JOURNAL OF FUNCTIONAL ANALYSIS 110, 410–433 (1992)

Spectral Factorization of Rectangular Rational Matrix Functions with Application to Discrete Wiener–Hopf Equations

MAREK RAKOWSKI*

Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210

Communicated by the Editors

Received January 31, 1991

The properties of a discrete Wiener–Hopf equation are closely related to the factorization of the symbol of the equation. We give a necessary and sufficient condition for existence of a canonical Wiener–Hopf factorization of a possibly nonregular rational matrix function W relative to a contour which is a positively oriented boundary of a region in the finite complex plane. The condition involves decomposition of the state space in a minimal realization of W and, if it is satisfied, we give explicit formulas for the factors. The results are generalized by means of centered realizations to arbitrary rational matrix functions. The proposed approach can be used to solve discrete Wiener–Hopf equations whose symbols are rational matrix functions which admit canonical factorization relative to the unit circle.

1. INTRODUCTION

Consider the discrete Wiener–Hopf equation

$$\sum_{k=0}^{\infty} A_{j-k} x_k = c_j \qquad (j=0, 1, 2, ...), \tag{1.1}$$

where A_j $(j = 0, \pm 1, \pm 2, ...)$ are complex $m \times n$ matrices with $\sum_{j=-\infty}^{\infty} \|A_j\| < \infty$, and $\{c_j\}_{j=0}^{\infty} \in l_p^m$. The function $A(z) = \sum_{j=-\infty}^{\infty} z^j A_j$ is called the symbol of the equation. It is well known that if $m = n$ and for every z on the unit circle $\det A(z) \neq 0$, then most properties of Eq. (1.1) can be deduced from the Wiener–Hopf factorization of the function A relative to the unit circle, where the Wiener–Hopf factorization is defined as follows. Let Γ be a rectifiable contour which forms the positively oriented boundary of a region on the Riemann sphere $\mathbf{C}_\infty$. A nonsingular matrix valued function A defined on Γ admits a (right) Wiener–Hopf factorization relative to Γ if

$$A = A_- D A_+, \tag{1.2}$$

* The author is partially supported by Department of Energy Grant DE-FGO2-92ER14270.

0022-1236/92 $5.00

where A_-, A_+, and D are matrix valued functions with the following properties. The function A_- is analytic outside Γ, continuous outside and on Γ, and $\det A_-$ does not vanish outside and on Γ. The function A_+ is analytic inside Γ, continuous inside and on Γ, and $\det A_+$ does not vanish inside and on Γ. The function D is equal to

$$\begin{bmatrix} \left(\frac{z-z_+}{z-z_-}\right)^{\kappa_1} & & & \\ & \left(\frac{z-z_+}{z-z_-}\right)^{\kappa_2} & & \\ & & \ddots & \\ & & & \left(\frac{z-z_+}{z-z_-}\right)^{\kappa_n} \end{bmatrix} \tag{1.3}$$

for some points z_+ inside and z_- outside Γ, and some integers κ_1, $\kappa_2, ..., \kappa_n$ with $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n$. Here, and in the sequel, the continuity of an $m \times n$ matrix valued function is understood in terms of the topology on $m \times n$ matrices induced by the (operator) norm of a matrix identified with an operator acting between the Euclidean spaces, and analyticity of a function at a point λ is understood in terms of the Laurent expansion of the function at λ. Equivalently, a matrix valued function A is continuous (analytic) at a point λ if each entry of A is continuous (analytic) at λ.

The integers $\kappa_1, \kappa_2, ..., \kappa_n$ above are uniquely determined by the function A and the contour Γ. They are called the indices of the factorization (or the (right) factorization indices). If all the indices are equal to zero, the factorization is said to be canonical. The factors A_- and A_+ in (1.2) are not unique. The possible nonuniqueness of A_- and A_+ is characterized in Theorem 7.2 in [17] (see also Theorem 1.2 in [10]).

We note that if the point at infinity is inside Γ, then a factorization $A_- D A_+$ with A_- and A_+ as above and

$$D(z) = \begin{bmatrix} (z-z_+)^{\kappa_1} & & & \\ & (z-z_+)^{\kappa_2} & & \\ & & \ddots & \\ & & & (z-z_+)^{\kappa_n} \end{bmatrix}, \tag{1.3'}$$

where z_+ is a point inside Γ and $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n$ are integers, is also called a Wiener–Hopf factorization relative to Γ. In fact, (1.3′) is a way of writing down (1.3) if $z_- = \infty$. If the point 0 is inside Γ, (1.3′) is usually simplified by choosing $z_+ = 0$. Also, if A_+ is analytic on $\mathbf{C}$ and $\det A_+$ does not vanish in the finite complex plane, a Wiener–Hopf factorization of A with the middle factor as in (1.3′) is called a Wiener–Hopf factorization at infinity.

The properties of a factorization (1.2) of a continuous nonsingular matrix valued function relative to the real axis and the unit circle have been obtained in [17] in the study of integral and discrete Wiener–Hopf equations. Suppose $A_- D A_+$ is a factorization of the symbol of Eq. (1.1) relative to the unit circle. Then the dimension of the solution set of the homogeneous equation

$$\sum_{k=0}^{\infty} A_{j-k} x_k = 0 \qquad (j = 0, 1, 2, \ldots)$$

equals the absolute value of the sum of the negative indices of the factorization. Also, the number of linearly independent elements $\{c_j\}_{j=0}^{\infty} \in l_p^n$ for which the equation is not solvable equals the sum of the positive indices of the factorization. Thus, the equation has a unique solution for every $\{c_j\}_{j=0}^{\infty} \in l_p^n$ if $D(z) \equiv I$, that is, if the factorization is canonical. A comprehensive treatment of a Wiener–Hopf factorization of nonsingular matrix valued functions can be found in [10]. We note that a Wiener–Hopf factorization is also called a standard [17, 10] or spectral [3] factorization in the literature.

The definition of a Wiener–Hopf factorization relative to a contour has been extended in [11] to the case of singular matrix valued functions as follows. Suppose a continuous matrix valued function A defined on Γ has constant rank equal to k. A factorization (1.2) is called a (right) Wiener–Hopf factorization of A relative to Γ if

(i) A_- is analytic outside and continuous outside and on Γ, and there exists a function $\tilde{A}_-$, analytic outside and continuous outside and on Γ, such that $\tilde{A}_-(z) A_-(z) = I$ for all z outside and on Γ,

(ii) A_+ is analytic inside and continuous inside and on Γ, and there exists a function $\tilde{A}_+$, analytic inside and continuous inside and on Γ, such that $A_+(z) \tilde{A}_+(z) = I$ for all z inside and on Γ,

(iii)

$$D(z) = \begin{bmatrix} \left(\frac{z - z_+}{z - z_-}\right)^{\kappa_1} & & & \\ & \left(\frac{z - z_+}{z - z_-}\right)^{\kappa_2} & & \\ & & \ddots & \\ & & & \left(\frac{z - z_+}{z - z_-}\right)^{\kappa_k} \end{bmatrix} \qquad (1.4)$$

for some points z_+ inside and z_- outside Γ, and some integers $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_k$.

If a function A admits a Wiener–Hopf factorization $A_- DA_+$ relative to Γ with D as in (1.4), then the integers $\kappa_1, \kappa_2, ..., \kappa_k$ are unique (see Theorem 2.1 in [11]). They are called the indices of the factorization. If all the indices are equal to zero, the factorization is said to be canonical. The definition of a Wiener–Hopf factorization at infinity of a singular matrix valued function is extended in the same way.

We note that the above definition of a Wiener–Hopf factorization of a singular matrix valued function differs from the definition in [14] in the size of factors. An $m \times n$ matrix valued function A is factored in [14] as $A_- DA_+$ where the sizes of A_- and A_+ are $m \times m$ and $n \times n$, respectively, and

$$D(z) = \begin{bmatrix} \operatorname{diag}(z^{\kappa_1}, z^{\kappa_2}, ..., z^{\kappa_k}) & 0 \\ 0 & 0 \end{bmatrix}.$$

While this difference does not affect factorization indices, the factorization according to our definition is "full-rank" (cf. [12, 18]). The idea of a factorization of a singular matrix valued function A which involves the rank of A has been used in [26].

The characterization of the possible nonuniqueness of factors in a Wiener–Hopf factorization relative to a contour extends to the singular case (see Theorem 2.3 in [11]).

THEOREM 1.1. *If a continuous matrix valued function A admits a Wiener–Hopf factorization $A_- DA_+$ relative to a contour Γ with D as in* (1.4), *then $B_- DB_+$ is a Wiener–Hopf factorization of A relative to Γ if and only if there exists a nonsingular $k \times k$ matrix valued function $Q = [q_{ij}]$ analytic on $\mathbf{C}_\infty \backslash \{z_-\}$ such that* $\det Q$ *does not vanish in $\mathbf{C}_\infty \backslash \{z_-\}$,*

$$B_+(z) = Q(z)\, A_+(z),$$
$$B_-(z) = A_-(z)\, D(z)\, [Q(z)]^{-1}\, [D(z)]^{-1},$$

and

(i) $q_{ij} = 0$ *if* $\kappa_i > \kappa_j$,

(ii) q_{ij} *is a constant if* $\kappa_i = \kappa_j$,

(iii) q_{ij} *is a polynomial in* $(z - z_+)/(z - z_-)$ *of degree at most* $\kappa_j - \kappa_i$ *if* $\kappa_i < \kappa_j$.

