From (2.7) and (2.8) it follows that

\[
(B - \lambda I) S^{-1} (A - \lambda I)^{-1} \Pi_1 = S^{-1} \Pi_1 W_A(\lambda). \tag{2.9}
\]

Notice now that according to the definitions (1.1), (1.4), (2.1), and (2.3) the right-hand side of (0.1) may be presented in the form

\[
F_m = (A - \lambda_m I)^{-1} \Pi_1 d_m, \tag{2.10}
\]

\[
\lambda_m = (m + \frac{1}{2})^2, \quad d_m = \begin{bmatrix} -\cos(\pi \theta(m + \frac{1}{2}))/m + \frac{1}{2} \\
\omega_m \cos(\pi \theta(m + \frac{1}{2}))/m + \frac{1}{2} \\
-1)^m \omega_m/(m + \frac{1}{2})^2 \end{bmatrix}.
\]
As in the proof of (2.10) we can show that for \( e_m = \text{col}[0 \cdots 0 1 0 \cdots] \) we have

\[
Se_m = (A - \lambda_m I)^{-1} \Pi_1 \left( d_m + 2 \cos\left(\frac{\pi\theta(m + \frac{1}{2})}{m + \frac{1}{2}}\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right).
\] (2.11)

Using (2.11) we prove an auxiliary lemma.

**Lemma 2.1.** The vectors \( e_m^* S^{-1} \Pi_1 \) never turn to zero.

**Proof.** Suppose that \( e_m^* S^{-1} \Pi_1 = 0 \). Then by (2.9) we get

\[
e_m^* S^{-1} (A - \lambda I)^{-1} \Pi_1 = 0.
\]

On the other hand formula (2.11) leads to the contradicting equality

\[
e_m^* S^{-1} (A - \lambda_m I)^{-1} \Pi_1 \left( d_m + 2 \cos\left(\frac{\pi\theta(m + \frac{1}{2})}{m + \frac{1}{2}}\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = 1.
\]

Relations (0.1), (2.10), and (2.9) yield

\[
(B - \lambda_m I) X_m = (B - \lambda_m I) S^{-1} F_m = S^{-1} \Pi_1 W_A(\lambda_m) d_m.
\] (2.12)

Notice that the \( m \)th element of the diagonal matrix \( B \) equals \( \lambda_m \), and \( B - \lambda_m I \) is not invertible. Therefore we introduce matrices \( B_m \) with +1 added to the \( m \)th element: \( B_m = B + e_m e_m^* \). The formula (2.12) is basic for the next proposition.

**Proposition 2.2.** The vector \( X_m \), with the elements that are the reflection coefficients \( R_m \) corresponding to the \( m \)th eigenwave, can be constructed by the formula

\[
X_m = (B_m - \lambda_m I)^{-1} S^{-1} \Pi_1 v_m + u_m e_m.
\] (2.13)

The coefficients \( v_m \in \mathbb{C}^2 \) and \( u_m \in \mathbb{C} \) in (2.13) are defined via 3 \times 2 matrices \( Z_m \) and \( \Psi_m = (\psi_{km})_{k=1}^3 \),

\[
Z_m = \alpha^* + \Pi_1^x (B_m - \lambda_m I)^{-1} S^{-1} \Pi_1 - \Pi_1^x e_m^* e_m S^{-1} \Pi_1,
\] (2.14)

\[
\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
\psi_{1m} = \begin{bmatrix} \omega_{m+1} & -1 & 0 \end{bmatrix} Z_m,
\]

\[
\psi_{2m} = \begin{bmatrix} 0 & 1 & (-1)^{m+1} (m + \frac{1}{2}) \cos(\pi \theta(m + \frac{1}{2})) \end{bmatrix} Z_m,
\] (2.15)

\[
\psi_{3m} = e_m^* S^{-1} \Pi_1.
\]
by the formulas

\[ v_m = - \frac{2 \omega_m \cos(\pi \theta(m + \frac{1}{2}))}{m + \frac{1}{2}} \left( \Psi_m^* \Psi_m \right)^{-1} \Psi_m^* \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \]  

(2.16)

\[ u_m = - \left( 1 + \frac{(m + \frac{1}{2})}{\cos(\pi \theta(m + \frac{1}{2}))} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} Z_m v_m. \]  

(2.17)

