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Combinatorics and invariant differential operators on multiplicity free spaces

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1. Introduction

Let G be a connected reductive group acting on a finite dimensional vector space U (everything defined over $\mathbb{C}$). We assume that U is a multiplicity free space, i.e., every simple G-module appears in $\mathcal{P}(U)$, the algebra of polynomial functions on U, at most once. Thus, as a G-module,

$$\mathcal{P}(U) \cong \bigoplus_{\lambda \in \Lambda_+} \mathcal{P}_\lambda \tag{1.1}$$

where Λ_+ is a set of dominant weights and $\mathcal{P}_\lambda$ is a simple G-module of *lowest* weight $-\lambda$. All elements of $\mathcal{P}_\lambda$ are homogeneous of the same degree, denoted $\ell(\lambda)$.

Now consider an invariant differential operator D on U. It will act on each irreducible constituent $\mathcal{P}_\lambda$ as a scalar, denoted by $c_D(\lambda)$. It can be shown that c_D extends to a polynomial function to V, the $\mathbb{C}$-span of Λ_+. Thus, $D \mapsto c_D$ is a homomorphism from $\mathcal{PD}(U)^G$, the algebra of invariant differential operators, into $\mathcal{P}(V)$. It is possible to determine the image of this map. One can show [7] that there is a “shift vector” $\rho \in V$ and a finite reflection group $W \subseteq GL(V)$ such that the following is an isomorphism:

$$\mathcal{PD}(U)^G \xrightarrow{\sim} \mathcal{P}(V)^W : D \mapsto p_D(z) := c_D(z - \rho). \tag{1.2}$$

Thus, the eigenvalues of D in $\mathcal{P}(U)$ are the values $p_D(\rho + \lambda)$, $\lambda \in \Lambda_+$.

The identification (1.2) works actually in the much wider context of G-varieties (see [6]) but only multiplicity free spaces have the following important feature: $\mathcal{PD}(U)^G$ has a

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distinguished basis D_λ, $\lambda \in \Lambda_+$. The construction of the D_λ goes back to Capelli. Via the identification (1.2), we get also a distinguished basis $p_\lambda := p_{D_\lambda}$ of $\mathcal{P}(V)^W$.

It is possible to characterize the elements of this basis purely in terms of V without any reference to U. Namely, p_λ is the unique W-invariant polynomial on V of degree $\ell(\lambda)$ which has the following interpolation property:

$$p_\lambda(\rho + \mu) = \delta_{\lambda\mu} \quad \text{for all } \mu \in \Lambda_+ \text{ with } \ell(\mu) \leqslant \ell(\lambda). \tag{1.3}$$

Note that this is a purely combinatorial description of p_λ: all we need to know are V, W, Λ_+, ℓ, and ρ. The first four of these data are rather rigid but there is some flexibility for ρ. In fact, there are many, quite different, examples of multiplicity free spaces for which V, W, Λ_+, and ℓ are the same but ρ is different. This is a motivation for using the characterization above to *define* a family of polynomials $p_\lambda(z;\rho)$ for an (almost) *arbitrary* $\rho \in V$ (a suggestion of Sahi, see [17]).

In general, not much can be said about $p_\lambda(z;\rho)$ but we showed in [9][1] that for ρ in a certain *non-trivial* subspace V_0 of V these polynomials have remarkable properties. The most important one is the existence of difference operators D_h, $h \in \mathcal{P}(V)^W$, for which all polynomials $p_\lambda = p_\lambda(z;\rho)$ are eigenfunctions. More precisely,

$$D_h(p_\lambda) = h(\rho + \lambda) p_\lambda. \tag{1.4}$$

Thus we can think of the polynomials $p_\lambda(z;\rho)$, $\rho \in V_0$, as a good deformation of the spectral polynomials $p_\lambda(z)$.

The central result of the present paper is the Transposition Formula for $p_\lambda(z;\rho)$. Again, it originates from differential operators. There, “transposition” is the unique *anti*-automorphism $D \mapsto {}^tD$ of $\mathcal{PD}(U)$ with

$${}^t x_i = x_i \quad \text{and} \quad {}^t\!\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}. \tag{1.5}$$

Transposition commutes with the G-action and induces an automorphism of $\mathcal{PD}(U)^G$. It is a natural problem to calculate its effect on $\mathcal{P}(V)^W$ under the identification (1.2). This is done in Section 2 and the answer is simply the map $h \mapsto h^-$ where $h^-(z) := h(-z)$ (Theorem 2.2).

From now on, we denote $\mathcal{P}(V)$ simply by $\mathcal{P}$. In Section 4, we compute the action of $h \mapsto h^-$ on $\mathcal{P}^W$ with respect to the p_λ-basis. The result is the *transposition formula* (Theorem 4.3):

$$q_\lambda(-z) = \sum_\mu (-1)^{\ell(\mu)} p_\mu(\rho + \lambda) q_\mu(z) \quad \text{for all } \lambda \in \Lambda_+. \tag{1.6}$$

[1] In fact, the present paper is as a continuation of [9]. For the convenience of the reader we recall all relevant results in Section 3.

Here, we used the renormalized polynomials

$$q_\lambda(z;\rho) := \frac{1}{p_\lambda(-\rho;\rho)}\, p_\lambda(z;\rho). \tag{1.7}$$

Its proof uses the difference operators D_h, an idea which goes back to Okounkov [14][2] who proved it for shifted Jack polynomials.

A first consequence of the transposition formula is the *evaluation formula* (Corollary 4.6)

$$p_\lambda(-\rho;\rho) = (-1)^{\ell(\lambda)} d_\lambda \tag{1.8}$$

where

$$d_\lambda = \prod_{\alpha\in\Delta^+} \frac{\alpha(\rho+\lambda)}{\alpha(\rho)} \prod_{\omega\in\Phi^+} \frac{(\omega(\rho)+k_\omega)_{\omega(\lambda)}}{(\omega(\rho)-k_\omega+1)_{\omega(\lambda)}}. \tag{1.9}$$

Here $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol, Δ^+ and Φ^+ are certain finite sets of linear functions on V (positive roots and pseudoroots, respectively), and k_ω is the multiplicity function determined by ρ. The number d_λ is called the *virtual dimension* since, in the case when ρ comes from a multiplicity free space U, it computes the dimension of the irreducible G-module $\mathcal{P}_\lambda$ occurring in $\mathcal{P}(U)$ (Theorem 4.8). This result generalizes a formula of Upmeier [19] who considered multiplicity free spaces attached to Hermitian symmetric spaces (see below).

As already observed in [14], another consequence of the transposition formula is the *interpolation formula*. It gives the expansion of an arbitrary polynomial $h\in\mathcal{P}^W$ in terms of the p_λ's. More precisely, we show in Section 5 (Theorem 5.2):

$$h(z) = \sum_{\lambda\in\Lambda_+} (-1)^{\ell(\lambda)} \widehat{h}(\rho+\lambda)\, p_\lambda(z) \tag{1.10}$$

where

$$\widehat{h}(\rho+\lambda) := \sum_{\mu\in\Lambda_+} (-1)^{\ell(\mu)} p_\mu(\rho+\lambda) h(\rho+\mu). \tag{1.11}$$

Another consequence (also noticed in [14]) of the transposition formula is the symmetry

$$q_\lambda(-\rho-\nu) = q_\nu(-\rho-\lambda), \quad \lambda,\nu\in\Lambda_+ \tag{1.12}$$

(just substitute $z=\rho+\nu$ in (1.6)). In Section 6, we define a scalar product on $\mathcal{P}^W$ by

$$\langle p_\lambda, p_\mu\rangle := d_\lambda \delta_{\lambda\mu}. \tag{1.13}$$

Then (1.12) is explained by the fact $\langle q_\lambda^-, q_\nu^-\rangle = q_\lambda(-\rho-\nu)$.

[2] There, the "transposition formula" is called "binomial formula".

Let $\mathcal{A} \subseteq \operatorname{End}_{\mathbb{C}} \mathcal{P}^W$ be the algebra generated by all multiplication operators $h \in \mathcal{P}^W$ and all difference operators D_h, $h \in \mathcal{P}^W$. Then the transformation (1.11) can be used to define an involutory automorphism $X \to \widehat{X}$ of $\mathcal{A}$ which interchanges h and D_h (Theorem 5.3). Moreover, we show that $\mathcal{A}$ is stable under taking adjoints for the auxiliary scalar product

$$\langle f, g\rangle^- := \langle f^-, g^-\rangle.$$

More precisely, (h, D_{h^-}) is an adjoint pair (Theorem 6.3).

These results are extended in Section 7. For every operator X define X^- by $X^-(h) := X(h^-)^-$. Let $\mathcal{B}$ be the algebra generated by all h, D_h, and D_h^- with $h \in \mathcal{P}^W$. In other words, $\mathcal{B}$ is generated by $\mathcal{A}$ and $\mathcal{A}^-$. First we observe that $\mathcal{B}$ is stable under taking adjoints X^* with respect to the original scalar product (Theorem 7.1). The main result of Section 7 is the construction of a $PGL_2(\mathbb{C})$-action on $\mathcal{B}$ which incorporates the two automorphisms $X \mapsto \widehat{X}$ and $X \mapsto X^-$. For this, let

$$L := \ell - D_\ell. \tag{1.14}$$

Then we show that $(L, 2\ell, L^-)$ forms an sl_2-triple (Theorem 7.2). The $PGL_2(\mathbb{C})$-action is obtained by integrating the adjoint action of this triple (Theorem 7.3).

In Section 8, we study the effect of operators in $\mathcal{B}$ on the top homogeneous components of polynomials. More precisely, both $\mathcal{P}^W$ and $\mathcal{B}$ are filtered by degree. Denote their associated graded algebras by $\overline{\mathcal{P}}^W$ and $\overline{\mathcal{B}}$, respectively. Then the $\overline{\mathcal{B}}$-module $\overline{\mathcal{P}}^W$ is called the *differential limit* of the $\mathcal{B}$-module $\mathcal{P}^W$. While it is clear that $\overline{\mathcal{P}}^W \cong \mathcal{P}^W$ (since $\mathcal{P}$ is graded to begin with) we show that also $\overline{\mathcal{B}} \cong \mathcal{B}$ (Theorem 8.3). Therefore, the algebra $\mathcal{B}$ of *difference* operators can be replaced by $\overline{\mathcal{B}}$, an algebra of *differential* operators (Proposition 8.1). Unfortunately, so far it seems to be very hard to construct $\overline{\mathcal{B}}$ directly.

In Section 9, we study another limit, namely the infinitesimal neighborhood of a particular W-fixed point δ in V. The transposition formula (1.6) then becomes, in the limit, a *binomial formula* (Theorem 9.1):

$$\bar{q}_\lambda^{(\delta)}(z+\delta) = \sum_{\substack{\mu \in \Lambda_+ \\ \ell^\delta(\mu) = \ell^\delta(\lambda)}} p_\mu(\rho+\lambda)\bar{q}_\mu^{(\delta)}(z) \tag{1.15}$$

where $\bar{q}_\lambda^{(\delta)}(z)$ is a certain renormalization of the top homogeneous component of $p_\lambda(z)$.

Multiplicity free spaces have been classified by Kac [5], Benson and Ratcliff [1], and Leahy [13]. So far, basically only two classes have been studied in more detail. The case which got by far the most attention is the so-called classical case. It includes the spaces when G is the complexification of the isotropy group of a Hermitian symmetric space and U is the complexification of "$\mathfrak{p}^+$". Here, the polynomials $p_\lambda(z; \rho)$ are called *shifted Jack polynomials* since their top homogeneous components are the Jack polynomials. By now there is a rich literature on these polynomials, and most results of this paper have been previously obtained in that case, (see, e.g., [10–12,14–18]) even though the results of Section 7 on the $PGL_2(\mathbb{C})$-action seem to be new even in the classical case.

The other case, in which the present theory is (mostly) worked out is the semiclassical case, [8]. This includes, e.g., the action of $GL_n(\mathbb{C})$ on $\wedge^2\mathbb{C}^{n+1}$. Among the few papers which deal with general multiplicity free spaces are most notably [4,20], and [2].

We found it useful to illustrate most of our results with the case $\dim V = 1$. This case is pretty elementary but still quite interesting. We could have sprinkled specializations to this case all over the paper but found it more useful to gather everything in a separate section at the end of the paper. It is recommended to consult this section frequently in the course of reading this paper or to even start with it.

2. Transposition of differential operators on multiplicity free spaces

Let U be a finite dimensional complex vector space. Then the algebra $\mathcal{PD}(U)$ of linear differential operators with polynomial coefficients has the following presentation: it is generated by U (the directional derivatives) and U^* (the linear functions) which satisfy the following relations:

$$[\partial_1, \partial_2] = 0, \quad [x_1, x_2] = 0, \quad [\partial, x] = \partial(x)$$
$$\text{for all } \partial_1, \partial_2, \partial \in U,\ x_1, x_2, x \in U^*. \tag{2.1}$$

This implies that there is a unique *anti*automorphism $D \mapsto {}^tD$ of $\mathcal{PD}(U)$ with

$${}^t\partial = -\partial, \quad {}^tx = x \quad \text{for all } \partial \in U \text{ and } x \in U^*. \tag{2.2}$$

The operator ${}^tD$ is called the *transpose* of D.

Let G be an algebraic group G acting linearly on U. Then transposition is G-equivariant. It follows, that it induces an antiautomorphism of the algebra $\mathcal{PD}(U)^G$ of G-invariant differential operators.

Now assume that G is connected, reductive and U a multiplicity free space. This means that the algebra $\mathcal{P}(U)$ of regular functions is multiplicity free as a G-module. Then it is easy to show that $\mathcal{PD}(U)^G$ is commutative (in fact, this will be shown below). Thus, transposition is an automorphism of $\mathcal{PD}(U)^G$. The purpose of this section is to calculate this automorphism explicitly.