We will consider a Wiener–Hopf factorization of rational matrix functions, that is, meromorphic matrix valued functions on the Riemann sphere $\mathbf{C}_\infty$. A rational matrix function is said to be regular if it takes nonsingular matrix values at all but a finite number of points. There is an extensive literature of Wiener–Hopf factorization of regular rational matrix functions (see, e.g., [15]). The necessary and sufficient condition for existence of a canonical Wiener–Hopf factorization of a regular rational matrix function, together with the formulas for the factors, can be found in [3] (Theorems 4.9 and 1.5). The construction of a (not necessarily canonical) Wiener–Hopf factorization of a regular rational matrix function W, based on the realization of W, is presented in [4]. The formulas for factorization indices at infinity of matrix polynomials are given in [19]. A method to compute the factorization indices of a regular rational matrix function is presented in [1].

If a rational matrix function W admits a canonical Wiener–Hopf factorization relative to a contour, the factorization can be found by means of elementary column and row operations on the function W viewed as a matrix over the field of scalar rational functions. Below, assuming the system theoretic approach of [3] to Wiener–Hopf factorization, we prove a necessary and sufficient condition for existence of a canonical Wiener–Hopf factorization of an arbitrary rational matrix function W in terms of decomposition of the state space in a minimal realization of W. We also give formulas for the factors if the condition is satisfied. In Section 5, these results are applied to discrete Wiener–Hopf equations with rational symbols.

2. Preliminaries on Rational Matrix Functions

We will denote by $\mathcal{R}$ the field of scalar rational functions, and by $\mathcal{R}^{m\times n}$ the $\mathcal{R}$-linear space of $m\times n$ rational matrix functions. One of the basic tools in studying the properties of a function $W\in\mathcal{R}^{m\times n}$ is a Smith–McMillan factorization of W (see [22, 25]), that is, a factorization $W=EMF$ where E and F are unimodular matrix polynomials and

$$M(z)=\begin{bmatrix} \operatorname{diag}\left(\frac{p_1(z)}{q_1(z)}, \frac{p_2(z)}{q_2(z)}, ..., \frac{p_k(z)}{q_k(z)}\right) & 0 \\ 0 & 0 \end{bmatrix} \tag{2.1}$$

with the p_i's and q_j's monic polynomials such that $p_i \mid p_{i+1}$ $(i=1, 2, ..., k-1)$, $q_{j+1} \mid q_j$ $(j=1, 2, ..., k-1)$, and p_i, q_i are relatively prime $(i=1, 2, ..., k)$. We note that the function (2.1) is unique. It is called the Smith–McMillan form of W. The existence of a Smith–McMillan factorization of a function $W\in\mathcal{R}^{m\times n}$ follows immediately from the existence and uniqueness of a Smith normal form of a matrix over a principal ideal (or, more generally, Bezout) domain. Note that the number of nonzero elements in D determines the rank of $W(\lambda)$ at all but a finite number of points λ. This rank is called the normal rank of W.

If the function W has a pole at a point $\lambda\in\mathbf{C}$ then, clearly, some of

$q_1, q_2, ..., q_k$ vanish at λ. The orders of zeros of $q_1, q_2, ..., q_k$ at λ are called the partial multiplicities of the pole of W at λ. The sum of partial multiplicities of the pole of W at λ is called the (total) multiplicity of the pole of W at λ. The sum of multiplicities of all poles of W is called the McMillan degree of W (see, e.g., [8]). We say that W has a zero at λ if some of $p_1, p_2, ..., p_k$ vanish at λ. The orders of the zeros of $p_1, p_2, ..., p_k$ at λ are called the partial multiplicities of the zero of W at λ. The sum of partial multiplicities of the zero of W at λ is called the (total) multiplicity of the zero of W at λ. The multiplicities of the pole of W at infinity, and the zero and the multiplicities of the zero of W at infinity, are defined to be the multiplicites of the pole at $z=0$, and the zero and the multiplicities of the zero at $z=0$, of the function $H(z)=W(z^{-1})$.

The preceding definitions are standard in systems theory. We emphasize that while the definition of a pole of W at a point $z=\lambda$ coincides with the definition of a pole based on the Laurent expansion of W at λ, the above definition of the zero of W at λ does not coincide with the definition based on the Laurent expansion. In particular, W may have a zero at λ without vanishing there. Also, W may have a zero and a pole at the same point. The zeros of a function W defined above are sometimes called the directional zeros.

If a function $W\in\mathscr{R}^{m\times n}$ admits a Wiener–Hopf factorization relative to a contour Γ then, plainly, W has neither poles nor zeros on Γ. The converse statement is also true (see Theorem 2.1 in [10] for the regular case and Theorem 3.1 in [11] for the adaptation of the proof in [10] to the nonregular case).

THEOREM 2.1. *A nonzero function $W\in\mathscr{R}^{m\times n}$ admits a Wiener–Hopf factorization relative to a contour Γ if and only if no point of Γ is a pole or a zero of W.*

A function $W\in\mathscr{R}^{m\times n}$ can be represented as an $m\times n$ matrix with entries in $\mathscr{R}$. Another representation, commonly used in systems theory, is in terms of realizations. Suppose the poles of W in the finite plane are located at $\lambda_1, \lambda_2, ..., \lambda_r$. The principal part in the Laurent expansion of W at $z=\lambda_i$ can be represented (see [6]) as $C_i(z-A_i)^{-1}B_i$ with A_i, B_i, C_i matrices. Hence

$$W(z)=D(z)+C(z-A)^{-1}B, \tag{2.2}$$

where A, B, C are matrices and $D(z)$ is a matrix polynomial. The representation (2.2) has been used in [25]. After finding a realization $D_\infty+C_\infty(z-A_\infty)^{-1}B_\infty$ for $D(z^{-1})$, with the matrix A_∞ nilpotent, we obtain (see [9])

$$W(z)=D_\infty+C(z-A)^{-1}B+C_\infty(z^{-1}-A_\infty)^{-1}B_\infty. \tag{2.3}$$

The representation (2.3) of W is called a realization (see [7]). We note that if W is proper, that is, analytic at infinity, then the last term on the right-hand side of (2.3) does not occur and

$$W(z) = D + C(z - A)^{-1}B, \tag{2.4}$$

where $D = D_\infty$. Also, the function (2.4) does not have a zero at infinity if and only if the rank of D is equal to the normal rank of W.

The representation (2.4) of a proper function $W \in \mathscr{R}^{m \times n}$ is denoted by (A, B, C, D). It is called a state-space realization of W. The matrices A, B, C, D are sometimes identified with linear operators acting between finite dimensional spaces. The domain A is called the state space of the realization. A realization $\Theta = (A, B, C, D)$ of W is called minimal if the dimension of its state space is minimal. It can be shown that the dimension of the state space in a minimal realization of W is equal to the McMillan degree of W.

If $W \in \mathscr{R}^{m \times n}$, let

$$W^{ol} = \{\phi \in \mathscr{R}^{1 \times m} : \phi W = 0\},$$

and let

$$W^{or} = \{\psi \in \mathscr{R}^{n \times 1} : W\psi = 0\}.$$

Then W^{ol} and W^{or} are $\mathscr{R}$-linear subspaces of $\mathscr{R}^{1 \times m}$ and $\mathscr{R}^{n \times 1}$, respectively. If $\lambda \in \mathbf{C}_\infty$ and Λ is a subspace of $\mathscr{R}^{i \times j}$, we will denote by $\Lambda(\lambda)$ the $\mathbf{C}$-linear subspace of $\mathbf{C}^{i \times j}$ formed by the values at λ of those functions in Λ which are analytic at λ.

Choose a point $\lambda \in \mathbf{C}_\infty$. If $r \in \mathscr{R}$ is a nonzero function, let $|r|_\lambda = e^\gamma$ where γ is an integer such that $(z - \lambda)^\gamma r(z)$ (or $z^{-\gamma} r(z)$ if $\lambda = \infty$) is analytic, and does not vanish, at λ. Then $|\cdot|_\lambda$ determines a real non-Archimedean valuation of $\mathscr{R}$. If $x = (x_1, x_2, ..., x_n) \in \mathscr{R}^n$, let

$$\|x\|_\lambda = \max\{|x_1|_\lambda, |x_2|_\lambda, ..., |x_n|_\lambda\}.$$

Then $(\mathscr{R}^n, \|\cdot\|_\lambda)$ is a non-Archimedean normed space. Subspaces X and Y of $(\mathscr{R}^n, \|\cdot\|_\lambda)$ are said to be orthogonal (see [21]) if

$$\|x + y\|_\lambda = \max\{\|x\|_\lambda, \|y\|_\lambda\}$$

for all $x \in X$ and $y \in Y$. By Proposition 2.3 in [8], X and Y are orthogonal in $(\mathscr{R}^n, \|\cdot\|_\lambda)$ if and only if

$$X(\lambda) \cap Y(\lambda) = (0),$$

where $X(\lambda)$ and $Y(\lambda)$ are as defined in the preceding paragraph.