**Proof.** We shall prove at first the equality

\[ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Pi_2 S^{-1} \begin{bmatrix} h \\ 0 \end{bmatrix} = 0 \]  

(2.18)

for all vectors \( h \in l_2 \). As

\[ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Pi_2 D_1^{-1} D_2^{1/2} = Y \]

\( Y = \begin{bmatrix} y_0 & y_1 & \ldots \end{bmatrix}, y_i = (l + \frac{1}{2})^{-(1/2)} \cos(\pi \theta(l + \frac{1}{2})), \)

so by Proposition 1.1 and (1.8) we have

\[ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Pi_2 S^{-1} \begin{bmatrix} h \\ 0 \end{bmatrix} \]

\[ = Y D_1 D_2^{-1} D_3^{1/2} S^{-1} \begin{bmatrix} h \\ 0 \end{bmatrix} \]

\[ = Y (S_0 - (4\theta^3/\pi^2) D_3^{1/2} C_0^* D_2 C_1 D_3^{1/2}) D_3^{1/2} S^{-1} \begin{bmatrix} h \\ 0 \end{bmatrix}. \]

Taking into account (1.15) one can simplify this equality:

\[ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Pi_2 S^{-1} \begin{bmatrix} h \\ 0 \end{bmatrix} = Y (S_0 D_3^{1/2} S^{-1}) \begin{bmatrix} h \\ 0 \end{bmatrix}. \]

According to (1.6), (1.8), and (1.9) we get \( S_0 D_3^{1/2} S^{-1} = D_3^{-(1/2)}V \). Hence from (1.7) it follows that

\[ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Pi_2 S^{-1} \begin{bmatrix} h \\ 0 \end{bmatrix} = Y D_3^{-(1/2)} V \begin{bmatrix} h \\ 0 \end{bmatrix} = (4\theta^3/\pi^2) Y D_3^{1/2} C_0^* h. \]

Applying now (1.14) we obtain (2.18). From (1.1), (1.2), (1.4), and (2.18) it follows that

\[ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Pi_2 S^{-1} F_m = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \Pi_2 \epsilon_m. \]  

(2.19)
By (2.7), (2.10), and (2.19) we get
\[ [0 \ 1 \ 0] W_A(\lambda_m) d_m = 0; \text{ i.e.,} \]
\[ W_A(\lambda_m) d_m = \alpha^* A W_A(\lambda_m) d_m. \]  
\( (2.20) \)
Formulas (2.12) and (2.20) yield
\[ (B - \lambda_m I) X_m = S^{-1} \tilde{H} \alpha W_A(\lambda_m) d_m. \]  
\( (2.21) \)
Formula (2.21) implies that if we put
\[ \nu_m = \alpha W_A(\lambda_m) d_m, \]  
\( (2.22) \)
there exists a coefficient \( u_m \) such that (2.13) is true. Let us prove that definitions (2.22) and (2.16) coincide. Consider matrix functions \( W_A \) and \( W_B \). From (2.2) and (2.7) one easily draws (see [11]) that
\[ W_B(\lambda) W_A(\lambda) = I_3. \]  
\( (2.23) \)
Moreover matrix function \( W_B \) has a first order pole at \( \lambda_m \) and admits a representation
\[ W_B(\lambda) = \Omega_1(\lambda) + \frac{\Omega_2}{\lambda - \lambda_m}, \]  
\( (2.24) \)
where \( \Omega_1 \) is the analytic in the neighborhood of the \( \lambda_m \) matrix function given by
\[ \Omega_1(\lambda) = I_3 + \Pi^*_2 (B_m - \lambda I)^{-1} S^{-1} \Pi_1 - (1 + \lambda_m - \lambda)^{-1} \Pi^*_2 e_m e^*_m S^{-1} \Pi_1, \]  
\( (2.25) \)
and \( \Omega_2 \) is the matrix given by
\[ \Omega_2 = -\Pi^*_2 e_m e^*_m S^{-1} \Pi_1. \]  
\( (2.26) \)
According to (2.23) and (2.24) we have
\[ \Omega_2 W_A(\lambda_m) = 0, \quad \Omega_1(\lambda_m) W_A(\lambda_m) + \Omega_2 \frac{dW_A}{d\lambda}(\lambda_m) = I_3. \]  
\( (2.27) \)
By virtue of (2.26) and the first of the equalities (2.27) we obtain
\[ e_m^* S^{-1} \Pi_1 W_A(\lambda_m) = 0. \]  
\( (2.28) \)
From (2.26) and the second of the equalities (2.27) we obtain
\[ \eta_m \Omega_1(\lambda_m) W_A(\lambda_m) = \eta_m \]
for \( \eta_m = \begin{bmatrix} \omega_m & -1 & 0 \\ 0 & 1 & (-1)^{m+1}(1 + \frac{1}{2}) \cos \left( \pi \theta \left( m + \frac{1}{2} \right) \right) \end{bmatrix} \).  
\( (2.29) \)
as one can see that \( \eta_m \Omega_2 = 0 \). Since by Lemma 2.1 \( e_m^s S^{-1} \Pi_1 \neq 0 \), formulas (2.28) and (2.29) yield the linear independence of the vector \( e_m^s S^{-1} \Pi_1 \) and the rows of \( \eta_m \Omega_1(\lambda_m) \). Notice now that definitions (2.14) and (2.15) and formula (2.25) imply

