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Characteristic classes of Lie algebroid morphisms

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ABSTRACT

We extend R. Fernandes' construction of secondary characteristic classes of a Lie algebroid to the case of a base-preserving morphism between two Lie algebroids. Like in the case of a Lie algebroid, the simplest characteristic class of our construction coincides with the modular class of the morphism.

In [6] R. Fernandes has constructed a sequence of secondary characteristic classes of a Lie algebroid whose first element coincides with the modular class. In this note we extend Fernandes' construction and use the general definition of D. Lehmann [12] in order to produce secondary characteristic classes of a base-preserving morphism of two Lie algebroids. In particular, like in [6], we get a sequence of secondary characteristic classes whose first element coincides with the modular class of the morphism [7,9]. We assume that the reader is familiar with Lie algebroids and Lie-algebroid connections and will consult [6,13,12,14] whenever needed. The framework of the paper is the C^∞-category. We mention that other constructions of secondary characteristic classes of Lie algebroids may also be found in the literature e.g., [2,3,11].

1. Selected topics on A-connections

Let $(A, \sharp_A, [\,,\,]_A)$ be a Lie algebroid and V a vector bundle with the same base manifold M^m ($m = \dim M$). By an A-connection we shall understand an A-covariant derivative $\nabla : \Gamma A \times \Gamma V \to \Gamma V$ (Γ denotes the space of cross sections of a vector bundle), written as $(a, v) \mapsto \nabla_a v$, which is $\mathbb{R}$-bilinear and has the properties

$$\nabla_{fa} v = f \nabla_a v, \quad \nabla_a (fv) = f \nabla_a v + \sharp_A a(f) v \quad \big(f \in C^\infty(M)\big). \tag{1.1}$$

Accordingly, the value $\nabla_a v(x)$ depends only on $a(x)$ and on $v|_{U_x}$ where U_x is a neighborhood of $x \in M$. In order to write down the local expression of ∇, we take a local basis $(b_i)_{i=1}^s$ ($s = \operatorname{rank} A$) of ΓA, with the dual basis (b^{*i}) of ΓA^* and a local basis $(w_u)_{u=1}^r$ ($r = \operatorname{rank} V$) of ΓV. Then, with the notation $\Omega^k(A) = \Gamma \wedge^k A^*$ for the space of A-*forms* of degree k and using the Einstein summation convention, we get

$$\nabla_{b_i} w_u = \omega_u^t(b_i) w_t, \quad \omega_u^t = \gamma_{iu}^t b^{*i} \in \Omega^1(A), \tag{1.2}$$

and we say that (ω_u^t) is the *local connection matrix*. Correspondingly, the curvature

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$$R_\nabla(a_1,a_2)w=\nabla_{a_1}\nabla_{a_2}w-\nabla_{a_2}\nabla_{a_1}w-\nabla_{[a_1,a_2]_A}w$$

gets the local expression

$$R_\nabla(b_i,b_j)w_u=\Omega^t_u(b_i,b_j)w_t,\quad \Omega^t_u=d_A\omega^t_u-\omega^s_u\wedge\omega^t_s\in\Omega^2(A), \tag{1.3}$$

where d_A denotes the A-exterior differential [13]. We will say that (Ω^t_u) is the *local curvature matrix* and a change of the basis (w_u) implies an $ad(Gl(r,\mathbb{R}))$-transformation of (Ω^t_u). Like in classical differential geometry, one has the covariant derivative machinery of V-tensors and tensor valued A-forms and the computation of the $d_A\Omega^t_u$ produces the Bianchi identity that may be written under the form $\nabla\Omega^t_u=0$.

In the study of characteristic classes we shall need the direct product of two Lie algebroids $p_c:A_c\to M_c$ $(c=1,2)$ and we recall its definition given in [13]. Consider the pullback bundles $\pi_c^{-1}A_c$, where π_c is the projection of $M_1\times M_2$ on M_c. Identify

$$\Gamma\big(\pi_c^{-1}A_c\big)\equiv\{\sigma:M_1\times M_2\to A_c/p_c\circ\sigma=\pi_c\} \tag{1.4}$$

and notice that local bases $(b^{(c)}_i)$ of ΓA_c have natural lifts to local bases of $\Gamma(\pi_c^{-1}A_c)$, which will also be denoted by $(b^{(c)}_i)$. Take local cross sections

$$\sigma_{(c)}=\sigma^i_{(c)}b^{(c)}_i,\quad \kappa_{(c)}=\kappa^i_{(c)}b^{(c)}_i\quad \big(\sigma^i_{(c)},\kappa^i_{(c)}\in C^\infty(M_1\times M_2)\big)$$

(there is no summation on indices in parentheses) and define the following anchors and brackets

$$\sharp_{(c)}\sigma_{(c)}=\sigma^i_{(c)}\sharp_{A_c}b^{(c)}_i,\quad [\sigma_{(c)},\kappa_{(c)}]_{(c)}=\sigma^i_{(c)}\kappa^j_{(c)}\big[b^{(c)}_i,b^{(c)}_j\big]_{A_c}+\big\{\sigma^j_{(c)}\big[\big(\iota_{*c}\sharp_{A_c}b^{(c)}_j\big)\kappa^i_{(c)}\big]-\kappa^j_{(c)}\big[\big(\iota_{*c}\sharp_{A_c}b^{(c)}_j\big)\sigma^i_{(c)}\big]\big\}b^{(c)}_i,$$

where ι_{*c} is the natural injection of TM_c in $T(M_1\times M_2)$. In these operations, the $M_{(c-1 \bmod 2)}$-variable is just a passive parameter and, since the anchor and bracket of each A_c are invariant, the results are independent of the choice of the bases. Thus, the vector bundles $\pi_c^{-1}A_c$ are Lie algebroids over $M_1\times M_2$ and the direct product of the Lie algebroids A_c is the Whitney sum $\mathcal{A}=\pi_1^{-1}A_1\oplus\pi_2^{-1}A_2$ endowed with the direct sum of the anchors and brackets of the two pullbacks (in particular, $[b^{(1)}_i,b^{(2)}_j]=0$).

Proposition 1.1. *Let $q:V\to M_1$ be a vector bundle on M_1. Then, any $\mathcal{A}$-connection $\tilde\nabla$ on the pullback $\pi_1^{-1}(V)$ defines a differentiable family $\nabla^{(x_2)}$ $(x_2\in M_2)$ of A_1-connections on V. Conversely, any x_2-parameterized, differentiable family of A_1-connections on V is induced by an $\mathcal{A}$-connection on $\pi_1^{-1}(V)$.*

Proof. Assume that we have the covariant derivatives $\tilde\nabla_{\sigma^i_{(1)}b^{(1)}_i+\sigma^j_{(2)}b^{(2)}_j}(\nu^u w_u)$, where $\sigma^i_{(c)},\nu^u$ are local, differentiable functions on $M_1\times M_2$. Then, the required family of connections on V is given by the covariant derivatives

$$\big(\nabla^{(x_2)}_{\xi^ib^{(1)}_i}\big(\eta^u w_u\big)\big)(x_1)=\big(\tilde\nabla_{\xi^ib^{(1)}_i}\big(\eta^u w_u\big)\big)(x_1,x_2),$$

where ξ^i,η^u are local, differentiable functions on M_1, $x_1\in M_1, x_2\in M_2$ and we use an identification like (1.4) for V. Notice that, if the local connection matrices of $\tilde\nabla$ are

$$\tilde\omega^v_u=\gamma^v_{(1)iu}(x_1,x_2)b^{*i}_{(1)}+\gamma^v_{(2)ju}(x_1,x_2)b^{*j}_{(2)}, \tag{1.5}$$

the connection $\nabla^{(x_2)}$ has the matrices $\gamma^v_{(1)iu}(x_1,x_2)b^{*i}_{(1)}$ with the fixed value of x_2. Conversely, if the family $\nabla^{(x_2)}$ is given, we get an $\mathcal{A}$-connection $\tilde\nabla$ by adding the local equations $\tilde\nabla_{b^{(2)}_j}w_u=0$. The local matrices of this connection $\tilde\nabla$ are the same as the matrices of $\nabla^{(x_2)}$ where x_2 is allowed to vary in M_2. □

In particular, we may apply Proposition 1.1 for $M_1=M, M_2=I=\{0\leqslant\tau\leqslant1\}, A_1=A, A_2=TI$. Then, an $\mathcal{A}$-connection $\tilde\nabla$ on $\pi_1^{-1}(V)$ is called a *link* between the A-connections ∇^0,∇^1 on V. Formula (1.5) shows that the local connection forms of $\tilde\nabla$ are given by

$$\tilde\omega^v_u=\omega^v_{(\tau)u}+\lambda^v_u(x,\tau)\,d\tau, \tag{1.6}$$

where $\omega_{(\tau)}$ is the local connection matrix at the fixed value τ and $\lambda^v_u\in C^\infty(M\times I)$. A simple calculation gives the corresponding local curvature forms

$$\tilde\Omega^v_u=\Omega^v_{(\tau)u}+\Lambda^v_u\wedge d\tau, \tag{1.7}$$

where

$$\Lambda_u^v = d_A\lambda_u^v + \lambda_u^w\omega_{(\tau)w}^v - \lambda_w^v\omega_{(\tau)u}^w + \frac{\partial\omega_{(\tau)u}^v}{\partial\tau}$$

(the partial derivative with respect to τ is applied to the coefficients of the form).