More generally, if σ is a subset of $\mathbf{C}_\infty$, we will say that subspaces X and Y of $\mathscr{R}^n$ are orthogonal on σ if X and Y are orthogonal in $(\mathscr{R}^n, \|\cdot\|_\lambda)$ for each $\lambda \in \sigma$. We will denote the orthogonality of X and Y on σ by $X \oplus_\sigma Y$. Also, a subspace of $\mathscr{R}^n$ generated by constant functions will be called a constant subspace of $\mathscr{R}^n$. The map Θ which sends constant subspaces of $\mathscr{R}^n$ to subspaces of $\mathbf{C}^n$ via the formula $\Theta(X) = X(\lambda)$, where $\lambda \in \mathbf{C}_\infty$ is arbitrary, is bijective. Thus, we can identify subspaces of $\mathbf{C}^n$ with constant subspaces of $\mathscr{R}^n$. Consequently, the definition of orthogonality of subspaces of $\mathscr{R}^n$ on σ extends to subspaces of $\mathscr{R}^n$ and $\mathbf{C}^n$.

Let V be a subspace of $\mathscr{R}^n$. One can choose a basis for V consisting of vectors polynomials $v_1, v_2, ..., v_k$ so that $\sum_{1 \leqslant i \leqslant k} \deg v_i$ is minimal. In any such basis, the degrees of $v_1, v_2, ..., v_k$ are unique up to a permutation (see [13]). If $W \in \mathscr{R}^{m \times n}$ and $V = W^{ol}$ (resp. $V = W^{or}$),

$$\deg v_1, \deg v_2, ..., \deg v_k$$

are called the left (resp. right) Forney indices of W. The sum of left (resp. right) Forney indices of W measures how much the column (resp. row) span of W (over $\mathscr{R}$) differs from a constant subspace of $\mathscr{R}^{m \times 1}$ (resp. $\mathscr{R}^{1 \times n}$).

One of the basic results on rational matrix functions (see [27] or [28]) is that the McMillan degree of a function $W \in \mathscr{R}^{m \times n}$ differs from the sum of multiplicities of all the zeros of W by the sum of its left and right Forney indices. The sum of left and right Forney indices of W is also called the defect of W in the literature (see [20]).

3. Factorization of Functions without a Pole or Zero at Infinity

In this section we will consider functions in $\mathscr{R}^{m \times n}$ which have neither a pole nor a zero at infinity. In Theorem 3.7 we will in addition assume that the contour Γ is a positively oriented boundary of a region in the finite plane $\mathbf{C}$. The results will be generalized to an arbitrary case in the next section. By a generalized inverse of a matrix D we will understand a (1, 2)-inverse of D, that is, any matrix $D^\ddagger$ such that $DD^\ddagger D = D$ and $D^\ddagger DD^\ddagger = D^\ddagger$.

Lemma 3.1. *Let (A, B, C, D) be a realization of a function $W \in \mathscr{R}^{m \times n}$ without a zero at infinity, and let $D^\ddagger$ be a generalized inverse of D. Then $\sigma(A - BD^\ddagger C)$ contains all the zeros of W.*

Proof. Let k be the normal rank of W. Since W does not have a zero at infinity, rank $D = k$. Choose $D_1 \in \mathbf{C}^{m \times k}$ and $D_2 \in \mathbf{C}^{k \times n}$ such that $D = D_1 D_2$. By Lemma 3.8 in [23], there exist a left inverse D_1^{-L} of D_1 and a right inverse D_2^{-R} of D_2 such that $D^{\ddagger} = D_2^{-R} D_1^{-L}$. Let $H(z) = D_1^{-L} W(z) D_2^{-R}$. The function H is square and takes the value I at infinity. Since $(A, BD_2^{-R}, D_1^{-L}C, I)$ is a realization of H, the function $G \in \mathscr{R}^{k \times k}$ with a realization $(A - BD_2^{-R}D_1^{-L}C, BD_2^{-R}, -D_1^{-L}C, I)$ satisfies the identity $G(z)\,H(z) = I$. Hence the zeros of H are the poles of G, of the same multiplicities. It remains to show that the zeros of W are the zeros of H.

Let EMF be a Smith–McMillan factorization of W. Partition E, M, and F so that

$$W = [E_1\ E_2] \begin{bmatrix} \tilde{M} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$
$$= E_1 \tilde{M} F_1$$

with $\tilde{M}$ regular, and let $\tilde{E}(z) = D_1^{-L} E_1(z)$, $\tilde{F}(z) = F_1(z) D_2^{-R}$. Then $H = \tilde{E}\tilde{M}\tilde{F}$ and the functions $\tilde{E}_1, \tilde{F}_1 \in \mathscr{R}^{k \times k}$ are regular. Suppose the ith diagonal entry of $\tilde{M}$ vanishes at $z = \lambda$, and let $\phi \in \mathscr{R}^{k \times 1}$ be such that $\tilde{F}(z)\,\tilde{\phi}(z) = e_i$ (cf. [6]), where e_i is the standard vector with 1 in the ith position and zeros elsewhere. Since $\tilde{F}$ is analytic at $z = \lambda$, either ϕ has a pole at $z = \lambda$ or $\phi(\lambda) \neq 0$. Let $\psi(z) = (z - \lambda)^{\eta} \phi(z)$ where η is a nonnegative integer such that ψ is analytic and does not vanish at λ. Then $(H\psi)(\lambda) = 0$, so H has a zero at $z = \lambda$. ∎

In fact, if we choose a generalized inverse $D^{\ddagger}$ of D appropriately, then the spectrum of $A^{\times} = A - BD^{\ddagger}C$ can give us a more complete information about the zeros of W.

LEMMA 3.2. *Let (A, B, C, D) be a minimal realization of a function $W \in \mathscr{R}^{m \times n}$ without a zero at infinity, let $\lambda \in \mathbf{C}$, and suppose $D^{\ddagger}$ is a generalized inverse of D such that*

(i) *the row span of $D^{\ddagger}$ intersects trivially with $W^{ol}(\lambda)$,*
(ii) *the column span of $D^{\ddagger}$ intersects trivially with $W^{or}(\lambda)$.*

Then the function W has a zero at λ if and only if $\lambda \in \sigma(A - BD^{\ddagger}C)$. Moreover, the partial multiplicities of the zero of W at λ coincide with the multiplicities of λ as an eigenvalue of $A - BD^{\ddagger}C$.

Proof. We use the notation introduced in the proof of Lemma 3.1. Since conditions (i) and (ii) hold, the matrix polynomials $\tilde{E}$ and $\tilde{F}$ have nonsingular values at λ. Hence the function H has a zero (resp. a pole) at λ if and only if W has a zero (resp. a pole) at λ. Also, the partial multiplicities of the zero (resp. a pole) of the function H at λ are equal to the partial multiplicities of the zero (resp. a pole) of the function W at λ. Choose a nonsingular matrix S such that

$$SAS^{-1}=\begin{bmatrix} A_1 & * & * \\ 0 & A_0 & * \\ 0 & 0 & A_2 \end{bmatrix}, \quad SBD_2^{-R}=\begin{bmatrix} * \\ B_0 \\ 0 \end{bmatrix}, \quad D_1^{-L}CS^{-1}=[0 \;\; C_0 \;\; *] \tag{3.1}$$

and (A_0, B_0, C_0, D) is a minimal realization of H (cf. Theorem 3.2 in [3]). Since the total multiplicity of the pole of H at λ is equal to the total multiplicity of the pole of W at λ, and (A, B, C, D) is a minimal realization of W, $\lambda \notin \sigma(A_1) \cup \sigma(A_2)$. Hence the Jordan blocks with the eigenvalue λ in the Jordan forms of $A - BD^{\ddagger}C$ and $A_0 - B_0C_0$ coincide. Since the partial multiplicities of the zero of H at λ are equal to the multiplicities of λ as an eigenvalue of $A_0 - B_0C_0$ (see, e.g., [7]), the assertion follows. ▮

Lemma 3.2 has the following corollary.

PROPOSITION 3.3. *Let (A, B, C, D) be a minimal realization of a function $W \in \mathcal{R}^{m\times n}$ without a zero at infinity, let $\sigma \subset \mathbf{C}$, and suppose $D^{\ddagger}$ is a generalized inverse of D such that*

(i) *the row span of $D^{\ddagger}$ is orthogonal to W^{ol} on σ,*

(ii) *the column span of $D^{\ddagger}$ is orthogonal to W^{or} on σ.*

Then the function W has a zero at a point $\lambda \in \sigma$ if and only if $\lambda \in \sigma(A - BD^{\ddagger}C)$. Moreover, the partial multiplicities of the zero of W at λ coincide with the multiplicities of λ as an eigenvalue of $A - BD^{\ddagger}C$.