\[
Z_m = \Omega_1(\lambda_m) \alpha^*, \quad \begin{bmatrix} \psi_{1m} \\ \psi_{2m} \end{bmatrix} = \eta_m \Omega_1(\lambda_m) \alpha^*, \quad \psi_{3m} = e_m^s S^{-1} \Pi_1 \alpha^*.
\]

(2.30)

Hence matrices \( \Psi_m \) have rank 2 and therefore matrices \( \Psi_m^* \Psi_m \) are invertible. We see that the right hand side of (2.16) is well defined. Taking into account (2.20) and (2.30) from (2.28) and (2.29) we obtain

\[
\Psi_m \alpha W_A(\lambda_m) d_m = \begin{bmatrix} \eta_m d_m \\ 0 \end{bmatrix}.
\]

(2.31)

Now rewrite (2.31) as

\[
- \frac{2 \omega_{mm} \cos(\pi \theta(m + \frac{1}{2}))}{m + \frac{1}{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \Psi_m \alpha W_A(\lambda_m) d_m.
\]

(2.32)

Substitute (2.32) into the right-hand side of (2.16). We see that (2.16) is equivalent to (2.22), and representation (2.13) with \( v_m \) given by (2.22) was already proved.

From (2.20), (2.22), and (2.23) it follows that \( e_m^* (B_m - \lambda_m I)^{-1} S^{-1} \tilde{\Pi} v_m = 0 \). Therefore in (2.13) we have \( u_m = e_m^* X_m \); i.e., \( u_m \) equals the \( m \)th element of \( X_m \). To prove (2.17) it remains to show that

\[
e_m^* X_m = -\left(1 + \frac{(m + \frac{1}{2})[1 \quad 0 \quad 0]}{\cos(\pi \theta(m + \frac{1}{2}))}\right) Z_{m' \mu}^m.
\]

(2.33)

For this purpose we need some preparation. By (0.1) and (2.10) we have

\[
e_m^* X_m = e_m^* S^{-1}(A - \lambda_m I)^{-1} \Pi_1 d_m.
\]

(2.34)

Recall that \( B = \text{diag}(\lambda_0, \ldots, \lambda_m, \ldots) \) and hence

\[
e_m^* (B - \lambda I)^{-1} S^{-1}(A - \lambda I)^{-1} \Pi_1 = (\lambda_m - \lambda) e_m^* S^{-1}(A - \lambda I)^{-1} \Pi_1.
\]

Then multiplying both sides of the equality (2.9) by \( e_m^* \) and taking derivatives of the results at \( \lambda = \lambda_m \) we obtain

\[
e_m^* S^{-1}(A - \lambda_m I)^{-1} \Pi_1 = -e_m^* S^{-1} \Pi_1 \frac{dW_A}{d\lambda}(\lambda_m).
\]

(2.35)
Relations (2.34) and (2.35) yield
\[ e_m^* X_m = -e_m^* S^{-1} \Pi_1 \frac{dW_A}{d\lambda} (\lambda_m) d_m. \] (2.36)