To do this, we need first an explicit description of the algebra $\mathcal{PD}(U)^G$ itself. Fix a Borel subgroup B of G and a maximal torus T of B. By assumption, the algebra $\mathcal{P}(U)$ decomposes as a G-module as $\mathcal{P}(U) = \bigoplus_{\lambda \in \Lambda_+} \mathcal{P}^\lambda$ where $\Lambda_+ \subseteq (\operatorname{Lie} T)^*$ is a certain set of integral dominant weights and $\mathcal{P}^\lambda$ is a simple G-module with *lowest* weight $-\lambda$.

Every $D \in \mathcal{PD}(U)^G$ acts on $\mathcal{P}^\lambda$ as multiplication by a scalar, which is denoted by $c_D(\lambda)$. Let $V \subseteq (\operatorname{Lie} T)^*$ be the $\mathbb{C}$-span of Λ_+. Then c_D is the restriction of a unique polynomial function on V (also denoted by c_D) to Λ_+ (see [7, Corollary 4.4]). Thus, we obtain an embedding

$$\mathcal{PD}(U)^G \hookrightarrow \mathcal{P}(V) : D \mapsto c_D \tag{2.3}$$

which shows, in particular, that $\mathcal{PD}(U)^G$ is commutative.

To describe the image of this embedding we need some more notation. Let $P \supseteq B$ be the largest parabolic subgroup such that all elements of Λ_+, considered as characters of T, extend to characters of P. Let β be the sum of all roots in the unipotent radical of P and let χ be the sum of all weights of U. Then it is shown in [9, Section 7] that $\rho := \frac{1}{2}(\beta + \chi)$ is an element of V. Using this weight, we define a new embedding

$$\mathcal{PD}(U)^G \hookrightarrow \mathcal{P}(V) : D \mapsto p_D \tag{2.4}$$

where

$$p_D(v) := c_D(v - \rho). \tag{2.5}$$

Then we have the following (Harish Chandra) isomorphism:

Theorem 2.1 [7, Theorem 4.8]. *There is a unique finite subgroup $W \subseteq GL(V)$ such that $D \mapsto p_D$ establishes an isomorphism between $\mathcal{PD}(U)^G$ and $\mathcal{P}(V)^W$.*

Now we can make transposition of invariant differential operators explicit:

Theorem 2.2. *Let U be a multiplicity free space for G. Then*

$$p_{{}^tD}(v) = p_D(-v) \quad \text{for every } D \in \mathcal{PD}(U)^G \text{ and } v \in V. \tag{2.6}$$

Proof. Using (2.5) we have to prove

$$c_{{}^tD}(v) = c_D(-v - \chi - \beta). \tag{2.7}$$

Let $\mathfrak{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra of G. The action of G on U induces a homomorphism $\Psi : \mathfrak{U}(\mathfrak{g}) \to \mathcal{PD}(U)$ which maps $\mathfrak{Z}(\mathfrak{g})$ to $\mathcal{PD}(U)^G$. We are going to verify (2.7) first for operators in the image of Ψ.

Let u_i be a basis of U where each u_i is a weight vector with weight χ_i. Let $x_i \in U^*$ be the dual basis and $\partial_i := \partial/\partial x_i$. Consider the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^-$. For $\eta \in \mathfrak{t}$ we have $\Psi(\eta) = -\sum_i \chi_i(\eta) x_i \partial_i$. Thus

$${}^t\Psi(\eta) = -\sum_i \chi_i(\eta)(-\partial_i)x_i = \sum_i \chi_i(\eta)(x_i\partial_i + 1) = -\Psi(\eta) + \chi(\eta). \tag{2.8}$$

If $\eta \in \mathfrak{n}^{\pm}$ then $\Psi(\eta) = \sum_{i \neq j} a_{ij} x_i \partial_j$, hence ${}^t\Psi(\eta) = -\Psi(\eta)$. Now observe that χ, being the sum of all weights of U, is actually a character of all of $\mathfrak{g}$. Thus, we can define an *anti*automorphism τ of $\mathfrak{U}(\mathfrak{g})$ by $\tau(\eta) := -\eta + \chi(\eta)$ for all $\eta \in \mathfrak{g}$ and the discussion above showed

$${}^t\Psi(\xi) = \Psi\big(\tau(\xi)\big) \quad \text{for all } \xi \in \mathfrak{U}(\mathfrak{g}). \tag{2.9}$$

Let $\xi \in \mathcal{Z}(\mathfrak{g})$ and $D = \Psi(\xi)$. From the theorem of Poincaré–Birkhoff–Witt follows that ξ decomposes uniquely as $\xi = \xi_0 + \xi_1$ with $\xi_0 \in \mathfrak{U}(\mathfrak{t})$ and $\xi_1 \in \mathfrak{n}^-\mathfrak{U}(\mathfrak{g})\mathfrak{n}$. Since $\mathfrak{U}(\mathfrak{t}) = S(\mathfrak{t})$, we can regard ξ_0 as a function on $\mathfrak{t}^*$ and write $\xi_0(v)$ for its value at $v \in \mathfrak{t}^*$. In particular, we have

$$\tau(\xi_0)(v) = \xi_0(-v + \chi). \tag{2.10}$$

On the other hand $\tau(\xi) = \tau(\xi_0) + \tau(\xi_1)$ with $\tau(\xi_0) \in \mathfrak{U}(\mathfrak{t})$ and $\tau(\xi_1) \in \mathfrak{n}\mathfrak{U}(\mathfrak{g})\mathfrak{n}^-$. Let f be a lowest weight vector of $\mathcal{P}^\lambda$. By definition, it has weight $-\lambda$. Thus $\Psi(\tau(\xi_1))f = 0$, and we have

$${}^tDf = \Psi\big(\tau(\xi_0)\big)f = \tau(\xi_0)(-\lambda)f = \xi_0(\lambda + \chi)f. \tag{2.11}$$

Thus

$$c_{{}^tD}(\lambda) = \xi_0(\lambda + \chi) \quad \text{for all } \lambda \in \Lambda_+. \tag{2.12}$$

Let w_0 be the longest element of the Weyl group $\overline{W}$ of G. Then the highest weight of $\mathcal{P}^\lambda$ is $-w_0\lambda$. Thus $c_D(\lambda) = \xi_0(-w_0\lambda)$. Let $\bar{\rho}$ be the half-sum of positive roots of G. By the (original) Harish Chandra isomorphism, the function $v \mapsto \xi_0(v - \bar{\rho})$ is $\overline{W}$-invariant. From $w_0\bar{\rho} = -\bar{\rho}$, we get

$$c_D(\lambda) = \xi_0(-w_0\lambda) = \xi_0\big(w_0(-\lambda - \bar{\rho}) - \bar{\rho}\big) = \xi_0(-\lambda - 2\bar{\rho}). \tag{2.13}$$

Let L be the Levi complement of P and w_L the longest element of its Weyl group. Then we have the relation $\beta = \bar{\rho} + w_L\bar{\rho}$. Since Λ_+ is Zariski dense in V, Equation (2.13) is valid for all $\lambda \in V$. In particular, we can replace λ by $-\lambda - \chi - \beta$. Thus,

$$c_D(-\lambda - \chi - \beta) = \xi_0(\lambda + \chi + \beta - 2\bar{\rho}) = \xi_0(\lambda + \chi + w_L\bar{\rho} - \bar{\rho}). \tag{2.14}$$

Now we use the fact that $\lambda \in V$ and χ are w_L-fixed. Hence

$$\xi_0(\lambda + \chi + w_L\bar{\rho} - \bar{\rho}) = \xi_0\big(w_L(\lambda + \chi + \bar{\rho}) - \bar{\rho}\big) = \xi_0(\lambda + \chi). \tag{2.15}$$

Equations (2.12), (2.14), and (2.15) imply (2.7) for $D = \Psi(\xi)$.

Now we consider the general case. Clearly, there is a unique automorphism σ of $\mathcal{P}^W$ such that

$$\sigma(c_D)(v) = c_{{}^tD}(-v) \quad \text{for all } D \in \mathcal{PD}(U)^G \tag{2.16}$$

and we have to show that σ is the identity. By what we proved above, σ fixes the subalgebra $\mathcal{P}_0 := \{c_D \mid D \in \Psi\mathfrak{Z}(\mathfrak{g})\}$ pointwise. Since $\mathcal{P}(\mathfrak{t}^*)$ is finitely generated as a $\mathfrak{Z}(\mathfrak{g}) = \mathcal{P}(V)^{\overline{W}}$-module, also $\mathcal{P}(V)^W$ is a finitely generated $\mathcal{P}_0$-module. Let $\mathcal{K}$ be the quotient field of $\mathcal{P}(V)^W$. Then we see that $[\mathcal{K}\colon \mathcal{K}^{\langle\sigma\rangle}]$ is finite which implies that σ has finite order.

In the last step, we use that transposition is filtration preserving. More precisely, $\mathcal{PD}(U)^G$ is filtered by the order of a differential operator and $\mathcal{P}(V)^W$ is filtered by degree. The associated graded ring of $\mathcal{PD}(U)^G$ is $\mathcal{P}(U \oplus U^*)^G$ and transposition induces on the latter the action $(u, \alpha) \mapsto (u, -\alpha)$. The map $D \mapsto c_D$ is degree preserving. Thus transposition acts on $\operatorname{gr}\mathcal{P}(V)^W$ by $v \mapsto -v$. This shows that σ acts on $\operatorname{gr}\mathcal{P}(V)^W$ as identity. But σ has finite order, hence is linearly reductive. This implies that σ is the identity on $\mathcal{P}(V)^W$. □

3. Capelli polynomials

This section is a synopsis of the essential parts of [9]. We have seen that to every multiplicity free space there is attached a finite dimensional vector space V, a finite reflection group W acting on it and a finitely generated monoid Λ_+ of dominant weights. Additionally, we have a linear function $\ell : V \to \mathbb{C}$ such that $\ell(\lambda) = \deg f$ for any non-zero $f \in \mathcal{P}^\lambda$. These data are by no means unrelated and in [9] we proposed a set of axioms which we are not going to repeat since we rarely need them directly. From now on we forget about multiplicity free spaces and consider just structures[3] (V, W, Λ_+, ℓ) satisfying these axioms. Note that all multiplicity free actions are classified [1,5,13]. The ensuing combinatorial structures are described in [9, Section 8].

Inside V we are going to consider the following objects:

$$\begin{array}{ccccccc} \Sigma^\vee & & \Lambda_1 & & & & \\ \cap\vert & & \cap\vert & & & & \\ \Lambda_+ & \subseteq & \Lambda & \subseteq & \Gamma^\vee & \subseteq & V \end{array} \tag{3.1}$$

Here $\Gamma^\vee$ is the lattice generated by Λ_+ and Λ is the submonoid generated by all $w\eta$ with $w \in W$ and $\eta \in \Lambda_+$. The minimal set of generators of Λ_+ is denoted by $\Sigma^\vee$. It forms a basis of $\Gamma^\vee$ and V. Also Λ has a minimal set of generators which is denoted by Λ_1. It coincides with the set of all $w\eta$ with $w \in W$, $\eta \in \Sigma^\vee$ and $\ell(\eta) = 1$.

Inside the dual space $V^\vee$ we need the following objects:

$$\begin{array}{ccccc} \Sigma & & \Delta & & \\ \cap\vert & & \cap\vert & & \\ \Phi & \subseteq & \Gamma & \subseteq & V^\vee \end{array} \tag{3.2}$$

Here Γ is the lattice dual to $\Gamma^\vee$ and Σ is the dual basis of $\Sigma^\vee$. The elements of $\Phi := \bigcup_{w \in W} w\Sigma$ are called *pseudoroots*. Attached to the reflection group W there is a unique root system Δ such that all roots are primitive vectors.

Let $\pm W$ be the group generated by W and -1. Then we define

$$V_0 := \bigl\{\rho \in V \mid \text{for all } \omega_1, \omega_2 \in \Sigma \text{ with } \omega_1 \in \pm W\omega_2 \text{ holds } \omega_1(\rho) = \omega_2(\rho)\bigr\}. \tag{3.3}$$

Thus, for $\rho \in V_0$ and for every $\omega \in \Phi \cup (-\Phi)$ we can define $k_\omega := \omega_1(\rho)$ where $\omega_1 \in \pm W\omega \cap \Sigma$. In particular we have $k_\omega = k_{-\omega}$ for all $\omega \in \Phi$.

Examples. For the rank one case see Section 10. Here we illustrate the notation above in two examples. In the *classical case* we have: $V := \mathbb{C}^n$, $W := S_n$ (symmetric group),

[3] Actually, in [9] we found it more convenient to state the axioms in terms of the equivalent data $(\Gamma, \Sigma, W, \ell)$.

$\Lambda_+ := \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}$ (partitions), and $\ell(\lambda) := \sum_i \lambda_i$. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$ and $z_1, \ldots, z_n \in (\mathbb{C}^n)^\vee$ its dual basis. Then we get the following derived data:

Subsets of V	Subsets of $V^\vee$
$\Gamma^\vee = \mathbb{Z}^n$	$\Gamma = \mathbb{Z}^n$
$\Sigma^\vee = \{e_1 + \cdots + e_i \mid 1 \leqslant i \leqslant n\}$	$\Sigma = \{z_i - z_{i+1} \mid 1 \leqslant i < n\} \cup \{z_n\}$
$\Lambda_1 = \{e_i \mid 1 \leqslant i \leqslant n\}$	$\Phi = \{z_i - z_j \mid 1 \leqslant i \neq j \leqslant n\} \cup \{z_i \mid 1 \leqslant i \leqslant n\}$
$\Lambda = \mathbb{N}^n$	$\Delta = \{z_i - z_j \mid 1 \leqslant i \neq j \leqslant n\}$
$V_0 = \{\sum_i [(n-i)r + s]e_i \mid r, s \in \mathbb{C}\}$	

Observe that Δ is a subset of Φ. This makes the classical case rather exceptional. It has been the topic of the papers [10] and [15] among others.