Now, we present another "selected topic". Let $V \to M$ be a vector bundle of rank r endowed with either a positive, symmetric tensor $g_+ \in \Gamma \odot^2 V^*$ or a 2-form $g_- \in \Gamma \wedge^2 V^*$. We shall say that $(V, g_\pm)$ is a *quasi-(skew)-metric vector bundle*. Notice that we do not ask $\operatorname{rank} g_\pm$ to be constant on M. An A-connection ∇ on V such that $\nabla g_\pm = 0$ will be called a *quasi-(skew)-metric connection*. If $g_\pm$ is non-degenerate, the particle "quasi" will not be used and the connection is called *orthogonal* for g_+ and *symplectic* for g_-. For a Lie algebroid A over M we shall denote by L a generic, integral leaf of the distribution $\operatorname{im}\sharp_A$ and by L_x the leaf through the point $x \in M$. In what follows we establish properties of a quasi-(skew)-metric connection that are relevant to the construction of characteristic classes.

Proposition 1.2. *Assume that there exists a quasi-(skew)-metric connection ∇ on $(V, g_\pm)$. Then, the following properties hold.* 1. *If $x \in M$ and $k \in \Gamma V$ is such that $k|_{L(x)} \in K|_{L(x)}$ ($K = \operatorname{ann} g_\pm$), then $\nabla_a k(x) \in K_x$, $\forall a \in \Gamma A$.* 2. *$q = \operatorname{rank} g_\pm|_L$ is constant along each leaf L and $\forall x \in M$ there exists an open neighborhood U_x where V has a local basis of cross sections of the form (s_h, t_l) $(h = 1, \dots, q, l = 1, \dots, r-q)$ such that $t_l|_{U_x\cap L_x} \in K|_{U_x\cap L_x}$ and the projections $[s_h] = s_h \pmod K$ define a canonical basis of the (skew)-metric vector bundle $((V/K)|_{U_x\cap L_x}, g'_\pm)$, where $g'_\pm$ is non-degenerate and induced by $g_\pm$.* 3. *With respect to this basis, the A-connection ∇ has local equations*

$$\nabla s_h = \varpi_{(1)h}^k s_k + \varpi_{(2)h}^p t_p, \qquad \nabla t_l = \varpi_{(3)l}^k s_k + \varpi_{(4)l}^p t_p, \tag{1.8}$$

where the coefficients are local 1-A-forms, $\varpi_{(3)l}^k(x) = 0$ and $(\varpi_{(1)h}^k(x)) \in o(q)$, the orthogonal Lie algebra, in the g_+-case, $(\varpi_{(1)h}^k(x)) \in sp(q, \mathbb{R})$, the symplectic Lie algebra, in the g_--case. 4. *The curvature of ∇ has the local expression*

$$R_\nabla s_h = \Phi_{(1)h}^k s_k + \Phi_{(2)h}^p t_p, \qquad R_\nabla t_l = \Phi_{(3)l}^k s_k + \Phi_{(4)l}^p t_p, \tag{1.9}$$

where the coefficients are local 2-A-forms and $\Phi_{(3)l}^k(x) = 0$, $(\Phi_{(1)h}^k(x)) \in o(q)$ in the g_+-case, $(\Phi_{(1)h}^k(x)) \in sp(q, \mathbb{R})$ in the g_--case.

Proof. 1. For any $a \in \Gamma A$, $v \in \Gamma V$ one has

$$(\nabla_a g_\pm)_x\big(v(x), k(x)\big) = (\sharp_A a)_x\big(g_\pm(v, k)\big) - g_{\pm,x}\big(\nabla_a v(x), k(x)\big) - g_{\pm,x}\big(v(x), \nabla_a k(x)\big) = 0. \tag{1.10}$$

Since $(\sharp_A a)_x(g_\pm(v, k))$ depends only on $k|_{L(x)} \in \operatorname{ann} g_\pm$, it vanishes, and we get the required result.

2. $\nabla g_\pm = 0$ is equivalent with the fact that $g_\pm$ is preserved by parallel translations along paths in a leaf L [6], therefore, $g_\pm$ has a constant rank q along L_x. This implies the existence of bases with the required properties on a neighborhood $U_x \cap L$ of x. (In the metric case canonical means orthonormal and in the skew-metric case canonical means symplectic.) Then, take any extension of such a basis to U_x and shrink the neighborhood U_x as needed to ensure the linear independence of the extended cross sections.

3. The equality $\varpi_{(3)l}^k(x) = 0$ is an immediate consequence of part 1. Then, in (1.10), replace v, k by s_h, s_k. Since the canonical character of the basis $(s_h|_L)$ implies $g_\pm(s_h|_L, s_k|_L) = \text{const.}$, we get

$$g_{\pm,x}\big(\nabla_a s_h(x), s_k(x)\big) + g_{\pm,x}\big(s_h(x), \nabla_a s_k(x)\big) = 0,$$

whence, $(\varpi_{(1)i}^j(x)) \in o(q), sp(q, \mathbb{R})$, respectively.

4. The (skew)-metric condition (1.10) also implies

$$\sharp_A[a_1, a_2]_A\big(g_\pm(v_1, v_2)\big) = g_\pm(\nabla_{[a_1,a_2]_A} v_1, v_2) + g_\pm(v_1, \nabla_{[a_1,a_2]_A} v_2),$$

where $a_1, a_2 \in \Gamma A$, $v_1, v_2 \in \Gamma V$, whence, after some obvious cancellations we get

$$\sharp_A[a_1, a_2]_A\big(\omega(v_1, v_2)\big) = -\omega\big(R_\nabla(a_1, a_2)v_1, v_2\big) - \omega\big(v_1, R_\nabla(a_1, a_2)v_2\big). \tag{1.11}$$

Like in the proof of 3, (1.11) for s_h, s_k implies $(\Phi_{(1)i}^j(x)) \in o(q), sp(q, \mathbb{R})$, respectively. Then, (1.11) for s_h, t_l together with part 1 of the proposition implies $\Phi_{(3)l}^k(x) = 0$. □

In the theory of characteristic classes we need the Weil algebra $I(Gl(r, \mathbb{R})) = \bigoplus_{k\geqslant 0} I^k(Gl(r, \mathbb{R}))$, where $I^k(Gl(r, \mathbb{R}))$ is the space of real, ad-invariant, symmetric, k-multilinear functions (equivalently, invariant, homogeneous polynomials of degree k) on the Lie algebra of the general, linear group ($r = \operatorname{rank} V$). Using the exterior product $\wedge$, such functions may be evaluated on arguments that are local matrices of $\wedge$-commuting A-forms on M with transition functions of the adjoint type and the result is a global A-form on M (e.g., [14]). Secondary characteristic classes appear as a consequence of vanishing phenomena encountered in the evaluation process described above. We shall need the following vanishing phenomenon (see [6]):

Proposition 1.3. *If the bundle V endowed with the form $g_\pm$ has a connection ∇ such that $\nabla g_\pm = 0$ and if $R_\nabla(a_1, a_2)k_x = 0$ for $x \in M$, $a_1, a_2 \in \Gamma A$, $k \in \ker g_{\pm,x}$, then, $\forall \phi \in I^{2k-1}(Gl(r,\mathbb{R}))$, one has $\phi(\Phi) = 0$, where Φ is the local curvature matrix of the connection ∇.*

Proof. By $\phi(\Phi)$ we understand the evaluation of ϕ where all the arguments are equal to Φ. It is known that (with a harmless abuse of terminology and notation) the required functions ϕ are spanned by the Chern polynomials

$$c_h(F) = \frac{1}{h!}\delta^{v_1\dots v_h}_{u_1\dots u_h} f^{u_1}_{v_1}\dots f^{u_h}_{v_h} \tag{1.12}$$

($\delta^{\cdots}_{\cdots}$ is the multi-Kronecker index), which are the sums of the principal minors of order h in $\det(F - \lambda\, Id)$ $(F \in gl(r,\mathbb{R}))$. With the notation of Proposition 1.2 and since $R_\nabla(a_1, a_2)k_x = 0$, we have to take

$$F = \begin{pmatrix} \Phi_{(1)} & 0 \\ \Phi_{(2)} & 0 \end{pmatrix}.$$

Therefore, $\forall x \in M$, we have $c_h(F) = c_h(\Phi_{(1)x})$. It is known that the polynomials c_{2l-1} vanish on $o(q)$ and on $sp(q,\mathbb{R})$ (in the first case $\Phi_{(1)}$ is skew-symmetric; for the second case see Remark 2.1.10 in [14], for instance). □

2. Secondary characteristic classes

A brief exposition of the classical theory of real characteristic classes may be found in [14]. In this section, we present a Lie algebroid version of the basic facts of the theory.