Notice that the second equality in (2.27) implies
\[ \left[ \begin{array}{cc} 1 & 0 \\ \end{array} \right] \Omega_2 \frac{dW_A}{d\lambda} (\lambda_m) d_m \\
-\frac{\cos(\pi\theta(m + \frac{1}{2}))}{m + \frac{1}{2}} - \left[ \begin{array}{cc} 1 & 0 \\ \end{array} \right] \Omega_1 (\lambda_m) W_A(\lambda_m) d_m. \] (2.37)

Substitute (2.26) into the formula (2.37) and take into account the definition of \( \Pi_2 \) in (2.4):
\[ \frac{\cos(\pi\theta(m + \frac{1}{2}))}{m + \frac{1}{2}} - e_m^* S^{-1} \Pi_1 \frac{dW_A}{d\lambda} (\lambda_m) d_m \\
-\frac{\cos(\pi\theta(m + \frac{1}{2}))}{m + \frac{1}{2}} - \left[ \begin{array}{cc} 1 & 0 \\ \end{array} \right] \Omega_1 (\lambda_m) W_A(\lambda_m) d_m. \] (2.38)

According to (2.20), (2.22), and (2.30) we rewrite the right-hand side of (2.38) as
\[ \frac{\cos(\pi\theta(m + \frac{1}{2}))}{m + \frac{1}{2}} - e_m^* S^{-1} \Pi_1 \frac{dW_A}{d\lambda} (\lambda_m) d_m \\
-\frac{\cos(\pi\theta(m + \frac{1}{2}))}{m + \frac{1}{2}} - \left[ \begin{array}{cc} 1 & 0 \\ \end{array} \right] Z_m v_m. \] (2.39)

Equalities (2.36) and (2.39) prove (2.33).

It is essential that the matrices \( Z_m \) and \( \Psi_m \) in (2.14) and (2.15) are expressed via \( S^{-1} \Pi \). Therefore the vectors \( v_m \) in (2.16) and the constants \( u_m \) in (2.17) are expressed via \( S^{-1} \Pi \) also. Thus Proposition 2.1 states that to define all the reflection coefficients \( R_{m\ell} \) we must only know the action of \( S^{-1} \) on the two columns of \( \Pi \).

3. THE INVERSE OF \( S \) ON TWO VECTORS

Proposition 1.1 gives us a good way of calculating \( S^{-1} \Pi \). Namely by (1.8) and (1.9) we have
\[ S^{-1} \Pi = D_0^{-1/2} S_0^{-1} D_3^{-1/2} V \Pi. \] (3.1)
The other way of calculating $S^{-1} \check{\Pi}$ is to obtain two columns $X_k$ and $X_l$ with some fixed values of $k$ and $l$ as an experimental data and to express $S^{-1} \check{\Pi}$ and all other columns $X_m$ via these two columns. Put

$$G(k,l) = \alpha \left[ d_k - \Pi^+_k X_k \quad d_l - \Pi^+_l X_l \right]. \quad (3.2)$$

**Proposition 3.1.** In the case $4k \neq 2m + 1$ there always exist values $k$ and $l$ such that the $2 \times 2$ matrix $G(k,l)$ is invertible. Then the matrix $S^{-1} \check{\Pi}$ may be constructed by the formula

$$S^{-1} \check{\Pi} = \left[ (B - \lambda_k I) X_k \quad (B - \lambda_l I) X_l \right] G(k,l)^{-1}. \quad (3.3)$$

**Proof.** According to (0.1) and (2.10) we have

$$S^{-1}(A - \lambda_m I)^{-1} \Pi_d m = X_m \quad (m \geq 0). \quad (3.4)$$

By (2.7) and (3.4) we get

$$W_A(\lambda_m) d_m = d_m - \Pi^+_m X_m \quad (m \geq 0). \quad (3.5)$$

Taking into account (3.2), (3.5), and (2.21) we see that

$$G(k,l) = \left[ \alpha W_A(\lambda_k) d_k \quad \alpha W_A(\lambda_l) d_l \right]
\left[ (B - \lambda_k I) X_k \quad (B - \lambda_l I) X_l \right] = S^{-1} \check{\Pi} G(k,l). \quad (3.6)$$