The second example is the *semiclassical case*. Then: $V := \mathbb{C}^n$, $W := \{w \in S_n \mid w(i) - i$ even for all $i\}$ (semisymmetric group), $\Lambda_+ := \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}$ (partitions), and $\ell(\lambda) := \sum_{i \text{ odd}} \lambda_i$. We get the following derived data:

Subsets of V	Subsets of $V^\vee$
$\Gamma^\vee = \mathbb{Z}^n$	$\Gamma = \mathbb{Z}^n$
$\Sigma^\vee = \{e_1 + \cdots + e_i \mid 1 \leqslant i \leqslant n\}$	$\Sigma = \{z_i - z_{i+1} \mid 1 \leqslant i < n\} \cup \{z_n\}$
$\Lambda_1 = \{e_i \mid i \text{ odd}\} \cup \{e_i + e_j \mid i \text{ odd}, j \text{ even}\}$	$\Phi = \{z_i - z_j \mid i - j \text{ odd}\} \cup \{z_i \mid n - i \text{ even}\}$
$\Lambda = \{\lambda \in \mathbb{N}^n \mid \sum_{i \text{ odd}} \lambda_i \geqslant \sum_{i \text{ even}} \lambda_i\}$	$\Delta = \{z_i - z_j \mid i \neq j,\ i - j \text{ even}\}$
$V_0 = \{\sum_i [(n-i)r + s]e_i \mid r, s \in \mathbb{C}\}$	

The semiclassical case has been investigated in [8].

We are going to need the following non-degeneracy conditions for $\rho \in V_0$. Let $\Delta^+ := \{\alpha \in \Delta \mid \alpha(\Sigma^\vee) \geqslant 0\}$. Then

$$\rho \in V_0 \text{ is } \begin{Bmatrix} \textit{dominant} \\ \textit{non-integral} \end{Bmatrix} \quad \text{if } \alpha(\rho) \notin \begin{Bmatrix} \mathbb{Z}_{<0} \\ \mathbb{Z} \end{Bmatrix} \text{ for all } \alpha \in \Delta^+. \tag{3.4}$$

Let $\mathcal{P}$ denote the algebra of polynomial functions on V. The next theorem introduces one of the main objects of the theory: a distinguished basis of $\mathcal{P}^W$ whose elements are sometimes called Capelli polynomials since they are related to the Capelli identities.

Theorem 3.1 [9, Theorem 3.6]. *Let $\rho \in V_0$ be dominant.*

(a) *For every $\lambda \in \Lambda_+$ there is a unique polynomial $p_\lambda \in \mathcal{P}^W$ with $\deg p_\lambda \leqslant \ell(\lambda)$ and $p_\lambda(\rho + \mu) = \delta_{\lambda\mu}$ (Kronecker delta) for all $\mu \in \Lambda_+$ with $\ell(\mu) \leqslant \ell(\lambda)$.*
(b) *For every $d \in \mathbb{N}$, the set of p_λ with $\ell(\lambda) \leqslant d$ forms a basis of the space of $p \in \mathcal{P}^W$ with $\deg p \leqslant d$.*

The polynomials vanish, in fact, in many more points than they are supposed to. This is the content of the Extra Vanishing Theorem:

Theorem 3.2 [9, Corollary 3.9]. *Let $\rho \in V_0$ be dominant. Then for any $\lambda, \mu \in \Lambda_+$ holds $p_\lambda(\rho + \mu) = 0$ unless $\mu \in \lambda + \Lambda$.*

For $d \in \mathbb{Z}$ define the following variant of the falling factorial polynomial:

$$[z \downarrow d] := \begin{cases} z(z-1)\cdots(z-d+1) & \text{if } d > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.5}$$

Then, for every $\tau \in \Gamma$ we define the rational function

$$f_\tau(z) := \frac{\prod_{\omega \in \Phi}[\omega(z) - k_\omega \downarrow \omega(\tau)]}{\prod_{\alpha \in \Delta}[\alpha(z) \downarrow \alpha(\tau)]}. \tag{3.6}$$

One of its main features are the following cut-off properties:

Lemma 3.3 [9, Lemmas 3.2 and 5.2]. *Let $\rho \in V_0$ be non-integral and $\tau \in \Lambda$.*

(a) *Assume $\lambda \in \Lambda_+$ but $\mu := \lambda - \tau \notin \Lambda_+$. Then $f_\tau(\rho + \lambda) = 0$.*
(b) *Assume $\mu \in \Lambda_+$ but $\lambda := \mu + \tau \notin \Lambda_+$. Then $f_\tau(-\rho - \mu) = 0$.*

For any $\eta \in V$ we define the shift operator T_η on $\mathcal{P}$ by $(T_\eta f)(z) = f(z - \eta)$. Then the difference operator

$$L := \sum_{\eta \in \Lambda_1} f_\eta(z) T_\eta$$

has very remarkable properties. Since its coefficients are rational functions, it doesn't act on $\mathcal{P}$ but is does on $\mathcal{P}^W$.

Examples. 1. Classical case:

$$L = \sum_{i=1}^{n} \left[\prod_{j \neq i} \frac{z_i - z_j - r}{z_i - z_j}\right](z_i - s) T_{e_i}. \tag{3.7}$$

2. Semiclassical case:

$$\begin{aligned} L = {} & \sum_{i \text{ odd}} \left[\frac{\prod_{j \text{ even}}(z_i - z_j - r)}{\prod_{j \neq i \text{ odd}}(z_i - z_j)}\right](z_i - s) T_{e_i} \\ & + \sum_{\substack{i \text{ odd} \\ j \text{ even}}} \left[\frac{\prod_{k \neq j \text{ even}}(z_i - z_k - r)\prod_{k \neq i \text{ odd}}(z_j - z_k - r)}{\prod_{k \neq i \text{ odd}}(z_i - z_k)\prod_{k \neq j \text{ even}}(z_j - z_k)}\right](z_i - s) T_{e_i + e_j}. \end{aligned} \tag{3.8}$$

One of the main properties of L is:

Theorem 3.4 [9, Corollary 5.7]. *Consider $h \in \mathcal{P}^W$ as multiplication operator on $\mathcal{P}^W$. Then $(\operatorname{ad} L)^n(h) = 0$ for $n > \deg h$.*

Thus, for every $h \in \mathcal{P}^W$ we can define the difference operator

$$D_h := \exp(\operatorname{ad} L)(h). \tag{3.9}$$

The most important special case is the *difference Euler operator* $E := D_\ell = \ell - L$. All these operators are diagonalized by the p_λ. More precisely:

Theorem 3.5 [9, Theorem 5.8]. *Let $h \in \mathcal{P}^W$. Then*

$$D_h(p_\lambda) = h(\rho + \lambda) p_\lambda \quad \text{for all } \lambda \in \Lambda_+. \tag{3.10}$$

In the classical and semiclassical case, these difference operators have been first constructed explicitly in [10] and [8], respectively. In general, much less is known. The rough structure of D_h is explained by the following lemma.

Lemma 3.6. *There is an expansion*

$$D_h = \sum_\eta b_\eta^h(z) T_\eta \tag{3.11}$$

where $b_\eta^h(z)$ is a rational function and $\eta \in \Lambda$ with $\ell(\eta) \leqslant \deg h$.

Proof. That $b_\eta^h(z)$ is rational is obvious from the definition. Let $d = \deg h$. Then

$$D_h = \sum_\eta b_\eta^h(z) T_\eta = \sum_{n=0}^{d} \frac{1}{n!} (\operatorname{ad} L)^n(h) \tag{3.12}$$

by Theorem 3.4. Thus $b_\eta^h = 0$ unless η is the sum of at most d elements of Λ_1. But this implies $\eta \in \lambda$ with $\ell(\eta) \leqslant d$. □

There is a strong connection between the difference operators D_h and Pieri-type formulas. For this we define for every $\lambda \in \Lambda_+$ the *virtual dimension*[4] as

$$d_\lambda := (-1)^{\ell(\lambda)} \frac{f_\lambda(-\rho)}{f_\lambda(\rho + \lambda)}. \tag{3.13}$$

It can be rewritten as

$$d_\lambda = \prod_{\alpha \in \Delta^+} \frac{\alpha(\rho + \lambda)}{\alpha(\rho)} \prod_{\omega \in \Phi^+} \frac{(\omega(\rho) + k_\omega)_{\omega(\lambda)}}{(\omega(\rho) - k_\omega + 1)_{\omega(\lambda)}} \tag{3.14}$$

[4] See Theorem 4.8 for an explanation of this term.

where $\Phi^+ := \{\omega \in \Phi \mid \omega(\Sigma^\vee) \geqslant 0\}$. Thus, the following condition on ρ is designed to make sure that d_λ is defined and non-zero: we call ρ *strongly dominant* if for all $\alpha \in \Delta^+$ and $\omega \in \Phi^+$:

$$\alpha(\rho) \notin \mathbb{Z}_{\leqslant 0}, \qquad \omega(\rho) - k_\omega \notin \mathbb{Z}_{<0}, \qquad \omega(\rho) + k_\omega \notin \mathbb{Z}_{\leqslant 0}. \tag{3.15}$$

Remark. All ρ's coming from multiplicity free actions are strongly dominant.

Theorem 3.7. *Let ρ be strongly dominant and non-integral. Let $h \in \mathcal{P}^W$. Then*

$$h(-z)p_\mu(z) = \sum_\tau (-1)^{\ell(\tau)} \frac{d_\mu}{d_{\mu+\tau}} b_\tau^h(-\rho-\mu) p_{\mu+\tau}(z) \quad \textit{for every } \mu \in \Lambda_+. \tag{3.16}$$

Here, the sum runs over those $\tau \in \Lambda$ with $\mu + \tau \in \Lambda_+$.

Proof. This is the combination of formulas (7), (8), and (13) of [9]. □

Later, we are also going to need the following more explicit Pieri type formula.

Theorem 3.8 [9, Corollary 3.11]. *Let $\lambda \in \Lambda_+$ and $k \in \mathbb{N}$. Then*

$$\binom{\ell(z) - \ell(\rho+\lambda)}{k} p_\lambda(z) = \sum_{\substack{\mu \in \Lambda_+ \\ \ell(\mu-\lambda)=k}} p_\lambda(\rho+\mu) p_\mu(z). \tag{3.17}$$

4. The transposition formula

In Section 2, we showed the representation theoretic meaning of the transformation $h(z) \mapsto h(-z)$ on $\mathcal{P}^W$. Now, we would like to express it in terms of the basis p_λ.

Difference operators act naturally on function from the *left*. Now, we consider also their action on (finite) measures on the *right*. More precisely, for any $v \in V$ let $\delta_v : \mathcal{P} \to \mathbb{C} : f \mapsto f(v)$ be the evaluation map (a.k.a. Dirac measure). Then the difference operator $D = \sum_\eta a_\eta(z) T_\eta$ acts on δ_v by

$$\delta_v D := \sum_\eta a_\eta(v) \delta_{v-\eta}, \tag{4.1}$$

provided the coefficient functions a_η are defined in $z = v$. In that case, we have $(\delta_v D)(f) = \delta_v(D(f))$. We are interested in measures supported in points of the form $-\rho - \mu$, $\mu \in \Lambda_+$. Therefore, we define for every $d \in \mathbb{N}$ the space

$$\overline{M}_d := \bigoplus_{\substack{\mu \in \Lambda_+ \\ \ell(\mu) \leqslant d}} \mathbb{C}\delta_{-\rho-\mu} \quad \text{and} \quad \overline{M} := \bigcup_d \overline{M}_d. \tag{4.2}$$

Proposition 4.1. *Let ρ be strongly dominant and non-integral. Then $\overline{M}$ is D_h-stable for all $h \in \mathcal{P}^W$. Moreover, the map*

$$\varphi : \mathcal{P}^W \to \overline{M} : h \mapsto \delta_{-\rho} D_h \tag{4.3}$$

is an isomorphism of filtered $\mathbb{C}$-vector spaces.

Proof. The non-integrality of ρ makes sure that δD_h is defined for every $\delta \in \overline{M}$. Clearly, the space $\overline{M}$ is stable for the multiplication operator h. Thus it suffices to show that $\overline{M}$ is L-stable. Since $\delta_{-\rho-\mu} L = \sum_\eta f_\eta(-\rho-\mu)\delta_{-\rho-\mu-\eta}$, we have to show: for every $\mu \in \Lambda_+$ holds $\mu + \eta \in \Lambda_+$ or $f_\eta(-\rho-\mu) = 0$. But this is a special case of Lemma 3.3(b).

Lemma 3.6 implies that φ preserves filtrations. Since the filtration spaces on both sides are of the same finite dimension (Theorem 3.1(b)) it suffices to show that φ is injective. If $\varphi(h) = 0$ then $b^h_\tau(-\rho) = 0$ for all $\tau \in \Lambda_+$. Theorem 3.7, applied to $\mu = 0$, then implies $h = 0$. □

The following consequence is needed in the proof of Theorem 4.3. A much stronger result will proved later on (Corollary 4.6).

Corollary 4.2. *Let ρ be strongly dominant and non-integral.*[5] *Then $p_\lambda(-\rho) \neq 0$ for all $\lambda \in \Lambda_+$.*

Proof. Suppose $p_\lambda(-\rho) = 0$. Using the bijectivity of φ we get for every $\mu \in \Lambda_+$ a function $h \in \mathcal{P}^W$ with $\delta_{-\rho} D_h = \delta_{-\rho-\mu}$. Hence

$$p_\lambda(-\rho-\mu) = \delta_{-\rho-\mu} p_\lambda = \delta_{-\rho} D_h(p_\lambda) = h(\rho+\lambda) p_\lambda(-\rho) = 0. \tag{4.4}$$

Since $-\rho - \Lambda_+$ is Zariski dense in V we conclude $p_\lambda = 0$ which is not true. □

It is convenient to renormalize p_λ such that its value at $-\rho$ becomes 1. Therefore, put

$$q_\lambda(z) := \frac{p_\lambda(z)}{p_\lambda(-\rho)}. \tag{4.5}$$

Then we can formulate the *transposition formula*:

Theorem 4.3. *Let ρ be strongly dominant and non-integral.*[6] *Then*

$$q_\lambda(-z) = \sum_\mu (-1)^{\ell(\mu)} p_\mu(\rho+\lambda) q_\mu(z) \quad \textit{for all } \lambda \in \Lambda_+. \tag{4.6}$$

[5] See Corollary 4.7.
[6] See Corollary 4.7.