Consider the direct product Lie algebroid $\mathcal{A} = A \times T\Delta^k \to M \times \Delta^k$, where

$$\Delta^k = \left\{ (t_0, t_1, \dots, t_k) \in \mathbb{R}^{k+1} / t_h \geqslant 0, \sum_{h=0}^{k} t_h = 1 \right\}$$

is the standard k-simplex, A is a Lie algebroid over M and $T\Delta^k$ is the tangent bundle of Δ^k endowed with the standard orientation $\kappa = dt^1 \wedge \dots \wedge dt^k$. Then, $\forall \Phi \in \Omega^*(\mathcal{A})$, the fiber-integral $\int_{\Delta^k} \Phi$ is defined as zero except for the case

$$\Phi = \alpha \wedge \kappa, \quad \alpha = \frac{1}{p!}\alpha_{i_1\dots i_p}(x,t) b^{*i_1} \wedge \dots \wedge b^{*i_p} \quad \left(x \in M, t \in \Delta^k\right)$$

when

$$\int_{\Delta^k} \Phi = \frac{1}{p!}\left(\int_{\Delta^k} \alpha_{i_1\dots i_p}(x,t)\kappa \right) b^{*i_1} \wedge \dots \wedge b^{*i_p} \in \Omega^p(A)$$

(b_i is a local basis of cross sections of A). The same proof as in the classical case (e.g., [14], Theorem 4.1.6) yields the Stokes formula:

$$\int_{\Delta^k} d_{\mathcal{A}}\Phi - d_A \int_{\Delta^k} \Phi = (-1)^{\deg \Phi - k} \int_{\partial\Delta^k} \iota^*\Phi, \qquad \iota : \partial\Delta^k \subseteq \Delta^k. \tag{2.1}$$

Assume that we have $k+1$ A-connections $\nabla^{(s)}$ on the vector bundle $V \to M$ that have the local connection matrices $\omega_{(\alpha)}$ $(\alpha = 0, \dots, k)$ with respect to the local basis (w_u) of V. Then, the convex combination

$$\nabla^{(t)} = \sum_{\alpha=0}^{k} t^\alpha \nabla^\alpha, \quad t = \left(t^0, \dots, t^k\right) \in \Delta^k, \tag{2.2}$$

defines a family of A-connections parameterized by Δ^k with the corresponding $\mathcal{A}$-connection $\tilde{\nabla}$ on $\pi_1^{-1}(V) \to M \times \Delta^k$ $(\pi_1 : M \times \Delta^k \to M)$. The connection and curvature matrices of $\tilde{\nabla}$ will be denoted by $\tilde{\omega}, \tilde{\Omega}$; generally, the curvature matrix of a connection will be denoted by the upper case of the letter that denotes the connection matrix. There exists a homomorphism

$$\Delta\left(\nabla^0, \dots, \nabla^k\right) : I^h\left(Gl(r,\mathbb{R})\right) \to \Omega^{2h-k}(A),$$

defined by R. Bott in the classical case, given by

$$\Delta\left(\nabla^0, \dots, \nabla^k\right)\phi = (-1)^{[\frac{k+1}{2}]} \int_{\Delta^k} \phi(\tilde{\Omega}), \quad \phi \in I^h\left(Gl(r,\mathbb{R})\right). \tag{2.3}$$

Moreover, Bott's proof in the classical case [14, Proposition 4.2.3] also holds in the Lie algebroid version and yields the following formula

$$d_A\big(\Delta\big(\nabla^0,\dots,\nabla^k\big)\phi\big)=\sum_{\alpha=0}^{k}(-1)^{\alpha}\Delta\big(\nabla^0,\dots,\nabla^{\alpha-1},\nabla^{\alpha+1},\dots,\nabla^k\big)\phi. \tag{2.4}$$

Let ∇ be an A-connection on the vector bundle $V\to M$. As a consequence of the Bianchi identity, $\forall\phi\in I^h(Gl(r,\mathbb{R}))$, $\Delta(\nabla)\phi\in\Omega^{2h}(A)$ is a d_A-closed A-form and the A-cohomology classes defined by the A-forms $\Delta(\nabla)\phi$ are called the *A-principal characteristic classes of V* [6]. If ∇^0,∇^1 are two A-connections, formula (2.4) yields

$$\Delta\big(\nabla^1\big)\phi-\Delta\big(\nabla^0\big)\phi=d_A\Delta\big(\nabla^0,\nabla^1\big)\phi. \tag{2.5}$$

Therefore, the principal characteristic classes do not depend on the choice of the connection.

The $\mathcal{A}$-connection $\tilde{\nabla}$ to be used in definition (2.3) of $\Delta(\nabla^0,\nabla^1)\phi$ is the link between ∇^0,∇^1 given by the family of A-connections

$$\nabla^{(\tau)}=(1-\tau)\nabla^0+\tau\nabla^1=\nabla^0+\tau D,\quad D=\nabla^1-\nabla^0\quad(\tau\in I).$$

For this link, we have (1.6) and (1.7) where $\lambda^{\nu}_{u}=0,\partial\omega^{\nu}_{(\tau)u}/\partial\tau=\alpha$, the local matrix of the connection difference D, and formula (2.3) yields

$$\Delta\big(\nabla^0,\nabla^1\big)\phi=h\int_0^1\phi(\alpha,\underbrace{\Omega_{(\tau)},\dots,\Omega_{(\tau)}}_{(h-1)\text{-times}})\,d\tau, \tag{2.6}$$

where

$$\Omega_{(\tau)}=(1-\tau)\Omega_{(0)}+\tau\Omega_{(1)}+\tau(1-\tau)\alpha\wedge\alpha \tag{2.7}$$

is the local curvature matrix of the connection $\nabla^{(\tau)}$.

We shall use the Lehmann version of the theory of secondary characteristic classes [12,14]. Let (J_0,J_1) be two (proper) homogeneous ideals of $I=I(Gl(r,\mathbb{R}))$. Define the algebra

$$W(J_0,J_1)=(I/J_0)\otimes(I/J_1)\otimes\big(\wedge\big(I^+\big)\big)\quad\Big(I^+=\bigoplus_{k>0}I^k\Big), \tag{2.8}$$

with the graduation

$$\deg[\phi]_{J_0}=\deg[\phi]_{J_1}=2h,\qquad\deg\hat{\phi}=2h-1,$$

and the differential

$$d[\phi]_{J_0}=d[\phi]_{J_1}=0,\qquad d\hat{\phi}=[\phi]_{J_1}-[\phi]_{J_0},$$

where we refer to the three elements defined by $\phi\in I^h$ in the factors of W.

Now, take a vector bundle $V\to M$ and two A-connections ∇^0,∇^1 on V such that $J_c\subseteq\ker\Delta(\nabla^c)$, $c=0,1$. By putting

$$\rho[\phi]_{J_0}=\Delta\big(\nabla^0\big)\phi,\qquad\rho[\phi]_{J_1}=\Delta\big(\nabla^1\big)\phi,\qquad\rho\hat{\phi}=\Delta\big(\nabla^0,\nabla^1\big)\phi, \tag{2.9}$$

we get a homomorphism of differential graded algebras

$$\rho(\nabla_0,\nabla_1):W(J_0,J_1)\to\Omega(A)$$

with an induced cohomology homomorphism

$$\rho^*\big(\nabla^0,\nabla^1\big):H^*\big(W(J_0,J_1)\big)\to H^*(A).$$

The cohomology classes in $\operatorname{im}\rho^*$ that are not principal characteristic classes are called *A-secondary characteristic classes*.