We can find a generalized inverse of D which satisfies the hypotheses of Proposition 3.3 whenever the set σ is finite.

LEMMA 3.4. *Let X be a k-dimensional subspace of $\mathcal{R}^n$ and let $\sigma = \{\lambda_i \in \mathbf{C}_{\infty} : 1 \leqslant i \leqslant r\}$. Then there exists an $(n-k)$-dimensional subspace A of $\mathbf{C}^n$ such that $A \oplus_{\sigma} X$.*

Lemma 3.4 (cf. Corollary 3.4 in [2]) follows immediately from the Baire Category Theorem. One can find A by picking $n-k$ linearly independent vectors in $\mathbf{C}^n$ whose span intersects trivially with $\bigcup_{i=1}^{r} X(\lambda_i)$. Lemma 3.4 has the following corollary.

LEMMA 3.5. *Let $W \in \mathcal{R}^{m\times n}$ be a function without a pole or a zero at infinity, let $D = W(\infty)$, and let $\sigma = \{\lambda_i \in \mathbf{C} : 1 \leqslant i \leqslant r\}$ be a finite set. Then there exists a generalized inverse $D^{\ddagger}$ of D such that*

(i) *the row span of $D^{\ddagger}$ and W^{ol} are orthogonal on σ,*

(ii) *the column span of $D^{\ddagger}$ and W^{or} are orthogonal on σ.*

Proof. Let k be the normal rank of W. By Lemma 3.4, we can find an $(m-k)$-dimensional subspace Λ_{row} of $\mathbf{C}^{1\times m}$ and an $(n-k)$-dimensional subspace Λ_{col} of $\mathbf{C}^{n\times 1}$ such that $\Lambda_{row} \oplus_{\sigma\cup\{\infty\}} W^{ol}$ and $\Lambda_{col} \oplus_{\sigma\cup\{\infty\}} W^{or}$. Then the generalized inverse $D^{\ddagger}$ of D with the column and row spans equal to Λ_{col} and Λ_{row}, respectively, satisfies conditions (i) and (ii). ∎

Remark. If $D^{\ddagger}$ is any generalized inverse of D, then the row span of $D^{\ddagger}$ is orthogonal to W^{ol}, and the column span of $D^{\ddagger}$ is orthogonal to W^{or}, on the whole Riemann sphere $\mathbf{C}_{\infty}$ except for a finite number of points. This follows from the fact that the row and column spans of $D^{\ddagger}$ are orthogonal to W^{ol} and W^{or} at infinity, and hence the rational matrix functions

$$\begin{bmatrix} P_{\kappa_l}(z) \\ D^{\ddagger} \end{bmatrix} \quad \text{and} \quad [P_{\kappa_r}(z)\, D^{\ddagger}],$$

where P_{κ_l} and P_{κ_r} are such that the rows of P_{κ_l} form a basis for W^{ol} and the columns of P_{κ_r} form a basis for W^{or}, are regular. In Lemma 3.5, a finite set σ is given a priori, and we can actually compute a generalized inverse $D^{\ddagger}$ of D which satisfies conditions (i) and (ii).

If (A, B, C, D) is a minimal realization of a function $W \in \mathcal{R}^{m\times n}$, the zeros of W are precisely (see Theorem 4.1 in [25, Chap. 3]) the points of the complex plane where the singular pencil

$$\begin{bmatrix} z-A & B \\ -C & D \end{bmatrix}$$

loses rank. If W does not have a zero at infinity, Lemmas 3.1, 3.4, and 3.5 imply the following equivalent characterization of the set of zeros of W.

PROPOSITION. 3.6. *Let (A, B, C, D) be a minimal realization of a function $W \in \mathcal{R}^{m\times n}$ without a zero at infinity. Then*

$$\bigcap \sigma(A - BD^{\ddagger}C), \tag{3.2}$$

with the intersection taken over all generalized inverses $D^{\ddagger}$ of D, is the set of zeros of W.

Proposition 3.6 generalizes the well known fact that if (A, B, C, D) is a minimal realization of a function $W \in \mathcal{R}^{n\times n}$ and the matrix D is invertible, then the zeros of W are precisely the points of spectrum of $A - BD^{-1}C$.

The following theorem gives a necessary and sufficient condition for existence of a canonical Wiener–Hopf factorization relative to a contour of a function $W \in \mathcal{R}^{m\times n}$ without a pole or zero at infinity. If Γ is a positively oriented boundary of a region in the finite plane $\mathbf{C}$ and A is a linear operator whose spectrum does not meet Γ, $P(A;\Gamma)$ will denote the projection induced by the part of the spectrum of A inside Γ according to the formula

$$P(A;\Gamma)=\frac{1}{2\pi i}\int_\Gamma (z-A)^{-1}\,dz.$$

THEOREM. 3.7. *Let $\Theta=(A, B, C, D)$ be a minimal realization of a function $W\in\mathscr{R}^{m\times n}$ and suppose all the zeros of W are contained in a set $\sigma=\{\lambda_i\in\mathbf{C}: 1\leqslant i\leqslant r\}$. Let Γ be a contour which is a positively oriented boundary of a region in the finite plane and which does not meet $\sigma\cup\sigma(A)$, and put $X_i=\operatorname{Im} P(A;\Gamma)$. Choose a generalized inverse $D^\ddagger$ of D such that the row span of $D^\ddagger$ is orthogonal on σ to W^{ol} and the column span of $D^\ddagger$ is orthogonal on σ to W^{or}. Let $A^\times=A-BD^\ddagger C$, let*

$$\Omega_{min}=\operatorname{Im}\sum\left\{\lim_{\varepsilon\to 0^+}\int_0^{2\pi}(\lambda_i+\varepsilon e^{\varphi i}-A^\times)^{-1}\,d\varphi: 1\leqslant i\leqslant r,\ \lambda_i \text{ is outside } \Gamma\right\},$$

and let

$$\Omega_{max}=\operatorname{Ker}\sum\left\{\lim_{\varepsilon\to 0^+}\int_0^{2\pi}(\lambda_i+\varepsilon e^{\varphi i}-A^\times)^{-1}\,d\varphi:\ 1\leqslant i\leqslant r,\ \lambda_i \text{ is outside } \Gamma\right\}.$$

Then the function W admits a canonical Wiener–Hopf factorization relative to Γ if and only if the state space X of Θ contains a subspace X_2 complementary to X_1 such that

(i) *X_2 is invariant under $A^\times$ and $\Omega_{min}\subset X_2\subset\Omega_{max}$,*

(ii) *the matrix representations of A, B, C with respect to the decomposition $X_1\dotplus X_2$*

$$\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\qquad \begin{bmatrix} B_1\\ B_2\end{bmatrix},\qquad [C_1\quad C_2]$$

are such that the row span of B_1 is contained in the row span of D and the column span of C_2 is contained in the column span of D.

Moreover, suppose conditions (i) *and* (ii) *hold for an appropiate X_2, and let k be the normal rank of W. Then W_1W_2 is a canonical Wiener–Hopf factorization of W relative to Γ and only if*

$$W_1(z)=D_1+C_1(z-A_{11})^{-1}B_1D_2^{-R} \tag{3.3}$$

and

$$W_2(z)=D_2+D_1^{-L}C_2(z-A_{22})^{-1}B_2, \tag{3.4}$$

where $D_1 \in \mathbf{C}^{m \times k}$ *and* $D_2 \in \mathbf{C}^{k \times n}$ *are such that* $D = D_1 D_2$, *and* $D_1^{-L} \in \mathbf{C}^{k \times m}$ *and* $D_2^{-R} \in \mathbf{C}^{n \times k}$ *are one-sided inverses of* D_1 *and* D_2 *such that* $D^\ddagger = D_2^{-R} D_1^{-L}$.

Proof. Let X_2 with $\Omega_{min} \subset X_2 \subset \Omega_{max}$ be such that $X = X_1 \dot{+} X_2$ and (i) and (ii) hold. Choose $D_1 \in \mathbf{C}^{m \times k}$ and $D_2 \in C^{k \times n}$ such that $D = D_1 D_2$. By Lemma 3.8 in [23], there exist one-sided inverses D_1^{-L} and D_2^{-R} of D_1 and D_2 such that $D^\ddagger = D_2^{-R} D_1^{-L}$. It follows from the definition of X_1 that $A(X_1) \subset X_1$. Hence, by Theorem 3.1 in [23], $W = W_1 W_2$ where W_1 and W_2 are as in (3.3) and (3.4). By Lemma 1.4 in [3], $\sigma(A_{11})$ is the part of the spectrum of A inside Γ and $\sigma(A_{22})$ is the part of the spectrum of A outside Γ. Thus, both W_1 and W_2 are analytic on Γ, W_1 is analytic outside Γ, and W_2 is analytic inside Γ.