Proof. The polynomials $q_\mu(-z)$ form also a basis of $\mathcal{P}^W$. Thus, every $f \in \mathcal{P}^W$ has an expansion

$$f(z) = \sum_{\mu\in\Lambda_+} a_\mu(f) q_\mu(-z) \tag{4.7}$$

where a_μ is a linear function on $\mathcal{P}^W$. We claim $a_\mu \in \overline{M}_{\ell(\mu)}$. To see that we evaluate (4.7) in $z = -\rho - \mu$ and get

$$\begin{aligned}\delta_{-\rho-\mu}(f) &= \sum_\tau a_\tau(f) q_\tau(\rho+\mu)\\ &= p_\mu(-\rho)^{-1} a_\mu(f) + \sum_{\ell(\tau)<\ell(\mu)} a_\tau(f) q_\tau(\rho+\mu).\end{aligned} \tag{4.8}$$

The second equation holds by Theorem 3.1(a). Now the claim follows by induction.

The claim and Proposition 4.1 imply that for every $\mu \in \Lambda_+$ there is $h_\mu \in \mathcal{P}^W$ with $\deg h_\mu \leqslant \ell(\mu)$ and $a_\mu(f) = \delta_\rho D_{h_\mu} f = (D_{h_\mu} f)(-\rho)$. Applying this to $f = q_\lambda$ yields

$$a_\mu(q_\lambda) = (D_\mu q_\lambda)(-\rho) = h_\mu(\rho+\lambda). \tag{4.9}$$

On the other hand, $q_\lambda(z)$ and $(-1)^{\ell(\lambda)} q_\lambda(-z)$ have the same top homogeneous component. Thus we get directly from (4.7) that

$$a_\mu(q_\lambda) = (-1)^{\ell(\mu)} \delta_{\lambda\mu} \quad \text{for all } \lambda, \mu \in \Lambda_+ \text{ with } \ell(\lambda) \leqslant \ell(\mu). \tag{4.10}$$

Thus $(-1)^{\ell(\mu)} h_\mu$ matches the definition of p_μ which implies $a_\mu(q_\lambda) = (-1)^{\ell(\mu)} p_\mu(\rho+\lambda)$. Inserting this into (4.7) and replacing z by $-z$ gives formula (4.6). □

Remark. In the classical case, the transposition formula was first proved by Okounkov in [14] and Lassalle [12] (even in the Macdonald polynomial setting). There it was called a "binomial theorem" but we prefer to reserve this term to the limiting case discussed in Section 9. We followed Okounkov's approach to the transposition formula with some substantial modifications. In particular, we do not need to know the difference operators very explicitly. The semiclassical case was done in [8].

A first consequence of the transposition formula is the following symmetry result:

Corollary 4.4. *Let ρ be strongly dominant and non-integral.*[7] *Then*

$$q_\lambda(-\rho-\nu) = q_\nu(-\rho-\lambda) \quad \textit{for all } \lambda, \nu \in \Lambda_+. \tag{4.11}$$

Proof. Evaluate the transposition formula (4.6) in $z = \rho + \nu$. Then the right-hand side is symmetric in λ and ν. □

[7] See Corollary 4.7.

From this, we derive a Pieri formula for the q_μ:

Theorem 4.5. *Let $\rho \in V_0$ be strongly dominant and non-integral. Then*

$$h(-z)q_\mu(z) = \sum_{\tau \in \Lambda} b_\tau^h(-\rho-\mu)q_{\mu+\tau}(z) \quad \textit{for every } h \in \mathcal{P}^W. \tag{4.12}$$

Proof. Consider the eigenvalue equation for D_h:

$$\sum_\tau b_\tau^h(z)q_\nu(z-\tau) = D_h(q_\nu) = h(\rho+\nu)q_\nu(z). \tag{4.13}$$

Now substitute $z = -\rho-\mu$ and apply (4.11) to both sides:

$$\sum_\tau b_\tau^h(-\rho-\mu)q_{\mu+\tau}(-\rho-\nu) = h(\rho+\nu)q_\mu(-\rho-\nu) \tag{4.14}$$

(if $\mu+\tau \notin \Lambda_+$ then $b_\tau^h(-\rho-\nu) = 0$, see [9, Proposition 6.3]). This implies (4.12) since $-\rho-\Lambda_+$ is Zariski dense in V. □

By comparing formulas (3.16) and (4.12) we obtain the *evaluation formula*:

Corollary 4.6. *Let $\rho \in V_0$ be strongly dominant. Then for all $\mu \in \Lambda_+$ holds*

$$p_\mu(-\rho) = (-1)^{\ell(\mu)}d_\mu. \tag{4.15}$$

Proof. Assume first that ρ is non-integral. We apply the Pieri formula (3.16) to $h(z) = p_\lambda(-z)$ and $\mu = 0$. Since $p_0 = 1$ we get

$$p_\lambda(-z)\cdot 1 = \frac{(-1)^{\ell(\lambda)}}{d_\lambda}b_\lambda^h(-\rho)p_\lambda(z) + \text{lower order terms}. \tag{4.16}$$

Doing the same thing with (4.12) gives

$$p_\lambda(-z)\cdot 1 = b_\lambda^h(-\rho)q_\lambda(z) + \text{lower order terms}. \tag{4.17}$$

Comparing these two formulas proves the evaluation formula. It follows from (3.14) that both sides of (4.15) are defined when ρ is just strongly dominant. Thus, we can drop the non-integrality assumption by a continuity argument. □

The last argument of the preceding proof gives:

Corollary 4.7. *In Corollary* 4.2*, Theorem* 4.3*, and Corollary* 4.4 *it suffices to assume that ρ is strongly dominant.*

Remark. This refinement is important since ρ-vectors coming from multiplicity free spaces are almost never non-integral.

A first consequence of the evaluation formula is the justification of the term "virtual dimension" for d_λ.

Theorem 4.8. *Let U be a multiplicity free space with ring of functions $\mathcal{P}(U)=\bigoplus_{\lambda\in\Lambda_+}\mathcal{P}^\lambda$ and associated ρ-vector as in Section* 2. *Then* $\dim\mathcal{P}^\lambda=d_\lambda$ *and* $\dim U=2\ell(\rho)$.

Proof. Let $\mathcal{D}(U)=\bigoplus_{\lambda\in\Lambda_+}\mathcal{D}_\lambda$ be the decomposition of the space of constant coefficient differential operators where $\mathcal{D}_\lambda$ is simple with highest weight λ. Fix $\lambda\in\Lambda_+$. If $D\in\mathcal{D}_\lambda$ and $f\in\mathcal{P}^\lambda$ then $D(f)$ is a polynomial of degree zero, hence a constant. This way we get a non-degenerate pairing $\mathcal{D}_\lambda\times\mathcal{P}^\lambda\to\mathbb{C}$. For any basis f_i of $\mathcal{P}^\lambda$ let $D_i\in\mathcal{D}_\lambda$ be the dual basis, i.e., $D_i(f_j)=\delta_{ij}$. Then $D:=\sum_i f_iD_i$ is G-invariant and acts as identity on $\mathcal{P}^\lambda$. By definition, the associated polynomial p_D is p_λ (see [7], or [9, Section 7]). We have ${}^tD=(-1)^{\ell(\lambda)}\sum_i D_if_i$, hence ${}^tD(1)=(-1)^{\ell(\lambda)}\sum_i D_i(f_i)=(-1)^{\ell(\lambda)}\dim\mathcal{P}^\lambda$. On the other hand, $p_{{}^tD}(v)=p_D(-v)=p_\lambda(-v)$ by Theorem 2.2. Thus ${}^tD(1)=p_{{}^tD}(\rho)=p_\lambda(-\rho)=(-1)^{\ell(\lambda)}d_\lambda$ which shows $\dim\mathcal{P}^\lambda=d_\lambda$.

The second formula is proved similarly. Here we choose a basis x_i of $U^\vee\subseteq\mathcal{P}(U)$. Let $\partial_i\in U\subseteq\mathcal{D}$ be its dual basis. Because $D=\sum_i x_i\partial_i$ is the Euler vector field we have $p_D(z)=\ell(z-\rho)$. As above we get $-\dim U={}^tD(1)=p_D(-\rho)=-2\ell(\rho)$. □

Remark. In the context of Hermitian symmetric spaces the dimension formula was proved by Upmeier [19].

5. The interpolation formula

In this section we state a formula which allows to expand an arbitrary W-invariant polynomial in terms of the basis p_μ. For this we need another immediate consequence of the transposition formula (4.6):

Theorem 5.1. *Let $\rho\in V_0$ be dominant. Then the matrix*

$$\big((-1)^{\ell(\mu)}p_\mu(\rho+\lambda)\big)_{\mu,\lambda\in\Lambda_+} \tag{5.1}$$

is an involutory.

Proof. By (4.6), the matrix expresses the involution $h(z)\mapsto h(-z)$ of $\mathcal{P}^W$ in the q_μ-basis. □

Let $\mathcal{C}(\rho+\Lambda_+)$ be the set of $\mathbb{C}$-valued functions on $\rho+\Lambda_+$. For $h\in\mathcal{C}(\rho+\Lambda_+)$ we define its *transform* $\widehat{h}\in\mathcal{C}(\rho+\Lambda_+)$ by

$$\widehat{h}(\rho+\mu):=\sum_{\tau\in\Lambda_+}(-1)^{\ell(\tau)}p_\tau(\rho+\mu)h(\rho+\tau). \tag{5.2}$$

The sum is finite since all summands with $\ell(\tau) > \ell(\mu)$ are zero. We consider two subspaces of $\mathcal{C}(\rho + \Lambda_+)$. First, let $\mathcal{C}_0(\rho + \Lambda_+)$ be the set of functions with finite support. Secondly, we consider, via restriction, $\mathcal{P}^W$ as subspace of $\mathcal{C}(\rho + \Lambda_+)$.

Theorem 5.2. *Let $\rho \in V_0$ be dominant. Then transformation $h \mapsto \widehat{h}$ has the following properties*:

(i) $\widehat{\widehat{h}} = h$.
(ii) $h \in \mathcal{P}^W \Leftrightarrow \widehat{h} \in \mathcal{C}_0(\rho + \Lambda_+)$.
(iii) *Interpolation formula*:

$$h(z) = \sum_{\mu \in \Lambda_+} (-1)^{\ell(\mu)} \widehat{h}(\rho + \mu) p_\mu(z) \quad \textit{for all } h \in \mathcal{P}^W. \tag{5.3}$$

Proof. Let $a_{\mu\lambda} := (-1)^{\ell(\mu)} p_\mu(\rho + \lambda)$. Then $\widehat{h}(\rho + \mu) = \sum_\tau a_{\tau\mu} h(\rho + \tau)$ and therefore

$$\begin{aligned} \widehat{\widehat{h}}(\rho + \lambda) &= \sum_\mu a_{\mu\lambda} \widehat{h}(\rho + \mu) = \sum_{\mu,\tau} a_{\mu\lambda} a_{\tau\mu} h(\rho + \tau) \\ &= \sum_\tau \left[\sum_\mu a_{\tau\mu} a_{\mu\lambda} \right] h(\rho + \tau). \end{aligned} \tag{5.4}$$

By Theorem 5.1, the sum in brackets equals $\delta_{\tau\lambda}$ which implies (i).

Let $\chi_{\rho+\nu} \in \mathcal{C}_0(\rho + \Lambda_+)$ be the characteristic function of $\{\rho + \nu\}$. Then

$$\widehat{\chi}_{\rho+\nu} = (-1)^{\ell(\nu)} p_\nu. \tag{5.5}$$

Hence, $h \mapsto \widehat{h}$ maps a basis of $\mathcal{C}_0(\rho + \Lambda_+)$ to a basis of $\mathcal{P}^W$ which proves (ii).

Finally, (i) implies

$$h(\rho + \lambda) = \sum_{\mu \in \Lambda_+} (-1)^{\ell(\mu)} p_\mu(\rho + \lambda) \widehat{h}(\rho + \mu). \tag{5.6}$$

By (ii), $\widehat{h}$ is a function with finite support. Therefore, the sum (5.6) is over a finite set of μ's which is independent of λ. This implies (5.3) since $\rho + \Lambda_+$ is Zariski dense in V. □

The operator L acts naturally on $\mathcal{C}(\rho + \Gamma)$,

$$(Lh)(\rho + \lambda) = \sum_\eta f_\eta(\rho + \lambda) h(\rho + \lambda - \eta), \tag{5.7}$$

provided the coefficients $f_\eta(\rho + \lambda)$ are defined, i.e., ρ is non-integral. Then it follows from the first cut-off property of f_η, Lemma 3.3(a), that the quotient $\mathcal{C}(\rho + \Lambda_+)$ is L-stable. Let $\mathcal{A}$ be the algebra generated by $\mathcal{P}^W$ and L in $\operatorname{End}\mathcal{C}(\rho + \Lambda_+)$. It follows that $\mathcal{C}(\rho+\Lambda_+)$ is an $\mathcal{A}$-module. Moreover, $\mathcal{C}_0(\rho+\Lambda_+)$ and $\mathcal{P}^W$ are $\mathcal{A}$-submodules. For every $X \in \mathrm{End}_{\mathbb{C}}\,\mathcal{C}(\rho+\lambda_+)$ we define $\widehat{X}$ by $\widehat{X}(h) := \widehat{X(\widehat{h})}$.