If J is a homogeneous ideal of $I(Gl(r,\mathbb{R}))$, two A-connections ∇,∇' on V are called *J-homotopic connections* if there exists a finite chain of links $\tilde{\nabla}^0,\dots,\tilde{\nabla}^n$ that starts with ∇, ends with ∇' and is such that $J\subseteq\bigcap_{l=0}^{n}\ker\Delta(\tilde{\nabla}^l)$. By replacing the usual Stokes' formula by formula (2.1) in the proof of Theorem 4.2.28 of [14], one gets

Proposition 2.1. *(See [12].) The cohomology homomorphism $\rho^*(\nabla^0,\nabla^1)$ remains unchanged if ∇^0,∇^1 are replaced by J_0,J_1-homotopic connections ∇'^0,∇'^1, respectively ($J_c\subseteq\ker\Delta(\nabla^c)$, $J_c\subseteq\ker\Delta(\nabla'^c)$, $c=0,1$).*

Corollary 2.1. *The secondary characteristic classes are invariant by any $J_0 J_1$-homotopy of the connections.*

Denote by $J_{\text{odd}} \subseteq I(Gl(r,\mathbb{R}))$ the ideal spanned by $\{\phi \in I^{2h-1}(Gl(r,\mathbb{R})), h = 1, 2, \ldots\}$. As explained in Proposition 1.3, if ∇ is an orthogonal connection for some metric g on the vector bundle V, then $J_{\text{odd}} \subseteq \ker \Delta(\nabla)$. Notice that there always exist positive definite metrics g on V and corresponding metric A-connections ∇, $\nabla g = 0$ (e.g., take $\nabla_a = \nabla'_{\sharp_A a}$, where ∇' is a usual orthogonal connection on (V, g)). Furthermore, any two orthogonal A-connections on V are J_{odd}-homotopic. Indeed, if ∇, ∇' are orthogonal for the same metric g, then $(1-\tau)\nabla + \tau\nabla'$ $(0 \leqslant \tau \leqslant 1)$ defines an orthogonal link. If orthogonality is with respect to different metrics g, g', then $(1-\tau)g + \tau g'$ is a metric on the pullback of V to $M \times [0,1]$ and a corresponding metric connection provides an orthogonal link between two orthogonal connections $\bar{\nabla}, \bar{\nabla}'$ with the metrics g, g', respectively. Thus, there exists a chain of three orthogonal links leading from ∇ to $\bar{\nabla}$, from $\bar{\nabla}$ to $\bar{\nabla}'$ and from $\bar{\nabla}'$ to ∇', which proves the J_{odd}-homotopy of ∇, ∇'.

Now, let $(V, g_\pm)$ be a quasi-(skew)-metric vector bundle that has a K-flat quasi-(skew)-metric connection ∇^1 $(K = \operatorname{ann} g_\pm)$. Then, Proposition 1.3 tells us that $J_{\text{odd}} \subseteq \ker \Delta(\nabla^1)$. Accordingly (like in the case of the Maslov classes [14]), if we also take an orthogonal A-connection ∇^0 on V, we shall obtain secondary characteristic classes corresponding to the ideals $J_0 = J_1 = J_{\text{odd}}$.

Following [14], Theorem 4.2.26, we may replace the algebra $W(J_0, J_1)$ by the algebra

$$\tilde{W} = \mathbb{R}[c_2, c_4, \ldots] \otimes \mathbb{R}\big[c'_2, c'_4, \ldots\big] \otimes \wedge(\hat{c}_1, \hat{c}_3, \ldots), \tag{2.10}$$

where $c_\cdot$ are the Chern polynomials and the accent and hat indicate the place in the three factors of (2.10); the homomorphism $\rho(\nabla^0, \nabla^1)$ is defined like on $W(J_0, J_1)$, while using orthogonal and quasi-(skew)-metric A-connections, respectively, and we get the same set of characteristic classes. Then, by the same argument like for [14], Theorem 4.4.37 we get

Proposition 2.2. *The A-secondary characteristic classes of $(V, g_\pm)$ are the real linear combinations of cup-products of A-Pontrjagin classes of V [6] and classes of the form*

$$\mu_{2h-1} = \big[\Delta\big(\nabla^0, \nabla^1\big)c_{2h-1}\big] \in H^{4h-3}(A). \tag{2.11}$$

The classes μ_{2h-1} will be called *simple A-secondary characteristic classes*.

Remark 2.1. If we start with an arbitrary vector bundle $(V, g_\pm)$, a K-flat, quasi-(skew)-metric A-connection ∇^1 may not exist. Furthermore, if ∇^1 exists, it may happen that all the secondary characteristic classes vanish. For instance, if we have a non-degenerate form g_-, a usual connection on the bundle of g_--canonical frames produces an A-connection ∇^1 such that $\nabla^1 g_- = 0$ and, since $K = 0$, we get A-secondary characteristic classes. Because of the J_{odd}-homotopy of orthogonal connections, these classes do not depend on the choice of the orthogonal connection ∇^0. Moreover, these classes are independent of the skew-metric connection ∇^1 because of the existence of the link $(1-\tau)\nabla^1 + \tau\nabla'^1$ between two such connections. But, the structure group of V may be reduced from the symplectic to the unitary group [14] and a unitary connection $\bar{\nabla}$ on V will be skew-metric and orthogonal simultaneously. From (2.6), and taking $\nabla^0 = \nabla^1 = \bar{\nabla}$, we see that the secondary characteristic classes above vanish.

3. Characteristic classes of morphisms

Let A be an arbitrary Lie algebroid on M, V, W vector bundles with the same basis M and $\varphi : V \to W$ a morphism over the identity on M. The A-connections ∇^V, ∇^W on V, W, respectively, will be called φ-compatible if $\nabla^W \circ \varphi = \varphi \circ \nabla^V$. An equivalent way to characterize compatibility is obtained by considering the vector bundle $S = V \oplus W^*$, which is endowed with the 2-forms

$$g_\pm\big((v_1, \nu_1), (v_2, \nu_2)\big) = \big\langle \nu_2, \varphi(v_1)\big\rangle \pm \big\langle \nu_1, \varphi(v_2)\big\rangle, \tag{3.1}$$

$v_1, v_2 \in V$, $\nu_1, \nu_2 \in W^*$. It suffices to work with one of these forms, but it is nice to mention that both may be used with the same effect. The pair of A-connections ∇^V, ∇^W produces an A-connection $\nabla^S = \nabla^V \oplus \nabla^{W^*}$ on S, where ∇^{W^*} is defined by

$$\big\langle \nabla^{W^*}_a \nu, v\big\rangle = (\sharp_A a)\langle \nu, v\rangle - \big\langle \nu, \nabla^W_a v\big\rangle, \quad \nu \in W^*,\ v \in V.$$

A straightforward calculation shows that ∇^V, ∇^W are φ-compatible iff either $\nabla^S g_+ = 0$ or $\nabla^S g_- = 0$. We also notice that the forms $g_\pm$ have the same annihilator

$$K = \ker \varphi \times \ker^t \varphi \tag{3.2}$$

where the index t denotes transposition.

Proposition 3.1. *If $V = A$, if $W = A'$ is a second Lie algebroid and if φ is a base-preserving Lie algebroid morphism, then there exist K-flat, φ-compatible A-connections (∇, ∇').*

Proof. We may proceed like in [6]. Take a neighborhood of M where $\Gamma A, \Gamma A'$ have the fixed local bases $(b_i), (b'_u)$. Define local A-connections ∇^U, ∇'^U by asking that

$$\nabla^U_{b_i} b_j = [b_i, b_j]_A, \qquad \nabla'^U_{b_i} b'_u = \big[\varphi b_i, b'_u\big]_{A'}, \tag{3.3}$$

then, extending the operators to arbitrary local cross sections in accordance with the properties of a connection. Using the local expression $\varphi b_i = \varphi_i^u b'_u$, it is easy to check that $\varphi \circ \nabla^U = \nabla'^U \circ \varphi$. If we consider a locally finite covering $\{U_\sigma\}$ of M by such neighborhoods U and glue up the local connections by a subordinated partition of unity $\{\theta_\sigma \in C^\infty(M)\}$, we get φ-compatible, global A-connections ∇, ∇' defined by

$$\nabla_v a(x) = \sum_{x \in U_\sigma} \theta_\sigma(x) \nabla^{U_\sigma}_v a(x), \qquad \nabla'_v a'(x) = \sum_{x \in U_\sigma} \theta_\sigma(x) \nabla'^{U_\sigma}_v a'(x), \tag{3.4}$$

where $x \in M$, $v \in A_x$, $a \in \Gamma A$, $a' \in \Gamma A'$.