Suppose $\lambda \in \sigma$ is a point of spectrum of $A_{22} - B_2 D^\ddagger C_2$ inside Γ. Then for some $x \in X_2$

$$2\pi i x = \lim_{\varepsilon \to 0^+} \int_0^{2\pi} (\lambda + \varepsilon e^{\varphi i} - A^\times)^{-1} x d\varphi \neq 0$$

and $X_2 \not\subset \Omega_{max}$, a contradiction. Similarly, if $\lambda \in \sigma$ is a point of spectrum of $A_{11} - B_1 D^\ddagger C_1$ outside Γ, then $\Omega_{min} \not\subset X_2$, a contradiction. Since $W_1 W_2$ is a minimal factorization of W (see [26]), σ contains all the zeros of W_1 and W_2. Consequently, by Lemma 3.2, all the zeros of W_1 are inside Γ and all the zeros of W_2 are outside Γ. Assume without loss of generality that 0 is inside Γ and ∞ is outside Γ. Let EMF be a Smith–McMillan factorization of W_2, let $\tilde{E}$ and $\tilde{F}$ be multiplicative inverses of E and F, and, if $M = [\text{diag}(p_1/q_1, p_2/q_2, \ldots, p_k/q_k) 0]$, let

$$\tilde{M} = \begin{bmatrix} \text{diag}(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \ldots, \frac{q_k}{p_k}) \\ 0 \end{bmatrix}.$$

Then $\tilde{W}_2 = \tilde{F}\tilde{M}\tilde{E} \in \mathscr{R}^{n \times k}$ is analytic inside, and on, Γ and $W_2(z)\tilde{W}_2(z) = I$ for all z inside and on Γ. Similarly, after considering a Smith–McMillan factorization of a function $W_1(z^{-1})$, one can see that there is a function $\tilde{W}_1 \in \mathscr{R}^{k \times m}$ which is analytic outside, and on, Γ such that $\tilde{W}_1(z) W_1(z) = I$ for all z outside and on Γ. Thus, $W_1 W_2$ is a canonical Wiener–Hopf factorization of W relative to Γ.

If $\tilde{W}_1 \tilde{W}_2$ is another canonical Wiener–Hopf factorization of W relative to Γ then, by Theorem 1.1, $\tilde{W}_1(z) = W_1(z)S$ and $\tilde{W}_2(z) = S^{-1}W_2(z)$ for some nonsingular matrix $S \in \mathbf{C}^{k \times k}$. So

$$\tilde{W}_1(z) = D_1 S + C_1(z - A_{11})^{-1} B_1 D_2^{-R} S$$

and

$$\tilde{W}_2(z) = S^{-1} D_2 + S^{-1} D_1^{-L} C_2(z - A_{22})^{-1} B_2.$$

Clearly, $D=(D_1S)(S^{-1}D_2)$ and $D^{\ddagger}=(D_2^{-R}S)(S^{-1}D_1^{-L})$. Thus, the second assertion in the theorem is valid. It remains to show that if W admits a canonical Wiener–Hopf factorization relative to the contour Γ then there exists a subspace X_2 of the state space Θ which has the required properties.

Suppose W_1W_2 is a canonical Wiener–Hopf factorization of W relative to the contour Γ. Let $\Theta_i=(A_i, B_i, C_i, D_i)$ be a minimal realization of W_i $(i=1, 2)$. Then (see e.g., [5]) Θ is similar to

$$\left(\begin{bmatrix} A_1 & B_1C_2 \\ 0 & B_2 \end{bmatrix}, \begin{bmatrix} B_1D_2 \\ A_2 \end{bmatrix}, [C_1 \quad D_1C_2], D\right). \tag{3.5}$$

In fact, the first three matrices in (3.5) represent A, B, C with respect to the decomposition $X=X_1 \dotplus X_2$, where $X_1=\operatorname{Im} P(A;\Gamma)$ and X_2 can be expressed in terms of some similarity matrix S. Clearly, condition (ii) holds and, by Theorem 3.1 in [23], the subspace X_2 is invariant under $A^{\times}$. Let D_1^{-L} be a left inverse of D_1 and let D_2^{-R} be a right inverse of D_2 such that $D^{\ddagger}=D_2^{-R}D_1^{-L}$. Since W_2 has no zeros inside Γ and W and W_2 have the same column span (over $\mathscr{R}$), by Proposition 3.3, σ does not meet $\sigma(A_2-B_2D_2^{-R}C_2)$ inside Γ. Hence $X_2\subset\Omega_{max}$. Since W_1 has no zeros outside Γ, by Proposition 3.3, σ does not meet $\sigma(A_1-B_1D_1^{-L}C_1)$ outside Γ. Hence

$$\frac{1}{2\pi i}\sum\left\{\lim_{\varepsilon\to 0^+}\int_0^{2\pi}(\lambda_i+\varepsilon e^{\varphi i}-A^{\times})^{-1}\,d\varphi : 1\leqslant i\leqslant r, \lambda_i \text{ is outside } \Gamma\right\} \tag{3.6}$$

maps X_1 into X_2. Since $A^{\times}(X_2)\subset X_2$, the projection (3.6) maps X_2 into X_2. Thus $\Omega_{min}\subset X_2$. ∎

Note that the dimension of the space X_2 in Theorem 3.7 is uniquely determined by the function W and the contour Γ. Indeed, W and Γ determine uniquely the dimensions of Ω_{min} and Ω_{max}, and $\dim X_2$ is related to $\dim\Omega_{min}$ (and $\dim\Omega_{max}$) as follows. Let N_L be the sum of the left Forney indices of W and let N_R be the sum of right Forney indices of W. Suppose W_1W_2 with W_1 and W_2 as in (3.3) and (3.4) is a canonical Wiener–Hopf factorization of W relative to Γ. Then $W_1^{ol}=W^{ol}$ and $W_2^{or}=W^{or}$, and so N_L is the sum of all (left and right) Forney indices of W_1 and N_R is the sum of all Forney indices of W_2. Hence N_L is the difference between the McMillan degree of W_1 and the sum of multiplicities of all zeros of W_1, and N_R is the difference between the McMillan degree of W_2 and the sum of multiplicities of all zeros of W_2. Now, by Proposition 3.3, $\dim\Omega_{min}$ is equal to the sum of multiplicities of the zeros of W outside Γ and $(\dim X-\dim\Omega_{max})$ is equal to the sum of multiplicities of the zeros of W inside Γ. Consequently,

$$\dim X_2-\dim\ \Omega_{min}=N_R$$

and

$$\dim \Omega_{max} - \dim X_2 = \dim X_1 - (\dim X - \dim \Omega_{max}) = N_L.$$

The result of this argument is summarized in the following proposition.

PROPOSITION. 3.8. *Suppose a function $W \in \mathscr{R}^{m \times m}$ without a pole or a zero at infinity admits a Wiener–Hopf factorization relative to a countour Γ. Then, in the notation of Theorem* 3.7,

$$\dim X_2 - \dim \Omega_{min} = N_R$$

and

$$\dim \Omega_{max} - \dim X_2 = N_L,$$

where N_L and N_R are the sums of left and right Forney indices of W, respectively.

Consequently, Theorem 3.7 can be specialized as follows.

THEOREM 3.9. *Let $\Theta = (A, B, C, D)$ be a minimal realization of a function $W \in \mathscr{R}^{m \times n}$ and suppose all the zeros of W are contained in a set $\sigma = \{\lambda_i \in \mathbf{C} : 1 \leqslant i \leqslant r\}$. Let N_L and N_R be the sums of left and right Forney indices of W, and suppose $N_L N_R = 0$. Let Γ be a contour which is a positively oriented boundary of a region in the finite plane and which does not meet $\sigma \cup \sigma(A)$, and put $X_1 = \operatorname{Im} P(A; \Gamma)$. Choose a generalized inverse $D^\ddagger$ of D such that the row span of $D^\ddagger$ is orthogonal on σ to W^{ol} and the column span of $D^\ddagger$ is orthogonal on σ to W^{or}. Let $A^\times = A - BD^\ddagger C$, let*

$$X_2 = \operatorname{Im} \sum \left\{ \lim_{\varepsilon \to 0^+} \int_0^{2\pi} (\lambda_i + \varepsilon e^{\varphi i} - A^\times)^{-1} \, d\varphi : 1 \leqslant i \leqslant r, \lambda_i \text{ is outside } \Gamma \right\}$$

if $N_R = 0$, and let

$$X_2 = \operatorname{Ker} \sum \left\{ \lim_{\varepsilon \to 0^+} \int_0^{2\pi} (\lambda_i + \varepsilon e^{\varphi i} - A^\times)^{-1} \, d\varphi : 1 \leqslant i \leqslant r, \lambda_i \text{ is inside } \Gamma \right\}$$

otherwise. Then the function W admits a canonical Wiener–Hopf factorization relative to Γ if and only if X_2 complements X_1, in the state space X of Θ and the matrix representations of A, B, C with respect to the decomposition $X_1 \dotplus X_2$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad [C_1 \quad C_2]$$

are such that the row span of B_1 is contained in the row span of D and the column span of C_2 is contained in the column span of D.