Theorem 5.3. *Assume ρ is non-integral. Then $X \mapsto \widehat{X}$ induces an involutory automorphism of $\mathcal{A}$. More precisely, we have $\widehat{m}_h = D_h$ and $\widehat{L} = -L$. Here, m_h is the operator*[8] *"multiplication by h."*

Proof. The equality $\widehat{m}_h = D_h$ is equivalent to

$$\widehat{h\widehat{p}}(\rho+\lambda) = D_h(p)(\rho+\lambda) \quad \text{for all } p \in \mathcal{C}(\rho+\Lambda), \lambda \in \Lambda_+. \tag{5.8}$$

Now we fix λ. Then both sides of (5.8) depend only on the values of p in finitely many points, more precisely, in points $\rho+\mu$ with $\ell(\mu) \leqslant \ell(\lambda)$. Since there is a W-invariant polynomial which has the same values at these points we may assume $p \in \mathcal{P}^W$. By linearity, we may assume $p = p_\nu$. Then, by (5.5),

$$h\widehat{p}_\nu = (-1)^{\ell(\nu)} h\chi_{\rho+\nu} = (-1)^{\ell(\nu)} h(\rho+\nu)\chi_{\rho+\nu} = h(\rho+\nu)\widehat{p}_\nu \tag{5.9}$$

and therefore

$$\widehat{m}_h(p_\nu) = h(\rho+\nu)\widehat{\widehat{p}}_\nu = h(\rho+\nu)p_\nu = D_h(p_\nu). \tag{5.10}$$

This proves $\widehat{m}_h = D_h$. But then $\widehat{L} = (\ell - D_\ell)^\wedge = D_\ell - \ell = -L$. This shows in particular that $X \mapsto \widehat{X}$ maps $\mathcal{A}$ into itself. $\square$

Remark. The non-integrality of ρ is needed to make sense of the action of $\mathcal{A}$ on $\mathcal{C}(\rho+\Lambda_+)$. As already mentioned, the element ρ attached to a multiplicity free representation is never non-integral. It will be a consequence of Proposition 7.4 that $X \mapsto \widehat{X}$ is, in fact, defined for every $\rho \in V_0$.

6. The scalar product

Assume ρ is strongly dominant. The symmetry property (4.11) indicates the presence of a scalar product on $\mathcal{P}^W$. In fact, we define a non-degenerate scalar product on $\mathcal{P}^W$ by

$$\langle p_\lambda, p_\mu\rangle = d_\lambda \delta_{\lambda\mu} \quad \text{for all } \lambda, \mu \in \Lambda_+. \tag{6.1}$$

Thus $\langle p_\lambda, q_\mu\rangle = (-1)^{\ell(\lambda)}\delta_{\lambda\mu}$ and $\langle q_\lambda, q_\mu\rangle = d_\lambda^{-1}\delta_{\lambda\mu}$. For any function $h(z)$ let $h^-(z) := h(-z)$. Then we have

[8] Since $\widehat{m}_h \neq m_{\widehat{h}}$ we are forced to use this notation.

Theorem 6.1. *For all $\lambda \in \Lambda_+$ and $h \in \mathcal{P}^W$ holds*

$$\langle q_\lambda, h\rangle = \widehat{h}(\rho+\lambda) \quad \textit{and} \quad \langle q_\lambda^-, h\rangle = h(\rho+\lambda). \tag{6.2}$$

Proof. From the interpolation formula (5.3) we obtain

$$\langle q_\lambda, h\rangle = \sum_\mu (-1)^{\ell(\mu)} \widehat{h}(\rho+\mu)\langle q_\lambda, p_\mu\rangle = \widehat{h}(\rho+\lambda). \tag{6.3}$$

Moreover, from (4.6), (6.2), and (5.3) we get

$$\begin{aligned}\langle q_\lambda^-, h\rangle &= \sum_\mu (-1)^{\ell(\mu)} p_\mu(\rho+\lambda)\langle q_\mu, h\rangle \\ &= \sum_\mu (-1)^{\ell(\mu)} \widehat{h}(\rho+\mu) p_\mu(\rho+\lambda) = h(\rho+\lambda).\end{aligned} \tag{6.4}$$

Remark. In particular, we have $\langle q_\lambda^-, q_\mu^-\rangle = q_\mu(-\rho-\lambda)$ which explains the symmetry in λ and μ.

There is also a general expression for the scalar product:

Theorem 6.2. *For all $g, h \in \mathcal{C}^W$:*

$$\langle g, h\rangle = \sum_{\mu\in\Lambda_+} d_\mu\, \widehat{g}(\rho+\mu)\widehat{h}(\rho+\mu). \tag{6.5}$$

Proof. Just apply the interpolation formula (5.3) to g and h. $\square$

The algebra $\mathcal{A}$ is not quite closed under taking adjoints for the scalar product. Therefore, these will be studied in the next section. Here, we use a slightly modified scalar product:

$$\langle g, h\rangle^- := \langle g^-, h^-\rangle. \tag{6.6}$$

The adjoint of an operator X with respect to the scalar product (6.6) will be denoted by X'.

Theorem 6.3. *Let $\rho \in V_0$ be strongly dominant. Then for every $X \in \mathcal{A}$ the adjoint X' exists and is again in $\mathcal{A}$. More precisely, $h' = D_{h^-}$, and $L' = L$. In particular, $X \mapsto X'$ induces an involutory antiautomorphism of $\mathcal{A}$. Moreover, $(\widehat{X})' = (X')^\wedge$ for all $X \in \mathcal{A}$.*

Proof. By (6.2) we have $\langle q_\lambda, h\rangle^- = h(-\rho-\lambda)$ for all $h \in \mathcal{P}^W$. Then I claim

$$\big\langle D_{h^-}(f), g\big\rangle^- = \langle f, hg\rangle^- \tag{6.7}$$

for all $f, g, h \in \mathcal{P}^W$. Indeed, it suffices to prove this for $f = q_\lambda$. Then

$$\left\langle D_{h^-}(q_\lambda), g\right\rangle^- = h(-\rho-\lambda)\langle q_\lambda, g\rangle^- = h(-\rho-\lambda)g(-\rho-\lambda) = \langle q_\lambda, hg\rangle^-. \tag{6.8}$$

Thus the adjoint operator of h is D_h. Then we also have

$$L' = (\ell - D_\ell)' = (-\ell^- - D_\ell)' = -D_\ell - \ell^- = L.$$

Finally, $(\widehat{L})' = -L = (L')^\wedge$ and $(\widehat{m}_h)' = D_h' = m_{h^-} = \widehat{D}_{h^-} = (m_h')^\wedge$ which shows the last claim. □

7. The PGL_2-action

For any operator $X \in \operatorname{End}_{\mathbb{C}}(\mathcal{P}^W)$ define the operator X^- by $X^-(g) = X(g^-)^-$. In particular, if $X = \sum_\tau a_\tau(z)T_\tau$ is a difference operator then $X^- = \sum_\tau a_\tau(-z)T_{-\tau}$ is again a difference operator. For multiplication operators we have $m_h^- = m_{h^-}$. On the other side, L^- is new. Therefore, let $\mathcal{B}$ be the algebra generated by $\mathcal{P}^W$, L, and L^-. It contains $\mathcal{A}$ as a subalgebra. Moreover, $X \mapsto X^-$ induces an involutive automorphism of $\mathcal{B}$. Observe that $\mathcal{A}$ contains only operators composed of shifts by $\tau \in \Lambda$ while in $\mathcal{B}$ arbitrary shifts $\tau \in \Gamma^\vee$ are possible.

For any $X \in \operatorname{End}_{\mathbb{C}}(\mathcal{P}^W)$ let X^* be the adjoint operator (if it exists) with respect to the scalar product $\langle\cdot,\cdot\rangle$ defined in (6.1). Its relation to the adjoint X' is $X^* = X^{-\prime -}$. Indeed

$$\langle Xf, g\rangle = \langle X^- f^-, g^-\rangle^- = \left\langle f^-, X^{-\prime} g^-\right\rangle^- = \left\langle f, X^{-\prime -} g\right\rangle. \tag{7.1}$$

Theorem 7.1. *Let $\rho \in V_0$ be strongly dominant. Then for every $X \in \mathcal{B}$ the adjoint operator X^* exists and is again in $\mathcal{B}$. More precisely, the following formulas hold* (*with $h \in \mathcal{P}^W$*):

$$\begin{aligned} h^* &= D_h^- = \exp(\operatorname{ad} L^-)(h^-), \\ L^* &= E^- - E = L - 2\ell - L^-, \\ (L^-)^* &= L^-, \\ D_h^* &= D_h. \end{aligned} \tag{7.2}$$

In particular, $X \mapsto X^*$ induces an involutive antiautomorphism of $\mathcal{B}$.

Proof. Since D_h has an orthogonal eigenbasis, p_λ, it is self-adjoint: $D_h^* = D_h$. By Theorem 6.3 we have $h^* = ((h^-)')^- = D_h^-$. Moreover,

$$\begin{aligned} L^* &= (\ell - D_\ell)^* = D_\ell^- - D_\ell = E^- - E = (\ell - L)^- - (\ell - L) \\ &= L - 2\ell - L^-. \end{aligned} \tag{7.3}$$

Finally, $(L^-)^* = L'^- = L^-$. □

Remark. Of course, $\mathcal{B}$ is still preserved under the other adjoint $X \mapsto X'$ with $(L^-)' = (L^-)^{-*-} = L^{*-} = E - E^- = -L^-$.

Recall that three elements (e, h, f) of a (Lie) algebra are called an sl_2-triple if the relations $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$ hold.

Theorem 7.2. *Both* $(L, 2\ell, L^-)$ *and* $(-L, 2E, L^*)$ *are* sl_2*-triples.*

Proof. For every $\eta \in \Gamma^\vee$ holds $[\ell, T_\eta] = \ell(\eta)T_\eta$. Hence, by definition of L, we have $[2\ell, L] = 2L$. We also get $[2E, L] = [2\ell - 2L, L] = 2L$. The equation $[2\ell, L^-] = -2L^-$ follows by applying $X \mapsto X^-$ to both sides of $[2\ell, L] = 2L$. Moreover, if we apply $X \mapsto X^*$ to $[2E, L] = 2L$ we get, according to (7.2), $[2E, L^*] = -[2E, L]^* = -2L^*$. Moreover,

$$\begin{aligned}[L, L^-] &= [L, L - 2\ell - L^*] = 2L - [\ell - E, L^*] = 2L - [\ell, L - 2\ell - L^-] - L^* \\ &= 2L - L - L^- - (L - 2\ell - L^-) = 2\ell. \end{aligned} \tag{7.4}$$

Finally, $[-L, L^*] = [-L, L - 2\ell - L^-] = -2L + 2\ell = 2E$. $\square$

Of course, the two triples span the same three dimensional subspace $\mathfrak{s}$ inside $\mathcal{B}$ which we identify with the Lie algebra $sl_2(\mathbb{C})$ by using the second triple:

$$-L \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 2E \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L^* \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{7.5}$$

Then we also have

$$2\ell \mapsto \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}, \quad L^- \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{7.6}$$

Now we would like to integrate the inner $\mathfrak{s}$-action on $\mathcal{B}$. For this, let $S := \operatorname{Aut}\mathfrak{s}$. Its Lie algebra is $\mathfrak{s}$. Moreover, if we identify $\mathfrak{s}$ with $sl_2(\mathbb{C})$ as above then S gets identified with $PGL_2(\mathbb{C})$. Its elements are invertible 2×2-matrices modulo scalar multiplication which we write in square brackets. Of particular interest is the involution

$$\sigma := \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \in S \tag{7.7}$$

which maps the two sl_2-triples into each other:

$$(L, 2\ell, L^-) = \sigma(-L, 2E, L^*). \tag{7.8}$$

Theorem 7.3. *The adjoint action of* $\mathfrak{s}$ *on* $\mathcal{B}$ *can be integrated to an algebraic* S*-action.*

Proof. First, we show that $\operatorname{ad}\mathfrak{s}$ acts locally finitely on $\mathcal{B}$. By Poincaré–Birkhoff–Witt it suffices to show that for L, 2ℓ, and L^-, separately.

We claim that the elements L and L^- act locally nilpotently. It suffices to show this on the generators $h \in \mathcal{P}^W$, L, and L^-. For L, the assertion follows from Theorem 3.4 (for h) and Theorem 7.2 (for L^-). For L^- we apply the automorphism $X \mapsto X^-$.

The action of $\operatorname{ad} 2\ell$ on difference operators is clearly diagonalizable. This shows already that $\operatorname{ad}\mathfrak{s}$ integrates to an $SL_2(\mathbb{C})$-action. The possible eigenvalues of $\operatorname{ad} 2\ell$ are $2\ell(\tau)$, $\tau \in \Gamma^\vee$. Since these are all even, the action of $SL_2(\mathbb{C})$ descends to an action of $PGL_2(\mathbb{C}) = S$. □

Now we compute the effect of some particular elements of S on $\mathcal{B}$.

Proposition 7.4. *The effect of σ on the generators of $\mathcal{B}$ are*

$$\sigma(L) = -L, \quad \sigma(h) = D_h, \quad \sigma(L^-) = L^*. \tag{7.9}$$

Proof. We already know $(L, 2\ell, L^-) = \sigma(-L, 2E, L^*)$. Thus it remains to calculate $\sigma(h)$. To this end, write $\sigma = \alpha\beta$ where

$$\alpha = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} = \sigma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sigma^{-1}. \tag{7.10}$$

The matrix β lies in the Cartan subgroup whose Lie algebra is $\mathbb{C}\ell$. Therefore, it fixes every element of $\mathcal{B}$ which commutes with ℓ. This implies $\beta(h) = h$. The matrix α acts by $\exp(\operatorname{ad} L)$ on $\mathcal{B}$. Hence it sends, by definition, h to D_h. We conclude $\sigma(h) = D_h$. □

Next we investigate the effect of $\mathfrak{s}$ on the $\mathcal{B}$-module $\mathcal{P}^W$.

Theorem 7.5. *Let $\rho \in V_0$ be strongly dominant. Then for all $\lambda \in \Lambda_+$ and $d \in \mathbb{N}$ holds*

$$\frac{1}{d!} L^d(p_\lambda) = \sum_{\substack{\mu \in \Lambda_+ \\ \ell(\mu) = \ell(\lambda) + d}} p_\lambda(\rho + \mu) p_\mu, \tag{7.11}$$

$$\frac{1}{d!} (-L^*)^d(q_\lambda) = \sum_{\substack{\mu \in \Lambda_+ \\ \ell(\mu) = \ell(\lambda) - d}} p_\mu(\rho + \lambda) q_\mu. \tag{7.12}$$

Proof. By Theorem 7.2 we have $[E, L] = L$, hence $[E, L^d] = dL^d$. For every $\lambda \in \Lambda_+$ follows that $L^d(p_\lambda)$ is a linear combination of those p_μ with $\ell(\mu) = \ell(\lambda) + d$. On the other hand, we have $L^d(p_\lambda) = (\ell - E)^d(p_\lambda) = \ell^d p_\lambda$ plus lower order terms. Then (7.11) follows from (3.17).