Now, we notice that the local connections (3.3) satisfy the following properties

$$\nabla^U_{b_i} a = [b_i, a]_A, \qquad \nabla'^U_{b_i} a' = \big[\varphi b_i, a'\big]_{A'}. \tag{3.5}$$

Indeed, if we put $a = f^j b_j, a' = h^u b'_u$, (3.3) and the properties of the Lie algebroid bracket imply (3.5). Furthermore, using (3.5), it is easy to check the following properties of the global compatible connections (3.4)

$$\nabla_v k(x) = [\tilde{v}, k]_A(x), \tag{3.6}$$

$$\big\langle \nabla'^*_v \alpha'(x), a'(x) \big\rangle = (\sharp_A v)\big\langle \alpha', a' \big\rangle - \big\langle \alpha'(x), \big[\varphi \tilde{v}, a'\big]_{A'}(x) \big\rangle, \tag{3.7}$$

$\forall x \in M, k \in \Gamma(\ker \varphi), a' \in \Gamma A', \alpha' \in \Gamma(\ker^t \varphi)$ and $\tilde{v} = v^i b_i$ is a cross section of ΓA that extends $v \in A_x$. The restrictions put on k, α' ensure the correctness of the passage from the covariant derivative to the Lie algebroid bracket and the independence of the result on the choice of $\tilde{v}$. Formulas (3.6), (3.7) imply $\varphi(\nabla_v k) = 0$, $\nabla'^*_v \alpha' \circ \varphi = 0$, which means that $\ker \varphi$ and $\ker^t \varphi$ are preserved by the connections ∇, ∇', respectively.

Finally, if we denote $S = A \oplus A'^*$ and $\nabla^S = \nabla \oplus \nabla'^*$, we can compute the curvature $[R_{\nabla^S}(a_1, a_2)(k, \alpha')](x)$, which has components on A and A'^*. The component on A is

$$(\nabla_{a_1}\nabla_{a_2} - \nabla_{a_2}\nabla_{a_1} - \nabla_{[a_1,a_2]_A})\tilde{k}(x) \overset{(3.6)}{=} \big(\big[\tilde{a}_1, [\tilde{a}_2, \tilde{k}]_A\big]_A - \big[\tilde{a}_2, [\tilde{a}_1, \tilde{k}]_A\big]_A - \big[[\tilde{a}_1, \tilde{a}_2]_A, \tilde{k}\big]_A\big)(x) = 0,$$

where tilde denotes extensions to cross sections and the final result holds because of the Jacobi identity. For the component on A'^* we get the following evaluation on any $a' \in \Gamma A'$:

$$\begin{aligned}\big\langle (\nabla'^*_{a_1}\nabla'^*_{a_2} - \nabla'^*_{a_2}\nabla'^*_{a_1} - \nabla'^*_{[a_1,a_2]_A})\tilde{\alpha}', a' \big\rangle(x) &\overset{(3.7)}{=} \big\langle \tilde{\alpha}', \big[\varphi\tilde{a}_2, [\varphi\tilde{a}_1, \tilde{k}]_{A'}\big]_{A'} - \big[\varphi\tilde{a}_1, [\varphi\tilde{a}_2, \tilde{k}]_{A'}\big]_{A'} - \big[[\varphi\tilde{a}_1, \varphi\tilde{a}_2]_{A'}, \tilde{k}\big]_{A'} \big\rangle(x) \\ &= 0,\end{aligned}$$

where the annulation is justified by the Jacobi identity again. Therefore,

$$\big[R_{\nabla^S}(a_1, a_2)\big(k, \alpha'\big)\big](x) = 0,$$

which is the meaning of K-flatness. □

Remark 3.1. During the proof of Proposition 3.1 we saw that $\ker \varphi$ is preserved by ∇, hence, it is preserved by the parallel translation along the paths in the leaves L of A. This shows that $\operatorname{rank} \varphi$ is constant along the leaves L.

Remark 3.2. If we use the definition of ∇'^* in the left hand side of (3.7) and take into account the relation $\operatorname{ann} \ker^t \varphi = \operatorname{im} \varphi$ we obtain the following equivalent form of (3.7):

$$\nabla'_v a'(x) = \big[\varphi \tilde{v}, a'\big]_{A'}(x) \ (\operatorname{mod} \operatorname{im} \varphi_x) \quad \forall x \in M,\ v \in A_x. \tag{3.8}$$

Definition 3.1. A pair of φ-compatible A-connections that satisfy the properties (3.6), (3.8) will be called a *distinguished pair* (in [6] one uses the term basic connections).

Now, we see that we may use Proposition 2.2 in order to get secondary characteristic classes for the bundle $S = A \oplus A'^*$ endowed with the quasi-(skew)-metrics (3.1), with a connection $\nabla^1 = \nabla \oplus \nabla'^*$, where (∇, ∇') is a distinguished pair of A-connections, and with an orthogonal connection $\nabla^0 = \nabla^{g_A} \oplus \nabla^{g_{A'}*}$, where $g_A, g_{A'}$ are metrics on the bundles A, A' and $\nabla^{g_A}, \nabla^{g_{A'}}$ are corresponding orthogonal connections on A, A'.

Definition 3.2. The above constructed secondary characteristic classes of $A \oplus A'^*$ will be called the *characteristic classes* of the base-preserving morphism φ. In particular, one has the *simple characteristic classes* $\mu_{2h-1}(\varphi) \in H^{4h-3}(A)$.

The secondary characteristic classes of the Lie algebroid A defined in [6] are the simple characteristic classes of the morphism $\varphi = \sharp_A : A \to TM$.

Proposition 3.2. *All the characteristic classes of a base-preserving isomorphism $\varphi : A \to A'$ are zero.*

Proof. If φ is an isomorphism, then g_- is non-degenerate and we are in the situation discussed in Remark 2.1. □

Thus, the characteristic classes of a morphism may be seen as a measure of its non-isomorphic character.

Proposition 3.3. *The characteristic classes of a base preserving morphism $\varphi : A \to A'$ of Lie algebroids do not depend on the choice of the orthogonal connection and of the distinguished pair of compatible connections required by their definition.*

Proof. The proposition is a consequence of Corollary 2.1. In the previous section we have seen that two orthogonal A-connections are J_{odd}-homotopic. On the other hand, take two φ-distinguished pairs of A-connections $\nabla, \nabla'; \tilde{\nabla}, \tilde{\nabla}'$. Then, it is easy to check that, $\forall t \in [0,1]$, $(1-t)\nabla + t\tilde{\nabla}, (1-t)\nabla' + t\tilde{\nabla}'$ is a φ-distinguished pair again. Therefore, J_{odd}-homotopy also holds for the corresponding quasi-(skew)-metric connections on S and we are done. □

We also have another consequence of Corollary 2.1:

Proposition 3.4. *Two homotopic, base-preserving morphisms $\varphi_0, \varphi_1 : A \to A'$ of Lie algebroids have the same secondary characteristic classes.*

Proof. By homotopic morphisms we understand morphisms φ_0, φ_1 that are linked by a differentiable family of morphisms $\varphi_\tau : A \to A'$ $(0 \leqslant \tau \leqslant 1)$. The corresponding forms $g_{+,\tau}$ on $S = A \oplus A'^*$ are different, but, still, all the connections $\nabla^{1,\tau}$ required in the construction of the secondary classes have skew-symmetric local connection and curvature matrices. Therefore, the J_{odd}-homotopy holds and we are done. □

Remark 3.3. In the case of an arbitrary pair of morphisms $\varphi_0, \varphi_1 : A \to A'$ we can measure the difference between the secondary characteristic classes as follows. Notice the existence of the *bi-characteristic classes* $\bar{\mu}_{2h-1}(\varphi_1, \varphi_2) = [\Delta(\nabla^1, \nabla^2)c_{2h-1}] \in H^{4h-3}(A)$ where ∇^1, ∇^2 are A-connections defined on $S = A \oplus A'^*$ by distinguished, $\varphi_{1,2}$-compatible connections respectively. Then, formula (2.4) yields

$$d_A \Delta\big(\nabla^0, \nabla^1, \nabla^2\big)c_{2h-1} = \Delta\big(\nabla^0, \nabla^1\big)c_{2h-1} + \Delta\big(\nabla^1, \nabla^2\big)c_{2h-1} + \Delta\big(\nabla^2, \nabla^0\big)c_{2h-1},$$

where ∇^0 is an orthogonal connection on S. Accordingly, we get

$$\mu_{2h-1}(\varphi_1) - \mu_{2h-1}(\varphi_2) = \bar{\mu}_{2h-1}(\varphi_1, \varphi_2). \tag{3.9}$$

In what follows we give explicit local expressions of A-forms that represent the characteristic classes $\mu_{2h-1}(\varphi)$. Take a point $x \in M$ and an open neighborhood U of x diffeomorphic to a ball. Assume that (∇^U, ∇'^U) and (∇, ∇') are pairs of local, respectively global, distinguished, φ-compatible A-connections on A, A'. Then, if $0 \leqslant \chi \in C^\infty(M)$ is equal to 1 on the compact closure $\bar{V}$ of the open neighborhood $V \subseteq U$ of x and equal to 0 on $M \backslash U$, then the convex combinations

$$\bar{\nabla} = \chi \nabla^U + (1-\chi)\nabla, \qquad \bar{\nabla}' = \chi \nabla'^U + (1-\chi)\nabla'$$

define a global pair of distinguished A-connections that coincides with (∇^U, ∇'^U) on V.