Moreover, suppose W admits a canonical Wiener–Hopf factorization relative to Γ, and let k be the normal rank of W. Then $W_1 W_2$ is a canonical Wiener–Hopf factorization of W relative to Γ if and only if

$$W_1(z) = D_1 + C_1(z - A_{11})^{-1} B_1 D_2^{-R} \tag{3.7}$$

and

$$W_2(z) = D_2 + D_1^{-L} C_2(z - A_{22})^{-1} B_2, \tag{3.8}$$

where $D_1 \in \mathbf{C}^{m \times k}$ and $D_2 \in \mathbf{C}^{k \times n}$ are such that $D = D_1 D_2$, and $D_1^{-L} \in \mathbf{C}^{k \times m}$ and $D_2^{-R} \in \mathbf{C}^{n \times k}$ are one-sided inverses of D_1 and D_2 such that $D^{\ddagger} = D_2^{-R} D_1^{-L}$.

4. Factorization of Functions with Poles or Zeros at Infinity

Suppose that a function $W \in \mathscr{R}^{m \times n}$ with a realization (2.3) has a pole or a zero at infinity. Choose a point $\alpha \in \mathbf{C}$ such that the matrices $(\alpha - A)$ and $(I - \alpha A_\infty)$ are invertible. Then (see [24]; cf. [16])

$$W(z) = D_\alpha + C_\alpha \left(\frac{1}{z - \alpha} - A_\alpha \right)^{-1} B_\alpha, \tag{4.1}$$

where

$$A_\alpha = \begin{bmatrix} -(\alpha - A)^{-1} & 0 \\ 0 & (I - \alpha A_\infty)^{-1} A_\infty \end{bmatrix}, \qquad B_\alpha = \begin{bmatrix} -(\alpha - A)^{-2} B \\ (I - \alpha A_\infty)^{-2} B_\infty \end{bmatrix},$$

$$C_\alpha = [C \quad C_\infty], \qquad D_\alpha = W(\alpha).$$

The representation (4.1) of W is called a centered realization and is denoted by $(A_\alpha, B_\alpha, C_\alpha, D_\alpha, \alpha)$. A realization $\Theta = (A, B, C, D, \alpha)$ of a function $W \in \mathscr{R}^{m \times n}$ is said to be minimal if the size of the matrix A is $\delta(W) \times \delta(W)$, where $\delta(W)$ is the McMillan degree of W. We will call the domain of the operator corresponding to the matrix A in the realization Θ the state space of the realization.

Since (A, B, C, D, α) is a minimal centered realization of a function $W \in \mathscr{R}^{m \times n}$ if and only if (A, B, C, D) is a minimal realization of a function $H(z) = W((\alpha z + 1)/z)$, all the results of Section 3 can be immediately extended to an arbitrary rational matrix function W and a contour which is a positively oriented boundary of a region on the Riemann sphere. We state the generalization of Theorem 3.7. Below, the symbol $T_{1/(z-\alpha)}$ will denote the Möbius transformation which sends z to $1/(z - \alpha)$.

THEOREM 4.1. *Let $\Theta=(A, B, C, D, \alpha)$ be a minimal realization of a function $W \in \mathscr{R}^{m \times n}$, and let Γ be a contour such that α is outside Γ. Suppose all the zeros of W are contained in a set $\sigma=\{\lambda_i \in \mathbf{C}_\infty \backslash \{\alpha\} : 1 \leqslant i \leqslant r\}$ and Γ does not meet $\sigma \cup \{\lambda \in \mathbf{C}_\infty : 1/(\lambda-\alpha) \in \sigma(A)\}$. Let $\tilde{\Gamma}=T_{1/(z-\alpha)}(\Gamma)$, and put $X_1=\operatorname{Im} P(A; \tilde{\Gamma})$. Choose a generalized inverse $D^{\ddagger}$ of D such that the row span of $D^{\ddagger}$ is orthogonal on σ to W^{ol} and the column span of $D^{\ddagger}$ is orthogonal on σ to W^{or}. Let $A^{\times}=A-BD^{\ddagger}C$, let*

$$\Omega_{min}=\operatorname{Im} \sum \left\{ \lim_{\varepsilon \to 0^+} \int_0^{2\pi} \left(\frac{1}{\lambda_i-\alpha}+\varepsilon e^{\varphi i}-A^{\times} \right)^{-1} \delta\varphi : \right.$$
$$\left. 1 \leqslant i \leqslant r, \lambda_i \text{ is outside } \Gamma \right\},$$

and let

$$\Omega_{min}=\operatorname{Ker} \sum \left\{ \lim_{\varepsilon \to 0^+} \int_0^{2\pi} \left(\frac{1}{\lambda_i-\alpha}+\varepsilon e^{\varphi i}-A^{\times} \right)^{-1} \delta\varphi : \right.$$
$$\left. 1 \leqslant i \leqslant r, \lambda_i \text{ is outside } \Gamma \right\},$$

Then the function W admits a canonical Wiener–Hopf factorization relative to Γ if and only if the the state space X of Θ contains a subspace X_2 complementary to X_1 such that

(i) *X_2 is invariant under $A^{\times}$ and $\Omega_{min} \subset X_2 \subset \Omega_{max}$,*

(ii) *the matrix representations of A, B, C with respect to the decomposition $X_1 \dotplus X_2$*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad [C_1 \quad C_2]$$

are such that the row span of B_1 is contained in the row span of D and the column span of C_2 is contained in the column span of D.

Moreover, suppose conditions (i) *and* (ii) *hold for an appopiate X_2, and let k be the normal rank of W. Then $W_1 W_2$ is a canonical Wiener–Hopf factorization of W relative to Γ if and only if*

$$W_1(z)=D_1+C_1\left(\frac{1}{z-\alpha}-A_{11}\right)^{-1} B_1 D_2^{-R}$$

and

$$W_2(z)=D_2+D_1^{-L} C_2\left(\frac{1}{z-\alpha}-A_{22}\right)^{-1} B_2,$$

where $D_1 \in \mathbf{C}^{m\times k}$ *and* $D_2 \in \mathbf{C}^{k\times n}$ *are such that* $D = D_1 D_2$, *and* $D_1^{-L} \in \mathbf{C}^{k\times m}$ *and* $D_2^{-R} \in \mathbf{C}^{n\times k}$ *are one-sided inverses of* D_1 *and* D_2 *such that* $D^{\ddagger} = D_2^{-R} D_1^{-L}$.

Proof. A point λ is a pole of a rational matrix function $w(z)$ inside (resp. outside) and on the contour Γ if and only if the point $\bar{\lambda} = 1/(\lambda - \alpha)$ is a pole of the function

$$h(z) = w\left(\frac{\alpha z + 1}{z}\right)$$

inside (resp. outside) and on the contour $\tilde{\Gamma}$. So the theorem follows from Theorem 3.7. ∎

We note that Theorem 3.9 can be restated in the setting of Theorem 4.1.

5. Discrete Wiener–Hopf Equations

We consider now the equation

$$\sum_{k=0}^{\infty} A_{j-k} x_k = c_j \qquad (j = 0, 1, 2, ...), \tag{5.1}$$

where A_j $(j = 0, \pm 1, \pm 2, ...)$ are complex $m \times n$ matrices such that

$$\sum_{j=-\infty}^{\infty} \|A_j\| < \infty, \tag{5.2}$$

and $\{c_j\}_{j=0}^{\infty} \in l_p^m$. Here l_p^m is the product of m copies of l_p $(p \geqslant 1)$, with the norm of $(x_1, x_2, ..., x_m)$ equal to the sum of the norms of the components. Suppose $\{\xi_j\}_{j=0}^{\infty} \in l_p^n$ is a solution of (5.1). Putting $\xi_j = 0$ $(j = -1, -2, ...)$, we obtain

$$\sum_{k=-\infty}^{\infty} A_{j-k} \xi_k = c_j \qquad (j = 0, \pm 1, \pm 2, ...), \tag{5.3}$$

where $c_{-1}, c_{-2}, ...$ are defined by (5.3). Multiplying both sides of (5.3) by z^j and summing over j, we obtain

$$A(z)\, \xi_+(z) - c_-(z) = c_+(z), \tag{5.4}$$

where

$$\begin{aligned} A(z) &= \sum_{j=-\infty}^{\infty} z^j A_j, \qquad & c_+(z) &= \sum_{j=0}^{\infty} z^j c_j, \\ \xi_+(z) &= \sum_{j=0}^{\infty} z^j \xi_j, \qquad & c_-(z) &= \sum_{j=-\infty}^{-1} z^j c_j. \end{aligned} \tag{5.5}$$

We will identify an element $\{c_j\}_{j=0}^{\infty} \in l_p^i$, i a positive integer, with the series $\sum_{j=0}^{\infty} z^j c_j$. The image of l_p^i under this identification will be denoted by l_p^{i+}. The space of series $\sum_{j=-\infty}^{-1} z^j c_j$ such that $\{c_j\}_{j=1}^{\infty} \in l_p^i$ will be denoted by l_p^{i-}. The projection

$$\sum_{j=-\infty}^{\infty} z^j c_j \to \sum_{j=0}^{\infty} z^j c_j$$

will be denoted by π_+.