Let us prove now that $G(k,l)$ is invertible for some $k$ and $l$. Notice that if $v_m = 0$, from (2.13) and (2.17) we draw $X_m = -e_m$. Substitute $-e_m$ and (2.10) into (0.1). The result contradicts the definitions (1.1)–(1.4) of $S$ and $F_m$; i.e., the vectors $v_m = \alpha W_A(\lambda_m) d_m$ cannot turn into zero. Notice further that the columns of $\check{\Pi}$ are linearly independent. Hence there exist values $k$ and $l$ such that the vectors $e_k^S S^{-1} \check{\Pi}$ and $e_l^S S^{-1} \check{\Pi}$ are linearly independent. As according to (2.20) and (2.28)

$$e_m^S S^{-1} \check{\Pi} v_m = e_m^S S^{-1} \Pi_1 \alpha^e \alpha W_A(\lambda_m) d_m = 0,$$

we see that the nonzero vectors $v_k$ and $v_l$ from $\mathbb{C}^2$ are linearly independent also. Therefore $\det G(k,l) \neq 0$ and the statement of the proposition follows from (3.6).
4. TRANSMISSION COEFFICIENTS

The matching method gives the following connection between the reflection and transmission coefficients:

\[
T_{km} = (-1)^k \frac{2 \theta^2}{\pi} \left( k + \frac{1}{2} \right)^{-2} \left[ 1 \ 0 \ 0 \right] \Pi_2 B \left( \frac{\theta^2}{(k + \frac{1}{2})^2} \right) \left[ 1 \ 0 \ 0 \right] X_m
\]

\[
= \left( m + \frac{1}{2} \right) c_{1km}, \quad (4.1)
\]

The operator identity (2.2) yields (see [12] and references therein) the equality

\[
\Pi_2 (B - \lambda I)^{-1} S^{-1} (A - zI)^{-1} \Pi_1 = (\lambda - z)^{-1} (W_B(\lambda)W_A(z) - I_2).
\]

(4.2)

From (3.4), (4.1), and (4.2) it follows that

\[
T_{km} = (-1)^k \frac{2 \theta^2}{\pi} \left( m + \frac{1}{2} \right)^2 \theta^2 \left( m + \frac{1}{2} \right)^2 - \left( k + \frac{1}{2} \right)^2 \left[ 1 \ 0 \ 0 \right]
\]

\[
\times \left( W_B \left( \frac{\theta^2}{(k + \frac{1}{2})^2} \right) W_A \left( \frac{1}{(m + \frac{1}{2})^2} \right) d_m - d_m \right)
\]

\[
= \left( m + \frac{1}{2} \right) c_{1km} . \quad (4.3)
\]

In view of definitions (1.3) and (2.10) and of formula (2.20), from (4.3) we draw

\[
T_{km} = \frac{(-1)^k \theta(m + \frac{1}{2})^2}{\theta^2 (m + \frac{1}{2})^2 - (k + \frac{1}{2})^2} \frac{2}{\pi} \left[ 1 \ 0 \ 0 \right] W_B \left( \frac{\theta^2}{(k + \frac{1}{2})^2} \right)
\]

\[
\times \alpha^* \alpha W_A \left( \frac{1}{(m + \frac{1}{2})^2} \right) d_m . \quad (4.4)
\]

Taking into account (2.22) and (4.4) and putting

\[
\gamma_k = (-1)^k \theta(2/\pi) \alpha W_B(\theta^2 \lambda_k)^* \text{col}[1 \ 0 \ 0],
\]
we obtain Ambartsumjan–Chandrasekhar–Sobolev type formulas for the transmission coefficients

$$T_{km} = \frac{(m + \frac{1}{2})^2}{\theta^2 (m + \frac{1}{2})^2 - (k + \frac{1}{2})^2} \gamma_k^m v_m,$$

with vectors depending on only one parameter on the right-hand side.

5. CONCLUSION

We have proposed a rigorous and analytically and computationally optimal approach to the diffraction problems in simple waveguides. The formulas are extended for the case of rational valued $\theta$ after the evident modifications $c_{1kl} = (-1)^{k+1} \pi/(2k+1)$ if $\theta(2l+1) = 2k+1$ and $c_{2kl} = (-1)^l/(2k+1)$ if $(1-\theta)(2l+1) = 2k+1$.

Analogous results were obtained for the case of the eigenwave in the narrow part of the planar waveguide with a cross-sectional jump and can be obtained for the waveguides mentioned in [13]. It would be interesting to apply this approach to different kinds of structured singular integral operators which are also used in similar situations. For other fields where the inversion of the structured matrices and operators is of great importance see [2–4, 6, 12].

REFERENCES