Accordingly, in formula (2.11) for $S = A \oplus A'^*$ we may always use a connection ∇^1 such that the expressions (3.3) hold on the neighborhood V. Then, if we denote

$$\begin{aligned} &[b_i, b_j]_A = \gamma_{ij}^k b_k, \qquad \big[b'_u, b'_v\big]_{A'} = \gamma'^{w}_{uv} b'_w, \\ &\sharp_A b_i = \rho_i^j \frac{\partial}{\partial x^j}, \qquad \sharp_{A'} b'_u = \rho'^{j}_{u} \frac{\partial}{\partial x^j} \varphi(b_i) = \varphi_i^s b'_s \end{aligned} \tag{3.10}$$

(remember that we use the Einstein summation convention), where x^i are local coordinates on M and $(b_i), (b'_u)$ are the bases used in (3.3), we get the following connection matrix of ∇^1 on the neighborhood V

$$\begin{pmatrix} \gamma_{ij}^k b^{*i} & 0 \\ 0 & (-\varphi_i^t \gamma'^{s}_{tu} + \rho'^{j}_{u} \frac{\partial \varphi_i^s}{\partial x^j}) b^{*i} \end{pmatrix} \tag{3.11}$$

(in (3.11), b^{*i} is the dual basis of b_i).

Furthermore, let g^U, g'^U be local metrics on A, A' such that $(b_i), (b'_u)$ are orthonormal bases and g, g' arbitrary, global metrics on A, A'. Then, define the metrics

$$\chi g^U + (1-\chi)g,\ \chi g'^U + (1-\chi)g'$$

and take an orthogonal connection ∇^0 whose components are corresponding orthogonal connections. The connection matrix of ∇^0 on the neighborhood V, with respect to the same local bases like in (3.11), will be of the form

$$\begin{pmatrix} \varpi_i^j & 0 \\ 0 & -\varpi_s'^t \end{pmatrix}, \tag{3.12}$$

where $(\varpi_i^j), (\varpi_s'^t)$ are skew-symmetric matrices of local 1-A-forms.

If these connections ∇^0, ∇^1 are used, then, along V, the difference matrix α of formula (2.6) is the difference between the matrices (3.11) and (3.12). Furthermore, we can compute the matrix $\Omega_{(\tau)}$ by using formula (2.7), where $\Omega_{(0)}$ is a skew-symmetric matrix. The final result may be formulated as follows

Proposition 3.5. *If a point $x \in M$ is fixed, there exist global representative A-forms $\Xi_{2h-1} \in \Omega^{4h-3}(A)$ of the characteristic classes μ_{2h-1} such that*

$$\Xi_{2h-1}|_V = \frac{1}{(2h-2)!}\int_0^1 \left(\delta^{\sigma_1\ldots\sigma_{2h-1}}_{\kappa_1 l\kappa_{2h-1}}\alpha^{\kappa_1}_{\sigma_1}\wedge\Omega^{\kappa_2}_{(\tau),\sigma_2}\wedge\cdots\wedge\Omega^{\kappa_{2h-1}}_{(\tau),\sigma_{2h-1}}\right)d\tau, \tag{3.13}$$

for some neighborhood V of x. In (3.13), the factors are the entries of the matrices $\alpha, \Omega_{(\tau)}$ given by formulas (3.11), (3.12) and Greek indices run from 1 to $\dim A + \dim A'$.

Proof. Use the expression (1.12) of the Chern polynomials and the connections ∇^0, ∇^1 constructed above. □

The difficulty in using Proposition 3.5, besides its complexity in the case $h > 1$, consists in the fact that formula (3.13) does not define global A-forms; for neighborhoods of different points $x_1 \neq x_2$ we have different pairs of distinguished connections $\bar\nabla, \bar\nabla'$. However, we can use Proposition 3.5 in order to extend a result proven for a Lie algebroid A ($\varphi = \sharp_A$) in [6]:

Proposition 3.6. *The secondary class $\mu_1(\varphi)$ is equal to the modular class of the morphism φ.*

Proof. Recall that the modular class of a morphism is defined by $\mu(\varphi) = \mu(A) - \varphi^*\mu(A') \in H^1(A)$, where $\mu(A), \mu(A')$ are the modular classes of the Lie algebroids A, A', respectively, [7,9,10]. Furthermore, the modular class $\mu(A)$ is defined as follows [5,6,8,10]. The line bundle $\bigwedge^s A \otimes \bigwedge^m T^*M$ ($s = \operatorname{rank} A$) has a flat A-connection defined, by means of local bases, as follows

$$\begin{aligned}\nabla_{b_i}\left(\left(\bigwedge_{j=1}^s b_j\right)\otimes\left(\bigwedge_{h=1}^m dx^h\right)\right) &= \sum_{j=1}^s b_1\wedge\cdots\wedge[b_i,b_j]_A\wedge\cdots\wedge b_s\otimes\left(\bigwedge_{h=1}^m dx^h\right)\\ &\quad+\left(\bigwedge_{j=1}^s b_j\right)\otimes L_{\sharp_A b_i}\left(\bigwedge_{h=1}^m dx^h\right),\end{aligned} \tag{3.14}$$

where L is the Lie derivative. Then, for $\sigma \in \Gamma(\bigwedge^s A \otimes \bigwedge^m T^*M)$ (which exists if the line bundle is trivial; otherwise we go to its double covering), one has $\nabla_a\sigma = \lambda(a)\sigma$ where λ is a d_A-closed 1-A-form and defines the cohomology class $\mu(A)$, which is independent on the choice of σ.

From (3.14) it follows easily that $\mu(A), \mu(A')$ are represented by the A-forms

$$\lambda = \sum_{i,k,j}\left(\gamma^k_{ik} + \frac{\partial\rho_i^j}{\partial x^j}\right)b^{*i}, \qquad \lambda' = \sum_{s,t,h}\left(\gamma'^t_{st} + \frac{\partial\rho_s'^j}{\partial x^j}\right)b'^{*s} \tag{3.15}$$

where the notation is that of (3.10). Notice that, even though the expressions (3.15) are local, the forms λ, λ' are global A-forms because the connection that was used in their definition is global.

On the other hand, using formulas (3.11), (3.12) and since the trace of a skew-symmetric matrix is zero, we may see that the A-form Ξ_1 defined in Proposition 3.5 is such that $\Xi|_{1V} = (\lambda - \varphi^*\lambda')|_V$, where V is a neighborhood of a fixed point $x \in M$. Accordingly, there exists a locally finite, open covering $\{V_\alpha\}$ of M and there exists a family of pairs of A-connections $(\nabla^{0\alpha}, \nabla^{1\alpha})$ that provide representative 1-A-forms $\Xi_{1\alpha}$ of the characteristic class $\mu_1(\varphi)$ such that

$$\Xi_{1\alpha}|_{V_\alpha} = \big(\lambda - \varphi^*\lambda'\big)|_{V_\alpha}. \tag{3.16}$$

Then, if we take a partition of unity $\{\theta_\alpha \in C^\infty(M)\}$ subordinated to $\{V_\alpha\}$ and glue up the families $\nabla^{0\alpha}, \nabla^{1\alpha}$, like in (3.4), we get connections ∇^0, ∇^1 that define the representative A-form

$$\Xi_1(x) = \sum_{x\in V_\alpha} \theta_\alpha(x)\Xi_{1\alpha}(x) = \big(\lambda - \varphi^*\lambda'\big)(x), \quad x \in M$$

of $\mu_1(\varphi)$. This justifies the required conclusion. □

Example 3.1. An interesting example appears on a Poisson–Nijenhuis manifold (M, P, N), where P is a Poisson bivector field and N is a Nijenhuis tensor. Then ${}^tN : (T^*M, N\circ\sharp_P) \to (T^*M, \sharp_P)$ is a morphism of cotangent Lie algebroids. The modular class of the morphism ${}^tN$ was studied in [4] and it would be interesting to get information about other characteristic classes of this morphism.

The calculation of the classes μ_{2h-1} for $h > 1$ is much more complicated. One of the difficulties is the absence of a global construction of a distinguished pair of connections.

Example 3.2. Let $\varphi : A \to A$ be an endomorphism of the Lie algebroid A and assume that there exists an A-connection ∇ on A that satisfies condition (3.6) and whose torsion

$$T_\nabla(a_1, a_2) = \nabla_{a_1}a_2 - \nabla_{a_2}a_1 - [a_1, a_2]_A, \quad a_1, a_2 \in \Gamma A,$$

takes values in $K = \ker\varphi$. Then, it is easy to check that the formula

$$\nabla'_{a_1}a_2 = [\varphi a_1, a_2]_A + \varphi\nabla_{a_2}a_1$$

defines a second A-connection that is φ-compatible with ∇ and satisfies condition (3.8). Therefore, (∇, ∇') is a distinguished pair.