The case when the symbol A of Eq.(5.1) is square, and its determinant does not vanish at any point of the unit circle $\mathscr{T}$, has been considered in [17] (see also [3]). Here we assume that the symbol A is a rational matrix function with a constant rank on $\mathscr{T}$. We characterize first elements $\{c_j\}_{j=0}^{\infty}$ for which Eq. (5.1) is consistent. Let T_A denote the Toeplitz operator defined by (5.1), that is,

$$T_A(\{\xi_j\}_{j=0}^{\infty}) = \left\{ \sum_{k=0}^{\infty} A_{j-k} \xi_k \right\}_{j=0}^{\infty}.$$

PROPOSITION 5.1. *Suppose the symbol A of Eq. (5.1) is a rational matrix function with a constant rank on the unit circle. Then the range of T_A is closed.*

Proof. It follows from (5.4) that

$$T_A = \pi_+ M_A, \tag{5.6}$$

where M_A is the multiplication operator with symbol A. Plainly, T_A is continuous. Let $A_- D A_+$ with $D(z) = \operatorname{diag}(z^{\kappa_1}, z^{\kappa_2}, ..., z^{\kappa_k})$ be a Wiener–Hopf factorization of A relative to the unit circle $\mathscr{T}$. Then M_{A_+} maps l_p^{m+} onto l_p^{k+}, and we may assume that $k = m$ and $A = A_- D$. Suppose that all the indices of the factorization are nonnegative, and let A_-^{-L} be a rational matrix function analytic outside the unit disc such that $A_-^{-L}(z) A_-(z) = I$. Then, by (5.4),

$$\xi_+(z) = M_{(D(z))^{-1}} \pi_+ M_{A_-^{-L}}(c_+(z))$$

for every c_+ in the range of T_A and T_A is an open map from l_p^n onto its range R in l_p^m. So R is a closed subspace of l_p^m. If some indices of the factorization are negative, the range of T_A differs from R by at most a finite dimensional space, and so it is closed. ∎

The characterization of the range of T_A in [17] carries over to our setting. If $x, y \in \mathbf{C}^m$, let (x, y) denote the sum of the products of the corresponding coordinates of x and y. Also, let $q \geqslant 1$ be such that $1/p + 1/q = 1$.

PROPOSITION 5.2. *Suppose the symbol of Eq.* (5.1) *is a rational matrix function with a constant rank on the unit circle. Then Eq.* (5.1) *has a solution in* l_p^n *if and only if*

$$\sum_{j=0}^{\infty} (c_j, u_j) = 0 \tag{5.7}$$

for every $\{u_j\}_{j=0}^{\infty} \in l_q^m$ *such that*

$$\sum_{j=0}^{\infty} A_{k-j}^T u_k = 0 \qquad (j = 0, 1, 2, ...). \tag{5.8}$$

Proof. Plainly, the elements in the range of T_A satisfy the condition. Suppose $\{c_j\}_{j=0}^{\infty} \in l_p^m$ is not in the range of T_A. Then, since the range of T_A is closed, there exists a solution $\{u_j\}_{j=0}^{\infty} \in l_q^m$ of (5.8) such that $\sum_{j=0}^{\infty} (c_j, u_j) \neq 0$. ∎

If the symbol of Eq. (5.1) is square and its determinant does not vanish at any point of the unit circle, then Eq. (5.8) has a finite number of linearly independent solutions. This number determines the defect of T_A in l_p^m. In the general case, when Eq. (5.8) has infinitely many linearly independent solutions, the characterization of $\{c_j\}_{j=0}^{\infty} \in l_p^m$ for which Eq. (5.1) is solvable provided by Proposition 5.2 is practically less significant. We give now a different characterization. It follows from (5.6) that the elements $\{c_j\}_{j=0}^{\infty} \in l_p^m$ for which Eq. (5.1) is solvable are contained in

$$\pi_+ M_A(l_p^{n+} \dot{+} l_p^{n-}). \tag{5.9}$$

We will call members of the set (5.9) *admissible elements*. The space of admissible elements is a closed subspace of l_p^m.

PROPOSITION 5.3. *Suppose the symbol A of Eq.* (5.1) *is a rational matrix function with a constant rank on the unit circle* $\mathscr{T}$. *Then the defect of the operator T_A in the space of admissible elements equals the sum of positive indices in a Wiener–Hopf factorization of A relative to* $\mathscr{T}$.

Proof. We construct the complement of the range of T_A in the space of admissible elements. Let $A_- D A_+$ with $D(z) = \operatorname{diag}(z^{\kappa_1}, z^{\kappa_2}, ..., z^{\kappa_k})$ and A_- and A_+ rational matrix functions be a Wiener–Hopf factorization of the symbol A relative to the unit circle, and suppose $\kappa_i > 0 \geqslant \kappa_{i+1}$. It follows from the definition of a Wiener–Hopf factorization that $A_-(z) = \sum_{j=-\infty}^{0} z^j A_j$, where $\sum_{j=\infty}^{0} \|A_j\| < \infty$ and A_0 has linearly independent columns. Hence the members of the set

$$\mathscr{C} = \{\pi_+ M_{A_-}(z^\gamma e_\beta) : \gamma = 0, 1, ..., \kappa_\beta - 1, \beta = 1, 2, ..., i\},$$

where e_β is a standard vector with 1 in the βth position, are linearly independent admissible elements. Also Span $\mathscr{C}$, the linear span of the members of $\mathscr{C}$, intersects trivially with $\pi_+ M_A(l_p^{n+})$. Hence, by (5.4), Span $\mathscr{C}$ intersects trivially with the range of T_A. Let ξ_- be an element of l_p^{n-}. Then

$$M_{DA_+}(\xi_-) \in l_p^{k-} + \operatorname{Span}\{z^\gamma e_\beta : \gamma = 0, 1, ..., \kappa_\beta - 1, \beta = 1, 2, ..., i\}$$
$$+ M_{DA_+}(l_p^{n+}).$$

Thus, $\pi_+ M_A \xi_- \in \operatorname{Span} \mathscr{C} + \operatorname{Ran} T_A$. ∎

Suppose the symbol A of Eq. (5.1) admits a Wiener–Hopf factorization $A_- DA_+$ relative to the unit circle. An argument similar to the one in the proof of Proposition 5.3 shows that the dimension of the kernel of T_A equals the sum of absolute values of negative indices of the factorization whenever the factor A_+ is square. If the factor A_+ is not square, T_A has an infinite dimensional kernel. However, the space

$$(\operatorname{Ker} T_A) \cap T_{A_+^{-R}}(l_p^{k+}) \tag{5.10}$$

is finite dimensional, where A_+^{-R} is a function analytic on the closed unit disc such that $A_+(z) A_+^{-R}(z) = I$. Clearly,

$$T_A(l_p^{n+}) = T_A(T_{A_+^{-R}}(l_p^{k+}))$$

and, for a fixed A_+^{-R}, we may call members of $T_{A_+^{-R}}(l_p^{k+})$ admissible solutions. The dimension of Ker T_A in the space of admissible solutions is equal to the absolute value of the sum of negative indices of the factorization.

The factorization results from the previous sections can be used to characterize the existence and uniqueness of solutions of Eq. (5.1). For simplicity, we formulate this characterization in the case when the values $A(z)$ of the symbol A have linearly independent columns for all z on the unit circle.

THEOREM 5.4. *Suppose the symbol A of Eq.* (5.1) *is a rational matrix function and $A(z)$ has linearly independent columns for all z on the unit circle. Choose a point $\alpha \in \mathbf{C}$ such that $|\alpha| > 1$ and α is neither a pole nor zero of A, and find a minimal realization $\Theta = (E, B, C, D, \alpha)$ of A. Suppose all the zeros of A are contained in a set $\sigma \subset \mathbf{C}_\infty$ and*

$$\left(\sigma \cup \left\{\lambda \in \mathbf{C}_\infty : \frac{1}{\lambda - \alpha} \in \sigma(E)\right\}\right) \cap \{\lambda \in \mathbf{C} : |\lambda| = 1\} = \varnothing.$$

Let $\Gamma(t)=1/(e^{it}-\alpha)$, $0\leqslant t\leqslant 2\pi$, *and put* $X_1=\operatorname{Im} P(E;\Gamma)$. *Choose a generalized inverse* $D^{\ddagger}$ *of* D *such that the row span of* $D^{\ddagger}$ *is orthogonal on* σ *to* A^{ol}. *Let* $E^{\times}=E-BD^{\ddagger}C$, *and let*

$$X_2=\operatorname{Im}\sum\left\{\lim_{\varepsilon\to 0^+}\int_0^{2\pi}\left(\frac{1}{\lambda-\alpha}+\varepsilon e^{\varphi i}-E^{\times}\right)^{-1}d\varphi : \lambda\in\sigma \text{ and } |\lambda|>1\right\}.$$

Then Eq. (5.1) *has a unique solution for each admissible* $\{c_j\}_{j=0}^{\infty}$ *if and only if* X_1+X_2 *is a direct sum decomposition of the state space of* Θ *and* $C(X_2)$ *is contained in the column span of* D.