Using the fact that the dual basis of the p_λ are the $(-1)^{\ell(\lambda)} q_\lambda$ we get from (7.11)

$$\frac{1}{d!} (L^*)^d((-1)^{\ell(\lambda)} q_\lambda) = \sum_{\substack{\mu \in \Lambda_+ \\ \ell(\mu) = \ell(\lambda) - d}} p_\mu(\rho + \lambda)(-1)^{\ell(\mu)} q_\mu \tag{7.13}$$

which is equivalent to (7.12). □

Formulas (7.11) and (7.12) can be expressed more conveniently as generating series:

$$\exp(tL)p_\lambda = \sum_{\mu\in\Lambda_+} t^{\ell(\mu)-\ell(\lambda)} p_\lambda(\rho+\mu)p_\mu \tag{7.14}$$

and

$$\exp(-tL^*)q_\lambda = \sum_{\mu\in\Lambda_+} t^{\ell(\lambda)-\ell(\mu)} p_\mu(\rho+\lambda)q_\mu. \tag{7.15}$$

There is a big difference between this two formulas in that the latter, (7.15), is a finite sum. This means that (7.15) defines an algebraic action of

$$\begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \tag{7.16}$$

on $\mathcal{P}^W$. There is also an action of the diagonal matrices on $\mathcal{P}$, defined by

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : q_\lambda \mapsto \left(\frac{a}{b}\right)^{\ell(\lambda)} q_\lambda. \tag{7.17}$$

Then (7.15) and (7.17) combine to an action of B, the subgroup of lower triangular matrices of $S = PGL_2(\mathbb{C})$. This action is compatible with that on $\mathcal{B}$:

$${}^b(Xh) = {}^bX\left({}^bh\right) \quad \text{for all } b\in B,\ X\in\mathcal{B},\ h\in\mathcal{P}^W. \tag{7.18}$$

Remark. The action of B on $\mathcal{P}^W$ is not quite the one which one would obtain by exponentiating the action of Lie $B \subset \mathfrak{s} \subset \mathcal{B}$ on $\mathcal{P}^B$. The reason is that q_λ is an eigenvector of E with eigenvalue $\ell(\lambda)+\ell(\rho)$ and not just $\ell(\lambda)$. Therefore, unless $\ell(\rho)$ is an integer, the exponentiated B-action is not algebraic. In the geometric case, i.e., when ρ comes from a multiplicity free action on a vector space U, we have that $\ell(\rho)=\frac{1}{2}\dim U$ (Theorem 4.8) is in $\frac{1}{2}\mathbb{Z}$. In that case, one can integrate the Lie B-action to an algebraic action of the lower triangular matrices in $SL_2(\mathbb{C})$.

Now we can locate the automorphism $X\mapsto X^-$ in S:

Theorem 7.6. *The matrix*

$$\gamma := \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \in B \tag{7.19}$$

acts as $h\mapsto h^-$ on $\mathcal{P}^W$ and as $X\mapsto X^-$ on $\mathcal{B}$.

Proof. We write $\gamma = \alpha\beta$ with

$$\alpha = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{7.20}$$

Then $\beta(q_\lambda) = (-1)^\ell q_\lambda$ (by (7.17)) and $\alpha(q_\lambda) = \sum_\mu (-1)^{\ell(\lambda)-\ell(\mu)} p_\mu(\rho+\lambda) q_\mu$ (by (7.15)). The transposition formula (4.6) implies $\gamma(q_\lambda) = q_\lambda^-$. We conclude $\gamma(h) = h^-$ by linearity. Finally, $\gamma(X)(h) = \gamma(X(\gamma(h))) = X(H^-)^- = X^-(h)$. $\square$

Remark. One consequence of Theorem 7.6 is the formula

$$\exp(L^*)(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda^-. \tag{7.21}$$

It has the advantage that it works for ρ which are just dominant.

Now we come back to the automorphism $X \to \widehat{X}$ of Section 5. Comparing Theorem 5.3 with Proposition 7.4 we see that σ induces on $\mathcal{A}$ exactly $X \to \widehat{X}$. Now we extend this to $\mathcal{B}$:

Theorem 7.7. *Let ρ be non-integral. Then $\mathcal{C}(\rho + \Lambda_+)$ is naturally a $\mathcal{B}$-module. Moreover the relation $(L^-)^\wedge = L^*$ holds. In particular, we have $\widehat{X} = \sigma(X)$ for all $X \in \mathcal{B}$.*

Proof. By definition, we have

$$L^-(h)(\rho + \mu) = \sum_{\eta \in \Lambda_1} f_\eta(-\rho - \mu) h(\rho + \mu + \eta). \tag{7.22}$$

Thus, it follows from Lemma 3.3(b) that L^- and therefore $\mathcal{B}$ acts on $\mathcal{C}(\rho + \Lambda_+)$.

For every fixed λ the values $(L^-)^\wedge(h)(\rho+\lambda)$ and $L^*(h)(\rho+\lambda)$ depend on only finitely many values of h which we may interpolate by a linear combination of p_λ's. This implies, that it suffices to prove $(L^-)^\wedge(h) = L^*(h)$ for $h = p_\lambda$. We have

$$\begin{aligned}(L^-)^\wedge(p_\lambda) &= (-1)^{\ell(\lambda)}(L^-)^\wedge(\widehat{\chi}_{\rho+\lambda}) = (-1)^{\ell(\lambda)}(L^- \chi_{\rho+\lambda})^\wedge \\ &= (-1)^{\ell(\lambda)} \sum_\eta f_\eta(-\rho - \lambda + \eta)\widehat{\chi}_{\rho+\lambda-\eta} \\ &= -\sum_\eta f_\eta(-\rho - \lambda + \eta) p_{\lambda-\eta}. \end{aligned} \tag{7.23}$$

Since $D_\ell = \ell - L$ we have $f_\eta(z) = -b_\eta^\ell(z)$. Therefore, if we compare (3.16) (with $h = \ell$) and (3.17) (with $k = 1$) we get

$$-f_\eta(-\rho - \lambda + \eta) = \frac{d_\lambda}{d_{\lambda-\eta}} p_{\lambda-\eta}(\rho + \lambda). \tag{7.24}$$

Thus, using (7.12) (with $d = 1$) we get

$$\begin{aligned}(L^-)^\wedge(p_\lambda) &= (-1)^{\ell(\lambda)-1} d_\lambda \sum_\eta p_{\lambda-\eta}(\rho+\lambda) q_{\lambda-\eta} \\ &= -(-1)^{\ell(\lambda)} d_\lambda(-L^*)(q_\lambda) = L^*(p_\lambda).\end{aligned} \tag{7.25}$$

8. The differential limit

In this section we consider the effect of our difference operators on the highest degree component of a polynomial. Let $\mathcal{P}_{\leqslant d} := \{h \in \mathcal{P} \mid \deg h \leqslant d\}$, $\overline{\mathcal{P}}_d := \mathcal{P}_{\leqslant d}/\mathcal{P}_{\leqslant d-1}$ and $\overline{\mathcal{P}} := \bigoplus_d \overline{\mathcal{P}}_d$, the associated graded algebra. Observe that $\overline{\mathcal{P}} \cong \mathcal{P}$ (even equivariantly) since $\mathcal{P}$ is a polynomial ring.

Now we introduce the degree of an operator $X \in \mathcal{B}$ as

$$\deg X := \max\bigl\{\deg X(h) - \deg h \mid h \in \mathcal{P}^W\bigr\}. \tag{8.1}$$

It is clear that the degree of any difference operator is finite. Let $\mathcal{B}_{\leqslant d} := \{X \in \mathcal{B} \mid \deg X \leqslant d\}$. This defines a filtration of $\mathcal{B}$, i.e., $\mathcal{B}_d$ is a subspace of $\mathcal{B}$ with $\mathcal{B} = \bigcup_d \mathcal{B}_d$ and $\mathcal{B}_d\mathcal{B}_e \subseteq \mathcal{B}_{d+e}$. Let $\overline{\mathcal{B}}_d := \mathcal{B}_{\leqslant d}/\mathcal{B}_{\leqslant d-1}$ and $\overline{\mathcal{B}} := \bigoplus_d \overline{\mathcal{B}}_d$, the associated graded algebra. The point is now that more or less by construction, $\overline{\mathcal{P}}^W$ is a faithful $\overline{\mathcal{B}}$-module. We call it the *differential limit* since:

Proposition 8.1. *Every $\overline{X} \in \overline{\mathcal{B}}$ acts as a differential operator on $\overline{\mathcal{P}}^W$.*

Proof. We may assume that $\overline{X} \in \overline{\mathcal{B}}_d$ is non-zero and that it is represented by a difference operator $X \in \mathcal{B}_{\leqslant d}$. Choose linear coordinates $z_1, \ldots, z_n \in V^\vee$. By Taylor's theorem, the translation operator T_η can be written as differential operator of infinite order:

$$T_\eta = \exp\left(-\sum_i z_i(\eta)\frac{\partial}{\partial z_i}\right). \tag{8.2}$$

Therefore, we can also expand X into an infinite order differential operator with coefficients of bounded degree.

Now let $h \in \mathcal{P}^W$ be a polynomial of degree e with highest degree component $\bar{h}$. For an indeterminate t let $h_t(z) := h(t^{-1}z)$. Then

$$h_t = \bar{h}t^{-e} + \cdots \tag{8.3}$$

where "$\cdots$" means "terms of higher order in t".

Correspondingly, we define X_t by $X_t(h) := X(h_{t^{-1}})_t$. This amounts to replacing all variables z_i by $t^{-1}z_i$ and all partial derivatives $\partial/\partial z_i$ by $t\partial/\partial z_i$. In particular, we have $X_t(h_t) = X(h)_t$. Now we develop X_t into a Laurent series in t. This is possible since the coefficients of X have bounded degree. Thus there is $N \in \mathbb{Z}$ with

$$X_t = \widetilde{X}t^{-N} + \cdots \tag{8.4}$$

where $\widetilde{X}$ is a non-zero differential operator. Hence

$$X(h)_t = X_t(h_t) = \widetilde{X}(\bar{h})t^{-e-N} + \cdots. \tag{8.5}$$

This shows that $\deg X(h) \leqslant \deg h + N$ with equality for most h. Therefore, $N = d$ and $\overline{X}(\bar{h}) = \widetilde{X}(\bar{h})$. Thus $\overline{X} = \widetilde{X}$ is a differential operator. $\square$

For the reminder of this section we assume that ρ is dominant. We show that the pair $(\mathcal{B}, \mathcal{P}^W)$ is isomorphic to $(\overline{\mathcal{B}}, \overline{\mathcal{P}}^W)$. For this we use the action of the difference Euler operator E. Its action on $\mathcal{P}^W$ is diagonalizable with eigenvalues of the form $d + \ell(\lambda)$, $d \in \mathbb{N}$. Therefore, let $\mathcal{P}_d^W := \{h \in \mathcal{P}^W \mid E(h) = (d + \ell(\rho))h\}$. Then $\mathcal{P}^W = \bigoplus_d \mathcal{P}_d^W$. A basis of $\mathcal{P}_d^W$ is formed by all p_λ with $\ell(\lambda) = d$. Thus Theorem 3.1 implies that

$$\mathcal{P}_{\leqslant d}^W = \bigoplus_{i \leqslant d} \mathcal{P}_i^W. \tag{8.6}$$

In particular, the projection $\mathcal{P}_d^W \to \overline{\mathcal{P}}_d^W$ is an isomorphism. This way, we get an isomorphism (of vector spaces)

$$\psi : \mathcal{P}^W = \bigoplus_d \mathcal{P}_d^W \xrightarrow{\sim} \bigoplus_d \overline{\mathcal{P}}_d^W = \overline{\mathcal{P}}^W. \tag{8.7}$$

Now we do the same thing with $\mathcal{B}$. We know from the last section that the action of $\operatorname{ad} E$ on $\mathcal{B}$ is diagonalizable with integral eigenvalues. Therefore, let $\mathcal{B}_d := \{X \in \mathcal{B} \mid [E, X] = dX\}$. Then $\mathcal{B} = \bigoplus_d \mathcal{B}_d$ is a grading of $\mathcal{B}$.

Lemma 8.2. *Let ρ be dominant. Then $\mathcal{B}_{\leqslant d} = \bigoplus_{i \leqslant d} \mathcal{B}_i$.*

Proof. Let $X \in \mathcal{B}_d$ and $h \in \mathcal{P}_e$. Then

$$EX(h) = [E, X](h) + XE(h) = dX(h) + eX(h) = (d + e)X(h) \tag{8.8}$$

implies $\mathcal{B}_d \mathcal{P}_e \subseteq \mathcal{P}_{d+e}$. In particular, we have $\mathcal{B}_d \mathcal{P}_{\leqslant e} \subseteq \mathcal{P}_{\leqslant d+e}$ which shows $\mathcal{B}_{\leqslant d} \supseteq \bigoplus_{i \leqslant d} \mathcal{B}_i$.