Another difficulty is produced by the complicated character of the expression (3.13). A simple example follows.

Example 3.3. If the Lie algebroids A, A' have anchors zero, the A-connections are tensors and formula (3.11) gives the local connection matrices of a global, flat A-connection ∇^1 as required in the definition of the characteristic classes (flatness is just Jacobi identity). In the simplest case $A = M\times\mathcal{G}$, $A' = M\times\mathcal{G}'$ where $\mathcal{G}$, $\mathcal{G}'$ are Lie algebras, we may take $\varpi_i^j = 0$ in (3.12), which gives a flat metric connection ∇^0. Then, formula (2.7) reduces to

$$\Omega_{(\tau)} = \tau(1-\tau)\alpha\wedge\alpha$$

where α is the matrix (3.11). Accordingly, like in [14], Theorem 4.5.11, we get the representative A-forms

$$\Xi_{2h-1} = \frac{1}{(2h-2)!}\nu_h\delta^{\sigma_1\ldots\sigma_{2h-1}}_{\kappa_1\ldots\kappa_{2h-1}}\alpha^{\kappa_1}_{\sigma_1}\wedge\alpha^{\lambda_2}_{\sigma_2}\wedge\alpha^{\kappa_2}_{\lambda_2}\wedge\cdots\wedge\alpha^{\lambda_{2h-1}}_{\sigma_2 h-1}\wedge\alpha^{\kappa_{2h-1}}_{\lambda_{2h-1}}$$

of the classes μ_{2h-1}, where $\alpha^{\cdot}_{\cdot}$ are the entries of the matrix (3.11) and

$$\nu_h = \int_0^1 \tau(1-\tau)\,d\tau = \sum_{i=1}^{2h-2}(-1)^{h+i+1}\frac{2^i}{4h-i-3}\binom{2h-2}{i}.$$

Remark 3.4. So far, we do not have a good definition of characteristic classes of a morphism between Lie algebroids over different bases. Using the terminology and notation of [10], let us consider a morphism

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array} \tag{3.17}$$

between the Lie algebroids A, B and assume that the mapping f is transversal to the Lie algebroid B. Then, Proposition 3.11 of [10] tells us that $\varphi = f^!_B\circ\varphi'$, where $f^!_B : f^!B \to B$, $\varphi' : A \to f^!B$ are the natural projections of the pullback Lie algebroid $f^!B$. Furthermore, Proposition 3.12 of [10] tells that the modular class of the non-base preserving morphism φ is equal to the modular class of the base preserving morphism φ'. This equality may be extended by definition to all the characteristic classes of φ, but it is not clear whether this definition is good (it does not loose information about φ) even in the indicated particular case.

4. Relative characteristic classes

From Proposition 3.6 and a known result on modular classes [10, formula (2.5)] we see that the first class $\mu_1(\varphi)$ has a nice behavior with respect to the composition of morphisms namely, for the morphisms $\varphi : A \to A'$, $\psi : A' \to A''$ one has

$$\mu_1(\psi \circ \varphi) = \mu_1(\varphi) + \varphi^*\big(\mu_1(\psi)\big). \tag{4.1}$$

In this section we give a proof of (4.1) by means of the definition of the characteristic classes of a morphism and we shall see why the result does not extend to the higher classes μ_{2h-1}, $h > 1$. The proof will use a kind of relative characteristic classes that are interesting in their own right; in particular, we will show that the relative classes defined by the jet Lie algebroid J^1A [3] are cohomological images of the absolute characteristic classes of a morphism $\varphi : A \to A'$.

Like in the definition of the characteristic classes of φ we can produce characteristic classes of $\psi : A' \to A''$ modulo $\varphi : A \to A'$ as follows. Take the Lehmann morphism $\rho^*(D^0, D^1)$ for an orthogonal A-connection D^0 on the vector bundle $A' \oplus A''^*$ associated with a sum of Euclidean metrics $g_{A'}, g_{A''}$ and an A-connection D^1 on $A' \oplus A''^*$, which is the sum of *distinguished A-connections* ∇', ∇'' on A', A'', respectively. Here by a distinguished pair we mean a pair of A-connections (∇', ∇'') that satisfies the following properties

$$\begin{aligned}
&\psi\nabla'_a a' = \nabla''_a\big(\psi a'\big), \quad a \in A_x\ (x \in M),\ a' \in \Gamma A',\\
&\nabla'_a k(x) = [\varphi\tilde{a}, k]_{A'}(x), \quad k \in \Gamma \ker\psi,\\
&\nabla''_a a''(x) = \big[\psi\varphi\tilde{a}, a''\big]_{A''}(x) \pmod{\operatorname{im}\psi},
\end{aligned} \tag{4.2}$$

where the sign tilde denotes the extension to a cross section. One can construct a ψ-distinguished pair of A-connections ∇', ∇'' by replacing the local formulas (3.3) by

$$\nabla'^{U}_{b_i} b'_{j'} = \big[\varphi b_i, b'_{j'}\big]_{A'}, \qquad \nabla''^{U}_{b_i} b''_{j''} = \big[\psi\varphi b_i, b''_{j''}\big]_{A''}, \tag{4.3}$$

then gluing the local connections via a partition of unity. (In (4.3) $(b_i), (b'_{i'}), (b''_{i''})$ are local bases of $\Gamma A, \Gamma A', \Gamma A''$, respectively.)

Definition 4.1. The characteristic A-cohomology classes in $\operatorname{im}\rho^*(D^0, D^1)$ will be called *relative characteristic classes* of ψ modulo φ. In particular,

$$\mu_{2h-1}(\psi \bmod \varphi) = \big[\Delta\big(D^0, D^1\big)\big] \in H^{4h-3}(A)$$

are the *simple relative characteristic classes*.

Proposition 4.1. *For $h = 1$, the relative and absolute characteristic class μ_1 of the morphism ψ are related by the equality*

$$\mu_1(\psi \bmod \varphi) = \varphi^*\mu_1(\psi). \tag{4.4}$$

Proof. By absolute classes we understand characteristic classes $\mu_{2h-1}(\psi) \in H^{4h-3}(A')$. The partition of unity argument given for (3.11) shows that we may assume the following local expressions of distinguished A'-connections on A', A''

$$\bar{\nabla}'^{U}_{b'_{i'}} b'_{j'} = \big[b'_{i'}, b'_{j'}\big]_{A'}, \qquad \bar{\nabla}''^{U}_{b'_{i'}} b''_{j''} = \big[\psi b'_{i'}, b''_{j''}\big]_{A''}. \tag{4.5}$$

Connections (4.5) induce A-connections $\tilde{\nabla}', \tilde{\nabla}''$ and we shall compute the local matrices of the induced connections. By definition, we have

$$\tilde{\nabla}'^{U}_{b_i} b'_{j'} = \bar{\nabla}'^{U}_{\varphi b_i} b'_{j'}, \qquad \tilde{\nabla}''^{U}_{b_i} b''_{j''} = \bar{\nabla}''^{U}_{\varphi b_i} b''_{j''}$$

and it is easy to check that the A-connections $\tilde{\nabla}'^{U}, \tilde{\nabla}''^{U}$ satisfy conditions (4.2). Therefore, $\tilde{\nabla}'^{U}, \tilde{\nabla}''^{U}$ may be used in the calculation of the relative characteristic classes of $\psi \bmod \varphi$. If we denote $\varphi b_i = \varphi_i^{j'} b'_{j'}$ and use expressions (4.5) and the properties of the Lie algebroid brackets we obtain the local connection matrices

$$\begin{aligned}
&\tilde{\omega}'^{k'}_{j'} = \varphi^*\bar{\omega}'^{k'}_{j'} - \langle d_{A'}\varphi_i^{k'}, b_{j'}\rangle b^{*i},\\
&\tilde{\omega}''^{k''}_{j''} = \varphi^*\bar{\omega}''^{k''}_{j''} - \langle d_{A''}\varphi_i^{k'}, \psi b_{j'}\rangle b^{*i}.
\end{aligned} \tag{4.6}$$

Formula (4.6) allows us to write down the local connection matrix of the connection $D^1 = \tilde{\nabla}' + \tilde{\nabla}''^*$ required by the definition of the relative classes. Furthermore, we may assume that the local matrix of the orthogonal connection D^0 that we use is skew-symmetric. Accordingly, and since ψ is a Lie algebroid morphism, (4.6) yields

$$\Delta\big(D^0, D^1\big)c_1 = \operatorname{tr}\begin{pmatrix} \tilde{\omega}'^{k'}_{j'} & 0 \\ 0 & -\tilde{\omega}''^{j''}_{k''} \end{pmatrix} = \varphi^* \operatorname{tr}\begin{pmatrix} \bar{\omega}'^{k'}_{j'} & 0 \\ 0 & -\bar{\omega}''^{j''}_{k''} \end{pmatrix} = \varphi^* \Delta\big(\bar{\nabla}^0, \bar{\nabla}^1\big)c_1,$$

where $\bar{\nabla}^1 = \bar{\nabla}' + \bar{\nabla}''^*$ and $\bar{\nabla}^0$ is an orthogonal A'-connection on $A' \oplus A''^*$. This result justifies (4.4). □