We note that the formulas for the factors in a Wiener–Hopf factorization relative to the unit circle of the symbol of Eq. (5.1) can be used to solve the equation. Indeed, suppose the equation is consistent and the functions $A_-^{-L}(z)=\sum_{j=-\infty}^{0} z^jA_{j-}$ and $A_+^{-R}(z)=\sum_{j=0}^{\infty} z^jA_{j+}$ in the definition of a canonical factorization of the symbol are such that $\sum_{j=-\infty}^{0}\|A_{j-}\|<\infty$ and $\sum_{j=0}^{\infty}\|A_{j+}\|<\infty$. Then, by (5.4),

$$\xi_+(z)=A_+^{-R}\pi_+A_-^{-L}c_+(z) \tag{5.11}$$

is a solution of the equation. Thus, the equation can be solved if we find the functions A_+^{-R} and A_-^{-L}. Now Theorem 4.1 provides the formulas for A_+ and A_-, and they can be used to find A_+^{-R} and A_-^{-L} as follows. Suppose (E, B, C, D, α) is a realization of A_+ and the matrix D has linearly independent rows. Let $\sigma_1\subset\mathbf{C}_\infty$ be a set which contains

$$\sigma=\left\{z\in\mathbf{C}_\infty : \frac{1}{z-\alpha}\text{ is a pole or zero of } A_+\right\},$$

and let $D_1^{\ddagger}$ be a generalized inverse of D whose column span is orthogonal on σ_1 to A^{or}. Let $\sigma_2\subset\mathbf{C}_\infty$ be a set which contains

$$\sigma\cup\left\{z\in\mathbf{C}_\infty : \frac{1}{z-\alpha}\in\sigma(E-BD_1^{\ddagger}C)\right\},$$

and let $D_2^{\ddagger}$ be a generalized inverse of D whose column span is orthogonal on σ_2 to A^{or}. Then

$$W_i(z)=D_i^{\ddagger}-D_i^{\ddagger}B\left(\frac{1}{z-\alpha}-E+BD_i^{\ddagger}C\right)^{-1}CD_i^{\ddagger}\qquad (i=1,2)$$

are functions such that $A_+(z)\,W_i(z)=I$ and a point λ with $|\lambda|<1$ is a pole of the corresponding rows of W_1 and W_2 if and only if λ is a zero of A_+. Without loss of generality, assume A_+ consists of a single row. Let p_i be a scalar polynomial of least degree such that p_iW_i is analytic on the closed unit disc ($i = 1, 2$). By Proposition 3.3, p_1 and p_2 are relatively prime, and we can find scalar polynomials q_1 and q_2 such that $p_1 q_1 + p_2 q_2 = 1$. Consequently, the function

$$A_+^{-R} = p_1 q_1 W_1 + p_2 q_2 W_2$$

is analytic on the closed unit disc and $A(z) A_+^{-R}(z) = I$. Similarly, we can find the function A_-^{-L}. In fact, we can compute realizations

$$A_+^{-R}(z) = D_+ + C_+(z^{-1} - E_+)^{-1} B_+$$

and

$$A_-^{-L}(z) = D_- + C_-(z - E_-)^{-1} B_- \tag{5.12}$$

(cf. Proposition 2.3 in [24]). Formulas (5.11) and (5.12) provide a solution $\{\xi_i\}_{i=0}^{\infty} \in l_p^n$ of Eq. (5.1) such that $\xi_i = \sum_{j=0}^{\infty} \gamma_{ij} c_j$ with

$$\gamma_{ij} = \sum_{k=0}^{\min\{i,j\}} \gamma_{i-k}^+ \gamma_{j-k}^-,$$

where

$$\gamma_i^+ = \begin{cases} D_+, & \text{if } i = 0, \\ C_+(E_+)^{i-1} B_+, & \text{if } i > 0 \end{cases}$$

and

$$\gamma_j^- = \begin{cases} D_-, & \text{if } j = 0, \\ C_-(E_-)^{j-1} B_-, & \text{if } j > 0. \end{cases}$$

References

1. J. A. Ball and K. Clancey, An elementary description of partial indices of rational matrix functions, *Integral Equations Operator Theory* **13** (1990), 316–322.
2. J. A. Ball, N. Cohen, and L. Rodman, Zero data and interpolation problems for rectangular matrix polynomials, *Linear and Multilinear algebra* **29** (1991), 53–78.
3. H. Bart, I. Gohberg, and M. A. Kaashoek, "Minimal Factorization of Matrix and Operator Functions," OT 1, Birkhäuser-Verlag, Basel/Boston/Stuttgart, 1979.
4. H. Bart, I. Gohberg, and M. A. Kaashoek, Explicit Wiener–Hopf factorization and realization, *in* "Constructive Methods of Wiener–Hopf Factorization," Birkhäuser-Verlag, Basel/Boston/Stuttgart, 1986.
5. H. Bart, I. Gohberg, M. A. Kaashoek, and P. Van Dooren, Factorization of transfer functions, *SIAM J. Control Optim.* **18** (1990), 675–696.
6. J. A. Ball, I. Gohberg, and L. Rodman, Realization and interpolation of rational matrix functions, *in* "Topics in Interpolation Theory of Rational Matrix Functions" (I. Gohberg, Ed.), OT 33, pp. 1–72, Birkhäuser-Verlag, Basel/Boston/Berlin, 1988.

7. J. A. BALL, I. GOHBERG, AND L. RODMAN, "Interpolation of Rational Matrix Functions," OT 45, Birkhäuser-Verlag, Basel/Boston/Berlin, 1990.
8. J. A. BALL AND M. RAKOWSKI, Zero-pole structure of nonregular rational matrix functions, *in* "Extension and Interpolation of Linear Operators and Matrix Functions," OT 47, pp. 137–193, Birkhäuser-Verlag, Basel/Boston/Berlin, 1990.
9. N. COHEN, "On Spectral Analysis and Factorization of Rational Matrix Functions," Ph.D. Dissertation, Weizmann Institute of Science, Rehovot, Israel, August 1984.
10. K. CLANCEY AND I. GOHBERG, "Factorization of Matrix Functions and Singular Integral Operators," OT 3, Birkhäuser-Verlag, Basel/Boston/Stuttgart, 1981.
11. K. CLANCEY AND M. RAKOWSKI, Factorization of rectangular matrix functions relative to a contour, unpublished note.
12. H. DYM AND I. GOHBERG, Unitary interpolants, factorization indices and infinite Hankel block matrices, *J. Funct. Anal.* **54** (1983), 229–289.
13. G. D. FORNEY, JR., Minimal bases of rational vector spaces, with applications to multivariable linear systems, *SIAM J. Control* **13** (1975), 493–520.
14. P. A. FUHRMANN AND J. C. WILLEMS, Factorization indices at infinity for rational matrix functions, *Integral Equations Operator Theory* **2/3** (1979), 287–301.
15. I. GOHBERG AND M. A. KAASHOEK (EDS.), "Constructive Methods of Wiener-Hopf Factorization," Birkhäuser-Verlag, Basel/Boston/Stuttgart, 1986.
16. I. GOHBERG AND M. A. KAASHOEK, Regular rational matrix functions with prescribed pole and zero structure, *in* "Topics in Interpolation Theory of Rational Matrix Functions" (I. Gohberg, Ed.), OT 33, pp. 109–122, Birkhäuser-Verlag, Basel/Boston/Berlin, 1988.
17. I. GOHBERG AND M. G. KREIN, Systems of integral equations on a half line with kernels depending on the differences of arguments, English translation, *Amer. Math. Soc. Trans. Ser. 2* **14** (1960), 217–287.
18. I. GOHBERG, M. A. KAASHOEK, L. LERER, AND L. RODMAN, Common multiples and common divisors of matrix polynomials. II. Vandermonde and resultant matrices, *Linear and Multilinear Algebra* **12** (1982), 159–203.
19. I. GOHBERG, L. LERER, AND L. RODMAN, Factorization indices for matrix polynomials, *Bull. Amer. Math. Soc.* **84** (1978), 275–277.
20. T. KAILATH, "Linear Systems," Prentice-Hall, Englewood Cliffs, NJ, 1980.
21. A. F. MONNA, "Analyse non-archimédienne," Springer-Verlag, Berlin/Heidelberg/New York, 1970.
22. B. MCMILLAN, Introduction to formal realizability theory, II, *Bell System Tech. J.* **31** (1952), 541–600.
23. M. RAKOWSKI, Generalization pseudoinverses of matrix valued functions, *Integral Equations Operator Theory* **14** (1991), 564–585.
24. M. RAKOWSKI, Minimal factorization of rational matrix functions, *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications* **39** (1992), 440–445.
25. H. H. ROSENBROCK, "State-space and Multivariable Theory," Wiley, New York, 1970.
26. P. VAN DOOREN, Factorization of a rational matrix: the singular case, *Integral Equations Operator Theory* **7** (1984), 704–741.
27. G. VERGHESE, P. VAN DOOREN, AND T. KAILATH, Properties of the system matrix of a generalized state-space system, *Internat. J. Control* **30** (1979), 235–243.
28. B. F. WYMAN, M. K. SAIN, G. CONTE, AND A. M. PERDON, On the zeros and poles of a transfer function, *Linear Algebra Appl.* **122/123/124** (1989), 123–144.