Conversely, let $X \in \mathcal{B}_{\leqslant d}$ and $X = \sum X_n$ with $X_n \in \mathcal{B}_n$ and $N = \max\{n \mid X_n \neq 0\}$. Choose $h \in \mathcal{P}_e$ with $X_N(p) \neq 0$. Since $X_i(p) \subseteq \mathcal{P}_{i+e}$ is either zero or has precisely the degree $i + e$ we conclude $\deg X(p) = N + e$. From the assumption $\deg X(p) \leqslant d + e$ follows $N \leqslant d$. This proves $X \in \bigoplus_{i \leqslant d} \mathcal{B}_i$. $\square$

An immediate consequence of the lemma is $\mathcal{B}_d \xrightarrow{\sim} \overline{\mathcal{B}}_d$ which gives rise to a map

$$\Psi : \mathcal{B} = \bigoplus_d \mathcal{B}_d \xrightarrow{\sim} \bigoplus_d \overline{\mathcal{B}}_d = \overline{\mathcal{B}}. \tag{8.9}$$

Theorem 8.3. *Let ρ be dominant. Then the map Ψ in* (8.9) *is an isomorphism of algebras. Moreover, under this isomorphism the $\mathcal{B}$-module $\mathcal{P}^W$ corresponds to the $\overline{\mathcal{B}}$-module $\overline{\mathcal{P}}^W$. More precisely,*

$$\psi(Xh) = \Psi(X)\psi(h) \quad \textit{for all } X \in \mathcal{B}, h \in \mathcal{P}^W. \tag{8.10}$$

Proof. The relation $\Psi(XY) = \Psi(X)\Psi(Y)$ has to be proven only for $X \in \mathcal{B}_d$, $Y \in \mathcal{B}_e$. But then it follows from $\mathcal{B}_d\mathcal{B}_e \subseteq \mathcal{B}_{d+e}$. Similarly, for (8.9) we may assume $X \in \mathcal{B}_d$ and $h \in \mathcal{P}_e$. Then it follows from $\mathcal{B}_d\mathcal{P}_e \subseteq \mathcal{P}_{d+e}$. $\square$

In view of this theorem it is probably more adequate to call $\overline{\mathcal{B}}$ the differential "picture" as opposed the differential "limit" of $\mathcal{B}$. It shows that the difference operators are just represented differently namely by differential operators.

Next, we study the maps ψ and Ψ more closely. Given $h \in \mathcal{P}^W$, there are two ways to produce an element of $\overline{\mathcal{P}}^W$: first $\bar{h}$, its top homogeneous component, and then $\psi(h)$. We have $\psi(h) = \bar{h}$ precisely if h is an E-eigenvector. Therefore, consider $\bar{p}_\lambda$, the top homogeneous component of p_λ. These polynomials are also of high representation theoretic interest. (See, e.g., [7]. In the classical case they are the Jack polynomials.) They form a basis of $\overline{\mathcal{P}}^W$. Since p_λ is an E-eigenvector we could *define* ψ by the property $\psi(p_\lambda) = \bar{p}_\lambda$.

The same thing works for $\mathcal{B}$: every $X \in \mathcal{B}$ gives rise to two elements in $\overline{\mathcal{B}}$ namely its top homogeneous component $\overline{X}$ and $\Psi(X)$. Moreover, $\Psi(X) = \overline{X}$ if and only if X is an $\operatorname{ad} E$-eigenvector. This holds in particular for $\mathcal{B}_0$, the commutant of E. Hence we have $\Psi(D_h) = \overline{D}_h$ where $\overline{D}_h$ are certain differential operators. In the classical case, they are the Sekiguchi–Debiard operators [3,18]. They are simultaneously diagonalized by the $\bar{p}_\lambda$:

$$\overline{D}_h(\bar{p}_\lambda) = h(\rho + \lambda)\bar{p}_\lambda \quad \text{for all } h \in \mathcal{P}^W. \tag{8.11}$$

Next, we compute the image of the sl_2-triple $(-L, 2E, L^*)$.

Proposition 8.4. *We have $\Psi(L) = m_{\bar{\ell}}$ (multiplication by $\bar{\ell} \in \overline{\mathcal{P}}^W$) and $\Psi(E) = \overline{E} = \xi + \ell(\rho)$ where ξ is the Euler vector field. The differential operator $\overline{L}^* := \Psi(L^*)$ is of order* 2 *and of degree* -1.

Proof. We have $L \in \mathcal{B}_1$, hence $\Psi(L) = \overline{L}$. From $\deg(m_\ell - L) = \deg E < 1$ it follows $\overline{L} = \overline{m_\ell} = m_{\bar{\ell}}$. Since $\overline{E}$ acts on $\overline{\mathcal{P}}_d^W$ by multiplication with $d + \ell(\rho)$ we have $\Psi(E) = \overline{E} = \xi + \ell(\rho)$. Since $L^* \in \mathcal{B}_{-1}$, the degree of $\overline{L}^*$ is -1. Expand L_t^* as a Laurent series in t as in the proof of Proposition 8.1. Since the coefficients of $L^* = L - 2\ell - L^-$ are rational functions of degree 1 we have

$$L_t^* = X_0 t^{-1} + X_1 + X_2 t + \cdots \tag{8.12}$$

where X_i is homogeneous of degree $1 - i$. Thus $X_0 = X_1 = 0$ and $X_2 = \overline{L}*$. By construction, X_i is a differential operator of order i. Therefore, the order of $\overline{L}^*$ is 2. $\square$

Now we compare the multiplication operators in $\mathcal{B}$ and $\overline{\mathcal{B}}$.

Theorem 8.5. (a) *Let $h \in \mathcal{P}^W$. Then*

$$\Psi(m_h) = \exp\bigl(-\operatorname{ad}\bar{\ell}\bigr)\bigl(\overline{D}_h\bigr). \tag{8.13}$$

(b) *Conversely, let $\bar{h} \in \overline{\mathcal{P}}_d^W$ and choose a lift $h \in \mathcal{P}_{\leqslant d}^W$. Then*

$$\Psi^{-1}(m_{\bar{h}}) = \frac{1}{d!}(-\operatorname{ad}L)^d(m_h). \tag{8.14}$$

Proof. (a) We have

$$m_h = \exp(-\operatorname{ad}L)(D_h) = \sum_i \frac{1}{i!}(-\operatorname{ad}L)^i(D_h). \tag{8.15}$$

Each summand is $\operatorname{ad}E$-homogeneous. Therefore,

$$\begin{aligned}\Psi(m_h) &= \sum_i \frac{1}{d!}\overline{(-\operatorname{ad}L)^d(D_h)} = \sum_i \frac{1}{d!}\bigl(-\operatorname{ad}\overline{L}\bigr)^d\bigl(\overline{D}_h\bigr)\\ &= \exp\bigl(-\operatorname{ad}\bar{\ell}\bigr)\bigl(\overline{D}_h\bigr).\end{aligned} \tag{8.16}$$

(b) Let R denote the right hand side of (8.14). The sum in (8.15) terminates at $i = d$. Moreover, the ith summand is $\operatorname{ad}E$-homogeneous of degree i. This implies $\Psi(R) = \overline{R} = \overline{m_h} = m_{\bar{h}}$. □

Thus we obtained besides the m_h and the D_h yet another commutative subalgebra of $\mathcal{B}$ formed by the $\Psi^{-1}(m_{\bar{h}})$.

Finally, we discuss the geometric situation: let U be a multiplicity free space as in Section 2. Since $\mathcal{P}^W$ can be identified with the algebra of G-invariant differential operators on U we can use the symbol map to identify $\overline{\mathcal{P}}^W$ with the algebra of G-invariant functions on the cotangent bundle, i.e., with $\mathcal{P}(U \oplus U^\vee)^G$. On the other hand we can think of $\overline{\mathcal{P}}^W$ as W-invariant functions on V, i.e., of functions on V/W.

Now consider $\mathcal{PD}(U \oplus U^\vee)^G$, the algebra of G-invariant differential operators on $U \oplus U^\vee$. These act on G-invariants and therefore we get a map

$$\Phi : \mathcal{PD}\bigl(U \oplus U^\vee\bigr)^G \to \mathcal{PD}(V/W). \tag{8.17}$$

(This is an analogue of the Harish Chandra homomorphism.) Observe that $\overline{\mathcal{B}} \subseteq \mathcal{PD}(V/W)$.

Theorem 8.6. *The algebra $\overline{\mathcal{B}}$ is in the image of Φ.*

Proof. The algebra $\mathcal{B}$ is generated by L, $\{m_h \mid h \in \mathcal{P}^W\}$, and L^*. Because of (8.15) we can replace m_h by D_h. Applying Ψ, we see that $\overline{\mathcal{B}}$ is generated by $m_{\bar{\ell}}$, $\{\overline{D}_h \mid h \in \mathcal{P}^W\}$, and $\overline{L}^*$. We show that these generators lie in the image of Φ.

Choose coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of $U \oplus U^\vee$ such that the natural pairing between U and $U^\vee$ is given by $q := \sum_i x_i y_i$. Then q is the symbol of the Euler vector field and therefore $\Phi(q) = \bar{\ell}$.

We have $\mathcal{PD}(U)^G \hookrightarrow \mathcal{PD}(U \oplus U^\vee)^G$ by letting operators act on the first factor. Thus we have a map $\mathcal{P}^W \to \mathcal{PD}(V/W)$ whose image are the differential operators $\overline{D}_h$ (see [7, Theorem 4.11]).

Finally, let $\Delta := -\sum_i (\partial^2/(\partial x_i \partial y_i))$ be the Laplace operator. Then it follows from [2, (1.8)][9] that $\Phi(\Delta)$ acts on the $\bar{q}_\lambda$ exactly as $\overline{L}^*$. $\square$

Question. Is the image of Φ exactly $\overline{\mathcal{B}}$?

9. The binomial formula

In this section we investigate another limiting case, namely, we are looking at the infinitesimal neighborhood of a point $\delta \in \Sigma^\vee \cap V^W$. We are going to prove a binomial type formula for $\bar{p}_\lambda(z + \delta)$.

The set $\Sigma^\vee \cap V^W$ has usually just one element but there are cases where it is empty[10] and there is one case where it consists of two points.[11] For the classical or semiclassical case see the example below.

Let $\omega_\delta \in \Sigma$ be the dual element for δ, i.e.,

$$\omega(\delta) = \begin{cases} 1 & \text{if } \omega = \omega_\delta, \\ 0 & \text{if } \omega \in \Sigma \text{ and } \omega \neq \omega_\delta. \end{cases} \tag{9.1}$$

Since δ is W-invariant we have $\omega(\delta) \in \{0, 1\}$ for all $\omega \in \Phi$. Moreover, $\omega(\delta) = 1$ if and only if $\omega \in W\omega_\delta$. This implies $\delta \in V_0$. Let $\ell_\delta := \sum W\omega_\delta$ and $\ell^\delta := \ell - \ell_\delta$. These are W-invariant linear functions on V.

Examples. 1. *Classical case*: Here $\delta = (1, \ldots, 1)$, $\omega_\delta(z) = z_n$, $W\omega_\delta = \{z_1, \ldots, z_n\}$, $l_\delta(z) = \sum_i z_i = \ell(z)$, and $\ell^\delta(z) = 0$.

2. *Semiclassical case*: Here $\delta = (1, \ldots, 1)$, $\omega_\delta(z) = z_n$, $W\omega_\delta = \{z_i \mid n - i \text{ even}\}$, $l_\delta(z) = \sum_{i:n-i \text{ even}} z_i$, and

$$\ell^\delta(z) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \sum_{i=1}^n (-1)^{i-1} z_i & \text{if } n \text{ is even.} \end{cases} \tag{9.2}$$

9 The p_α in that paper is our $\bar{q}_\alpha$.

10 Cases III (n odd), IVa, and IVc of [9, Section 8].

11 Case V of [9, Section 8].

For $\lambda \in \Lambda_+$ let

$$c_\lambda^{(\delta)} = c_\lambda^{(\delta)}(\rho) := \prod_{\omega \in W\omega_\delta} \big(\omega(\rho) + k_\omega\big)_{\omega(\lambda)}, \tag{9.3}$$

where $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer symbol. Up to a sign, this is just the contribution of $W\omega_\delta$ to $f_\lambda(-\rho)$. Now we renormalize $\bar{p}_\lambda$ as follows:

$$\bar{q}_\lambda^{(\delta)}(z) := \frac{c_\lambda^{(\delta)}}{d_\lambda} \bar{p}_\lambda(z) = (-1)^{\ell(\lambda)} c_\lambda^{(\delta)} \bar{q}_\lambda(z). \tag{9.4}$$

Then the generalized *binomial formula* is:

Theorem 9.1. *Let $\delta \in \Sigma^\vee \cap V^W$. Then*

$$\bar{q}_\lambda^{(\delta)}(z+\delta) = \sum_{\substack{\mu \in \Lambda_+ \\ \ell^\delta(\mu) = \ell^\delta(\lambda)}} p_\mu(\rho+\lambda)\, \bar{q}_\mu^{(\delta)}(z) \quad \textit{for every } \lambda \in \Lambda_+. \tag{9.5}$$

Proof. To emphasize dependence on ρ we will also write $p_\lambda(z;\rho)$, etc. Let $\rho' := \rho + (s/2)\delta$ with $s \in \mathbb{C}$. Then it follows from the definitions that

$$p_\lambda(z;\rho') = p_\lambda\left(z - \frac{s}{2}\delta; \rho\right) \quad \text{and} \quad f_\lambda(z;\rho') = f_\lambda\left(z - \frac{s}{2}\delta; \rho\right). \tag{9.6}$$

Hence

$$\begin{aligned} q_\lambda(z;\rho') &= \frac{f_\lambda(\rho'+\lambda;\rho')}{f_\lambda(-\rho';\rho')} p_\lambda(z;\rho') = \frac{f_\lambda(\rho+\lambda)}{f_\lambda(-\rho-s\delta)} p_\lambda\left(z - \frac{s}{2}\delta\right) \\ &= \frac{f_\lambda(-\rho)}{f_\lambda(-\rho-s\delta)} q_\lambda\left(z - \frac{s}{2}\delta\right). \end{aligned} \tag{9.7}$$

Since the contributions of $\omega \in \Phi \setminus W\omega_\delta$ and $\alpha \in \Delta$ cancel out, we have

$$\frac{f_\lambda(-\rho)}{f_\lambda(-\rho-s\delta)} = \frac{c_\lambda^{(\delta)}(\rho)}{c_\lambda^{(\delta)}(\rho+s\delta)}. \tag{9.8}$$

Now we apply the transposition formula (4.6) with ρ' instead of ρ. We also replace z by $z + (s/2)\delta$. Then we obtain:

$$c_\lambda^{(\delta)}(\rho) q_\lambda(-z-s\delta) = \sum_\mu (-1)^{\ell(\mu)} p_\mu(\rho+\lambda) \frac{c_\lambda^{(\delta)}(\rho+s\delta)}{c_\mu^{(\delta)}(\rho+s\delta)} c_\mu^{(\delta)}(\rho) q_\mu(z). \tag{9.9}$$