Proposition 4.2. *For $h = 1$, the relative and absolute characteristic class μ_1 of the morphisms φ, ψ are related by the equality*

$$\mu_1(\psi \circ \varphi) = \mu_1(\varphi) + \mu_1(\psi \bmod \varphi). \tag{4.7}$$

Proof. In the computation of $\mu_1(\psi \circ \varphi)$ we may use an A-connection $\nabla + \nabla''^*$ on $A \oplus A''^*$ where, on the specified neighborhood U, ∇ is given by (3.3) and ∇'' is given by (4.3), while in the computation of $\mu_1(\psi \bmod \varphi)$ we shall use the connections ∇', ∇'' of (4.3). Thus, the non-zero blocks of the local difference matrix α that enters into the expression of the representative 1-A-form of $\mu_1(\psi \circ \varphi)$ are given by the local matrix of

$$\nabla'' - \nabla = \nabla'' - \nabla' + \nabla' - \nabla \tag{4.8}$$

and the opposite of its transposed matrix (in spite of the notation, calculation (4.8) is for the connection matrices not for the connections). Then, if we use orthogonal connections of metrics where the bases used in (4.3) are orthonormal bases (therefore, with trace zero), formula (4.8) justifies (4.7). □

Corollary 4.1. *The characteristic class μ_1 of a composed morphism $\psi \circ \varphi$, ψ is given by formula (4.1).*

Proof. The result is an obvious consequence of formulas (4.4) and (4.7). □

Remark 4.1. Formulas (4.1), (4.4), do not hold for $h > 1$ because of the more complicated expression of the polynomials c_{2h-1} (there is no nice formula for the determinant of a sum of matrices).

We finish by showing the relation between the characteristic classes of the base-preserving Lie algebroid morphism $\varphi : A \to A'$ and the relative classes defined by the first jet Lie algebroid J^1A; for $\varphi = \sharp_A : A \to TM$ this relation was established in [3].

The first jet bundle J^1A may be defined as follows. Let D be a TM-connection on A and let Da denote the covariant differential of a cross section $a \in \Gamma A$ (i.e., $Da(X) = D_X a$, $X \in \Gamma TM$). The properties of a connection tell us that $Da \in \operatorname{Hom}(TM, A)$ and, if $a(x_0) = 0$ for some point $x_0 \in M$, then $Da(x_0) : T_{x_0}M \to A_{x_0}$ is a linear mapping that is independent of the choice of the connection D. (This is not true if $a(x_0) \neq 0$.) If (x^h) are local coordinates of M around x_0 and (b_i) is a local basis of ΓA, and if $a = \xi^i(x^h)b_i$, the local matrix of $Da(x_0)$ is $(D_h\xi^i(x_0))$ (the covariant derivative tensor), which reduces to $(\partial\xi^i/\partial x^h(x_0))$ if $\xi^i(x_0) = 0$.

Now, for any point $x_0 \in M$, the space of 1-jets of cross sections of A at x_0 is

$$J^1_{x_0}A = \Gamma A / \big\{a \in \Gamma A / a(x_0) = 0, Da(x_0) = 0\big\} \tag{4.9}$$

and each $a \in \Gamma A$ defines an element $j^1_{x_0}a \in J^1_{x_0}A$ called the 1-jet of a at x_0. With the local coordinates and basis considered above, we may write

$$a = \xi^i\big(x^h\big)b_i = \left(\xi^i(x_0) + \frac{\partial\xi^i}{\partial x^h}(x_0)\big(x^h - x^h(x_0)\big) + o\Big(\big(x^h - x^h(x_0)\big)^2\Big)\right)b_i.$$

Hence,

$$j^1_{x_0}a = \xi^i(x_0)\, j^1_{x_0}b_i + \frac{\partial\xi^i}{\partial x^h}(x_0)\, j^1_{x_0}\big(\big(x^h - x^h(x_0)\big)b_i\big)$$

and

$$j^1_{x_0}b_i,\, j^1_{x_0}\big(\big(x^h - x^h(x_0)\big)b_i\big) = j^1_{x_0}\big(x^h b_i\big) - x^h(x_0)\, j^1_{x_0}b_i \tag{4.10}$$

is a basis of the vector space $J^1_{x_0}A$ such that $(\xi^i(x_0), \partial\xi^i/\partial x^h(x)(x_0))$ are coordinates with respect to this basis.

A change of the local coordinates and basis of A gives the transition formulas

$$\tilde{x}^h = \tilde{x}^h\big(x^k\big), \qquad \tilde{\xi}^i = \lambda^i_j\big(x^k\big)\xi^j, \qquad \frac{\partial\tilde{\xi}^i}{\partial\tilde{x}^h} = \frac{\partial x^k}{\partial\tilde{x}^h}\left(\frac{\partial\lambda^i_j}{\partial x^k}\xi^j + \lambda^i_j\frac{\partial\xi^j}{\partial x^k}\right) \tag{4.11}$$

and may be seen as the composition of the change of the coordinates (x^h) with the change of the basis (b_i), while the order of the two changes is irrelevant. This remark allows for an easy verification of the fact that the change of the coordinates

discovered above in $J^1_{x_0}A$ has the cocycle property. Accordingly, (4.11) shows that $J^1A = \bigcup_{x\in M} J^1_x A$ has a natural structure of a differentiable manifold and vector bundle $\pi : J^1A \to M$ over M called the first jet bundle of A.

From (4.10), we see that $(j^1b_i, j^1(x^h b_i))$ is a local basis of cross sections of J^1A at each point of the coordinate neighborhood where x^h are defined. This basis consists of 1-jets of local cross sections of A, therefore, the cross sections of J^1A are locally spanned by 1-jets of cross sections of A over $C^\infty(M)$. In the case of a Lie algebroid A, the previous remark allows us to define a Lie algebroid structure on J^1A by putting

$$\sharp_{J^1A}\big(j^1a\big) = \sharp_A a, \qquad \big[j^1a_1, j^1a_2\big]_{J^1A} = j^1[a_1, a_2]_A \tag{4.12}$$

and by extending the bracket to general cross sections via the axioms of a Lie algebroid. We refer the reader to Crainic and Fernandes [3] for details. A general, global expression of the Lie algebroid bracket of J^1A was given by Blaom [1].

Moreover, (4.12) shows that the natural projection $\pi^1 : J^1A \to A$, $\pi^1(j^1a) = a$ is a base-preserving morphism of Lie algebroids and, if $\varphi : A \to A'$ is a morphism of Lie algebroids, we may define relative characteristic classes of φ modulo π^1. Following [3], there exist flat J^1A-connections $\nabla^{j^1}, \nabla'^{j^1}$ on A, A', respectively, given by

$$\nabla^{j^1}_{fj^1a_1} a_2 = f[a_1, a_2]_A, \qquad \nabla'^{j^1}_{fj^1a} a' = f\big[\varphi a, a'\big]_{A'}, \tag{4.13}$$

where $a, a_1, a_2 \in \Gamma A, a' \in \Gamma A'$, $f \in C^\infty(M)$. These connections obviously satisfy conditions (4.2), hence, $D^1 = \nabla^{j^1} + \nabla'^{*j^1}$ is a J^1A-connection on $A \oplus A'^*$ that may be used in Definition 4.1 for the present case. We shall prove the following result

Proposition 4.3. *The relative characteristic classes of φ modulo π^1 are the images of the corresponding absolute characteristic classes of φ by the homomorphism $\pi^{1*} : H^*(A) \to H^*(J^1A)$.*

Proof. Here, we have the particular case of the situation that existed in Proposition 4.1 where π^1 comes instead of φ and φ comes instead of ψ. Therefore, we may construct connections that correspond to (4.5) and the induced J^1A-connections and get the corresponding formulas (4.6). If we use the local bases (4.10) of ΓJ^1A, the components $\varphi_i^{k'}$ that appear in (4.6) are constant and (4.6) simply tell us that the local connection forms of the induced connections are the pullback of the connection forms of the connections (4.5) by π^1. Of course, the same holds for the curvature forms, and, if we also use a J^1A-orthogonal connection of $A \oplus A'^*$ that is induced by an A-orthogonal connection, we see that π^{1*} commutes with the Lehmann morphism, which implies the required result. □

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