Let t be a formal parameter. In Equation (9.9), we replace z, s by $t^{-1}z$, t^{-1}, respectively, and multiply by $t^{\ell(\lambda)}$. Thus, we get

$$c_\lambda^{(\delta)}(\rho)t^{\ell(\lambda)}q_\lambda\bigl(-t^{-1}z-t^{-1}\delta\bigr) = \sum_\mu p_\mu(\rho+\lambda)A_\mu(t)\bigl[(-1)^{\ell(\mu)}c_\mu^{(\delta)}(\rho)t^{\ell(\mu)}q_\mu\bigl(t^{-1}z\bigr)\bigr] \tag{9.10}$$

with

$$A_\mu(t) := t^{\ell(\lambda-\mu)}\frac{c_\lambda^{(\delta)}(\rho+t^{-1}\delta)}{c_\mu^{(\delta)}(\rho+t^{-1}\delta)}. \tag{9.11}$$

Now, we take the limit for $t\to 0$. The left hand side of (9.10) becomes $\bar{q}_\lambda^{(\delta)}(z+\delta)$ while the expression in brackets on the right hand side tends to $\bar{q}^{(\delta)}(z)$. Finally, we have

$$\bigl(\omega\bigl(\rho+t^{-1}\delta\bigr)+k_\omega\bigr)_{\omega(\lambda)} = \bigl(t^{-1}+\omega(\rho)+k_\omega\bigr)_{\omega(\lambda)} = t^{-\omega(\lambda)}+\cdots \tag{9.12}$$

where again "$\cdots$" means "terms of higher order in t." Thus

$$c_\lambda^{(\delta)}\bigl(\rho+t^{-1}\delta\bigr) = t^{-\ell_\delta(\lambda)}+\cdots \tag{9.13}$$

and

$$A_\mu(t) = t^{\ell^\delta(\lambda-\mu)}+\cdots. \tag{9.14}$$

By the Extra Vanishing Theorem 3.2 only those μ in (9.10) have to be considered for which $\tau := \lambda-\mu\in\Lambda$. Thus the binomial formula (9.5) is proved when we show that $\ell^\delta(\tau)\geqslant 0$ for all $\tau\in\Lambda$.

Since ℓ^δ is linear we may assume $\ell(\tau)=1$ since those τ's generate Λ. Because ℓ^δ is W-invariant, we may moreover assume that $\tau\in\Sigma^\vee$. Now consider formula (9.10) with $\lambda=\tau$. Then the right-hand side has only two non-vanishing terms summands, corresponding to $\mu=\tau$ and $\mu=0$. Thus

$$c_\tau^{(\delta)}(\rho)tq_\tau\bigl(-t^{-1}z-t^{-1}\delta\bigr) = A_0(t)-A_\tau(t)c_\tau^{(\delta)}(\rho)tq_\tau\bigl(t^{-1}z\bigr). \tag{9.15}$$

Since the limit $\lim_{t\to 0}A_0(t)$ exists and $q_\tau(z)$ is a non-constant polynomial of degree 1 also $\lim_{t\to 0}A_\tau(t)$ exists. Therefore $\ell^\delta(\tau)\geqslant 0$ by (9.14). □

Putting $z=0$, we get as an immediate consequence an evaluation formula:

Corollary 9.2. *For all $\lambda\in\Lambda_+$ and $\delta\in\Sigma^\vee\cap V^W$ holds*

$$\bar{q}_\lambda^{(\delta)}(\delta) = \begin{cases} 1 & \text{if } \ell^\delta(\lambda)=0,\\ 0 & \text{otherwise.}\end{cases} \tag{9.16}$$

Remark. Consider the classical case. Then the binomial formula (9.5) is due to Okounkov and Olshanski [15]. Before that, Lassalle [11] used the binomial formula to *define* the "generalized binomial coefficients" $p_\mu(\rho+\lambda)$. We see now that this was only possible because $\ell^\delta = 0$. For arbitrary multiplicity free actions, Yan [20] took another approach to define $p_\mu(\rho+\lambda)$ from the homogeneous polynomials $\bar{p}_\lambda$, namely via the formula

$$\frac{1}{k!}\ell(z)^k \bar{p}_\lambda(z) = \sum_{\substack{\mu\in\Lambda_+\\ \ell(\mu-\lambda)=k}} p_\lambda(\rho+\mu)\bar{p}_\mu(z) \tag{9.17}$$

which follows readily from (3.17). Yet another construction can be found in [2]. Observe though, that none of these approaches give the polynomiality nor the W-invariance of p_λ. Also the latter two constructions work only for those $\rho \in V_0$ which actually come from a multiplicity free action.

10. Example: the rank one case

In this section, we illustrate the main assertions of this paper with the rank one case.

Section 2. Let $G = GL_n(\mathbb{C})$ and $U = \mathbb{C}^n$, the defining representation. Then $\mathcal{P}^\lambda = S^\lambda(\mathbb{C}^n)^\vee$, the space of homogeneous polynomials of degree $\lambda \in \mathbb{N}$. The algebra of invariant differential operators is generated by $\xi = \sum_i x_i(\partial/\partial x_i)$, the Euler vector field. The eigenvalue of ξ on $\mathcal{P}^\lambda$ is λ, hence $c_\xi(z) = z$.

The parabolic P is the stabilizer of the line $\mathbb{C}e_1 \subseteq \mathbb{C}^n$. Denote the weights of $\mathbb{C}^n$ by ε_i. Then the roots in the unipotent radical of P are $\varepsilon_1 - \varepsilon_i$. Thus,

$$\rho = \frac{1}{2}\left(\sum_{i=2}^n(\varepsilon_1-\varepsilon_i) + \sum_{i=1}^n \varepsilon_i\right) = \frac{n}{2}\varepsilon_1. \tag{10.1}$$

Thus $p_\xi(z) = z - n/2$. On the other hand, we have

$${}^t\xi = \sum_{i=1}^n\left(-\frac{\partial}{\partial x_i}\right)x_i = -\sum_{i=1}^n\left(x_i\frac{\partial}{\partial x_i}+1\right) = -\xi - n. \tag{10.2}$$

Thus

$$p_{{}^t\xi}(z) = p_{-\xi-n}(z) = -\left(z-\frac{n}{2}\right) - n = -z - \frac{n}{2} = p_\xi(-z). \tag{10.3}$$

Section 3. In the rank one case we have

$$V = \mathbb{C},\quad W = 1,\quad \Lambda_+ = \mathbb{N},\quad \text{and}\quad \ell(z) = z. \tag{10.4}$$

Moreover,

$$\Sigma^\vee = \Lambda_1 = \{1\}, \quad \Phi = \Phi^+ = \Sigma = \{z\}, \quad \text{and} \quad \Delta = \emptyset. \tag{10.5}$$

We have $V_0 = \mathbb{C}$ and put $\rho = s$. Thus every ρ is non-integral while “strongly dominant” means $s \notin -\frac{1}{2}\mathbb{N}$.

The polynomial $p_\lambda \in \mathcal{P} = \mathbb{C}[z]$ vanishes in $z = s, s+1, \ldots, s+\lambda-1$ and is 1 in $z = s+\lambda$. There is indeed only one such polynomial, namely

$$p_\lambda(z) = \binom{z-s}{\lambda}. \tag{10.6}$$

We have $f_\tau(z) = [z - s \downarrow \tau]$ for $\tau \in \mathbb{N}$. Thus

$$L = (z-s)T \quad \text{and} \quad E = z - (z-s)T = (z-s)\nabla + s \tag{10.7}$$

where T is the shift operator $T(h)(z) = h(z-1)$ and $\nabla := 1 - T$. Then an easy calculation shows

$$D_h = \sum_{d=0}^{\infty} (-1)^d \binom{z-s}{d} \big(\nabla^d h\big)(z) T^d \quad \text{for all } h \in \mathbb{C}[z]. \tag{10.8}$$

The equation $E(p_\lambda) = (s+\lambda)p_\lambda$ is equivalent to the well-known relation

$$z\nabla\binom{z}{\lambda} = \lambda\binom{z}{\lambda} \tag{10.9}$$

while $D_h(p_\lambda) = h(s+\lambda)p_\lambda$ gives, after using (10.8) and some easy manipulations, Newton’s interpolation formula:

$$h(x+z) = \sum_{d=0}^{\infty} \frac{1}{d!} \big(\nabla^d h\big)(z)\,(x)_d \tag{10.10}$$

(we substituted $s + \lambda = x + z$). Here $(x)_d = x(x+1)\cdots(x+d-1)$ is the Pochhammer symbol. This can be used to rewrite formula (10.8). Since $T = 1 - \nabla$, we get

$$D_h = \sum_{0 \leqslant d \leqslant m} (-1)^m \binom{z-s}{m} \big(\nabla^m h\big)(z)(-1)^d \binom{m}{d} \nabla^d \tag{10.11}$$

$$= \sum_{d=0}^{\infty} \binom{z-s}{d} \left[\sum_{m=d}^{\infty} (-1)^{m-d} \binom{z-s-d}{m-d} \big(\nabla^m h\big)(z) \right] \nabla^d. \tag{10.12}$$

If we apply ∇_z^m on both sides of (10.10) and then substitute $x = s + d - z$ we get the expression in brackets of (10.12). Thus

$$D_h = \sum_{d=0}^{\infty} \big(\nabla^d h\big)(s+d)\binom{z-s}{d}\nabla^d. \tag{10.13}$$

Section 4. According to (3.13) we have

$$d_\lambda = (-1)^\lambda \frac{[-2s \downarrow \lambda]}{[\lambda \downarrow \lambda]} = (-1)^\lambda \binom{-2s}{\lambda} = \binom{2s-1+\lambda}{\lambda} \tag{10.14}$$

which affirms the evaluation formula (4.15). Moreover, in the geometric situation above with $GL_n(\mathbb{C})$ acting on $\mathbb{C}^n$ we check Theorem 4.8:

$$\dim S^\lambda(\mathbb{C}^n)^\vee = \binom{n-1+\lambda}{\lambda} \tag{10.15}$$

(since $s = n/2$). Furthermore,

$$q_\lambda(z) = \frac{[z-s \downarrow \lambda]}{[-2s \downarrow \lambda]} = \frac{(-z+s)_\lambda}{(2s)_\lambda}. \tag{10.16}$$

Thus, the transposition formula (4.6) reads

$$\frac{(z+s)_\lambda}{(2s)_\lambda} = \sum_{\mu=0}^{\lambda} (-1)^\mu \binom{\lambda}{\mu} \frac{(-z+s)_\mu}{(2s)_\mu}. \tag{10.17}$$

A direct proof boils down, after some manipulations, to the Chu–Vandermonde identity. Finally, the symmetry statement (4.11) becomes

$$\frac{(2s+\nu)_\lambda}{(2s)_\lambda} = \frac{(2s+\lambda)_\nu}{(2s)_\nu} \tag{10.18}$$

which is easily verified directly.

Section 5. The involutivity of the matrix (5.1)

$$\left((-1)^\mu \binom{\lambda}{\mu}\right)_{\lambda\mu} \tag{10.19}$$

is well known. The transformation $h \mapsto \widehat{h}$ can be rewritten as

$$\widehat{h}(s+\lambda) = \sum_{\mu=0}^{\lambda} (-1)^\mu \binom{\lambda}{\mu} h(s+\mu) = (-1)^\lambda \big(\Delta^\lambda h\big)(s) \tag{10.20}$$

where

$$\varDelta := T^{-1} - 1, \quad \text{i.e.,} \quad (\varDelta h)(z) = h(z+1) - h(z). \tag{10.21}$$

Then the interpolation formula (5.3) becomes another form of Newton interpolation (with $z = x + s$):

$$h(x+s) = \sum_{\mu=0}^{\infty} \bigl(\varDelta^{\mu} h\bigr)(s) \binom{x}{\mu}. \tag{10.22}$$

Section 6. The scalar product (6.5) is

$$\langle g, h\rangle = \sum_{\mu=0}^{\infty} \binom{2s-1+\mu}{\mu} \bigl(\varDelta^{\mu} g\bigr)(s) \bigl(\varDelta^{\mu} h\bigr)(s). \tag{10.23}$$

Section 7. We have

$$L^- = -(z+s)T^{-1}, \tag{10.24}$$

$$L^* = (z-s)T - 2z + (z+s)T^{-1} = z(\varDelta - \nabla) + s(\varDelta + \nabla). \tag{10.25}$$

Since $\mathcal{B}$ is the algebra generated by the sl_2-triple

$$\bigl((z-s)T, 2z, -(z+s)T^{-1}\bigr) \tag{10.26}$$

it is actually isomorphic to the universal enveloping algebra of $sl_2(\mathbb{C})$.

Section 8. We have

$$\bar{p}_\lambda(z) = \frac{z^\lambda}{\lambda!} \quad \text{and} \quad \bar{q}_\lambda(z) = \frac{(-1)^\lambda}{(2s)_\lambda} z^\lambda.$$

The algebra $\overline{\mathcal{B}}$ is generated by

$$\Psi(L) = z, \quad \Psi(E) = z\frac{\mathrm{d}}{\mathrm{d}z} + s, \quad \Psi(L^*) = z\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2s\frac{\mathrm{d}}{\mathrm{d}z}. \tag{10.27}$$

Moreover, according to (10.13):

$$\Psi(D_h) = \overline{D}_h = \sum_{m=0}^{\infty} \frac{1}{m!} \bigl(\nabla^m h\bigr)(s+m) z^m \frac{\mathrm{d}^m}{\mathrm{d}z^m}. \tag{10.28}$$

Section 9. We have $\delta = 1$, $c_\lambda^{(\delta)} = (2s)_\lambda$, and $\bar{q}_\lambda^{(\delta)}(z) = z^\lambda$. Thus, formula (9.5) just specializes to the classical binomial formula

$$(z+1)^\lambda = \sum_{\mu=0}^{\lambda} \binom{\lambda}{\mu} z^\mu. \tag{10.29}$$

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